Modern Portfolio Theory

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Risk and Variation

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Return Notation

notation	description	formula	example
r^i	return rate of asset i		
r^f	risk-free return rate		
$\tilde{r}^i$	excess return rate of asset i	$r^i - r^f$	

Return Notation Continued

Table: Portfolio example

	return	allocation weight
bonds	r^b	w
stocks	r^s	$1 - w$

Table: Return statistics notation

mean	variance	correlation
μ	σ^2	ρ

Where weight allocation is the percentage of each holding comprises in an investment portfolio

Portfolio Return Stats

$$\mu^p = w\mu^b + (1 - w)\mu^s$$

$$\sigma_p^2 = w^2\sigma_b^2 + (1 - w)^2\sigma_s^2 + 2w(1 - w)\rho\sigma_s\sigma_b$$

Imperfect Correlation

- The volatility function is convex

$$\sigma_\rho < w\sigma_b + (1 - w)\sigma_s$$

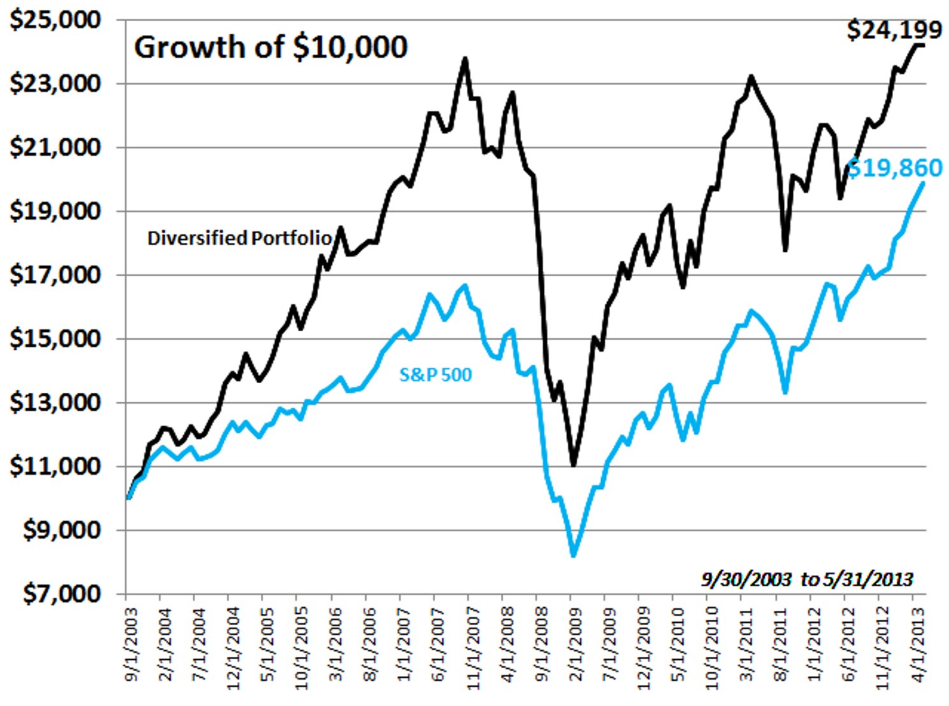
- Yet the mean return is still linear in the portfolio allocation

$$\mu^\rho = w\mu^b + (1 - w)\mu^s$$

Diversification

Portfolio diversification refers to the case where:

- Mean returns are linear in allocations
- While volatility of returns is less than linear in a
- However, this is only required when p < 1
  - Opportunity to diversify portfolio risk away



Portfolio Variance as Average Covariances

- Suppose that asset returns have
  - Identical volatilities
  - Identical correlations

$$Correlation = \frac{Cov\,(x, y)}{\sigma x * \sigma y}$$

- Using the following notation for averaging variances and covariances across the n assets

$$\overline{\sigma_i}^2 \equiv \frac{1}{n} \sum_{i=1}^{n} \sigma_i^2$$

$$\overline{\sigma_{i,j}} \equiv \frac{1}{n(n-1)} \sum_{j \neq i} \sum_{i=1}^{n} \sigma_{i,j}$$

Portfolio Variance Decomposition

If we take an equally weighted portfolio, $w_i = 1/n$.

$$\sigma_p^2 = \frac{1}{n}\overline{\sigma_i}^2 + \frac{n-1}{n}\overline{\sigma_{i,j}}$$

We now note that variance has a term that can be diversified to zero, and another term that remains

Systematic Risk

$$\sigma_p^2 = \frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2$$

$$\lim_{n\to\infty} \sigma_p^2 \to \underbrace{\rho\sigma^2}_{\text{systematic}}$$

- A fraction, p, of the variance is systematic

- No amount of diversification can get the portfolio variance lower

$$\sigma_p^2 \geq \rho\sigma^2$$

1. The inequality at the bottom holds for any n and any set of allocation weights w [i]

Idiosyncratic Risk

$$\sigma_p^2 = \frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2$$

- Idiosyncratic risk refers to the diversifiable part of $\sigma_p{}^2$
- An equally weighted portfolio has idiosyncratic risk equal to $\sigma^2/2$
- For general weights, w^i, remaining idiosyncratic is risk bounded my $\max_i w^i\sigma^2$

