Problem Set 1

Problem 1. [24 points]
Translate the following sentences from English to predicate logic. The domain that you are working over is X, the set of people. You may use the functions $S(x)$, meaning that "x has been a student of 6.042," $A(x)$, meaning that "x has gotten an 'A' in 6.042," $T(x)$, meaning that "x is a TA of 6.042," and $E(x, y)$, meaning that "x and y are the same person."

(a) [6 pts] There are people who have taken 6.042 and have gotten A's in 6.042

Solution. $\exists x, y \in X : S(x) \wedge S(y) \wedge A(x) \wedge A(y) \wedge \neg E(x, y)$ ■

(b) [6 pts] All people who are 6.042 TA's and have taken 6.042 got A's in 6.042

Solution. $\forall x \in X : S(x) \wedge T(x) \Rightarrow A(x)$ ■

(c) [6 pts] There are no people who are 6.042 TA's who did not get A's in 6.042.

Solution. $\neg\exists x \in X : (\neg A(x)) \wedge T(x)$ ■

(d) [6 pts] There are at least three people who are TA's in 6.042 and have not taken 6.042
Solution.

$\exists x, y, z \in X : (\neg E(x, y) \wedge \neg E(y, z) \wedge \neg E(x, z)) \wedge T(x) \wedge \neg S(x) \wedge T(y) \wedge \neg S(y) \wedge T(z) \wedge \neg S(z)$
■

Problem 2. [24 Points]
Use a truth table to prove or disprove the following statements:

(a) [12 pts]

$$\neg(P \vee (Q \wedge R)) = (\neg P) \wedge (\neg Q \vee \neg R)$$

Solution. Let $F(P, Q, R) = \neg(P \vee (Q \wedge R))$ (the left side of the equation) and let $S(P, Q, R) = (\neg P) \wedge (\neg Q \vee \neg R)$ (the right side of the equation). The following is the truth table.

P	Q	R	F(P, Q, R)	S(P, Q, R)
T	T	T	F	F
T	T	F	F	F
T	F	T	F	F
F	T	T	F	F
T	F	F	F	F
F	F	T	T	T
F	T	F	T	T
F	F	F	T	T

The left and right sides have the same outputs for all truth value combinations, thus we conclude that the statement is true.

■

(b) [12 pts]

$$\neg(P \land (Q \lor R)) = \neg P \lor (\neg Q \lor \neg R)$$

Solution. Let $F(P, Q, R) = \neg(P \land (Q \lor R))$ (the left side of the equation) and let $S(P, Q, R) = \neg P \lor (\neg Q \lor \neg R)$ (the right side of the equation). The following is the truth table.

P	Q	R	F(P, Q, R)	S(P, Q, R)
T	T	T	F	F
T	T	F	F	T
T	F	T	F	T
F	T	T	T	T
T	F	F	T	T
F	F	T	T	T
F	T	F	T	T
F	F	F	T	T

There are two instances where F and G do not produce the same value, which indicates that they are not equal. ■

Problem 3. [24 Points]
The binary logical connectives $\land$ (*and*), $\lor$ (*or*), and $\implies$ (*implies*) appear often in not only computer programs, but also everyday speech. In computer chip designs, however, it is considerably easier to construct these out of another operation, `nand`, which is simpler to represent in a circuit. Here is the truth table for `nand`:

P	Q	P nand Q
T	T	F
T	F	T
F	T	T
F	F	T

(a) [12 pts] For each of the following expressions, find an equivalent expression using only `nand` and $\neg$ (*not*), as well as grouping parentheses to specify the order in which the operations apply. You may use A, B, and the operators any number of times.

(i) $A \land B$

Solution. $\neg(A \texttt{nand} B)$ ■

(ii) $A \lor B$

Solution. We know that $\neg(P \lor Q) = \neg P \land \neg Q$, so we can do:

$$\begin{aligned} A \lor B &= \neg(\neg(A \lor B)) \\ &= \neg((\neg A) \land (\neg B)) \\ &= (\neg A) \texttt{ nand } (\neg B) \end{aligned}$$

■

(iii) $A \implies B$

Solution.

$$\begin{aligned} (A \implies B) &= ((\neg A) \lor B) \\ &= \neg(A \land (\neg B)) \\ &= A \texttt{ nand } (\neg B) \end{aligned}$$

■

(b) [4 pts] It is actually possible to express each of the above using only `nand`, without needing to use $\neg$. Find an equivalent expression for $(\neg A)$ using only `nand` and grouping parentheses.

Solution. If we take a look at the `nand` truth table, we notice that when both values P and Q are `true`, the output is `false`; conversely, when both values are `false`, the output it `true`. To represent $(\neg A)$ using only `nand` we can write it as:

$$(A \texttt{ nand } A).$$

■

(c) [8 pts] The constants true and false themselves may be expressed using only `nand`. Construct an expression using an arbitrary statement A and `nand` that evaluates to true regardless of whether A is `true` or `false`. Construct a second expression that always evaluates to `false`. Do not use the constants `true` and `false` themselves in your statements.

