DSGA 3001.001
PROBABILISTIC TIME SERIES ANALYSIS
HOMEWORK 1

* PROBLEM 1 :

Considering Sample Mean as a Linear combination of Random Variables, we can find the variance in it.

$$\hat{\mu} = \frac{1}{T} \sum_{t=1}^{T} x_t$$

$$\mathrm{Var}[\hat{\mu}] = \mathrm{Var}\left[\frac{1}{T} \sum_{t} x_t \right]$$

$$= \frac{1}{T^2} \mathrm{Var}\left[\sum_{t} x_t \right]$$

$$= \frac{1}{T^2} \sum_{i=1}^{T} \sum_{j=1}^{T} \mathrm{Cov}(x_i, x_j)$$

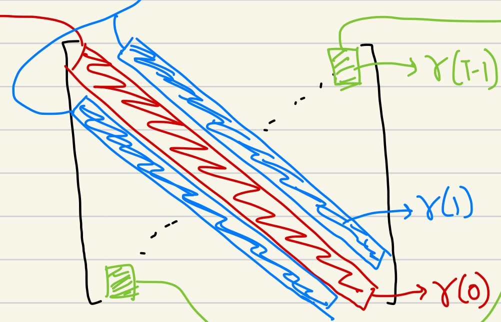
Since this value is the sum of elements in the autocovariance matrix $(R(i,j))$ of size $T \times T$, and we know that

① $\mathrm{Cov}(x_i, x_j) = \mathrm{Cov}(x_j, x_i)$

② $\gamma(h) = \mathrm{Cov}(x_i, x_{i+h})$

Since the series is stationary

$$\therefore \mathrm{Var}[\hat{\mu}] = \frac{1}{T^2}\left[\underbrace{T \times \gamma(0)} + \underbrace{2 \times \gamma(1) \times (T-1)} + \dots + \underbrace{2 \times \gamma(T-1) \times 1} \right]$$

$R(i,j) =$ 

$$\therefore \mathrm{Var}[\hat{\mu}] = \frac{1}{T^2}\left[T \times \gamma(0) + \sum_{h=1}^{T-1} 2 \times \gamma(h) \times (T-h) \right]$$

$$= \frac{\gamma(0)}{T} + \frac{2}{T^2} \sum_{h=1}^{T-1} \gamma(h) \times (T-h)$$

[NEXT PAGE]

* PROBLEM 2

Let $x_t = w_t$

This process satisfies the definition of linear process (1.12)

Definition 1.12 *A* **linear process**, *x_t, is defined to be a linear combination of white noise variates w_t, and is given by*

$$x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}, \qquad \sum_{j=-\infty}^{\infty} |\psi_j| < \infty. \tag{1.31}$$

where we can set $\mu = 0$ & $\psi_j = \begin{cases} 0 & ; j \neq 0 \\ 1 & ; j = 0 \end{cases}$

Hence, we can apply theorem A.7, which is

Theorem A.7 *If x_t is a stationary linear process of the form (1.31) satisfying the fourth moment condition (A.49), then for fixed K,*

$$\begin{pmatrix} \widehat{\rho}(1) \\ \vdots \\ \widehat{\rho}(K) \end{pmatrix} \sim \text{AN}\left[\begin{pmatrix} \rho(1) \\ \vdots \\ \rho(K) \end{pmatrix}, n^{-1}W\right],$$

where W is the matrix with elements given by

$$\begin{aligned} w_{pq} &= \sum_{u=-\infty}^{\infty} \Big[\rho(u+p)\rho(u+q) + \rho(u-p)\rho(u+q) + 2\rho(p)\rho(q)\rho^2(u) \\ &\qquad - 2\rho(p)\rho(u)\rho(u+q) - 2\rho(q)\rho(u)\rho(u+p)\Big] \\ &= \sum_{u=1}^{\infty}[\rho(u+p) + \rho(u-p) - 2\rho(p)\rho(u)] \\ &\qquad \times [\rho(u+q) + \rho(u-q) - 2\rho(q)\rho(u)], \end{aligned} \tag{A.54}$$

where the last form is more convenient.

