Inverse Kinematics of the General 6R Manipulator and Related Linkages

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This paper elaborates on a method developed by the authors for solving the inverse kinematics of a general 6R manipulator. The method is shown to be applicable to determining the joint variables associated with all series-chain manipulators and closed-loop linkages constructed in a single loop with revolute, prismatic, or cylindric joints. The method is shown to yield a single polynomial, of minimum degree, in terms of just one of the joint variables. Once the roots of this polynomial are found, the remaining variables are then usually determined from linear sets of equations. It is shown that this method works equally well for general geometries and for special geometries such as those chararcterized by intersecting or parallel joint axes.

Introduction

In this paper we describe a problem formulation and elimination technique which can be applied to a large class of manipulators and closed-loop mechanisms in order to determine: (*a*) for a manipulator, all possible joint-displacement values corresponding to a specified end-effector pose, (*b*) for a closed-loop mechanism, all possible joint-displacement values corresponding to a specified input displacement.

This paper uses the solution method developed by the authors originally for solving the inverse kinematics of 6R series-chain manipulators of general geometry, Raghavan and Roth (1989). We also rely on additional proofs presented by Raghavan (1990), and an extension of the original method to manipulators with prismatic joints, as given by Raghavan and Roth (1990).

The formulation we use produces a system of multivariate polynomials in terms of the unknown joint displacements. The method we have developed for the solution of these equations brings about the elimination of all the joint variables except for one, which we call the suppressed variable. The method always yields a single "characteristic" polynomial, of minimum degree, in terms of just this single suppressed joint variable. Once the roots of this polynomial are found, the variables that were eliminated can be determined from linear sets of equations. It is shown that this method works equally well for general geometries and for special geometries such as those characterized by intersecting or parallel joint axes.

Both the so-called inverse manipulator-kinematics problem and the closed-loop linkage analysis problem, problems (*a*) and (*b*) above, have been the subjects of many previous research publications. The fact that both problems are in fact connected seems to have first been published in Roth, Rastegar, and Scheinman (1973). It is now widely understood that if we consider the known position of the input link of a closed-loop linkage as the known position of a manipulator hand, the displacement (or position) analysis of any n-link closed-loop linkage is identical to the inverse kinematics of a corresponding n-link manipulator (including the end-effector), and vice versa. This idea is now used extensively, see for example the book by Duffy (1980).

Previous studies of linkage analysis can be traced back to the last century and are too numerous to list here. Interested readers can consult, for example, Duffy (1980), Yang and Freudenstein (1964), and De Groot (1970). The inverse manipulator-kinematics problem was first discussed in Pieper (1968). A bibliography for the research into the inverse 6R problem can be constructed from the papers by Albala and Angeles (1979), Duffy and Crane (1980), Tsai and Morgan (1985), Lee and Liang (1988a and 1988b) and Raghavan and Roth (1989). The inverse kinematics of series-chain manipulators with special geometries and with prismatic joints have been treated systematically in, for example, Pieper and Roth (1969), Duffy (1980) and Raghavan and Roth (1990), and in a more ad hoc manner in numerous analyses of specific commercial and experimental manipulators [see, for example, textbooks such as Paul (1981) and Craig (1989).]

Problem Formulation

In order to understand how to study any single-loop open or closed loop system with our method, it is important to understand how it is applied to the 6R manipulator. To acquaint the reader we repeat the essential elements of our development for the 6R manipulator problem in Raghavan and Roth (1990). Namely, using exactly the same variant of the Denavit and Hartenberg nomenclature as in Tsai and Morgan (1985), the links of the 6R manipulator are numbered from 1 to 7, the fixed or base link being 1, and the outermost link or hand being 7. A coordinate system is attached to each link to facilitate a mathematical description of the linkage and the relative arrangement of the links. The coordinate system at-

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tached to the ith link is numbered i. The 4×4 transformation matrix relating coordinate systems $i+1$ and i is as follows:

$$A_i = \begin{pmatrix} c_i & -s_i\lambda_i & s_i\mu_i & a_ic_i \\ s_i & c_i\lambda_i & -c_i\mu_i & a_is_i \\ 0 & \mu_i & \lambda_i & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

where

$s_i = \sin\theta_i$, $c_i = \cos\theta_i$,

$\lambda_i = \cos\alpha_i$, $\mu_i = \sin\alpha_i$,

a_i is the length of link $i+1$,
α_i is the twist angle between the axes of joints i and $i+1$,
d_i is the offset distance at joint i,
θ_i is the joint rotation angle at joint i.

The closure equation for the 6R manipulator is the following matrix equation

$$A_1A_2A_3A_4A_5A_6 = A_{\text{hand}} \tag{2}$$

A_{hand} is the 4×4 transformation matrix describing the Cartesian coordinate system 7, attached to the hand (or last link) with respect to coordinate system 1, attached to the base link. The entries of this matrix are known because the hand coordinates in the goal position are specified. The left-hand side of the above matrix equation describes coordinate system 7 with respect to coordinate system 1, in terms of the relative arrangements of the intermediate coordinate systems. The quantities a_i, d_i, λ_i, μ_i, $i = 1, \ldots, 6$, appearing in the matrices on the left-hand side of Eq. (2) are all known. The unknown quantities are θ_i, $i = 1, \ldots, 6$, and the above matrix equation must be solved for them. This is the inverse kinematics problem for the 6R series manipulator.

Derivation of $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{l}}$

The matrix Eq. (2), is equivalent to 12 scalar equations. Of these only 6 equations are independent, because the submatrix comprised of the first 3 rows and columns is orthogonal. We perform operations on these multivariate equations and eliminate all but one variable, thus reducing the problem to the solution of a single equation in one variable. This is in the spirit of Gauss Elimination applied to systems of linear equations, except that the equations in this problem are nonlinear.

