Solving the Kinematics of the Most General Six- and Five-Degree-of-Freedom Manipulators by Continuation Methods

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This paper presents a unique approach to the kinematic analysis of the most general six-degree-of-freedom, six-revolute-joint manipulators. Previously, the problem of computing all possible configurations of a manipulator corresponding to a given hand position was approached by reducing the problem to that of solving a high degree polynomial equation in one variable. In this paper it is shown that the problem can be reduced to that of solving a system of eight second-degree equations in eight unknowns. It is further demonstrated that this second-degree system can be routinely solved using a continuation algorithm. To complete the general analysis, a second numerical method—a continuation heuristic—is shown to generate partial solution sets quickly. Finally, in some special cases, closed form solutions were obtained for some commonly used industrial manipulators. The results can be applied to the analysis of both six and five-degree-of-freedom manipulators composed of mixed revolute and prismatic joints. The numerical stability of continuation on small second-degree systems opens the way for routine use in off-line robot programming applications.

Introduction

Two different types of problems exist in the kinematic analysis of manipulators. The first type is known as the direct-position problem, in which all the relative joint displacements are given and the positions of every link including the free end, the hand, are to be found. This type of problem can be easily solved by the matrix method of analysis [1]. The second type is known as the indirect-position problem, in which the position and orientation of the hand are given and the relative joint displacements are to be found. The indirect-position problem is more difficult to solve because the governing equations are very complicated and nonlinear.

In solving the indirect-position problem, we are always interested in obtaining a closed-form solution, i.e. an algebraic equation relating the given position and orientation of the hand to only one unknown joint displacement. In this manner all the possible solutions can be found. To achieve this goal, many different methods of analysis have been used. Pieper and Roth [2] and Pieper [3] pointed out that the analysis of an open-loop manipulator is related to the displacement analysis of a closed-loop spatial mechanism. In particular, the analysis of a six-degree-of-freedom, 6-revolute-joint (6-R) manipulator is equivalent to that of a single-loop, 7-revolute-joint (7-R) spatial mechanism. Therefore, all the methods of analysis used for spatial mechanisms, and the results, can be applied to the analysis of manipulators. The methods that are most commonly used are: screw algebra, dual numbers, dual matrices, dual quarternians, vector methods, (4×4) matrices, (3×3) matrices, etc. [4–11]. However, the ability to obtain closed-form solutions seems to be limited to manipulators having special geometry. For example, Pieper and Roth [2] applied the (4×4) matrix method to the six-degree-of-freedom manipulator with revolute and prismatic joints. They found the sufficient condition for a closed-form solution is to have any three adjacent joint axes intersecting at a common point. Duffy and Rooney [12] and Duffy [13] applied the sine and cosine laws of spatial triangles to the equivalent spatial mechanism of a manipulator and obtained closed-form solutions of many special manipulators. The most general 6-R manipulator problem, once referred to as the "Mount Everest" of kinematic problems [14], remained unsolved.

Nevertheless, Roth, et al. [15] concluded by deductive reasoning that, corresponding to each given position and orientation of the hand, there are at most 32 solutions, and that the degree of the polynomial relating one joint displacement to the hand position cannot be less than 32.

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Recently, Duffy and Crane [16] derived the solutions of the general 7-R single-loop spatial mechanism. The solution was obtained in an implicit form of a (16×16) determinant equated to zero. The determinant, when expanded, should yield a polynomial equation of 32nd degree. An attempt to expand the determinant was not successful. Albala [17, 18], using indicial notation, was able to derive the solutions of the general 7-R single-loop spatial mechanism similar to that given by Duffy [16]. Because of the complexity of the equation, the analysis of the general 6-R manipulator and 7-R spatial mechanism can only be solved by numerical techniques. Many different numerical iteration techniques have been investigated [11, 15, 18, 19], but the success of these numerical methods seems to depend on the initial guesses. For the analysis of closed-loop spatial mechanisms, where the iteration can be started from a known mechanism position, the methods work well. For the indirect-position problem of a manipulator, where the desired position of the hand is sometimes very far away from a known position of the manipulator or where there is no knowledge of the state of the manipulator, the numerical methods applied to date most probably will not work. In addition, even if they do work, there is almost no way of finding all the possible configurations.

Recently, however, there has been a breakthrough in the numerical solution of small systems of polynomial equations. Using certain continuation techniques, algorithms can be constructed that are *guaranteed* to find all isolated solutions to systems of n polynomial equations in n unknowns [20–26]. For background on continuation in general, see [27, 28]. For small systems, these algorithms can be implemented to run in practical CPU times.