Covariance Matrix of ACF is given by

$$n^{-1} W = \frac{1}{T} W \quad (\because \text{ we have } T \text{ data points})$$

$$w_{pq} = \sum_{u=1}^{\infty} [\rho(u+p) + \rho(u-p) - 2\rho(p)\rho(u)] \times [\rho(u+q) + \rho(u-q) - 2\rho(q)\rho(u)]$$

For the given stationary process, we know that

$$\rho(h) = \begin{cases} 1 & ; h = 0 \\ 0 & ; \text{otherwise} \end{cases}$$

Using the above conditions on A.54 we get

$$w_{pq} = \begin{cases} 1 & ; \ p = q \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$\therefore \quad W = Id_T$$

$\therefore$ Covariance matrix of ACF

$$= \frac{1}{T} W = \frac{1}{T} Id_T$$

$$\therefore \ Cov\left(ACF(i), ACF(j)\right) = \begin{cases} \frac{1}{T} & ; \ i = j \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$\boxed{\therefore \ Var\left(ACF(i)\right) = \frac{1}{T} \quad \forall i}$$

PROOF:

$$X_t = W_t \sim N(0, \sigma^2)$$

$$Cov(X_i, X_j) = \begin{cases} \sigma^2 & ; \ i = j \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$\rho_X(i, j) = \begin{cases} \frac{\sigma^2}{\sigma^2} & ; \ i = j \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$ACF(h) = \begin{cases} 1 & ; \ h = 0 \\ 0 & ; \ \text{otherwise} \end{cases}$$

In [1]:
```
import numpy as np
from statsmodels.tsa.stattools import acf
from statsmodels.graphics.tsaplots import plot_acf
import matplotlib.pyplot as plt
```

In [2]:
```
ts = []
var_acfs = []
by_ts = []

for t in range(500, 2500, 3):
    acfs = []
    for i in range(500):
        y = np.random.normal(0, 1, t)
        acfs.append(acf(y))

    acfs = np.array(acfs)
    var_acfs.append(np.var(acfs, axis=0))

    by_ts.append(1/t)
    ts.append(t)
```

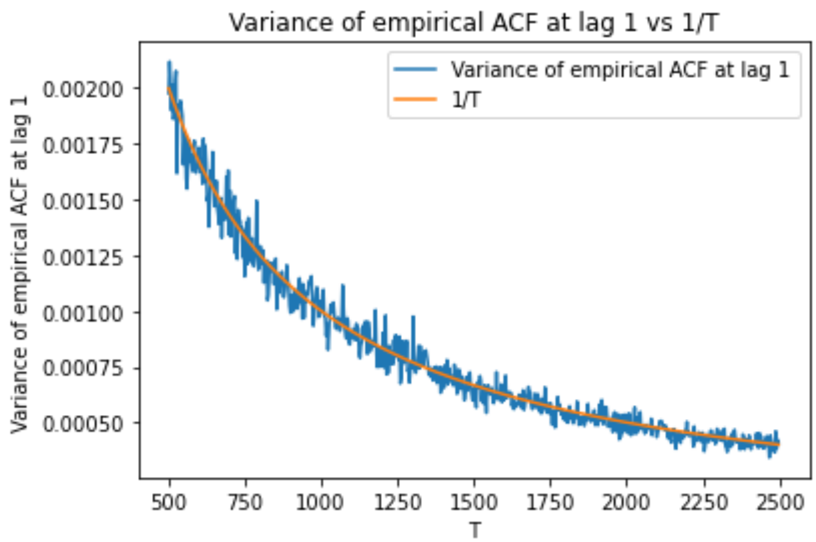
```
/usr/local/anaconda3/envs/pTSA/lib/python3.8/site-packages/statsmodels/
tsa/stattools.py:652: FutureWarning: The default number of lags is chan
ging from 40 tomin(int(10 * np.log10(nobs)), nobs - 1) after 0.12is rel
eased. Set the number of lags to an integer to  silence this warning.
  warnings.warn(
/usr/local/anaconda3/envs/pTSA/lib/python3.8/site-packages/statsmodels/
tsa/stattools.py:662: FutureWarning: fft=True will become the default a
fter the release of the 0.12 release of statsmodels. To suppress this w
arning, explicitly set fft=False.
  warnings.warn(
```