Correlation and Diversified Portfolios

- For p =1, there is no possible diversification, regardless of n.

$$\sigma_\rho^2 = \sigma^2$$

- For p = 0, there is no systematic risk, only variance or idiosyncratic risk is remaining

$$\sigma_\rho^2 = \frac{1}{n}\sigma^2$$

- In this case, as n get large, the portfolio can theoretically become riskless

$$\lim_{n\to\infty} \sigma_\rho^2 = 0$$

Portfolio Irrelevance of Individual Security Variance

- As the number of securities in portfolio, n, gets large,

$$\lim_{n \to \infty} \sigma_p^2 = \overline{\sigma_{i,j}}$$

- Individual security variance is unimportant and therefore, overall portfolio variance is the average of individual security covariance

Mean Variance

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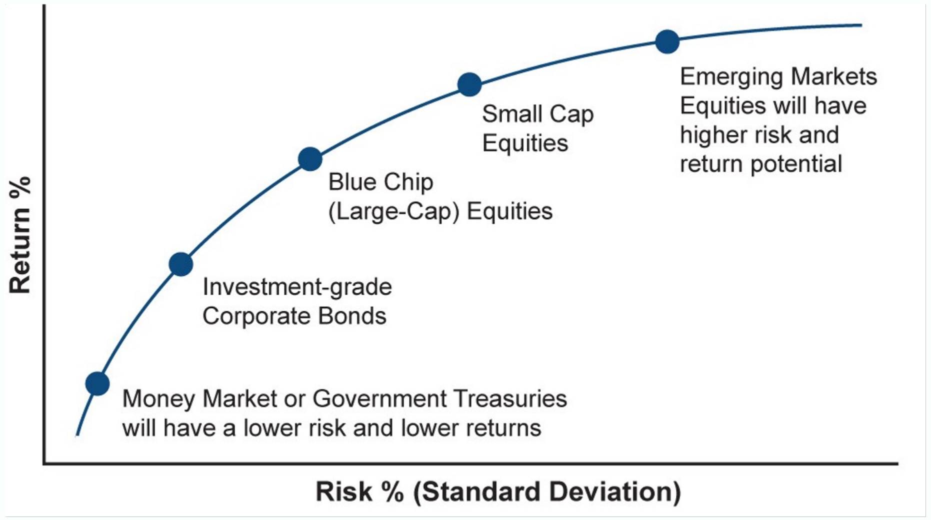
Mean-Variance Portfolio Allocation

- We want to create a portfolio that given any N assets, it optimizes the risk to return profile of that portfolio
  - Mean excess returns as a measurement of portfolio benefit
  - Average variance to measure risk

$$\text{Sharpe Ratio} = \frac{\mu^p - r^f}{\sigma^p} = \frac{\tilde{\mu}^p}{\sigma^p}$$



Risk and Return Tradeoff

- Traditional risk and return analysis states that higher return must always equal higher risk
- According to modern portfolio theory, risk and return tradeoff follows a hyperbolic path

