Recitation 2

1 Problem: A Geometric Sum

Perhaps you encountered this classic formula in school:

$$1 + r + r^2 + r^3 + ... + r^n = \frac{1 - r^{n+1}}{1 - r}$$

First use the well ordering principle, and then induction, to prove that this formula is correct for all real values $r \neq 1$.

Solution. Proof by Well Ordering Principle

Proof. The proof is by contradiction and the use of the Well Ordering Principle. Assume that the theorem is false; then, some nonnegative integers serve as counterexamples to it. Let these counterexamples be collected in a set: $C ::= \{n \in \mathbb{N} | 1 + r + r^2 + r^3 + ... + r^n \neq \frac{1-r^{n+1}}{1-r}\}$.

By our assumption that the theorem is false, it must admit counterexamples; therefore C must be a nonempty set of nonnegative integers. So, by the Well Ordering Principle, C has a minimum element, which we'll call c, a nonnegative integer representing the smallest counterexample.

Since c is the smallest counterexample, we know that the theorem is false for $n = c$, but true for all nonnegative numbers $n < c$. The formula is true for $n = 0$ because $1 = \frac{1-r^{0+1}}{1-r}$, so $c > 0$. $c - 1$ must then be a nonnegative integer, and since it is less than c, the formula is true for $c - 1$.

$$1 + r + r^2 + r^3 + ... + r^{c-1} = \frac{1 - r^c}{1 - r}$$

We can then add r^c to both sides of the equation to give us

$$\begin{aligned} 1 + r + r^2 + r^3 + ... + r^{c-1} + r^c &= \frac{1 - r^c}{1 - r} + r^c \\ &= \frac{1 - r^c + (1 - r)r^c}{1 - r} \\ &= \frac{1 - r^c + r^c - r^{c+1}}{1 - r} \\ &= \frac{1 - r^{c+1}}{1 - r} \end{aligned}$$

meaning the formula holds for $n = c$, contradicting the assumption that c is a counterexample.

Since the assumption that C is nonempty leads to a contradiction, the set C must be empty. Therefore, the formula holds for all $n >= 0$. □

Solution. Proof by Induction

Proof. We use induction. let $P(n)$ be the proposition that the following equation holds for all $r \neq 1$:

$$1 + r + r^2 + r^3 + ... + r^n = \frac{1 - r^{n+1}}{1 - r}$$

Base Case: The formula holds for $n = 0$ because both sides equal 1.

Inductive Step: Assume that $P(n)$ is true, where $n \in \mathbb{N}$. We can reason:

$$\begin{aligned} 1 + r + r^2 + r^3 + ... + r^n + r^{n+1} &= \frac{1 - r^{n+1}}{1 - r} + r^{n+1} \\ &= \frac{1 - r^{n+1} + (1 - r) \cdot r^{n+1}}{1 - r} \\ &= \frac{1 - r^{n+2}}{1 - r} \end{aligned}$$

The first equation follows the assumption that $P(n)$ is true, and simplification follows, indicating that $P(n+1)$ is also true. That is, $P(n)$ implies $P(n+1)$ for all $n \in \mathbb{N}$. By the principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$. ■

2 Problem: Surveyevor

In a new reality TV series called *Surveyevor*, a group of contestants is placed on a small island. Before the series begins, each contestant agrees to have a small purple or red tattoo, in the shape of an eye, applied to the middle of his or her forehead. In all, there are $p \geq 1$ purple eyes and $r \geq 0$ red eyes. However, none of the contestants knows the color of his or her third eye, nor how many total purple and red eyes there are. Furthermore, there are no mirrors and no one is allowed to discuss the tattoos ever. Therefore, everyone knows the colors of everyone else's third eye, but not their own. Good thing, because a contestant who learns that he or she has a purple eye must leave the island at the end of the show that day, and is therefore no longer eligible to win the $1 million cash prize at the end of the show!

The contestants live in uneasy ignorance for several weeks. As time goes on, however, most of them lose their fear of being exiled, adapt to island living, and even make friends with one another. Things are going quite well for the islanders, but as you might suppose, the television audience grows bored, and the show's ratings plummet. When the network threatens to cancel the series, the producer decides she needs to do something, fast: on the next show, to the surprise of the happy islanders, the producer herself appears and convenes a meeting. Very loudly, she proclaims, "I see that at least one person here has a purple eye." Assuming that all the contestants are master logicians, what happens?

Use induction to prove that your conclusion is correct. We suggest a hypothesis $P(n)$ that asserts all of the following are true on day n:

1. If $p > n$, then ____.

2. If $p = n$, then _____.

3. If $p > n$, then _____.

(We leave the task of filling in the blanks to you.)

Solution. Let's assume that the red-eyed islanders are in fact master logicians and always correctly reason what their eye-color is; thus no red-eyed islander should leave the island.

Theorem 1. *All the purple-eyed peopple leave the island on day p.*

Proof. This is a proof by induction. Let $P(n)$ be the proposition that asserts the following are true on day n:

1. If $p > n$, then all purple-eyed islanders stay that day.

2. If $p = n$, then all purple-eyed islanders leave the island.

3. If $p < n$, then all purple-eyed islanders are already gone.

Base Case: We verify that the three parts of $P(n)$ hold for base case day $n = 1$.

1. Suppose $p > 1$. A purple-eyed islander will look around and see that another contestant has a purple-eye, so this islander can have either a red or purple eye; they have solid reason to leave, and so they stay for the day.

2. Suppose $p = 1$. A purple-eyed islander will look around and see that no other contestant has a purple-eye, leading them to conclude that they are the purple-eyed islander. No one else has a reason to leave because they see a purple-eyed islander.

3. Suppose $p < 1$. This statement is always true. The statement restricts $p \geq 1$, because $p < 1$ would be false.

Therefore, $P(1)$ is true.

Inductive Step: Suppose that $P(n)$ is true where $n >= 0$. We must now verify that the three parts of $(P + 1)$.

1. Suppose $p > n+1$. Then $p > n$, so all purple-eyed contestants survive the preceding day by the first part of $P(n)$. A purple-eyed contestant can see at least $n + 1 > n$ other purple-eyed contestants, and therefore conclude that they themselves are either purple-eyed or red-eyed. Having no reason to leave, all purple-eyed contestants stay.

2. Suppose $p = n + 1$. Then $p > n$, so all purple-eyed contestants survive the preceding day by the first part of $P(n)$. On day $n + 1$, each purple-eyed contestant knows $p > n$, but sees n other people with a purple eye, and therefore conclude that they themselves have purple eye and leave.

3. Suppose $p < n + 1$. Then either $p = n$ (refer to part 2 of $P(n)$) or $p < n$ (refer to part 3 of P(n)). In either case, the purple-eyed contestants leave.

Therefore, $P(n)$ implies $P(n + 1)$ for all $n >= 0$. By the principle of induction, $P(n)$ is true for all $n \geq 0$ and the theorem follows.

All the purple-eyed contestants leave the island at the end of the pth day. ■

References

Lehman, Eric, F Thomson Leighton, and Albert R Meyer, *Mathematics for Computer Science*, Suwanee, GA: 12th Media Services, June 2017.

Leighton, Tom and Marten van Dijk, "6.042 Mathematics for Computer Science," 2010. License: Creative Commons BY-NC-SA.