INSTITUTE OF MATHEMATICS

Generic Tool for Preliminary Performance Analysis of Iterative Parallel-in-Time Methods

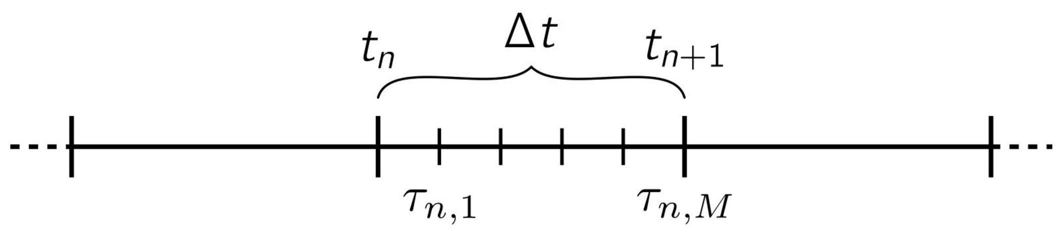
T. Lunet (TUHH), J. Hahne (University of Wuppertal)

Toward a common formalism ...

Focus on an elementary ODE :

$$\frac{du}{dt} = \lambda u, \quad \lambda \in \mathbb{C}, \quad t \in [0, T]$$

1) Decompose time domain in N blocks :



Block Variable : $\boldsymbol{u}_n = [u_{n,1}, u_{n,2}, ..., u_{n,M}]^T$

2) Define the **Block Operators**

$$\phi(\boldsymbol{u}_{n+1}) = \chi(\boldsymbol{u}_n)$$

ϕ is a bijective operator
- Runge-Kutta type
- Multistep
- Collocation based, SDC, …

χ builds the initial solution for next block

3) Build the **Block Problem**

$$\begin{pmatrix} \phi & & & \\ -\chi & \phi & & \\ & \ddots & \ddots & \\ & & -\chi & \phi \end{pmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_N \end{bmatrix} = \begin{bmatrix} \chi(u_0 \mathbb{1}) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \quad \mathbf{A}\boldsymbol{u} = \boldsymbol{f}$$

Iterative Parallel-in-Time algorithms simply solve a Block Problem iteratively

Main Idea : for a preconditioned iteration

$$\boldsymbol{u}^{k+1} = \boldsymbol{u}^k + \mathbf{P}^{-1}(\boldsymbol{f} - \mathbf{A}\boldsymbol{u}^k)$$

→write the block update component wise : **Block Iteration**

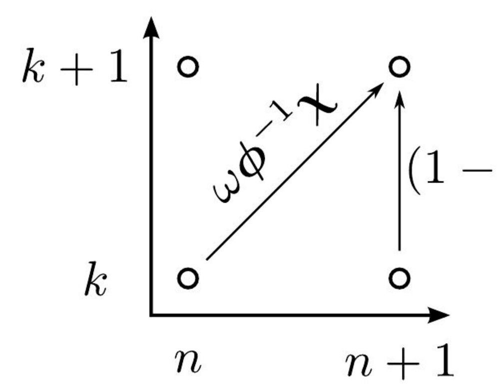
$$\boldsymbol{u}_{n+1}^{k+1} = \mathbf{B}_0^0 \boldsymbol{u}_n^k + \mathbf{B}_1^0 \boldsymbol{u}_{n+1}^k + \mathbf{B}_0^1 \boldsymbol{u}_n^{k+1} + \dots$$

→same approach for time multigrid with coarse grid correction

Example : Block Jacobi Smoothing

$$\mathbf{P}_{BJ} = \frac{1}{\omega} \begin{pmatrix} \phi & & \\ & \ddots & \\ & & \phi \end{pmatrix}, \quad \omega > 0$$

KN-graph :