Figure 1: Traditional Risk-Return Tradeoff

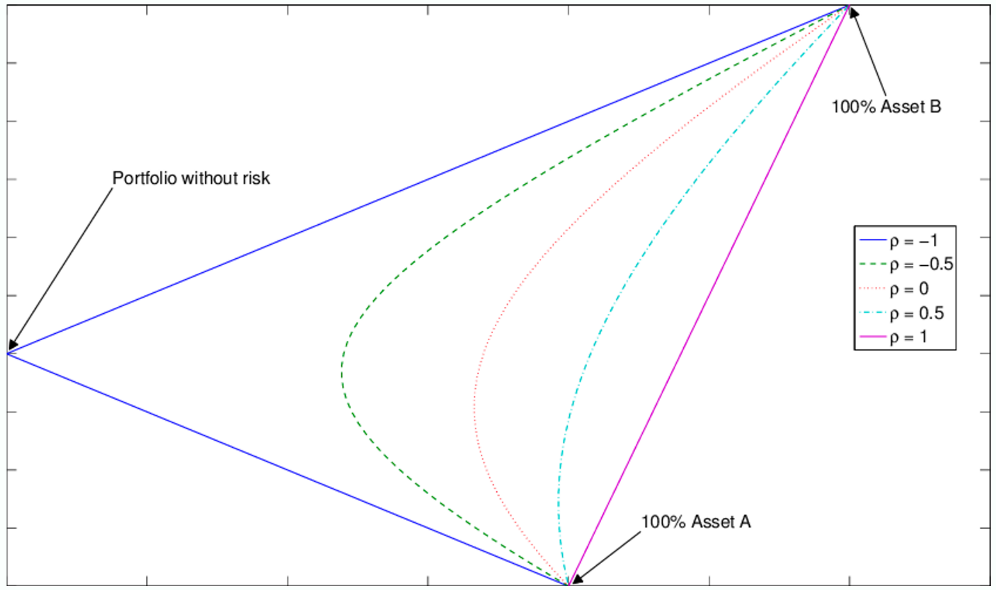
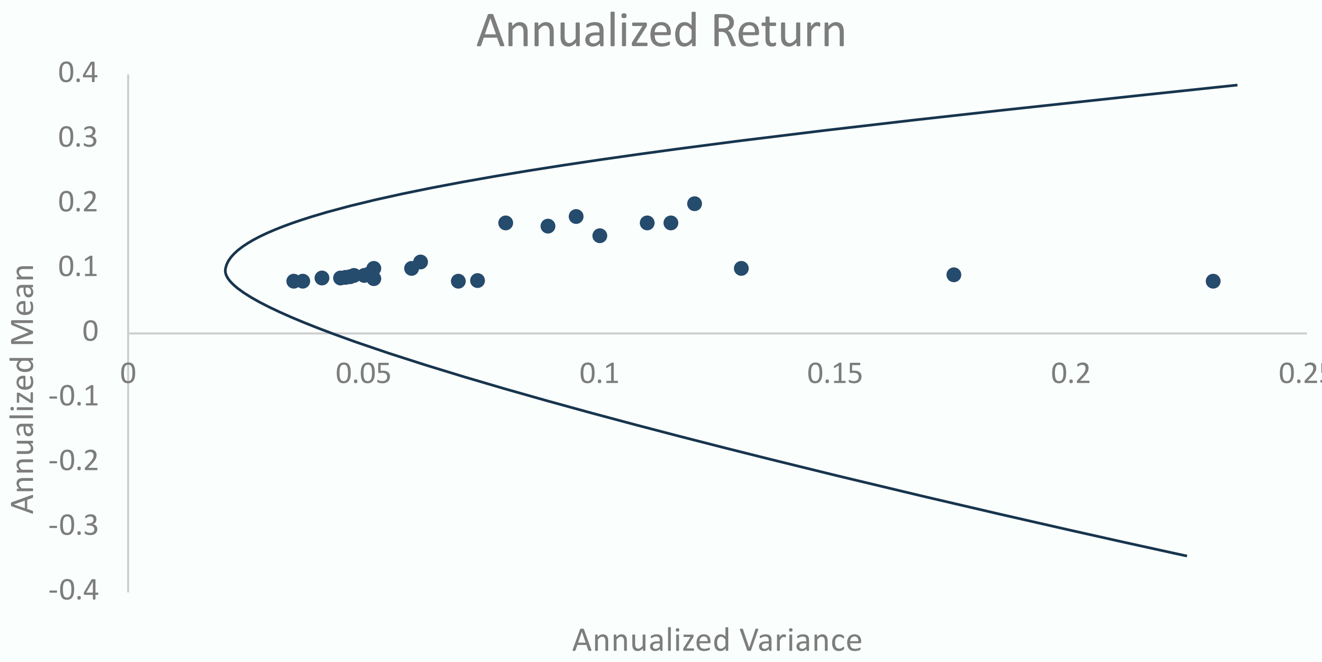


