Trigonometric Functions and the Chain Rule

The “Chain Rule versions” of the derivatives of the six trigonometric functions are shown below.

$$\frac{d}{dx}[\sin u] = (\cos u)\,u' \qquad \frac{d}{dx}[\cos u] = -(\sin u)\,u'$$

$$\frac{d}{dx}[\tan u] = (\sec^2 u)\,u' \qquad \frac{d}{dx}[\cot u] = -(\csc^2 u)\,u'$$

$$\frac{d}{dx}[\sec u] = (\sec u \tan u)\,u' \qquad \frac{d}{dx}[\csc u] = -(\csc u \cot u)\,u'$$

Example 10

The Chain Rule and Trigonometric Functions

a. $y = \sin \overbrace{2x}^{u}$ $\qquad y' = \overbrace{\cos 2x}^{\cos u}\,\overbrace{\frac{d}{dx}[2x]}^{u'} = (\cos 2x)(2) = 2\cos 2x$

b. $y = \cos \overbrace{(x-1)}^{u}$ $\qquad y' = \overbrace{-\sin(x-1)}^{-(\sin u)}\,\overbrace{\frac{d}{dx}[x-1]}^{u'} = -\sin(x-1)$

c. $y = \tan \overbrace{3x}^{u}$ $\qquad y' = \overbrace{\sec^2 3x}^{(\sec^2 u)}\,\overbrace{\frac{d}{dx}[3x]}^{u'} = (\sec^2 3x)(3) = 3\sec^2 3x$

Be sure you understand the mathematical conventions regarding parentheses and trigonometric functions. For instance, in Example 10(a), $\sin 2x$ is written to mean $\sin(2x)$.

Example 11

Parentheses and Trigonometric Functions

a. $y = \cos 3x^2 = \cos(3x^2)$ $\quad y' = (-\sin 3x^2)(6x) = -6x \sin 3x^2$

b. $y = (\cos 3)\,x^2$ $\quad y' = (\cos 3)\,(2x) = 2x \cos 3$

c. $y = \cos(3x)^2 = \cos(9x^2)$ $\quad y' = (-\sin 9x^2)\,(18x) = -18x \sin 9x^2$

d. $y = \cos^2 x = (\cos x)^2$ $\quad y' = 2\,(\cos x)\,(-\sin x) = -2 \cos x \sin x$

e. $y = \sqrt{\cos x} = (\cos x)^{1/2}$ $\quad y' = \dfrac{1}{2}(\cos x)^{-1/2}(-\sin x) = -\dfrac{\sin x}{2\sqrt{\cos x}}$

Remark

Another way to write the derivative in Example 11(d) is to use the double angle identity, $2 \sin x \cos x = \sin 2x$. Applying this identity, the result is

$$\begin{aligned} y' &= -2 \cos x \sin x \\ &= -(2 \sin x \cos x) \\ &= -\sin 2x. \end{aligned}$$

To find the derivative of a function of the form $k(x) = f\,(g\,(h\,(x)))$, you need to apply the Chain Rule twice, as shown in Example 12.

Example 12

Repeated Application of the Chain Rule

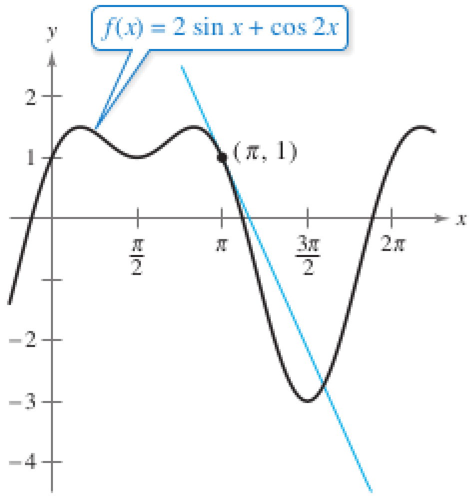
$$\begin{aligned} f\,(t) &= \sin^3 4t && \text{Original function} \\ &= (\sin 4t)^3 && \text{Rewrite.} \\ f'(t) &= 3(\sin 4t)^2 \frac{d}{dt}[\sin 4t] && \text{Apply Chain Rule once.} \\ &= 3(\sin 4t)^2 (\cos 4t) \frac{d}{dt}[4t] && \text{Apply Chain Rule a second time.} \\ &= 3(\sin 4t)^2 (\cos 4t)\,(4) && \\ &= 12 \sin^2 4t \cos 4t && \text{Simplify.} \end{aligned}$$

Example 13

Tangent Line of a Trigonometric Function

Find an equation of the tangent line to the graph of $f(x) = 2 \sin x + \cos 2x$ at the point $(\pi, 1)$, as shown in Figure 2.26. Then determine all values of x in the interval $(0, 2\pi)$ at which the graph of f has a horizontal tangent.

Figure 2.26



Solution

Begin by finding $f'(x)$.

$$f(x) = 2 \sin x + \cos 2x \qquad \text{Write original function.}$$

$$f'(x) = 2 \cos x + (-\sin 2x)(2) \qquad \text{Apply Chain Rule to } \cos 2x.$$

$$= 2 \cos x - 2 \sin 2x \qquad \text{Simplify.}$$

To find the slope of the tangent line at $(\pi, 1)$, evaluate $f'(\pi)$.

$$f'(\pi) = 2 \cos \pi - 2 \sin 2\pi \qquad \text{Substitute.}$$

$$= -2 \qquad \text{Slope of tangent line at } (\pi, 1)$$

Now, use the point-slope form of the equation of a line to write

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$

$$y - 1 = -2(x - \pi) \qquad \text{Substitute for } y_1, m, \text{ and } x_1.$$

$$y = -2x + 1 + 2\pi. \qquad \text{Equation of tangent line at } (\pi, 1)$$

Note that $f'(x) = 0$ in the interval $(0, 2\pi)$ when $x = \pi/6, \pi/2, 5\pi/6$, and $3\pi/2$. So, the graph of f has horizontal tangents at $x = \pi/6, \pi/2, 5\pi/6$, and $3\pi/2$.

This section concludes with a summary of the differentiation rules studied so far. To become skilled at differentiation, you should learn each rule in words, not symbols. As an aid to memorization, note that the cofunctions (cosine, cotangent, and cosecant) require a negative sign as part of their derivatives.

Summary of Differentiation Rules

General Differentiation Rules

Let c be a real number, let n be a rational number, let u and v be differentiable functions of x, and let f be a differentiable function of u.

Constant Rule:

$$\frac{d}{dx}[c] = 0$$

(Simple) Power Rule:

$$\frac{d}{dx}[x^n] = nx^{n-1},$$
$$\frac{d}{dx}[x] = 1$$

Constant Multiple Rule:

$$\frac{d}{dx}[cu] = cu'$$

Sum or Difference Rule:

$$\frac{d}{dx}[u \pm v] = u' \pm v'$$

Product Rule:

$$\frac{d}{dx}[uv] = uv' + vu'$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$$

Chain Rule:

$$\frac{d}{dx}[f(u)] = f'(u)\,u'$$

General Power Rule:

$$\frac{d}{dx}[u^n] = nu^{n-1}u'$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\cos x] = -\sin x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$