Stanford CS103 Lecture Notes

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Claim

A claim.

Proof for Claim.

A reference to Theorem 1.1.2

Proof. Veniam velit incididunt deserunt est proident consectetur non velit ipsum voluptate nulla quis. Ea ullamco consequat non ad amet cupidatat cupidatat aliquip tempor sint ea nisi elit dolore dolore.

Laboris labore magna dolore eiusmod ea ex et eiusmod laboris. Et aliquip cupidatat reprehenderit id officia pariatur. □

Example.

> Nostrud esse occaecat Lorem dolore laborum exercitation adipisicing eu sint sunt et. Excepteur voluptate consectetur qui ex amet esse sunt ut nostrud qui proident non. Ipsum nostrud ut elit dolor. Incididunt voluptate esse et est labore cillum proident duis.

Some remark.

Remark.

> Some more remark.

1.2 Pictures

Figure 1.1: Waterloo, ON

Chapter 2

Lecture 0: Introduction, Set Theory

2.1 Key Questions

Computability Theory: What problems can you solve with a computer?
Complexity Theory: Why are some problems harder to solve than others?
Discrete Mathematics: How can we be certain in our answers to those questions?

2.2 Introduction to Set Theory

Definition 2.2.1: Set

A **set** is an unordered collection of distinct objects, which may be anything, including other sets. For example, $\{a, b, c, d\}$.

Theorem 2.2.2: Set Equality

Two sets are equal when they have the same contents, ignoring the order. For example, these are two different descriptions of the same set: $\{a, b, c\} = \{c, b, a\}$.

Theorem 2.2.3: Set Duplicates

Sets cannot contain duplicate elements. Any repeated elements are ignored. For example, these are two different descriptions of the same set: $\{a, b, c\} = \{a, a, a, b, b, c\}$.

Definition 2.2.4: Elements

The objects that make up a set are called the **elements** of a set, and we use the $\in$ (\in) symbol to signal that an element is part of said set. For example: $a \in \{a, b, c, d\}$. We can also use the $\notin$ (\notin) symbol to indicate that an element is not a member of a set: $e \notin \{a, b, c, d\}$.

Theorem 2.2.5: Number of Elements

Sets can contain any number of elements. The **empty set** is the set with no elements, and we denote the empty set using the $\emptyset$ symbol: $\{\} = \emptyset$.

Theorem 2.2.6: Is x equal to the set of x?

However, is a number equal to the set of that number? In other words, is $1 = \{1\}$? The answer is no. No object x is equal to the set containing x. That is,$x \neq \{x\}$.

2.3 Infinite Sets

Some sets contain *infinitely many* elements. The set $\mathbb{N} = \{0, 1, 2, 3, ...\}$ is the set of all the **natural numbers**. The set $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ is the set of all the **integers**, taken from the German "Zahlen." The set $\mathbb{R}$ is the set of all **real numbers**: $e \in \mathbb{R}, \pi \in \mathbb{R}, 4 \in \mathbb{R}, -137 \in \mathbb{R}, \text{etc.}$

To describe complex sets mathematically, we use **set-builder notation**.

$$\{n \mid n \in \mathbb{N} \text{ and } n \text{ is even}\}$$

The first n tells us: "the set of all n"; the | means "where" or "such that"; and "n is even" just tells us n is even. This would build the set $\{0, 2, 4, 6, 8, 12, 14, 16, ...\}$.