Figure 2: MPT Risk-Return Tradeoff

Diversification Across N Assets

With n securities, there is further potential for diversification

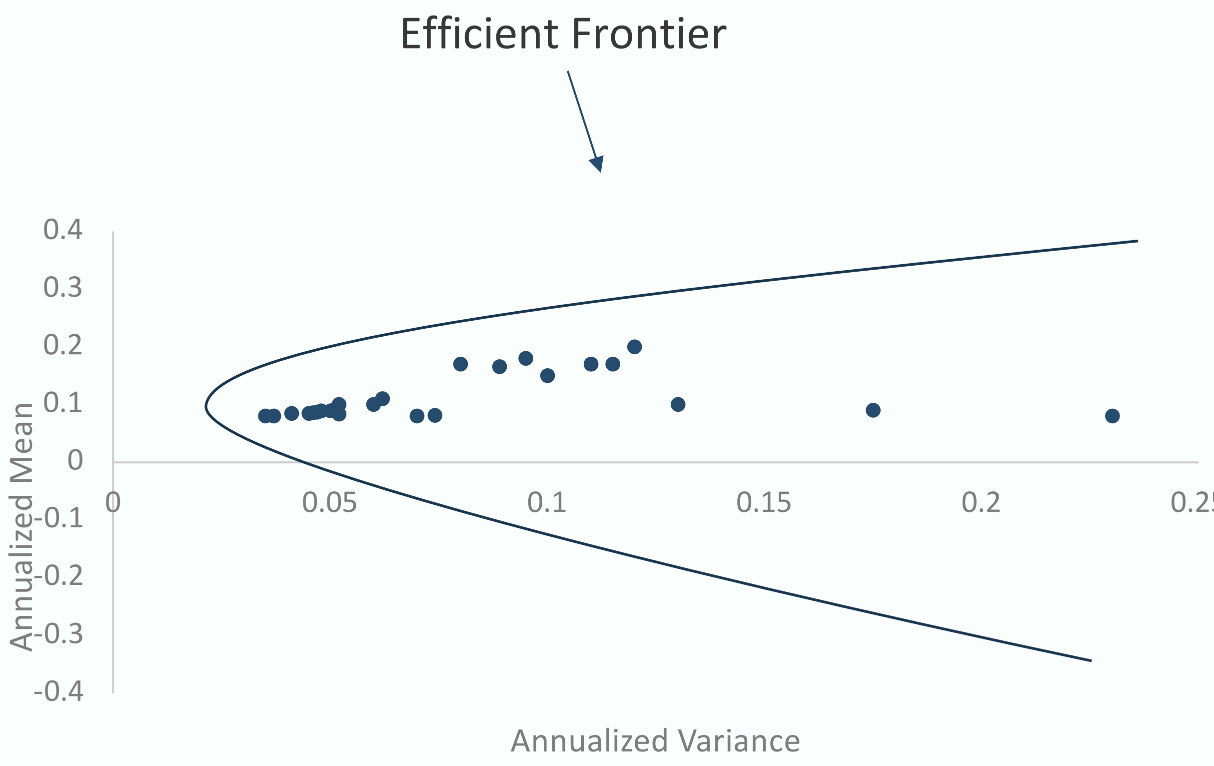
- A portfolio of $n > 1$ assets can be adjusted to result in different in-sample risk-return characteristics
- The set of all possible portfolios formed from this basis of assets forms a convex set in mean-variance space.
- The boundary of this set is known as the mean-variance frontier, and it forms a hyperbola.



Efficient Portfolios

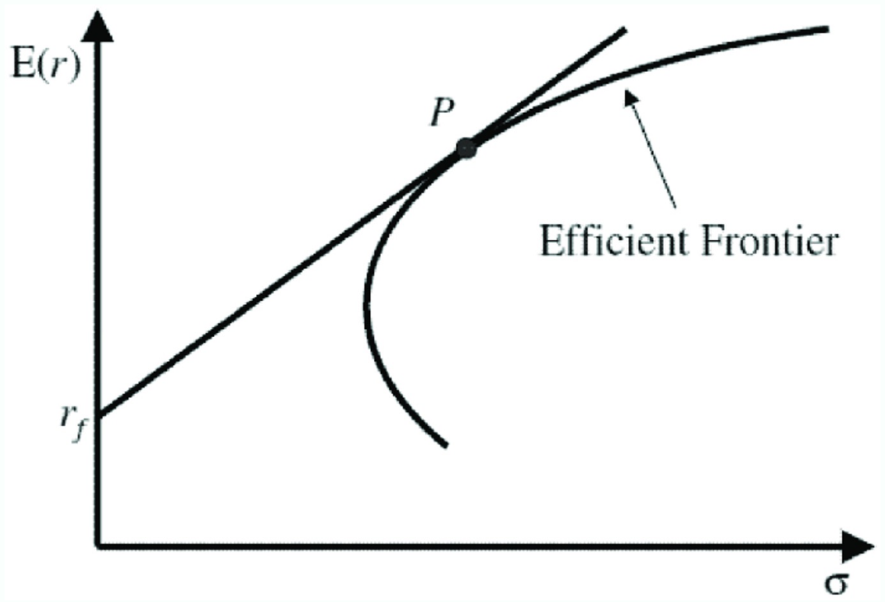
The top segment of the MV frontier is the set of efficient MV portfolios.

- Portfolios on this frontier have the maximum return at a given variance
  - You want to weight assets in your portfolio to have an overall portfolio on this efficient frontier
- Contrast this with the lower segment of the MV frontier, the inefficient MV portfolios.
  - The inefficient MV portfolios minimize mean return given the return variance.



Tangency Portfolio

- Assuming there exists a risk-free rate, there exists a portfolio on the efficient frontier that optimizes the in-sample portfolio Sharpe ratio.
  - This portfolio is the point where the capital market line is tangent to the efficient frontier
  - Capital market line shows risk-return tradeoff for MV investors
  - Slope of the Capital Market Line is the maximal Sharpe ratio which can be achieved by any portfolio
- This portfolio is called the tangency portfolio and assumes you invest 100% of the portfolio into risky assets.