In [3]:
```
np.array(var_acfs).shape
```

Out[3]:
```
(667, 41)
```

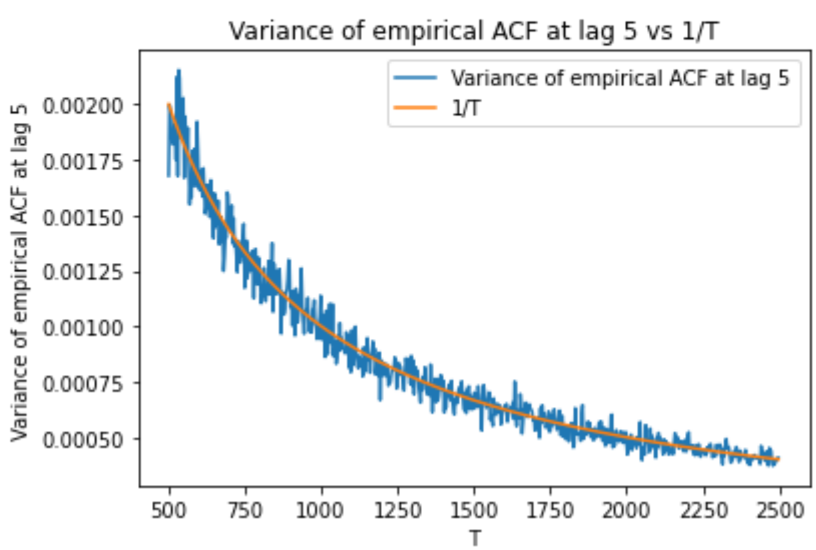
In [4]:
```
h=1
plt.plot(ts, np.array(var_acfs).T[h], label=f'Variance of empirical ACF at lag {h}')
plt.plot(ts, by_ts, label='1/T')
plt.title(f'Variance of empirical ACF at lag {h} vs 1/T')
plt.xlabel('T')
plt.ylabel(f'Variance of empirical ACF at lag {h}')
plt.legend()
```

Out[4]: <matplotlib.legend.Legend at 0x7f8ee4cc9d90>



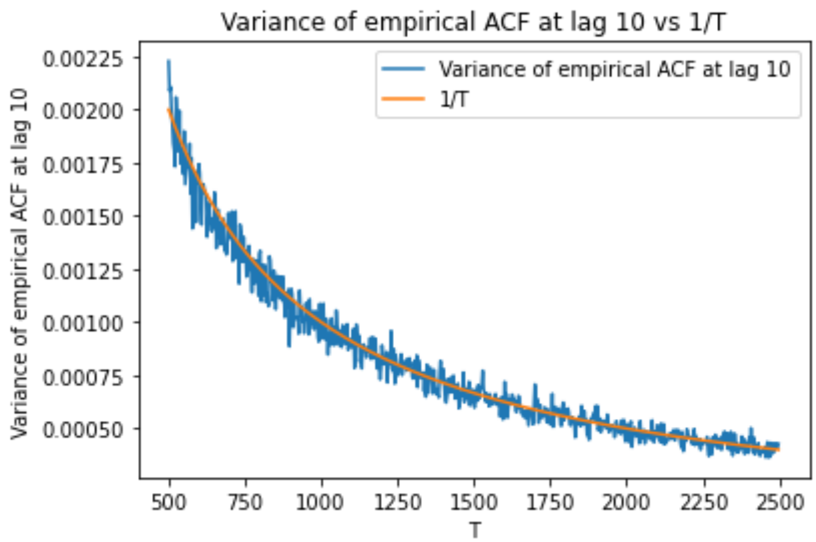
In [5]:
```
h=5
plt.plot(ts, np.array(var_acfs).T[h], label=f'Variance of empirical ACF at lag {h}')
plt.plot(ts, by_ts, label='1/T')
plt.title(f'Variance of empirical ACF at lag {h} vs 1/T')
plt.xlabel('T')
plt.ylabel(f'Variance of empirical ACF at lag {h}')
plt.legend()
```