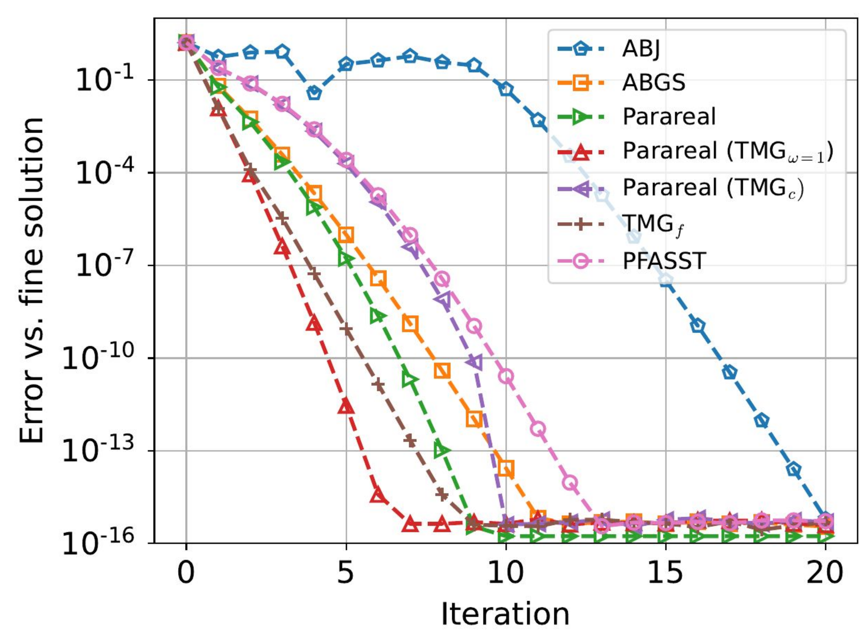
⇒ Block Iteration formula :

$$\boldsymbol{u}_{n+1}^{k+1} = (1-\omega)\boldsymbol{u}_{n+1}^k + \omega\phi^{-1}\chi\boldsymbol{u}_n^k$$

Possibility to write most iterative methods as Block Iterations (Parareal, MGRIT, PFASST, Time Multigrid, …)

Use of the block representation to analyze Parallel-in-Time error and estimate speedup