Let A_{hand} be equal to $\begin{pmatrix} l_x & m_x & n_x & \rho_x \\ l_y & m_y & n_y & \rho_y \\ l_x & m_z & n_x & \rho_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Equation (2) may be rewritten as

$$A_3A_4A_5 = A_2^{-1}A_1^{-1}A_{\text{hand}}A_6^{-1} \tag{3}$$

We do this so as to move θ_1, θ_2, and θ_6 to the right-hand side. This lowers the degrees of the equations and also reduces their complexity.

There is another preliminary concern which simplifies matters. It is useful to remember that an A_i matrix can be written as the product of two matrices: $A_i = A_{iv}A_{is}$, in which A_{is} contains only the three structural Denavit and Hartenberg parameters, and A_{iv} contains only the joint variable. We have for a revolute joint and prismatic joint, respectively:

$$A_{iv} = \begin{pmatrix} c_i & -s_i & 0 & 0 \\ s_i & c_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_{is} = \begin{pmatrix} 1 & 0 & 0 & a_i \\ 0 & \lambda_i & -\mu_i & 0 \\ 0 & \mu_i & \lambda_i & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$A_{iv} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_{is} = \begin{pmatrix} c_i & -s_i\lambda_i & s_i\mu_i & a_ic_i \\ s_i & c_i\lambda_i & -c_i\mu_i & a_is_i \\ 0 & \mu_i & \lambda_i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Substituting $A_i = A_{iv}A_{is}$ for the matrix at one end of the equation, makes it simple to have a link's motion parameter isolated from its structural parameters; the advantage is that the motion parameter then appears in a simple form. So we premultiply (3) by A_{2s} in order to simplify the structures of the θ_2 terms. The resulting equation is then

$$A_{2s}A_3A_4A_5 = A_{2v}^{-1}A_1^{-1}A_{\text{hand}}A_6^{-1}. \tag{4}$$

When the matrix multiplications are carried out, Eq. (4) has the form

$$\begin{pmatrix} (\theta_3,\theta_4,\theta_5) & (\theta_3,\theta_4,\theta_5) & (\theta_3,\theta_4,\theta_5) & (\theta_3,\theta_4,\theta_5) \\ (\theta_3,\theta_4,\theta_5) & (\theta_3,\theta_4,\theta_5) & (\theta_3,\theta_4,\theta_5) & (\theta_3,\theta_4,\theta_5) \\ (\theta_4,\theta_5) & (\theta_4,\theta_5) & (\theta_4,\theta_5) & (\theta_4,\theta_5) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (\theta_1,\theta_2,\theta_6) & (\theta_1,\theta_2,\theta_6) & (\theta_1,\theta_2) & (\theta_1,\theta_2) \\ (\theta_1,\theta_2,\theta_6) & (\theta_1,\theta_2,\theta_6) & (\theta_1,\theta_2) & (\theta_1,\theta_2) \\ (\theta_1,\theta_2,\theta_6) & (\theta_1,\theta_2,\theta_6) & (\theta_1,\theta_2) & (\theta_1,\theta_2) \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

Equation (5) shows the variables appearing in the individual entries in Eq. (3). The 6 scalar equations obtained from columns 3 and 4 of Eq. (3) are devoid of θ_6. These equations though linearly independent are governed by the constraint that the magnitude of the column 3 vector is unity. We work with these 6 equations with the goal of eliminating 4 of the 5 variables so as to obtain a univariate polynomial which will vanish at their common zeros. The 6 equations are:

$$\begin{pmatrix} c_2 & s_2 & 0 \\ s_2 & -c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{\mathbf{h}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\lambda_2 & \mu_2 \\ 0 & \mu_2 & \lambda_2 \end{pmatrix} \begin{pmatrix} c_3 & s_3 & 0 \\ s_3 & -c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{\mathbf{f}} + \begin{pmatrix} a_2 \\ 0 \\ d_2 \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} c_2 & s_2 & 0 \\ s_2 & -c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{\mathbf{n}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\lambda_2 & \mu_2 \\ 0 & \mu_2 & \lambda_2 \end{pmatrix} \begin{pmatrix} c_3 & s_3 & 0 \\ s_3 & -c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{\mathbf{r}} \tag{7}$$

where $\tilde{\mathbf{h}} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$, $\tilde{\mathbf{f}} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$, $\tilde{\mathbf{n}} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$,

$$f_1 = c_4g_1 + s_4g_2 + a_3$$
$$f_2 = -\lambda_3(s_4g_1 - c_4g_2) + \mu_3g_3$$
$$f_3 = \mu_3(s_4g_1 - c_4g_2) + \lambda_3g_3 + d_3$$
$$r_1 = c_4m_1 + s_4m_2$$
$$r_2 = -\lambda_3(s_4m_1 - c_4m_2) + \mu_3m_3$$
$$r_3 = \mu_3(s_4m_1 - c_4m_2) + \lambda_3m_3$$
$$g_1 = c_5a_5 + a_4$$
$$g_2 = -s_5\lambda_4a_5 + \mu_4d_5$$
$$g_3 = s_5\mu_4a_5 + \lambda_4d_5 + d_4$$
$$m_1 = s_5\mu_5$$
$$m_2 = c_5\lambda_4\mu_5 + \mu_4\lambda_5$$
$$m_3 = -c_5\mu_4\mu_5 + \lambda_4\lambda_5$$

$$h_1 = c_1 p + s_1 q - a_1$$
$$h_2 = -\lambda_1 (s_1 p - c_1 q) + \mu_1 (r - d_1)$$
$$h_3 = \mu_1 (s_1 p - c_1 q) + \lambda_1 (r - d_1)$$
$$n_1 = c_1 u + s_1 v$$
$$n_2 = -\lambda_1 (s_1 u - c_1 v) + \mu_1 w$$
$$n_3 = \mu_1 (s_1 u - c_1 v) + \lambda_1 w$$
$$p = -l_x a_6 - (m_x \mu_6 + n_x \lambda_6) d_6 + \rho_x$$
$$q = -l_y a_6 - (m_y \mu_6 + n_y \lambda_6) d_6 + \rho_y$$
$$r = -l_z a_6 - (m_z \mu_6 + n_z \lambda_6) d_6 + \rho_z$$
$$u = m_x \mu_6 + n_x \lambda_6$$
$$v = m_y \mu_6 + n_y \lambda_6$$
$$w = m_z \mu_6 + n_z \lambda_6$$