Out[5]: <matplotlib.legend.Legend at 0x7f8ee4e58b80>



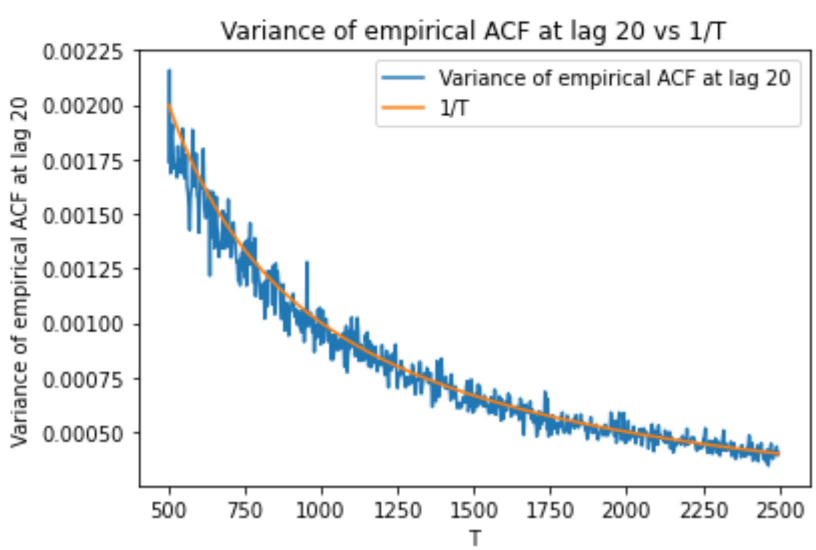
In [6]:
```
h=10
plt.plot(ts, np.array(var_acfs).T[h], label=f'Variance of empirical ACF
 at lag {h}')
plt.plot(ts, by_ts, label='1/T')
plt.title(f'Variance of empirical ACF at lag {h} vs 1/T')
plt.xlabel('T')
plt.ylabel(f'Variance of empirical ACF at lag {h}')
plt.legend()
```

Out[6]: <matplotlib.legend.Legend at 0x7f8ee4e6c700>



In [7]:
```
h=20
plt.plot(ts, np.array(var_acfs).T[h], label=f'Variance of empirical ACF
 at lag {h}')
plt.plot(ts, by_ts, label='1/T')
plt.title(f'Variance of empirical ACF at lag {h} vs 1/T')
plt.xlabel('T')
plt.ylabel(f'Variance of empirical ACF at lag {h}')
plt.legend()
```

Out[7]: <matplotlib.legend.Legend at 0x7f8ee523d8e0>



* PROBLEM 3

MA(1) : $\quad x_t = w_t + \theta w_{t-1}$

Now,

$$\begin{aligned}
Var(x_t) &= E[(x_t - E[x_t])^2] \\
&= E[x_t^2] \\
&= E[(w_t + \theta w_{t-1})^2] \\
&= E[w_t^2 + 2\theta w_t w_{t-1} + \theta^2 w_{t-1}^2] \\
&= E[w_t^2] + 2\theta E[w_t w_{t-1}] + \theta^2 E[w_{t-1}^2] \\
&= Var(w_t) - E^2[w_t] + 2\theta E[w_t] E[w_{t-1}] + \theta^2 (Var(w_{t-1}) - E^2[w_{t-1}]) \\
&= \sigma^2 - 0^2 + 2\theta(0)(0) + \theta^2(\sigma^2 - 0^2) \\
&= \sigma^2 + \theta^2\sigma^2 = (1+\theta^2)\sigma^2
\end{aligned}$$

$$\begin{aligned}
Cov(x_t, x_u) &= E[(x_t - E[x_t])(x_u - E[x_u])] \\
&= E[(x_t - 0)(x_u - 0)] = E[x_t x_u] \\
&= E[(w_t + \theta w_{t-1})(w_u + \theta w_{u-1})] \\
&= E[w_t w_u + \theta w_t w_{u-1} + \theta w_{t-1} w_u + \theta^2 w_{t-1} w_{u-1}] \\
&= E[w_t w_u] + \theta E[w_t w_{u-1}] + \theta E[w_{t-1} w_u] + \theta^2 E[w_{t-1} w_{u-1}]
\end{aligned}$$

$$\therefore Cov(x_t, x_u) = \begin{cases} E[w_t^2] + 0 + 0 + \theta^2 E[w_{t-1}^2] & ; \ t = u \\ 0 + \theta E[w_t^2] + 0 + 0 & ; \ t = u-1 \\ 0 + 0 + \theta E[w_u^2] + 0 & ; \ t = u+1 \\ 0 + 0 + 0 + 0 & ; \ \text{otherwise} \end{cases}$$

$$= \begin{cases} \sigma^2(1+\theta^2) & ; \ t = u \\ \theta\sigma^2 & ; \ t = u-1 \\ \theta\sigma^2 & ; \ t = u+1 \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$= \begin{cases} \sigma^2(1+\theta^2) & ; \ t = u \\ \theta\sigma^2 & ; \ |t-u| = 1 \\ 0 & ; \ \text{otherwise.} \end{cases}$$

$$\therefore \ \rho_x(h) = \begin{cases} \sigma^2(1+\theta^2) / \sigma^2(1+\theta^2) & ; \ h = 0 \\ \theta\sigma^2 / \sigma^2(1+\theta^2) & ; \ |h| = 1 \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & ; \ h = 0 \\ \theta/(1+\theta^2) & ; \ |h| = 1 \\ 0 & ; \ \text{otherwise.} \end{cases}$$