In this paper, a system of eight second-degree equations in eight unknowns is derived whose solution set includes all solutions of the most general 6-R manipulator. This system is small enough to solve using continuation. The ease with which this can be done in practice is illustrated by solving a selection of test samples using a generic continuation code, SYMPOL.

Having established that the problem can be solved completely using SYMPOL, the question of speeding up the numerical method led to the development of a special continuation "heuristic," SYMMAN. SYMMAN is faster than SYMPOL but generally computes only a subset of the total set of configurations. The performance of the SYMMAN heuristic is compared with that of the SYMPOL algorithm in the reported numerical tests.

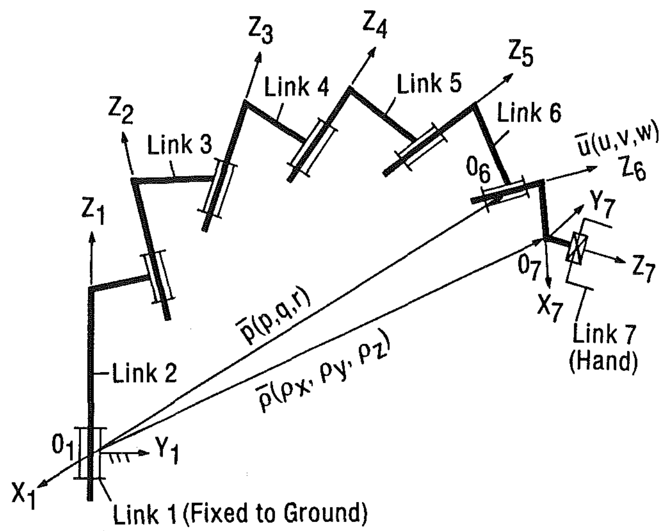

Fig. 1 A general 6-R manipulator

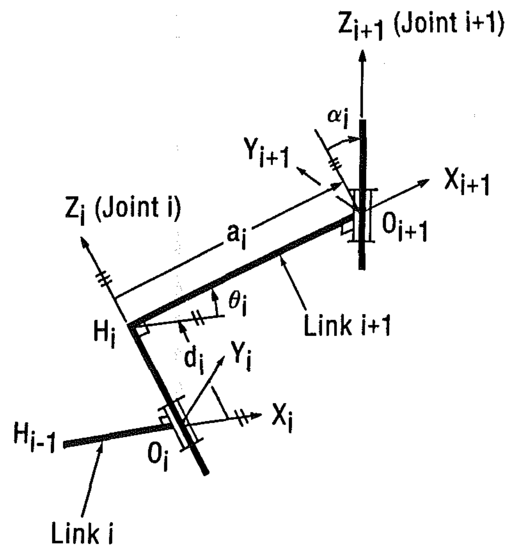

Fig. 2 The basic notation

The Fundamentals

A manipulator may be considered to consist of a group of rigid-bodies, or links, connected together by joints. The relative motion associated with each joint can be controlled such that the free-end, the hand, can be positioned in a desired manner. Although there are a variety of manipulators, we shall assume a manipulator has the form of an open-loop kinematic chain. Each link is connected to no more than two others, and the joints are either of the revolute or prismatic type. We shall begin our discussion with the general 6-revolute (6-R) manipulator and extend the results to manipulators with prismatic joints.

A 6-R manipulator has six moving links, numbered sequentially from 2 to 7, as shown in Fig. 1. Link 1 is designated as the base (fixed to ground) and link 7 as the hand of the manipulator. Every two neighboring links are connected by a joint. Every joint is associated with a joint axis Z_i, $i = 1 - 6$. Let Z_i and Z_{i+1} be two adjacent joint axes and H_iO_{i+1} be the directed common normal between Z_i and Z_{i+1}, where H_i is the intersection of H_iO_{i+1} and Z_i, and O_{i+1} is the intersection of H_iO_{i+1} and Z_{i+1}. Then, we can define the following linkage parameters as shown in Fig. 2 [1]:

a_i = the length of the common normal H_iO_{i+1} is the offset distance a_i

α_i = the angle to rotate the axis Z_i about the common normal H_iO_{i+1} so that Z_i is parallel to Z_{i+1}. The sign of rotation is given by the right-hand screw rule with the screw taken along the normal H_iO_{i+1}.

d_i = the distance between the two normals $H_{i-1}O_i$ and H_iO_{i+1} measured along Z_i. The sign of d_i is positive if O_iH_i points to the positive Z_i direction. Otherwise, d_i is negative.