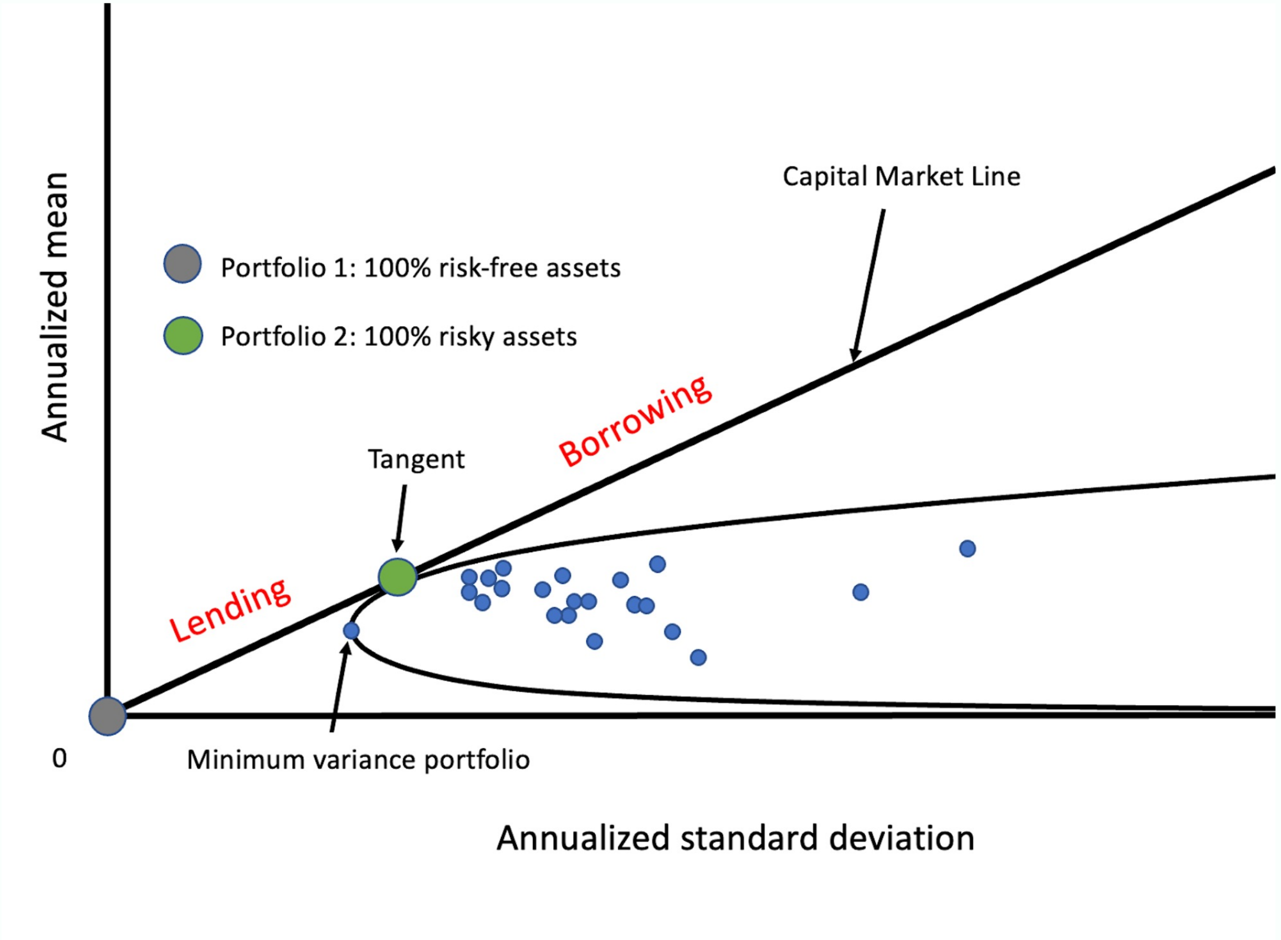
Two-Fund Separation

- Every Mean-Variance portfolio is the combination of the risky portfolio with the maximal Sharpe Ratio and the risk-free rate
- Thus, for a Mean-Variance investor, the asset allocation decision can be broken into two parts:
  - Find the tangency portfolio of risky assets, w_t
  - Choose an allocation between the risk-free rate and the tangency portfolio

Intuition of Asset Allocation

- The two-fund separation says that:
  - Any investment in risky assets should be in the tangency portfolio since it offers the maximum Sharpe Ratio.
  - One must decide the desired level of risk in the investment, which determines the split between the riskless asset and the tangency portfolio.

Note: Lending portion assumes a combination of the risk-free asset and the risky assets to form the portfolio. You're "lending" to the provider of the risk-free asset by incorporating it into your portfolio. Borrowing is the opposite; you're borrowing money from a riskless lender to invest more into risky assets to gain higher return through leverage but also higher risk



Notation

Suppose there are n risky assets

- r is an $n \times 1$ random vector. Each element is the return on one of the n assets.
- Let μ denote the $n \times 1$ vector of mean returns. Let Σ denote the $n \times n$ covariance matrix of returns.

$$\boldsymbol{\mu} = \mathbb{E}[\boldsymbol{r}]$$

$$\Sigma = \mathbb{E}\left[(\boldsymbol{r} - \boldsymbol{\mu})(\boldsymbol{r} - \boldsymbol{\mu})'\right]$$

$$\Sigma = \begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.6 \\ 0.4 & 0.5 & 0.2 & 0.8 \\ 0.2 & 0.8 & 0.5 & 0.9 \\ 0.7 & 0.3 & 0.3 & 0.4 \end{bmatrix} \end{array}$$

- Assume Σ is positive definite—no asset is a linear function of the others.