Now we have to prove :

$$|\rho_x(h)| \leq 0.5$$

$$-0.5 \leq \rho_x(h) \leq 0.5$$

$$-0.5 \leq \frac{\theta}{1+\theta^2} \leq 0.5$$

OR $\quad \theta^2 - 2\theta + 1 \geq 0 \quad$ AND $\quad \theta^2 + 2\theta + 1 \geq 0$

$$(\theta-1)^2 \geq 0 \quad \text{AND} \quad (\theta+1)^2 \geq 0$$

Both these conditions are always true, hence proved !

$\boxed{\ }$

$\underset{\theta}{\text{argmax}} / \underset{\theta}{\text{argmin}}$ of $\rho(1) = ?$

$$\rho(1) = \frac{\theta}{1+\theta^2}$$

To find maximizers / minimizers

$$\frac{d\rho(1)}{d\theta} = 0$$

$$\therefore \ \frac{d\left(\frac{\theta}{1+\theta^2}\right)}{d\theta} = 0$$

$$\frac{(1+\theta^2)(1) - \theta(2\theta)}{(1+\theta^2)^2} = 0$$

$\therefore \quad 1-\theta^2 = 0 \qquad$ [since $(1+\theta^2)^2$ cannot be equal to 0]

$\therefore \quad \theta^2 = 1$

$\therefore \quad \theta = \pm 1$

$$p_{x|\theta=1}(1) = \frac{1}{1+1^2} = 0.5$$

$$p_{x|\theta=-1}(1) = \frac{-1}{1+(-1)^2} = -0.5$$

$\therefore \; p_x(1)$ is maximum at $\theta = 1$ & minimum at $\theta = -1$

* PROBLEM 4

i] $x_t = 0.8\,x_{t-1} - 0.15\,x_{t-2} + w_t - 0.3\,w_{t-1}$

$$x_t - 0.8\,x_{t-1} + 0.15\,x_{t-2} = w_t - 0.3\,w_{t-1}$$

$$\therefore (1 - 0.8B + 0.15B^2)\,x_t = (1 - 0.3B)\,w_t$$

$$\therefore \theta(B) = (1 - 0.3B)$$
$$\phi(B) = (1 - 0.8B + 0.15B^2) = (1 - 0.3B)(1 - 0.5B)$$

After eliminating common factors

$$\theta(B) = 1$$
$$\phi(B) = (1 - 0.5B)$$

$$\therefore (1 - 0.5B)\,x_t = w_t$$

$$x_t - 0.5\,x_{t-1} = w_t$$

$$x_t = 0.5\,x_{t-1} + w_t$$

$\therefore$ ARMA(1, 0)

ii] $x_t = x_{t-1} - 0.5\,x_{t-2} + w_t - w_{t-1}$

$$\therefore x_t - x_{t-1} + 0.5\,x_{t-2} = w_t - w_{t-1}$$

$$(1 - B + 0.5B^2)\,x_t = (1 - B)\,w_t$$

Complex roots

$\therefore$ No common factors.

$\therefore$ ARMA(2, 1)

* PROBLEM 5

a] MA(3)

$$x_t = w_t + \phi_1 w_{t-1} + \phi_2 w_{t-2} + \phi_3 w_{t-3}$$

Given : $\{ x_1, x_2, \dots, x_t \}$

Assume that we know w_{-2}, w_{-1}, w_0 & parameters of model (ϕ_i)

then
$$w_1 = x_1 - \phi_1 w_0 - \phi_2 w_{-1} - \phi_3 w_{-2}$$
$$w_2 = x_2 - \phi_1 w_1 - \phi_2 w_0 - \phi_3 w_{-1}$$
$$\vdots$$
$$w_t = x_t - \phi_1 w_{t-1} - \phi_2 w_{t-2} - \phi_3 w_{t-3}$$

$$E[x_{t+1} | \{x_1, \dots, x_t ; w_1, \dots, w_t\}]$$
$$= E[w_{t+1} + \phi_1 w_t + \phi_2 w_{t-1} + \phi_3 w_{t-2} | \{x_1, \dots, x_t ; w_1, \dots, w_t\}]$$
$$= 0 + \phi_1 w_t + \phi_2 w_{t-1} + \phi_3 w_{t-2}$$