Definition 2.3.1: Set Builder Notation

A set may be specified in **set builder notation**:

$$\{x \mid \text{some property } x \text{ satisfies}\}$$
$$\{x \in S \mid \text{some property } x \text{ satisfies}\}$$

For example:

$$\{n \mid n \in \mathbb{N} \text{ and n is even}\}$$
$$\{C \mid C \text{ is a set of US currency}\}$$
$$\{r \in \mathbb{R} \mid r < 3\}$$
$$\{n \in \mathbb{N} \mid n < 3\}$$

2.4 Combining Sets

For the purpose of these examples, let:

$$A = \{1, 2, 3\}$$
$$B = \{3, 4, 5\}$$

The **union** of A and B is written as: $A \cup B$ and gives us a combination of sets A and B as a set with duplicates removed (remember Theorem 2.2.3: Set Duplicates 2.2.3):

$$A \cup B = \{1, 2, 3, 4, 5\}$$

The **intersection** of A and B is written as $A \cap B$ and gives us mutual elements in both sets:

$$A \cap B = \{3\}$$

The **difference** of A and B can be written as $A - B$ or $A \backslash B$, and gives us the different elements in A but not in B:

$$A \backslash B = \{1, 2\}$$

The **symmetric difference** of A and B can be written as $A \Delta B$ and gives us the different elements in both sets:

$$A \Delta B = \{1, 2, 4, 5\}$$

2.5 Subsets and Power Sets

Definition 2.5.1: Subset

A set S is called a **subset** of a set T (denoted $S \subseteq T$) if all elements of S are also elements of T. For example:

$$\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$$
$$\{b, c\} \subseteq \{a, b, c, d\}$$
$$\mathbb{N} \subseteq \mathbb{Z}$$
$$\mathbb{Z} \subseteq \mathbb{R}$$

We say that $S \in T$ if, among the elements of T, one of them is *exactly* the object S. We say that $S \subseteq T$ if S is a set and every element of S is also an element of T. (S has to be a set for the statement $S \subseteq T$ to be true). To put it simply: Not all elements of a set are subsets of that set and vice versa.

Definition 2.5.2: Power Set

The **power set** of S is the set of all subsets of S. We write this as

$$\wp(S) = \{T \mid T \subseteq S\}$$

The symbol $\wp$ is the Weierstrass, denoted with \wp.

$$\begin{aligned} S &= \{a, b\} \\ \wp(S) &= \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \end{aligned}$$

2.6 Cardinality

The **cardinality** of a set is the number of elements it contains. If S is a set, we denote its cardinality as $|S|$. For example:

$$\begin{aligned} |\{a, b\}| &= 2 \\ |\{\{a, b\}, \{c, d, e, f, g\}, \{h\}\}| &= 3 \\ |\{1, 2, 3, 3, 3, 3, 3\}| &= 3 \\ |\{n \in \mathbb{N} \mid n < 4\}| &= |\{0, 1, 2, 3\}| = 4 \\ |\emptyset| &= 0 \\ |\{\emptyset\}| &= 1 \end{aligned}$$

But what's $|\mathbb{N}|$? There are infinitely many natural numbers, and since $|\mathbb{N}|$ can't be a natural number (because it's infinitely large), we'll need a new term: $\aleph_0$ (pronounced "aleph-zero," "aleph-naught," or "aleph-null." Let's define $\aleph_0 = |\mathbb{N}|$.

Two sets have the same cardinality if there is a way to pair their elements off without leaving any elements uncovered.

Consider the set $S = \{n \mid n \in \mathbb{N} \text{ and } n \text{ is even}\}$. What is $|S|$?

$$\begin{array}{cccccccccccc} \mathbb{N} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \mathbb{S} & 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & \cdots \end{array}$$

We see here that each natural is paired to some number in the set S ($n \leftrightarrow 2n$). Thus, we can conclude (by rules of cardinality) that $|S| = |\mathbb{N}| = \aleph_0$.

Likewise, we can do the same process when comparing $\mathbb{N}$ and $\mathbb{Z}$:

$$\begin{array}{cccccccccccc} \mathbb{N} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \mathbb{Z} & 0 & -1 & 1 & -2 & 2 & -3 & 3 & -4 & 4 & \cdots \end{array}$$

And thus we conclude that $|\mathbb{N}| = |\mathbb{Z}| = \aleph_0$!