θ_i = the angle to rotate the extended line of $H_{i-1}O_i$ about Z_i so that the extended line of $H_{i-1}O_i$ is parallel to H_iO_{i+1}. The sign of rotation is given by the right-hand screw rule with the screw pointing along the positive Z_i-axis.

If the ith joint is revolute, then a_i, d_i, and α_i are constant while θ_i is variable. If the ith joint is prismatic, then a_i, α_i, and θ_i are constant while d_i is a variable.

A coordinate system (X_i, Y_i, Z_i) is attached to each link of the manipulator as shown in Fig. 2. In each coordinate system, the Z_i-axis is defined to align with the ith joint axis; the X_i-axis along the extended line of $H_{i-1}O_i$; and the Y_i-axis according to the right-hand screw rule. The first coordinate system is fixed to ground. Since the common normal H_0O_1 does not exist, the X_1-axis is chosen perpendicular to Z_1, in an arbitrary manner. Also, we have attached a seventh coordinate system to the free-end to specify the position of the hand. The Z_7-axis lies in the direction from which the hand would approach an object, as shown in Fig. 1. The X_7-axis is defined by the common normal between the Z_6 and Z_7 axes, and the Y_7-axis according to the right-hand screw rule.

Let the coordinates of a point P expressed in the ith coordinate system be (p_{xi}, p_{yi}, p_{zi}) and in the $(i+1)$th coordinate system be $(p_{x\,i+1}, p_{y\,i+1}, p_{z\,i+1})$. Then the vectors $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ can be written in the (4×1) matrix forms as follows:

$$\mathbf{p}_i = \begin{bmatrix} p_{xi} \\ p_{yi} \\ p_{zi} \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{p}_{i+1} = \begin{bmatrix} p_{x\,i+1} \\ p_{y\,i+1} \\ p_{z\,i+1} \\ 1 \end{bmatrix} \tag{1}$$

The transformation of coordinates from the $(i + 1)$th system to the ith system is [1, 2]

$$\mathbf{p}_i = A_i \mathbf{p}_{i+1} \tag{2}$$

where A_i is a (4×4) matrix defined as follows:

$$A_i = \begin{bmatrix} c_i & -s_i\lambda_i & s_i\mu_i & a_ic_i \\ s_i & c_i\lambda_i & -c_i\mu_i & a_is_i \\ 0 & \mu_i & \lambda_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3}$$

where $c_i = \cos\theta_i$, $s_i = \sin\theta_i$, $\lambda_i = \cos\alpha_i$, and $\mu_i = \sin\alpha_i$.

Also, the inverse transformation exists:

$$\mathbf{p}_{i+1} = A_i^{-1}\mathbf{p}_i \tag{4}$$

where

$$A_i^{-1} = \begin{bmatrix} c_i & s_i & 0 & -a_i \\ -s_i\lambda_i & c_i\lambda_i & \mu_i & -d_i\mu_i \\ s_i\mu_i & -c_i\mu_i & \lambda_i & -d_i\lambda_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

Similarly, let the components of a unit vector $\mathbf{u}$ expressed in the ith coordinate system be (u_{xi}, u_{yi}, u_{zi}) and in the $(i + 1)$th coordinate system be $(u_{x\,i+1}, u_{y\,i+1}, u_{z\,i+1})$. Then, the vectors $\mathbf{u}_i$ and $\mathbf{u}_{i+1}$ can be written as

$$u_i = \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{zi} \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_{i+1} = \begin{bmatrix} u_{x\,i+1} \\ u_{y\,i+1} \\ u_{z\,i+1} \\ 0 \end{bmatrix} \tag{6}$$

And the transformation of coordinates can be written as

$$\mathbf{u}_i = A_i \mathbf{u}_{i+1} \tag{7}$$

and

$$\mathbf{u}_{i+1} = A_i^{-1}\, \mathbf{u}_i \tag{8}$$

Applying the matrix transformation to each pair of coordinate systems between two successive links and proceeding from link 7 to link 1, we obtain