Henceforth we will refer to Eqs. (6) and (7) by the vectors $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{l}}$ respectively. By this we mean

$$\tilde{\mathbf{p}} = \begin{pmatrix} c_2 & s_2 & 0 \\ s_2 & -c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{\mathbf{h}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\lambda_2 & \mu_2 \\ 0 & \mu_2 & \lambda_2 \end{pmatrix} \begin{pmatrix} c_3 & s_3 & 0 \\ s_3 & -c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{\mathbf{f}} + \begin{pmatrix} a_2 \\ 0 \\ d_2 \end{pmatrix}$$

and

$$\tilde{\mathbf{l}} = \begin{pmatrix} c_2 & s_2 & 0 \\ s_2 & -c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{\mathbf{n}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\lambda_2 & \mu_2 \\ 0 & \mu_2 & \lambda_2 \end{pmatrix} \begin{pmatrix} c_3 & s_3 & 0 \\ s_3 & -c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{\mathbf{r}}$$

Let p_1, p_2, and l_1, l_2, l_3 be respectively the components of $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{l}}$. We now proceed to eliminate 4 of the 5 variables in $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{l}}$ by exploiting the structure of the ideal generated by the component equations of $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{l}}$.

The Ideal of $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{l}}$

It is noteworthy that each of f_1, f_2, f_3, r_1, r_2, r_3, is a linear combination of the terms s_4s_5, s_4c_5, c_4s_5, c_4c_5, s_4, c_4, s_5, c_5, 1. Similarly each of h_1, h_2, h_3, n_1 n_2, n_3 is a linear combination of the terms s_1, c_1, 1. Equations (6) and (7) taken together may be written in matrix form as:

$$(A) \begin{bmatrix} s_4s_5 \\ s_4c_5 \\ c_4s_5 \\ c_4c_5 \\ s_4 \\ c_4 \\ s_5 \\ c_5 \\ 1 \end{bmatrix} = (B) \begin{bmatrix} s_1s_2 \\ s_1c_2 \\ c_1s_2 \\ c_1c_2 \\ s_1 \\ c_1 \\ s_2 \\ c_2 \end{bmatrix} \quad (8)$$

where A is a 6×9 matrix whose entries are linear combinations of $s_3 c_3$, 1, and B is a 6×8 matrix whose entries are all constants.

The ideal generated by a set of polynomials $q_1, q_2, \ldots, q_r$, in the variables $x_1, x_2 \ldots, x_m$ is the set of all elements of the form $q_1\beta_1 + q_2\beta_2 + \ldots + q_r\beta_r$, where $\beta_1, \beta_2, \ldots, \beta_r$ are arbitrary elements of the set of all polynomials in $x_1, x_2, \ldots, x_m$.

The ideal generated by the component equations of $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{l}}$ has the following interesting properties:

Property 1: $\tilde{\mathbf{p}} \cdot \tilde{\mathbf{p}}$ has the same power products as $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{l}}$ (i.e., Eq. (8)). By power products we mean "terms" (e.g., the power products of the polynomial $5x^2y + 3xz + 9y^2 + 4z = 0$ are x^2y, xz, y^2 and z.) Therefore the equation $\tilde{\mathbf{p}} \cdot \tilde{\mathbf{p}}$ may be written in the form

$$(M) \begin{bmatrix} s_4s_5 \\ s_4c_5 \\ c_4s_5 \\ c_4c_5 \\ s_4 \\ c_4 \\ s_5 \\ c_5 \\ 1 \end{bmatrix} = (N) \begin{bmatrix} s_2s_2 \\ s_1c_2 \\ c_1s_2 \\ c_1c_2 \\ s_1 \\ c_1 \\ s_2 \\ c_2 \end{bmatrix}$$

where M is a 1×9 matrix whose entries are linear combinations of s_3, c_3, 1 and N is a 1×8 matrix whose entries are constants.

Property 2: $\tilde{\mathbf{p}} \cdot \tilde{\mathbf{l}}$ has the same power products as $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{l}}$.

Property 3: $\tilde{\mathbf{p}} \times \tilde{\mathbf{l}}$ has the same power products as $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{l}}$.

Property 4: $(\tilde{\mathbf{p}} \cdot \tilde{\mathbf{p}})\tilde{\mathbf{l}} - (2\tilde{\mathbf{p}} \cdot \tilde{\mathbf{l}})\tilde{\mathbf{p}}$ has the same power products as $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{l}}$.

A rigorous justification of properties 1–4 is presented in Raghavan and Roth (1989).

We now have the following set of linearly independent equations all of which have the same power products:

Vector	No. of Scalar Equations
$\tilde{\mathbf{p}}$	3
$\tilde{\mathbf{l}}$	3
$\tilde{\mathbf{p}} \cdot \tilde{\mathbf{p}}$	1
$\tilde{\mathbf{p}} \cdot \tilde{\mathbf{l}}$	1
$\tilde{\mathbf{p}} \times \tilde{\mathbf{l}}$	3
$(\tilde{\mathbf{p}} \cdot \tilde{\mathbf{p}})\tilde{\mathbf{l}} - (2\tilde{\mathbf{p}} \cdot \tilde{\mathbf{l}})\tilde{\mathbf{p}}$	3
	(Total) 14

These 14 equations may be written in matrix form as

$$(P) \begin{bmatrix} s_4s_5 \\ s_4c_5 \\ c_4s_5 \\ c_4c_5 \\ s_4 \\ c_4 \\ s_5 \\ c_5 \\ 1 \end{bmatrix} = (Q) \begin{bmatrix} s_1s_2 \\ s_1c_2 \\ c_1s_2 \\ c_1c_2 \\ s_1 \\ c_1 \\ s_2 \\ c_2 \end{bmatrix} \quad (9)$$

where P is a 14×9 matrix whose entries are linear combinations of s_3, c_3, 1 and Q is a 14×8 matrix whose entries are all constants. We now proceed to eliminate variables sequentially from the above equations.