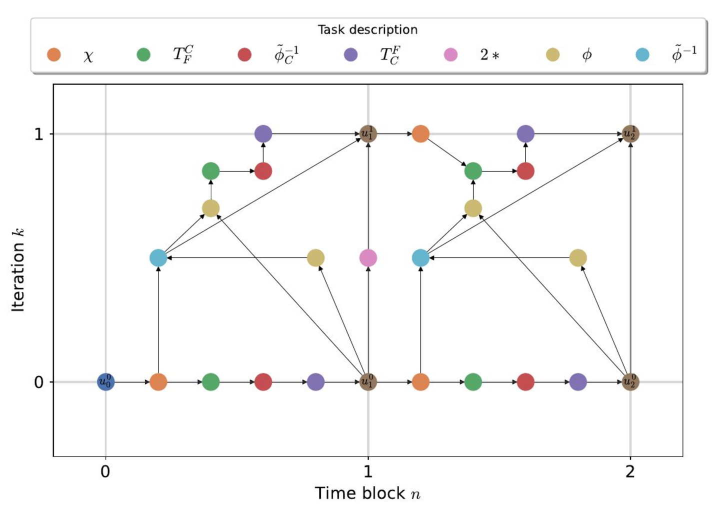
Error Analysis



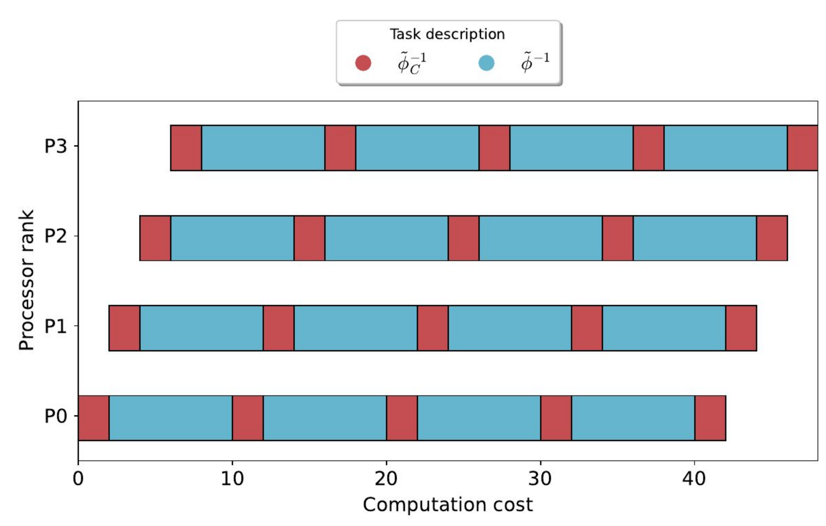
Example : convergence comparison of different algorithms with
- same time integration methods
- same problem parameters

In addition : generic error bounds for more in-depth analysis …

Task Graph and Scheduling

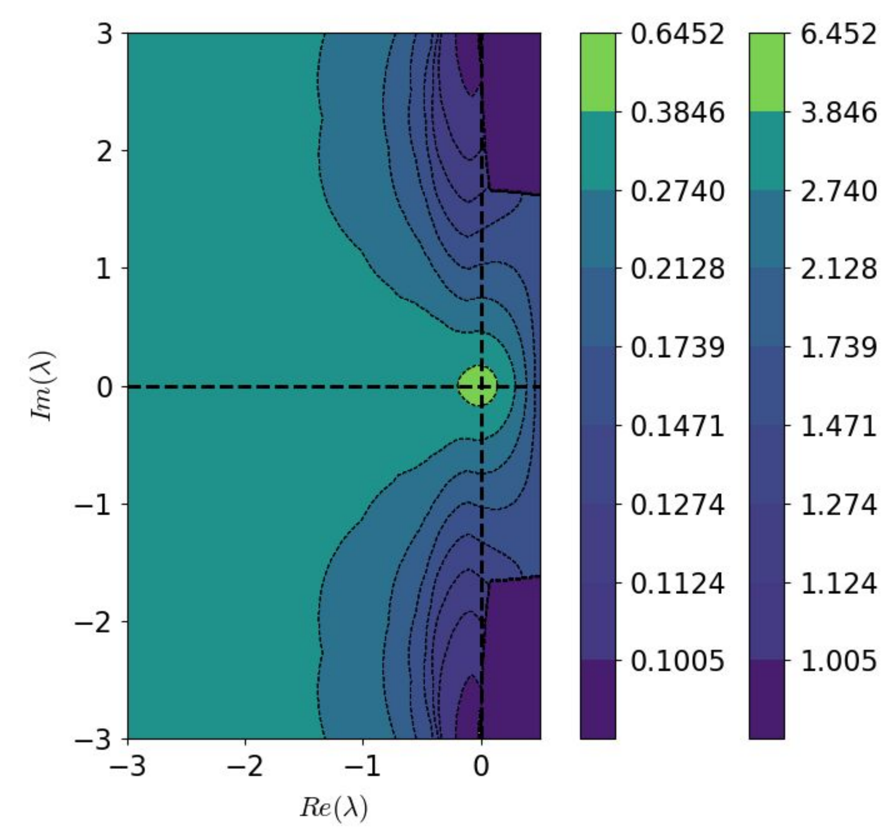


Task graph for one iteration of PFASST on 2 blocks



Schedule graph for 4 iterations of PFASST on 4 blocks

Speedup Estimation



Parallel efficiency and speedup for Parareal in the complex plane.
Use of Backward Euler with a 1/20 coarsening

Open Source Code & Web Application
https://github.com/Parallel-in-Time/time4apint

TUHH Hamburg University of Technology

Technische Universität Hamburg
Institut für Mathematik
Lehrstuhl Computational Mathematics
Am Schwarzenberg-Campus 3,
Gebäude E
21073 Hamburg

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