$$\mathbf{p}_1 = A_1\, A_2\, A_3\, A_4\, A_5\, A_6\, \mathbf{p}_7 \tag{9}$$

Pieper and Roth [2] defined the equivalent matrix of transformation A_{eq} as

$$A_{eq} = A_1\, A_2\, A_3\, A_4\, A_5\, A_6 \tag{10}$$

Therefore, equation (9) can be written as

$$\mathbf{p}_1 = A_{eq}\, \mathbf{p}_7 \tag{11}$$

Similarly, the transformation of the unit vector can be written as

$$\mathbf{u}_1 = A_{eq}\mathbf{u}_7 \tag{12}$$

Since the equivalent matrix of transformation defines the relationship between the coordinates of any point in the seventh system, $\mathbf{p}_7$, and that of the same point expressed in the first system, $\mathbf{p}_1$, the matrix A_{eq} is known when the position and orientation of the hand is specified. Let $\boldsymbol{\rho}(\rho_x, \rho_y, \rho_z)$[1] be the position vector from the origin of the first system to the origin of the seventh system as shown in Fig. 1; and $\mathbf{l}(l_x, l_y, l_z)$, $\mathbf{m}(m_x, m_y, m_z)$ and $\mathbf{n}(n_x, n_y, n_z)$ be the three mutually perpendicular unit vectors aligned with the X_7, Y_7, and Z_7 axes, respectively. Then, when $\boldsymbol{\rho}$, $\mathbf{l}$, $\mathbf{m}$, and $\mathbf{n}$ are given in the first system, the equivalent matrix is given by

$$A_{eq} = \begin{bmatrix} l_x & m_x & n_x & \rho_x \\ l_y & m_y & n_y & \rho_y \\ l_z & m_z & n_z & \rho_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

Six-Revolute Joint (6-R) Manipulators

Theoretically, we may use equation (10) to solve the indirect-position problem of a general 6-R manipulator. We may try to eliminate one unknown at a time to obtain one equation in one unknown. However, it is difficult to carry out this elimination explicitly. Further, solving a high-degree polynomial equation leads to unique numerical difficulties associated with stability of solutions with respect to sensitivity in the coefficients [29]. We choose a different approach. We derive instead a system of eight second-degree equations. Then we solve this system using the continuation algorithm, SYMPOL, and the continuation heuristic, SYMMAN, described below.

Let $\mathbf{p}_j(p_{xj}, p_{yj}, p_{zj})$ be the position vector measured from the origin of the jth coordinate system to the origin of the sixth coordinate system. Let $\mathbf{u}_j(u_{xj}, u_{yj}, u_{zj})$ be a unit vector attached to the Z_6-axis and expressed in the jth coordinate system. Then, by definition, we have

$$\mathbf{p}_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and } \mathbf{u}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{14}$$

These two vectors, when expressed in the first system, are related to the hand position and orientation vectors by the following two equations.

[1] For brevity we shall omit the subscript 1 when a vector is expressed in the first coordinate system.

$$\mathbf{p}_1 \equiv \begin{bmatrix} p \\ q \\ r \\ 1 \end{bmatrix} = \begin{bmatrix} -l_x a_6 - (m_x \mu_6 + n_x \lambda_6) d_6 + \rho_x \\ -l_y a_6 - (m_y \mu_6 + n_y \lambda_6) d_6 + \rho_y \\ -l_z a_6 - (m_z \mu_6 + n_z \lambda_6) d_6 + \rho_z \\ 1 \end{bmatrix} \quad (15)$$

and

$$\mathbf{u}_1 \equiv \begin{bmatrix} u \\ v \\ w \\ 0 \end{bmatrix} = \begin{bmatrix} m_x \mu_6 + n_x \lambda_6 \\ m_y \mu_6 + n_y \lambda_6 \\ m_z \mu_6 + n_z \lambda_6 \\ 0 \end{bmatrix} \quad (16)$$

Equations (15) and (16) imply that once the position and orientation of the hand are specified, the origin and the Z_6-axis of the sixth coordinate system are automatically defined.

The transformation of coordinates between $\mathbf{p}_6$ and $\mathbf{p}_1$, and between $\mathbf{u}_6$ and $\mathbf{u}_1$ can also be written as

$$\mathbf{p}_1 = A_1\ A_2\ A_3\ A_4\ A_5\ \mathbf{p}_6 \quad (17)$$

and

$$\mathbf{u}_1 = A_1\ A_2\ A_3\ A_4\ A_5\ \mathbf{u}_6 \quad (18)$$

Multiplying both sides of equations (17) and (18) by $A_2^{-1} A_1^{-1}$, we obtain