Elimination of θ_1 and θ_2

We use any 8 of the 14 equations in Eq. (9) to solve for the 8 right-hand side terms containing θ_1 and θ_2 in terms of the left-hand side which is a function of θ_3, θ_4, and θ_5. We use these to eliminate terms containing θ_1 and θ_2 from the remaining 6 equations, which then take the form:

$$(\Sigma) \begin{bmatrix} s_4s_5 \\ s_4c_5 \\ c_4s_5 \\ c_4c_5 \\ s_4 \\ c_4 \\ s_5 \\ c_5 \\ 1 \end{bmatrix} = 0 \quad (10)$$

where Σ is a 6×9 matrix whose entries are linear combinations of s_3, c_3, 1.

Elimination of θ_4 and θ_5

We make the following substitutions in Eq. (10):

$$s_4 \leftarrow \frac{2x_4}{1+x_4^2}, c_4 \leftarrow \frac{1-x_4^2}{1+x_4^2}, s_5 \leftarrow \frac{2x_5}{1+x_5^2}, c_5 \leftarrow \frac{1-x_5^2}{1+x_5^2}$$

where $x_4 = \tan\left(\frac{\theta_4}{2}\right), x_5 = \tan\left(\frac{\theta_5}{2}\right)$.

We then multiply each equation by $(1 + x_4^2)$ and $(1 + x_5^2)$ to clear denominators. Equation (10) then takes the form

$$(\Sigma') \begin{pmatrix} x_4^2x_5^2 \\ x_4^2x_5 \\ x_4^2 \\ x_4x_5^2 \\ x_4x_5 \\ x_4 \\ x_5^2 \\ x_5 \\ 1 \end{pmatrix} = 0 \qquad (11)$$

where Σ' is a 6×9 matrix whose entries are linear combinations of s_3, c_3, 1. We make the following substitutions in Eq. (11):

$$s_3 \leftarrow \frac{2x_3}{1+x_3^2}, c_3 \leftarrow \frac{1-x_3^2}{1+x_3^2}$$

We multiply the first 4 scalar equations in Eq. (11) by $(1 + x_3^2)$ to clear denominators. The resulting equation is of the form

$$(\Sigma'') \begin{pmatrix} x_4^2x_5^2 \\ x_4^2x_5 \\ x_4^2 \\ x_4x_5^2 \\ x_4x_5 \\ x_4 \\ x_5^2 \\ x_5 \\ 1 \end{pmatrix} = 0 \qquad (12)$$

where Σ'' is a 6×9 matrix. The entries in the first 4 rows of Σ'' are quadratic polynomials in x_3. The entries in the last 2 rows are rational functions of x_3, the numerators being quadratic polynomials in x_3, the denominators being $(1 + x_3^2)$. It is noteworthy that the determinant of the 6×6 array comprised of any set of 6 columns of Σ'' is always an 8th degree polynomial and not a rational function. This fact is proved rigorously in Raghavan and Roth (1989).

We now eliminate x_4 and x_5 dialytically [see Salmon (1885)] as follows: Multiplying Eq. (12) by x_4, we get the following equation

$$(\Sigma'') \begin{pmatrix} x_4^3x_5^2 \\ x_4^3x_5 \\ x_4^3 \\ x_4^2x_5^2 \\ x_4^2x_5 \\ x_4^2 \\ x_4x_5^2 \\ x_4x_5 \\ x_4 \end{pmatrix} = 0 \qquad (13)$$

Equations (12) and (13) taken together may be written in matrix form as:

$$\begin{pmatrix} \Sigma'' & 0 \\ 0 & \Sigma'' \end{pmatrix} \begin{pmatrix} x_4^3x_5^2 \\ x_4^3x_5 \\ x_4^3 \\ x_4^2x_5^2 \\ x_4^2x_5 \\ x_4^2 \\ x_4x_5^2 \\ x_4x_5 \\ x_4 \\ x_5^2 \\ x_5 \\ 1 \end{pmatrix} = 0 \qquad (14)$$

Equation (14) constitutes a set of 12 linearly independent equations in the 12 terms $x_4^3x_5^2$, $x_4^3x_5$, x_4^3, $x_4^2x_5^2$, $x_4^2x_5$, x_4^2, $x_4x_5^2$, x_4x_5, x_4, x_5^2, x_5, 1. This is clearly an overconstrained linear system. In order for this system to have a nontrivial solution, the coefficient matrix must be singular. The determinant of the coefficient matrix is a 16th degree polynomial in x_3. The roots of this polynomial give the values of x_3 corresponding to the 16 solutions of the inverse kinematics problem. For each value of x_3 thus obtained, the corresponding value of θ_3 may be computed using the formula $\theta_3 = 2\tan^{-1}x_3$.

The Remaining Joint Variables

For each value of θ_3, we may compute the remaining joint variables as follows: We substitute the numerical value of θ_3 in the coefficient matrix of Eq. (14). We then use 11 independent members of Eq. (14) to solve for the 11 terms $x_4^3x_5^2$, $x_4^3x_5$, x_4^3, $x_4^2x_5^2$, $x_4^2x_5$, x_4^2, $x_4x_5^2$, x_4x_5, x_4, x_5^2, x_5. The numerical values of x_4 and x_5 may be used to compute θ_4 and θ_5. We then substitute numerical values for θ_3, θ_4, and θ_5 in Eq. (9). We use 8 linearly independent members of the resulting equation to solve for s_1s_2, s_1c_2, c_1s_2, c_1c_2, s_1, c_1, s_2 and c_2. We then use the numerical values of s_1 and c_1 to obtain a unique value for θ_1. Similarly, θ_2 may be computed using the numerical values of s_2 and c_2. Finally, substituting values for θ_1, θ_2, θ_3, θ_4, and θ_5 in the (1, 1) and (2, 1) elements of the following equation

$$A_6 = A_5^{-1}A_4^{-1}A_3^{-1}A_2^{-1}A_1^{-1}A_{hand} \qquad (15)$$

yields 2 linear equations in s_6 and c_6. After solving for s_6 and c_6 we may use their values to determine a unique value for θ_6.