With a Riskless Asset

- There are n risky assets available, with returns as r
- An investor chooses a portfolio, defined as a $n \times 1$ vector of allocation weights, w, in those n risky assets
- Since the total portfolio allocations must add to one, we have allocation to the risk-free rate $= 1 - w'1$

Mean Excess Returns

- Let μp denote the mean return on a portfolio.

$$\mu^p = \left(1 - \mathbf{w}'\mathbf{1}\right) r^f + \mathbf{w}'\boldsymbol{\mu}$$

- Use the following notation for excess returns:

$$\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} - \mathbf{1} r^f$$

- Thus, the mean return and mean excess return of the portfolio are

$$\mu^p = r^f + \mathbf{w}'\tilde{\boldsymbol{\mu}}$$

$$\tilde{\mu}^p = \mathbf{w}'\tilde{\boldsymbol{\mu}}$$

- Let μ^p denote the mean return on a portfolio.

Variance of Return

- The risk-free rate has zero variance and zero correlation with any security.
- Let Σ continue to denote the n × n covariance matrix of risky assets
- The return variance of the portfolio, wp is

$$\sigma_p^2 = \mathbf{w}'\Sigma\mathbf{w}$$

Calculating Mean-Variance Portfolio Weightings

- w^t is a nx1 vector, where each value in the vector is the suggested weighting allocation to that individual risky asset.

$$\boldsymbol{w}^{\mathrm{t}} = \underbrace{\left(\frac{1}{\mathbf{1}'\Sigma^{-1}\tilde{\boldsymbol{\mu}}}\right)}_{\text{scaling}} \Sigma^{-1}\tilde{\boldsymbol{\mu}}$$

Two-Fund Allocation Adjustment

- A portfolio on the capital market line can be created through a scalar operation onto the tangency portfolio.

$$\boldsymbol{w}^* = \tilde{\delta}\ \boldsymbol{w}^{\mathrm{t}}$$

- Where the scalar value is calculated using the following formula

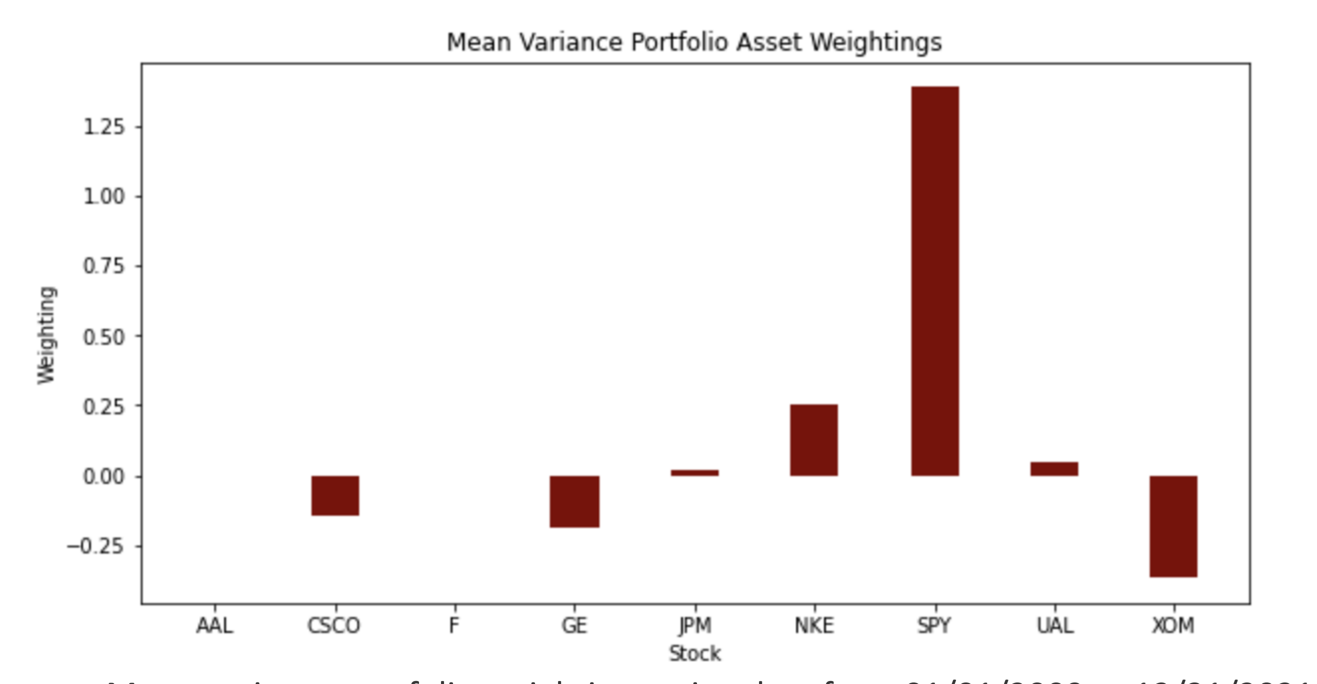
$$\tilde{\delta} = \left(\frac{\mathbf{1}'\Sigma^{-1}\tilde{\boldsymbol{\mu}}}{(\tilde{\boldsymbol{\mu}})'\Sigma^{-1}\tilde{\boldsymbol{\mu}}} \right) \tilde{\mu}^{p}$$