Similarly,

$$\Rightarrow E[x_{t+2} | \{x_1, \dots, x_t ; w_1, \dots, w_t\}]$$
$$= E[w_{t+2} + \phi_1 w_{t+1} + \phi_2 w_t + \phi_3 w_{t-1} | \{x_1, \dots, x_t ; w_1, \dots, w_t\}]$$
$$= 0 + \phi_1 (0) + \phi_2 w_t + \phi_3 w_{t-1} = \phi_2 w_t + \phi_3 w_{t-1}$$

$$\Rightarrow E[x_{t+3} | \{x_1, \dots, x_t ; w_1, \dots, w_t\}]$$
$$= E[w_{t+3} + \phi_1 w_{t+2} + \phi_2 w_{t+1} + \phi_3 w_t | \{x_1, \dots, x_t ; w_1, \dots, w_t\}]$$
$$= 0 + \phi_1 (0) + \phi_2 (0) + \phi_3 w_t = \phi_3 w_t$$

$$\Rightarrow E[x_{t+4} | \{x_1, \dots, x_t ; w_1, \dots, w_t\}]$$
$$= E[w_{t+4} + \phi_1 w_{t+3} + \phi_2 w_{t+2} + \phi_3 w_{t+1} | \{x_1, \dots, x_t ; w_1, \dots, w_t\}]$$
$$= 0 + \phi_1 (0) + \phi_2 (0) + \phi_3 (0) = 0$$

Error $= x_{t+h} - E[x_{t+h} \mid \{x_1, \ldots, x_t; w_1, \ldots, w_t\}]$

Variances in error for different h:

$\therefore \Rightarrow Var(x_{t+1} - E[x_{t+1} \mid \{x_1, \ldots, x_t; w_1, \ldots, w_t\}])$

$$= Var((w_{t+1} + \phi_1 w_t + \phi_2 w_{t-1} + \phi_3 w_{t-2}) - (\phi_1 w_t + \phi_2 w_{t-1} + \phi_3 w_{t-2}))$$

$$= Var(w_{t+1})$$

$$= \sigma^2$$

$\Rightarrow Var(x_{t+2} - E[x_{t+2} \mid \{x_1, \ldots, x_t; w_1, \ldots, w_t\}])$

$$= Var((w_{t+2} + \phi_1 w_{t+1} + \phi_2 w_t + \phi_3 w_{t-1}) - (\phi_2 w_t + \phi_3 w_{t-1}))$$

$$= Var(w_{t+2} + \phi_1 w_{t+1})$$

$$= \sigma^2 + \phi_1^2 \sigma^2$$

$$= (1 + \phi_1^2)\sigma^2$$

$\Rightarrow Var(x_{t+3} - E[x_{t+3} \mid \{x_1, \ldots, x_t; w_1, \ldots, w_t\}])$

$$= Var((w_{t+3} + \phi_1 w_{t+2} + \phi_2 w_{t+1} + \phi_3 w_t) - (\phi_3 w_t))$$

$$= (1 + \phi_1^2 + \phi_2^2)\sigma^2$$

$\Rightarrow Var(x_{t+4} - E[x_{t+4} \mid \{x_1, \ldots, x_t; w_1, \ldots, w_t\}])$

$$= Var(w_{t+4} + \phi_1 w_{t+3} + \phi_2 w_{t+2} + \phi_3 w_{t+1})$$

$$= (1 + \phi_1^2 + \phi_2^2 + \phi_3^2)\sigma^2$$

$\therefore Var(x_{t+h} - E[x_{t+h} \mid \{x_1, \ldots, x_t; w_1, \ldots, w_t\})$

$$= \begin{cases} \sigma^2 & ; h = 1 \\ (1 + \sum_{u=1}^{h-1} \phi_u^2)\sigma^2 & ; 1 < h \leq 4 \\ (1 + \sum_{u=1}^{3} \phi_u^2)\sigma^2 & ; \text{otherwise} \end{cases}$$