Characteristic Polynomial

It is important to notice that in the foregoing solution method all the joint variables are obtained from linear equations once the suppressed variable, θ_3, is known. The 16th degree polynomial we get by expanding the determinant $\begin{vmatrix} \Sigma'' & 0 \\ 0 & \Sigma'' \end{vmatrix} = 0$, determines the number of solution sets for each hand position. We call this polynomial the characteristic polynomial, since it characterizes the number of possible solution sets. The maximum number of possible values for the variable θ_3 is 16, i.e., the degree of the characteristic polynomial. In general not all of the characteristic polynomial's roots will be real, and the actual number of solutions will often be much less than 16. If the specified hand position is not reachable all of the roots will be imaginary (i.e., the roots are all complex numbers), signifying that it is not physically possible for the manipulator to place the hand in the specified position.

Although the number of real roots of the characteristic polynomial usually determines the actual number of solution sets, of the variables that were eliminated during the process of obtaining the characteristic polynomial, this is not always the case, and for certain types of manipulators the number of solution sets is actually double the number of real roots of the characteristic polynomial. This occurs when some joint variables become multivalued with respect to the suppressed variable. Mathematically this occurs whenever we have only a single linear equation in either the sine or the cosine of an angle. For an angle to be a single-valued function of the suppressed variable it is necessary for there to be two linear conditions on its sine and/or cosine, or a single linear condition on the tangent of its half-angle.

Alternative Equations

The physical meaning of our equations $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{l}}$ are immediately obvious from (4). Let $2'$ represent a coordinate system which differs from the Denavit and Hartenberg 2-system by having its x-axis rotated parallel to the x-axis of the 3-system. Clearly $\tilde{\mathbf{p}}$ represents the coordinates in the $2'$ system of a vector from the origin of the Denavit and Hartenberg coordinates in the 2 system to the origin in the 6 system, and $\tilde{\mathbf{l}}$ represents the directions of a unit vector parallel to the z-axis in the 6 system as measured along axes parallel to the $2'$ system. The right- and left-hand sides of $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{l}}$ simply give the measures of these vectors in terms of the variables denoted by the subscripts on the two sides of (4); clearly these vectors must have equal measures if we transform successive coordinates moving inward toward the base, or moving out toward the hand and then through the base outward toward the hand.

There is nothing special about the choice of the 2, 3, and 6 coordinate systems. The all-important power product properties 1–4 are a function of the structure of the A_i matrices, see Raghavan (1990). So it is possible to change (4) in a cyclic manner and still maintain the equality of the power products in the resulting sets of 14 scalar equations. In nonsymmetrical situations (for example, the 3rd and 5th joints are prismatic and all the others are revolutes) it may be convenient to rearrange (4) by pre- and post-multiplying by the A_i's or their inverses in order to move matrices from one side to the other. Of course if we change (4), the systems $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{l}}$ are measured in, and what they actually physically measure changes accordingly.

Finally, it is important to notice that a joint angle can always be eliminated from the equations for $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{l}}$ by simply rearranging terms so that the corresponding A_i matrix appears in inverse form at the right-hand most position, on either side; as A_6 does in (4).

Prismatic Joints

The foregoing method can be used for problem formulation and solution when determining the inverse kinematics of a series-chain manipulator which has one or more prismatic joints. This is because the basic mathematical structure of the 14 scalar equations remains the same regardless of whether a joint is revolute or prismatic. If the ith joint is a revolute joint then we have c_i and s_i as the variables which form the power products, while if it is prismatic we have d_i and d_i^2 instead. If we simply regroup the terms so that d_i and d_i^2 rather than c_i and s_i are treated as the variables, we obtain exactly the same number of power products. This means that the structure of our 14 equations is the same if we have revolute or prismatic joints. It is for this reason that changing one of the revolutes in the 6R to a prismatic joint does not alter the degree of the basic characteristic polynomial, it remains at 16.

However, the degree of the characteristic polynomial is reduced if we have two or more prismatic joints in a six-degree-of-freedom manipulator. The reason for this is clear if we note that the d_i variables cannot appear in the three components of the $\tilde{\mathbf{l}}$ equation. (Since these equations are from the rotation part of the matrix they must be independent of the joint translations.) Moreover, each d_i appears as a linear variable in the $\tilde{\mathbf{p}}$ equations and no products such as d_id_j are possible in $\tilde{\mathbf{p}}$ [by virtue of the structure of (1)]. It therefore follows that:

(a) $\tilde{\mathbf{l}}$ does not contain any of the d_i,

(b) $\tilde{\mathbf{p}}$, $\tilde{\mathbf{p}} \cdot \tilde{\mathbf{l}}$ and $\tilde{\mathbf{p}} \times \tilde{\mathbf{l}}$ contain only the d_i and not any d_i^2,

(c) $\tilde{\mathbf{p}} \cdot \tilde{\mathbf{p}}$ and $(\tilde{\mathbf{p}} \cdot \tilde{\mathbf{p}})\tilde{\mathbf{l}} - (2\tilde{\mathbf{p}} \cdot \tilde{\mathbf{l}})\tilde{\mathbf{p}}$ generally contain both d_i and d_i^2 terms, and can contain d_id_j terms [if the two prismatic-joint-matrices are both on the same side of the equal sign in (4)] but cannot contain any $d_i^2d_j$ or $d_i^2d_j^2$ terms.