- The resulting weights give a portfolio that is on the capital market line and has an adjusted level of risk and return

Problems with Mean Variance

- Sensitized to In-Sample
- Market movements change and portfolio is not great out of sample
- Unrealistic weightings

AAL	-0.004421
CSCO	-0.146570
F	-0.000828
GE	-0.189041
JPM	0.018942
NKE	0.252251
SPY	1.390724
UAL	0.048748
XOM	-0.369806

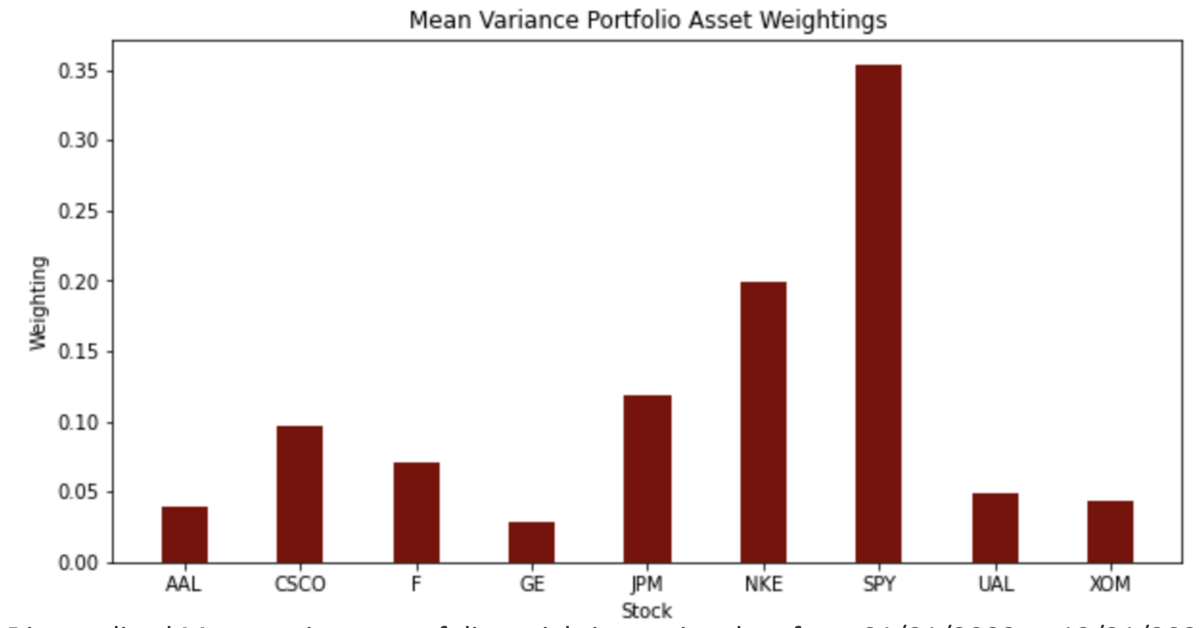


Mean-variance portfolio weightings using data from 01/01/2009 to 12/31/2021

Mean-Variance Alteration: Diagonalization

- Mean-variance relies on the pairwise covariance matrix as the measure of portfolio volatility
  - Using entire matrix makes it too sensitive to in-sample data
- You can diagonalize the covariance matrix to improve the allocation and performance out-of-sample
  - Only individual asset variances left (does not use covariance to other assets)
  - Less sensitive to in-sample data
  - No covariances to other assets means long-only weightings (formula can't use long-short paradigm to balance risk when there's no relational risk)

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.6 \\ 0.4 & 0.5 & 0.2 & 0.8 \\ 0.2 & 0.8 & 0.5 & 0.9 \\ 0.7 & 0.3 & 0.3 & 0.4 \end{bmatrix} \end{array} \longrightarrow \begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0.4 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.8 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix} \end{array}$$



Diagonalized Mean-variance portfolio weightings using data from 01/01/2009 to 12/31/2021

Factor Decomposition

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Factor Decomposition

- You can break down an asset or portfolio's return onto "factors" using a regression
  - Factors are portfolios or attributes that are said to explain individual asset returns
    - The entire market portfolio is an example of a factor
- Use Cases:
  - Performance breakdown
  - Tracking
  - Hedging

Performance Breakdown

Factor models can be used as a measurement of fund performance

- Fund managers or portfolios can have impressive returns, but we want to know how the returns were generated
  - Was it due to strong correlation to a fast-growing market
  - Did the portfolio take on excessive risk

Linear Factor Decomposition

$$r = \alpha + \beta x + \varepsilon$$

- Decompose the returns and variation by running a regression of the portfolio (regressand) onto the factor (regressor)
  - Alpha is the return not explained by the factor
  - Beta and ε are the variation explained and unexplained by the benchmark, respectively
  - R^2 states how well the factor explains the variation of returns
    - Should have a high R^2
    - Low R^2 means you're not decomposing the returns onto the proper benchmark (for example, a technology fund should use the Nasdaq as the benchmark and not the S&P 500).