Solution. We have already represented $(\neg A)$ as $(A \texttt{ nand } A)$. The `nand` table tells us that all combinations of P and Q produce `true` values except for when both are `true`. From here, we notice that an `nand` written like this always produces `true`:

$$(A \texttt{ nand } (\neg A))$$

`nand` only evaulate to `false` when both inputted values are `true`, so if we take the `nand` of A and its negated value $(\neg A)$, we will always have differing truth values, resulting in `true`. We can substitute back in $(\neg A) = (A \texttt{ nand } A$ into the equation to represent it with only `nand`:

$$(A \texttt{ nand } (A \texttt{ nand } A)$$

We can then apply this to make an expression that always evaluates to `false` by applying and `nand` to the equation above with itself. That is, since $(A \texttt{ nand } (A \texttt{ nand } A)$ always evaluates to `true`, and `nand` only produces `false` when both inputted are `true`, we will always get `false` when applying `nand` to itself $(A \texttt{ nand } (A \texttt{ nand } A)$:

$$((A \texttt{ nand } (A \texttt{ nand } A)) \texttt{ nand } ((A \texttt{ nand } (A \texttt{ nand } A))$$

■

Problem 4. [10 points] You have 12 coins and a balance scale, one of which is fake. All the real coins weigh the same, but the fake coin weighs less than the rest. All the coins visually appear the same, and the difference in weight is imperceptible to your senses. In at most 3 weighings, give a strategy that detects the fake coin. (Note: the scale in this problem is a scale with two dishes, which tips toward the side that is heavier. For clarification, do an image search for "balance scale").

Solution. One possible solution would be to separate the 12 coins into three groups of four coins each, which we will denote groups A, B, and C. We could first begin by weighing A and B against each other; if the balance tips in one direction, we know that the fake coin is in the other group of coins (slightly lighter). If the balance scale remains level, the fake coin must be in group C. For whichever group has the fake coin, we could split that specific group into two groups of two coins, then weigh these two subgroups against each other. For whichever subgroup of two coins is lighter (balance scale is higher on that side), we split this subgroup, and weigh those two coins. Whichever side is lighter (higher on the balance scale) is our fake coin. This process takes at most 3 weighings. ■

Problem 5. [6 points] Prove the following statement by proving its contrapositive: if r is irrational, then $r^{1/5}$ is irrational. (Be sure to state the contrapositive explicitly.)

Solution. We prove the contrapositive: if $r^{1/5}$, then r is rational. Assume that $r^{1/5}$ is rational. Then there exists integers m and n such that:

$$r^{1/5} = \frac{m}{n}$$

Take the fifth power of both sides:

$$r = \frac{m^5}{n^5}$$

Since m^5 and n^5 are integers, r is also rational. This proves the contrapositive, so the original proposition must be true. ■

Problem 6. [12 points] Suppose that $w^2 + x^2 + y^2 = z^2$, where w, x, y and z always denote positive integers. (Hint: It may be helpful to represent even integers as $2i$ and odd integers as $2j + 1$ where i and j are integers.

Prove the proposition: z is even if and only if w, x and y are even. Do this by considering all the cases of w, x, y being odd or even.

Solution. This is proof by case analysis. Here we consider two cases:

- w, x, y are all even. We can write w, x, y as $2i, 2j, 2k$, respectively, where $i, j, k \in \mathbb{Z}$. Substituting into the equation $w^2+x^2+y^2 = z^2$, we find $4i^2+4j^2+4k^2 = 4(i+j+k) = z^2$. This would imply that z^2 is divisible by 4, so in this case, z must be even. Let $z = 2l$. Then $i^2 + j^2 + k^2 = l^2$, which is satisfied when $l \in \mathbb{Z}$. Thus, z is necessarily even, and the equation holds for all even w, x, y.

- w, x, y are all odd. In this case, each of their squares is odd, and the sum of three odd numbers is odd. If z is even, then z^2 must also be even, since the square of an even number is always even. This leads to a contradiction because an odd number $(w^2 + x^2 + y^2)$ cannot equal and even number (z^2). Therefore, it cannot be true that $w^2 + x^2 + y^2 = z^2$, where w, x, y are all odd and z is even.

- One of w, x, y is odd. In this case, two squares are even and one square is odd, and the sum of an even and odd number is odd, which by definition cannot equal the sum of an even integer. Therefore z cannot be even in this case.

- Two of w, x, y are odd. Let's assume that w and x are odd, and that y is even. Then, we can write the sum as $w^2 + y^2 + z^2 = (2i + 1)^2 + (2j + 1)^2 + (2k)^2 = 4i^2 + 4i + 1 + 4j^2 + 4j + 1 + 4k^2 = 4(i^2 + j^2 + k^2 + i + j) + 2$. The sum of the squares is always 2 more than a multiple of 4. Since z^2 is a perfect square. Perfect squares modulo 4 are either 0 or 2. However, $4m + 2$ (the form of the sum $w^2 + x^2 + y^2$) cannot be a perfect square because it is 2 modulo 4. Thus, it is impossible for $w^2 + y^2 + z^2 = z^2$ to hold true when two of w, x, y are odd and the other is even.

By analyzing all possible cases of the parities of w, x, y, we have shown that the equation $w^2 + y^2 + z^2 = z^2$ holds only when w, x, y are all even. In this case, z must also be even, and the equation reduces to $i^2 + j^2 + k^2 = l^2$, where $i, j, k, l \in \mathbb{Z}$. For all other combinations of parity (all odd, one odd, or two odd), the equation leads to contradictions in the modular arithmetic of perfect squares. Therefore, the only valid solution occurs when w, x, y, z are all even. ■

References

Lehman, Eric, F Thomson Leighton, and Albert R Meyer, *Mathematics for Computer Science*, Suwanee, GA: 12th Media Services, June 2017.

Leighton, Tom and Marten van Dijk, "6.042 Mathematics for Computer Science," 2010. License: Creative Commons BY-NC-SA.