Thus if we have two prismatic joints we have less power products than in the 6R case, and this leads to a characteristic polynomial of lower degree. In Raghavan and Roth (1990) we show that when there are two prismatic joints and four revolute joints, the manipulator's characteristic polynomial is of degree eight and not sixteen. This is in agreement with the results in Duffy (1980), where most of these cases were first treated in a systematic manner. In Appendix 1 we briefly outline a procedure which yields the 8th degree characteristic polynomial for the RPRRPR manipulator. This analysis serves to illustrate one way in which the number of power products can diminish as we add a second prismatic joints.

For three prismatic joints the analysis becomes very simple since the $\tilde{\mathbf{l}}$ equations contain only the angular-displacement parameters. It always turns out that the $\tilde{\mathbf{l}}$ equation can be written so that one of its three components is a linear equation in the sine and/or cosine of one joint angle. This then can be rewritten as a second degree polynomial in the tangent of the half-angle, and it becomes the characteristic polynomial. Once the two roots of the characteristic polynomial are determined all the other variables follow linearly from the remaining two components of $\tilde{\mathbf{l}}$, from the three scalar components of $\tilde{\mathbf{p}}$, and from (4) as usual, Raghavan and Roth (1990).

Characteristic Polynomials of Lower Degree

We have seen that generally 6R and 5R,P manipulators have 16-degree characteristic polynomials, 4R,2P manipulators have 8-degree characteristic polynomials, and 3R,3P manipulators have characteristic polynomials of degree 2. (We have used a comma to signify that the prismatic joints can be located any place in the chain.) However, if a manipulator has special geometry, it is possible for its characteristic polynomial to be of lower degree than for the same joint types and number under a general geometry. This loss of degree is manifested in our solution method in three possible ways: a lowered number of power products in the initial equations for $\tilde{\mathbf{l}}$ and $\tilde{\mathbf{p}}$, a lowered number of power products during the elimination phase, and/or the coefficients of the highest order terms of the characteristic polynomial become zero.

Furthermore, it is also possible under special geometry to lose constraint equations such that some of the joint variables, which are generally single-valued functions, become instead double-valued for each real root of the characteristic polynomial in the suppressed variable.

We illustrate these effects by considering the case of a 6R manipulator with a wrist, i.e., the last three joint axes always intersect in a common point. If we substitute the special geometry conditions for this case, i.e., $a_4 = a_5 = 0$, $d_5 = 0$, we find that $\tilde{\mathbf{p}}$ becomes a function of only θ_1, θ_2 and θ_3. In terms of power products the third component of $\tilde{\mathbf{p}}$ yields

$$(A') = (B') \begin{pmatrix} s_1 \\ c_1 \end{pmatrix}, \tag{16}$$

where A' is a 1×1 matrix with entries that are linear combinations of s_3, c_3, 1, and B' is a 1×2 matrix with entries that are constants. Furthermore, $\tilde{\mathbf{p}} \cdot \tilde{\mathbf{p}}$ yields

$$(M') = (N') \begin{pmatrix} s_1 \\ c_1 \end{pmatrix}, \tag{17}$$

where M' is a 1×1 matrix with entries that are linear combinations of s_3, c_3, 1, and N' is a 1×2 matrix with entries that are constants. If we substitute the tangent of the half-angle formulas for s_1 and c_1 we obtain, after clearing the denominators

$$(A'')\begin{pmatrix} x_1^2 \\ x_1 \\ 1 \end{pmatrix} = 0, \text{ and } (M'')\begin{pmatrix} x_1^2 \\ x_1 \\ 1 \end{pmatrix} = 0, \tag{18}$$

respectively. Where $x_1 = \tan(\theta_1/2)$ and A'' and M'' are 1×3 matrices with entries that contain linear combinations of s_3, c_3, 1.

Using the dialytic elimination technique, we multiply each of these equations by x_1, and thereby obtain two additional equations. Now we have a set of four equations which can be written:

$$\begin{pmatrix} A'' & & 0 \\ 0 & A'' & \\ M'' & & 0 \\ 0 & M'' & \end{pmatrix}\begin{pmatrix} x_1^3 \\ x_1^2 \\ x_1 \\ 1 \end{pmatrix} = 0. \tag{19}$$

Setting the determinant of the coefficient matrix to zero and substituting the tangent of the half-angle functions for s_3 and c_3, and then clearing the denominator by multiplying by $(1 + x_3^2)^2$, yields a characteristic polynomial of (only) degree 4 in x_3.

Generally, for each real root of x_3, a unique value of θ_1 follows from (16) and (17), and a unique value of θ_2 follows from the other two components of $\tilde{\mathbf{p}}$. However, if either $a_1 = 0$ or $\alpha_1 = 0$ then N' or B' are respectively zero, and we have to determine θ_1 from, respectively, either (16) or (17). This means we get two values of θ_1 for each root of x_3. It turns out that in this case the characteristic polynomial degenerates to a quadratic, and we have at most two values for θ_3 at each hand position.

In order to obtain θ_4 and θ_5 we use $\tilde{\mathbf{l}}$. Since we know θ_1, θ_2, and θ_3, we can write $\tilde{\mathbf{l}}$ in the form

$$(A''')\begin{pmatrix} s_4 \\ c_4 \\ 1 \end{pmatrix} = 0, \tag{20}$$

where A''' is a 3×3 matrix with elements which are linear functions of s_5, c_5, 1. It follows that we require that $\mathrm{Det}(A''') = 0$. It can be shown, see Appendix 2, that this yields a quadratic polynomial in x_5. Once the values of θ_5 are obtained from this polynomial, we obtain a unique value of θ_4 (corresponding to each set of θ_1, θ_2, θ_3 and θ_4) from (20). Finally θ_6 follows as usual from (4).

Conclusions

We have presented a method for obtaining a characteristic polynomial for any series-manipulator with six joints which are combinations of revolute or prismatic joints. By virtue of the analogy to the closed-loop linkage analysis problem, this method will also determine the motion variables for closed-loop spatial mechanism. In spatial mechanisms one finds additional types of joints, other than simply revolute and prismatic. However by virtue of the fact that any lower pair joint can be modeled as a combination of revolute and prismatic joints, it is clear that this analysis can be applied fairly broadly.