Understanding the Results

- Alpha
  - Sensitive to which benchmark you use
  - High alpha could mean strong performance or using an improper benchmark
    - Low R^2 meaning model is not capturing risk properly
- Could be missing beta from the model, meaning not all risk was captured

Performance Metrics

These metrics measure return-to-risk breakdown and are useful for comparing different assets and portfolios

Sharpe Ratio	Information Ratio	Treynor Ratio
Performance metric using variance as the measure of volatility	Measures the non-factor performance of the regressand	Performance metric using beta as the measure of volatility

$$\text{Sharpe Ratio} = \frac{\tilde{\mu}^p}{\sigma^p}$$

$$\text{IR} = \frac{\alpha}{\sigma_\epsilon}$$

$$\text{Treynor Ratio} = \frac{\mathbb{E}\left[\tilde{r}^i\right]}{\beta^{i,m}}$$

Hedging

Suppose someone wants to invest in asset r but wants to remove the risk related to the overall market movements

- Run the regression of the asset Y onto the market portfolio x
  - $Y = \alpha + \beta x + \varepsilon$
- For every $1.00 invested into Y, the fund can short-sell β*$1.00 of x.
- The fund is then holding
  - $Y - \beta x = \alpha + \varepsilon$
  - Where the remaining returns are those unexplained by the market

Tracking

You can construct a portfolio that tracks the returns of another asset

- Assume you don't know what assets are in a portfolio but you have its return data.
  - You have K assets that you want to invest in to track the above portfolio
- Run the multivariate regression where y is the portfolio you want to track
  - $y = \alpha + \beta^1 x^1 + \beta^2 x^2 + \ldots \beta^k x^k + \varepsilon$
- To track the portfolio, invest β^i dollars into each of the assets x^i
- R^2 measures how well your tracking portfolio replicates the original portfolio

Linear Factor Pricing Models

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Factor Pricing

Via no-arbitrage arguments (all securities of same type are priced the same),

- The expected (excess) return of an asset is explained by the tangency portfolio and its relation (beta) to that portfolio

$$E(\tilde{r}^i) = \beta^{i,t} E(\tilde{r}^t)$$

$$\beta^{i,t} \equiv \frac{\mathrm{cov}(\tilde{r}^i, \tilde{r}^t)}{\mathrm{var}(\tilde{r}^t)}$$

Linear Factor Pricing Models (LFPM)

LFPMs are assertions about the identity of the tangency portfolio

- Investors don't allocate to the Mean Variance Portfolio
- Assumes mean-variance portfolio is for pricing expected returns
- However, even if you don't want to hold a MV portfolio, all E(r)s are calculated as covariances to the MV portfolio.

Capital Asset Pricing Model

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The Market Portfolio

CAPM identifies the market portfolio as the tangency portfolio.

- Market portfolio is the value-weighted portfolio of all available assets
- In practice, a broad equity index like SPY is generally used

The CAPM

- The most famous of these linear factor models is the CAPM, or Capital Asset Pricing Model

$$E\left(\tilde{r}^{i}\right) = \beta^{i,m} E\left(\tilde{r}^{m}\right)$$

$$\beta^{i,m} \equiv \frac{\operatorname{cov}(\tilde{r}^{i}, \tilde{r}^{m})}{\operatorname{var}(\tilde{r}^{m})}$$

1. The r with a tilda on top denotes the excess return of an asset

Expected Returns

The CAPM tells you about expected returns:

- E(r) for any asset is a function of the risk-free rate and the market risk premium.
- Beta is determined using a regression. CAPM doesn’t tell you how the risk-free rate or market risk premium are given.

The Role of Beta

- The CAPM says that the market beta is the **only** risk associated with higher average returns.
  - Because you can diversify, investors don't charge higher for non-correlated (idiosyncratic) risk.
  - Higher systematic risk demands higher return

Expected vs Realized Returns

- The CAPM implies that expected returns for any security are

$$E(\tilde{r}^i) = \beta^{i,m} E(\tilde{r}^m)$$

- which implies that realized returns can be written as

$$\tilde{r}_t^i = \beta^{i,m} \tilde{r}_t^m + \epsilon_t$$

- The CAPM implies that $\alpha i = 0$ for every single αi because the market tangency portfolio should perfectly predict all asset returns and no returns should be unexplained by the market