$\therefore$ As h increases, uncertainty in our prediction increases (prediction power decreases), because of increase in variance of the error. For $h \in \{5, 6, \ldots, \infty\}$, variance in error is constant but very high.

b) AR(1) model

$$x_t = \lambda x_{t-1} + \omega_t$$

Given : $\{x_1, x_2, \ldots, x_t\}$

$$E[x_{t+1} | \{x_1, x_2, \ldots, x_t ; \lambda\}] = E[\lambda x_t + \omega_{t+1} | \{x_1, \ldots, x_t ; \lambda\}]$$

$$= E[\lambda x_t] + 0 = \lambda x_t$$

$$E[x_{t+2} | \{x_1, \ldots, x_t ; \lambda\}] = E[\lambda x_{t+1} + \omega_{t+2} | \{x_1, \ldots, x_t ; \lambda\}]$$

$$= \lambda E[x_{t+1} | \{x_1, \ldots, x_t ; \lambda\}] + 0$$

$$= \lambda^2 x_t$$

$$E[x_{t+3} | \{x_1, \ldots, x_t ; \lambda\}] = E[\lambda x_{t+2} + \omega_{t+3} | \{x_1, \ldots, x_t ; \lambda\}]$$

$$= \lambda E[x_{t+2} | \{x_1, \ldots, x_t ; \lambda] + 0$$

$$= \lambda^3 x_t$$

$$\therefore E[x_{t+h} | \{x_1, \ldots, x_t ; \lambda\}] = \lambda^h x_t \quad ; \quad h \geqslant 1$$

$$\text{Error} = x_{t+h} - E[x_{t+h} | \{x_1, \ldots, x_t ; \lambda\}]$$

Var (Error) for different h :

$\Rightarrow$ $h = 1$

$$\text{Var(error)} = \text{Var}(x_{t+1} - E[x_{t+1} | \{x_1, \ldots, x_t\}])$$

$$= \text{Var}(\lambda x_t + \omega_{t+1} - \lambda x_t)$$

$$= \text{Var}(\omega_{t+1})$$

$$= \sigma^2$$

$\Rightarrow$ $h = 2$

$$\mathrm{Var}(\text{error}) = \mathrm{Var}\left(x_{t+2} - E[x_{t+2} \mid \{x_1, \ldots, x_t ; \lambda\}]\right)$$
$$= \mathrm{Var}\left(\lambda(\lambda x_t + \omega_{t+1}) + \omega_{t+2} - \lambda^2 x_t \right)$$
$$= \mathrm{Var}\left(\lambda \omega_{t+1} + \omega_{t+2} \right)$$
$$= \lambda^2 \sigma^2 + \sigma^2$$
$$= (1 + \lambda^2)\,\sigma^2$$

$\Rightarrow h = 3$

$$\mathrm{Var}(\text{error}) = \mathrm{Var}\left(x_{t+3} - E[x_{t+3} \mid \{x_1, \ldots, x_3 ; \lambda\}]\right)$$
$$= \mathrm{Var}\left(\lambda^3 x_t + \lambda^2 \omega_{t+1} + \lambda \omega_{t+2} + \omega_{t+3} - \lambda^3 x_t \right)$$
$$= \mathrm{Var}\left(\lambda^2 \omega_{t+1} + \lambda \omega_{t+2} + \omega_{t+3} \right)$$
$$= (1 + \lambda^2 + \lambda^4)\,\sigma^2$$

$$\therefore \quad \mathrm{Var}(\text{error}) = \sigma^2 \sum_{u=0}^{h-1} (\lambda^u)^2 \quad ; \quad h \geq 1$$

It converges to :

$$\mathrm{Var}(\text{error}) = \frac{\sigma^2}{1 - \lambda^2}$$

[By sum of ∞ geometric series & since $|\lambda| < 1$]

Again, the predictive power decrease with a bigger h since the variance in error increases and it converges to $\frac{\sigma^2}{1-\lambda^2}$ for a very big h.