Interestingly, joints such as cylindric joints and spherical joints represent special geometries, and for these the characteristic polynomials tend to reduce in degree. So for example the RPRRPR manipulator analysis presented in Appendix 1, also represents the analysis of a spatial 7-bar RRPRRPR. However if we take the P and a neighboring R axis as coincident, the same analysis also includes the spatial RCRCR 5-bar and the RRCCR 5-bar as well as the RRCRC and RCRRC 5-bars, and the RCRRPR, CRRPR, RPRCR and RPRRC 6-bars. The same computer program which determines the RPRRPR characteristic polynomial and manipulator analysis, immediately gives the linkage analysis for all of these linkages. Analogous results follow from all the manipulators discussed in this paper.

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References

Albala, H., and Angeles, J., 1979, "Numerical Solution to the Input-Output Displacement Equation of the General 7R Spatial Mechanism," *Proceedings of the Fifth World Congress on Theory of Machines and Mechanisms*, pp. 1008–1011.

Craig, J. J., 1989, *Introduction to Robotics*, 2nd edition, Addison Wesley, Reading, MA.

Duffy, J., 1980, *Analysis of Mechanisms and Manipulators*, Wiley, New York.

Duffy, J., and Crane, C., 1980, "A Displacement Analysis of the General Spatial 7R Mechanism," *Mechanisms and Machine Theory*, Vol. 15, pp. 153–169.

Lee, H-Y., and Liang, C-G., 1988a, "A New Vector Theory for the Analysis of Spatial Mechanisms," *Mechanisms and Machine Theory*, Vol. 23, No. 3, pp. 209–217.

Lee, H-Y., and Liang, C-G., 1988b, "Displacement Analysis of General Spatial 7-Link 7R Mechanism," *Ibid*, pp. 219–226.

Paul, R. P., 1981, *Robot Manipulators*, MIT Press, Cambridge, MA.

Pieper, D., 1968, *The Kinematics of Manipulators Under Computer Control*, Ph.D. Thesis, Stanford University.

Pieper, D. L., and Roth, B., 1969, "The Kinematics of Manipulators Under Computer Control," *Proceedings of the 2nd International Congress for the Theory of Machines and Mechanisms*, Zakopane, Poland, Vol. 2, pp. 159–168.

Raghavan, M., and Roth, B., 1989, "Kinematic Analysis of the 6R Manipulator of General Geometry," *Proceedings of the 5th International Symposium on Robotics Research*, H. Miura and S. Arimoto, eds., MIT Press, preprint 1989, final 1990.

Raghavan, M., 1990, "Manipulator Kinematics," *Proceedings of the AMS Sympqsium on Mathematical Questions in robotics*, R. Brockett, ed., Louisville, KY, PSAM Vol. 41, American Mathematical Society, Providence.

Raghavan, M., and Roth, B., 1990, "A General Solution for the Inverse Kinematics of all Series Chains," *Proceedings of the Eighth CISM-IFTOMM Symposium on Robots and Manipulators (ROMANSY-90)*, Cracow, Poland.

Roth, B., Rastegar, J., and Scheinman, V., 1973, "On the Design of Computer Controlled Manipulators," *On the Theory and Practice of Robots and Manipulators*, Vol. 1, First CISM-IFToMM Symposium, September, pp. 93–113.

Salmon, G., 1885, *Lessons Introductory to the Modern Higher Algebra*, Chelsea Publishing Co., New York.

Tsai, L-W., and Morgan, A., 1989, "Solving the Kinematics of the Most General Six- and Five-Degree-of-Freedom Manipulators by Continuation Methods," ASME JOURNAL OF MECHANISMS, TRANSMISSIONS, AND AUTOMATION IN DESIGN, Vol. 107, June, pp. 189–200.

Yang, A. T., and Freudenstein, F., 1964, "Application of Dual-Number Quaternion Algebra to the Analysis of Spatial Mechanisms," ASME *Journal of Applied Mechanics*, Vol. 86, pp. 300–308.

APPENDIX 1

Characteristic Polynomial for the RPRRPR Manipulator

For this case the d_5^2 and d_2^2 terms do not appear in the power products of the 10 scalar equations obtained from $\tilde{\mathbf{l}}$, $\tilde{\mathbf{p}}$, $\tilde{\mathbf{p}} \cdot \tilde{\mathbf{l}}$, $\tilde{\mathbf{p}} \times \tilde{\mathbf{l}}$. Writing these 10 equations in matrix form yields a system of the following type:

$$
\begin{bmatrix}
0 & 0 & 0 & (s_3,c_3) & (s_3,c_3) & (s_3,1) \\
0 & 0 & 0 & (s_3,c_3,1) & (s_3,c_3,1) & (c_3,1) \\
0 & 0 & 0 & (s_3,c_3,1) & (s_3,c_3,1) & (c_3,1) \\
s_3 & c_3 & s_3 & (s_3,c_3) & (s_3,c_3) & (s_3,c_3,1) \\
(c_3,1) & s_3 & (c_3,1) & (s_3,c_3,1) & (s_3c_3,1) & (s_3,c_3,1) \\
(c_3,1) & s_3 & (c_3,1) & (s_3,c_3,1) & (s_3,c_3,1) & (s_3,c_3,1) \\
1 & 0 & 0 & 1 & 1 & 1 \\
s_3 & (s_3,c_3) & (s_3,c_3) & (s_3,c_3) & (s_3,c_3) & (s_3,c_3,1) \\
(c_3,1) & (s_3,c_3,1) & (s_3,c_3,1) & (s_3,c_3,1) & (s_3,c_3,1) & (s_3,c_3,1) \\
(c_3,1) & (s_3,c_3,1) & (s_3,c_3,1) & (s_3,c_3,1) & (s_3,c_3,1) & (s_3,c_3,1)
\end{bmatrix}
\begin{pmatrix} d_5 \\ d_5s_4 \\ d_5c_4 \\ s_4 \\ c_4 \\ 1 \end{pmatrix}
=
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0
\end{bmatrix}
\begin{pmatrix} s_1 \\ c_1 \\ d_2s_1 \\ d_2c_1 \\ d_2 \end{pmatrix}.
$$

In the (i,j)th element $(s_3, c_3, 1)$ represents $K_{1ij}s_3 + K_{2ij}c_3 + K_{3ij}$, where $K_{1ij}, K_{2ij}, K_{3ij}$, are constants that depend upon the manipulator's structural parameters. All terms with 1 simply imply that $K_{1ij} = K_{2ij} = 0$ for that element.