* PROBLEM 6

AR(2) process $\longrightarrow$ $P(B) = (1-0.2B)(1-0.5B)$

$$\therefore P(B) = 1 - 0.5B - 0.2B + 0.1B^2$$

$$= 1 - 0.7B + 0.1B^2$$

$\therefore$ AR(2) process $\longmapsto$ $x_t - 0.7x_{t-1} + 0.1x_{t-2} = w_t$

$$\therefore x_t = 0.7x_{t-1} - 0.1x_{t-2} + w_t$$

$$\therefore x_t x_{t-h} = 0.7 x_{t-1} x_{t-h} - 0.1 x_{t-2} x_{t-h} + w_t x_{t-h}$$

$$E[x_t x_{t-h}] = 0.7 E[x_{t-1} x_{t-h}] - 0.1 E[x_{t-2} x_{t-h}] + E[w_t x_{t-h}]$$

$$\therefore \gamma(h) = 0.7\gamma(h-1) - 0.1\gamma(h-2) + 0$$

$$\therefore \rho(h) = 0.7\rho(h-1) - 0.1\rho(h-2)$$

OR

$$\rho(h) - 0.7\rho(h-1) + 0.1\rho(h-2) = 0$$

$$\therefore \rho(1) - 0.7\rho(0) + 0.1\rho(-1) = 0 \longrightarrow (1)$$

$$\rho(2) - 0.7\rho(1) + 0.1\rho(0) = 0 \longrightarrow (2)$$

We know that $\rho(0) = 1$ & $\rho(-h) = \rho(h)$

$\therefore$ From (1)

$1.1\rho(1) - 0.7 = 0$ $\longrightarrow$ $\boxed{\rho(1) = \frac{0.7}{1.1} = 0.6363}$

$\therefore$ From (2)

$$\rho(2) - (0.7)(0.6363) + 0.1 = 0$$

$$\boxed{\begin{aligned}\rho(2) &= (0.7)(0.6363) - 0.1 \\ &= 0.34541\end{aligned}}$$

$$x_t = 0.7 * x_{t-1} - 0.1 * x_{t-2} + w_t$$

In [1]:
```
import numpy as np
from statsmodels.tsa.stattools import acf
from statsmodels.graphics.tsaplots import plot_acf
import matplotlib.pyplot as plt
```

In [2]:
```
def acf_impl(x, nlags):
    """
    TODO
    @param x: a 1-d numpy array (data)
    @param nlags: an integer indicating how far back to compute the ACF
    @return a 1-d numpy array with (nlags+1) elements.
            Where the first element denotes the acf at lag = 0 (1.0 by d
efinition).
    """
    mean_x = x.mean()
    acfs = []

    acf_0 = 0
    for i in range(0, len(x)):
        acf_0 = acf_0 + ((x[i] - mean_x) * (x[i] - mean_x))
    acf_0 = acf_0 / len(x)

    for j in range(nlags+1):
        total = 0
        for i in range(0, len(x)-j):
            total = total + ((x[i+j] - mean_x) * (x[i] - mean_x))
        total = total / len(x)
        total = total / acf_0
        acfs.append(total)

    return acfs
```

In [3]:
```
# Analytical ACF calculations
phi_1 = 0.7
phi_2 = -0.1

rho_1 = phi_1 / (1 - phi_2)
rho_2 = ((phi_1 * phi_1)/(1 - phi_2)) + phi_2

print(f"Analytical -> rho_1 = {rho_1}, rho_2 = {rho_2}")

# Data simulation for empirical ACF
x_a, x_b = np.random.normal(0, 1), np.random.normal(0, 1) # Initially se
ries equal to white noise with variance 1

n = 10000
x = []
for i in range(n):
    x.append(0.7 * x_a - 0.1 * x_b + np.random.normal(0, 1))
    x_b = x_a
    x_a = x[-1]
x = np.array(x)
transient = 1000
x = x[transient::]

# Estimated ACF calculation
acf_val = acf_impl(x=x, nlags=2)

# Plots
print(f"Empirical -> rho_1 = {acf_val[1]}, rho_2 = {acf_val[2]}")
plt.figure()
plt.plot(acf_val, 'or', label='statsmodels empirical acf')
plt.plot([1, rho_1, rho_2], 'xb', label='own acf')
plt.legend();
plt.title('Empirical ACF vs Analytical ACF')
```

```
Analytical -> rho_1 = 0.6363636363636362, rho_2 = 0.34545454545454535
Empirical -> rho_1 = 0.6291197014298906, rho_2 = 0.33906375999370325
```

Out[3]:
```
Text(0.5, 1.0, 'Empirical ACF vs Analytical ACF')
```