If we use the first two rows we can solve for s_1 and c_1. With these results the sixth row can be used to determine d_2. Finally the seventh and eighth rows can now be used to determine d_2c_1.

Substituting these results into rows 3, 4, 5,9, and 10 yields only 5 equations in the 6 power products (d_5, d_5s_4, d_5c_4, s_4, c_4, 1). However a sixth equation can be obtained if we multiply the equation from the third row by d_5. By virtue of the fact that this row did not originally have any d_5 terms, the resulting power products for this sixth equation are only (d_5s_4, d_5c_4, d_5). We now have a 6×6 system of the following form

$$
(\Sigma) \begin{bmatrix} d_5 \\ d_5s_4 \\ d_5c_4 \\ s_4 \\ c_4 \\ 1 \end{bmatrix} = 0.
$$

Here Σ is a 6×6 matrix with entries which are linear in s_3, c_3, 1. Introducing the tangent of the half-angle substitution for s_3 and c_3, and then clearing the denominators yields

$$
(\Sigma') \begin{bmatrix} d_5 \\ d_5s_4 \\ d_5c_4 \\ s_4 \\ c_4 \\ 1 \end{bmatrix} = 0.
$$

The characteristic polynomial follows from $|\Sigma'| = 0$. After removing the factor $(1 + x_3^2)^2$, this determinental equation yields a polynomial of degree 8 in x_3. For each real root of the characteristic polynomial, the other joint variables follow in the usual manner from the linear systems developed during the elimination process.

APPENDIX 2

Determining Angles θ_4 and θ_5 from $\tilde{\mathbf{l}}$ when θ_1, θ_2, and θ_3 Are Known

From our equation for $\tilde{\mathbf{l}}$, (7), it is clear that everything is known but $\tilde{\mathbf{r}}$. Thus we can easily determine numerical values for $\tilde{\mathbf{r}}$. Now with $\tilde{\mathbf{r}}$ known we can turn to the definition of $\tilde{\mathbf{r}}$:

$$r_1 = c_4m_1 + s_4m_2 \tag{21}$$

$$r_2 = -\lambda_3(s_4m_1 - c_4m_2) + \mu_3m_3 \tag{22}$$

$$r_3 = \mu_3(s_4m_1 - c_4m_2) + \lambda_3m_3 \tag{23}$$

This can be rewritten as

$$
\begin{pmatrix} m_2 & m_1 & -r_1 \\ -\lambda_3m_1 & \lambda_3m_2 & \mu_3m_3 - r_2 \\ \mu_3m_1 & -\mu_3m_2 & \lambda_3m_3 - r_3 \end{pmatrix} \begin{pmatrix} s_4 \\ c_4 \\ 1 \end{pmatrix} = 0.
$$

Setting the determinant of the coefficient matrix to zero yields,

$$(m_1^2 + m_2^2)(m_3 - \mu_3r_2 - \lambda_3r_3) = 0.$$

Since $(m_1^2 + m_2^2) \neq 0$ we have

$$m_3 - \mu_3r_2 - \lambda_3r_3 = 0.$$

Substituting from the definition of m_3 yields

$$-c_5\mu_4\mu_5 + (\lambda_4\lambda_5 - \mu_3r_2 - \lambda_3r_3) = 0,$$

from which two values of θ_5 follow for each set of r_2, r_3. For each θ_5 one value of θ_4 follows from (21)–(23).

Numerical Example

For an RPRRPR manipulator with parameters:

$a_1 = 1.46$	$\alpha_1 = 135$ (deg.)	$d_1 = 0.21$
$a_2 = 0.56$	$\alpha_2 = 78$	$\theta_2 = 65$ (deg.)
$a_3 = 0.38$	$\alpha_3 = 23$	$d_3 = 0.29$
$a_4 = 0.56$	$\alpha_4 = 46$	$d_4 = -.54$
$a_5 = 1.08$	$\alpha_5 = 35$	$\theta_5 = 54$
$a_6 = 0.67$	$\alpha_6 = 47$	$d_6 = 0.48$

When

$$
A_{hand} = \begin{pmatrix} -0.4435 & -0.6171 & 0.6500 & -1.443 \\ -0.1837 & 0.7724 & 0.6080 & 0.4665 \\ -0.8773 & 0.1502 & -0.4559 & -2.0579 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

The 8th degree characteristic polynomial is:

$$1.077x_3^8 - 3.547x_3^7 + 5.495x_3^6 - 16.37x_3^5 + 14.72x_3^4 - 21.93x_3^3 + 16.68x_3^2 - 9.217x_3 + 6.500 = 0$$

This polynomial has two real roots:

$$x_3 = 0.2979 \text{ and } x_3 = 0.805$$

Using these roots yields, respectively, the following two sets of joint variables:

$\theta_1 = 165.0$, $d_2 = 0.170$, $\theta_3 = 77.7$, $\theta_4 = 42.0$, $d_5 = -1.08$, $\theta_6 = -9.00$;

$\theta_1 = 181.2$, $d_2 = 0.340$, $\theta_3 = 142.9$, $\theta_4 = -21.5$, $d_5 = -0.264$, $\theta_6 = 12.9$