$$A_2^{-1} A_1^{-1}\ \mathbf{p}_1 = A_3\ A_4\ A_5\ \mathbf{p}_6 \quad (19)$$

and

$$A_2^{-1} A_1^{-1}\ \mathbf{u}_1 = A_3\ A_4\ A_5\ \mathbf{u}_6 \quad (20)$$

For brevity, let us define

$$\mathbf{p}_3 = A_3\ A_4\ A_5\ \mathbf{p}_6 \quad (21)$$

$$\mathbf{p}_3' = A_2^{-1}\ A_1^{-1} \mathbf{p}_1 \quad (22)$$

$$\mathbf{u}_3 = A_3\ A_4\ A_5\ \mathbf{u}_6 \quad (23)$$

and

$$\mathbf{u}_3' = A_2^{-1}\ A_1^{-1} \mathbf{u}_1 \quad (24)$$

Then, $\mathbf{p}_3$ and $\mathbf{p}_3'$ represent the same position vector defined from the origin of the third system to the origin of the sixth system. Position vector $\mathbf{p}_3$ is calculated from the matrix product representing the transformation of coordinates from links $6 \rightarrow 5 \rightarrow 4$ to 3 while $\mathbf{p}_3'$ is calculated from links $6 \rightarrow 1 \rightarrow 2$ to 3. Similarly, $\mathbf{u}_3$ and $\mathbf{u}_3'$ represent the same unit vector that is attached to the Z_6-axis. Unit vector $\mathbf{u}_3$ is calculated from the linkage loop $6 \rightarrow 5 \rightarrow 4 \rightarrow 3$ while $\mathbf{u}_3'$ is calculated from the loop $6 \rightarrow 1 \rightarrow 2 \rightarrow 3$. Expanding the matrix multiplications of equations (21)-(24), the components of the vectors $\mathbf{p}$ and $\mathbf{u}$ can be expressed as follows:

$$p_{x3} = c_3(c_4 g_1 + s_4 g_2 + a_3) + s_3(-s_4 \lambda_3 g_1 + c_4 \lambda_3 g_2 + \mu_3 g_3) \quad (25)$$

$$p_{y3} = s_3(c_4 g_1 + s_4 g_2 + a_3) - c_3(-s_4 \lambda_3 g_1 + c_4 \lambda_3 g_2 + \mu_3 g_3) \quad (26)$$

$$p_{z3} = s_4 \mu_3 g_1 - c_4 \mu_3 g_2 + \lambda_3 g_3 + d_3 \quad (27)$$

$$p_{x3}' = c_2 h_1 + s_2 h_2 - a_2 \quad (28)$$

$$p_{y3}' = -s_2 \lambda_2 h_1 + c_2 \lambda_2 h_2 + \mu_2 h_3 - \mu_2 d_2 \quad (29)$$

$$p_{z3}' = s_2 \mu_2 h_1 - c_2 \mu_2 h_2 + \lambda_2 h_3 - \lambda_2 d_2 \quad (30)$$

$$u_{x3} = c_3(c_4 m_1 + s_4 m_2) + s_3(-s_4 \lambda_3 m_1 + c_4 \lambda_3 m_2 + \mu_3 m_3) \quad (31)$$

$$u_{y3} = s_3(c_4 m_1 + s_4 m_2) - c_3(-s_4 \lambda_3 m_1 + c_4 \lambda_3 m_2 + \mu_3 m_3) \quad (32)$$

$$u_{z3} = s_4 \mu_3 m_1 - c_4 \mu_3 m_2 + \lambda_3 m_3 \quad (33)$$

$$u_{x3}' = c_2 n_1 + s_2 n_2 \quad (34)$$

$$u_{y3}' = -s_2 \lambda_2 n_1 + c_2 \lambda_2 n_2 + \mu_2 n_3 \quad (35)$$

and

$$u_{z3}' = s_2 \mu_2 n_1 - c_2 \mu_2 n_2 + \lambda_2 n_3 \quad (36)$$

where

$$\left.\begin{aligned} g_1 &= c_5 a_5 + a_4 \\ g_2 &= -s_5 \lambda_4 a_5 + \mu_4 d_5 \\ g_3 &= s_5 \mu_4 a_5 + \lambda_4 d_5 + d_4 \end{aligned}\right\} \quad (37)$$

$$\left.\begin{aligned} h_1 &= c_1 p + s_1 q - a_1 \\ h_2 &= -s_1 \lambda_1 p + c_1 \lambda_1 q + \mu_1 (r - d_1) \\ h_3 &= s_1 \mu_1 p - c_1 \mu_1 q + \lambda_1 (r - d_1) \end{aligned}\right\} \quad (38)$$

$$\left.\begin{aligned} m_1 &= s_5 \mu_5 \\ m_2 &= c_5 \lambda_4 \mu_5 + \mu_4 \lambda_5 \\ m_3 &= -c_5 \mu_4 \mu_5 + \lambda_4 \lambda_5 \end{aligned}\right\} \quad (39)$$

and

$$\left.\begin{aligned} n_1 &= c_1 u + s_1 v \\ n_2 &= -s_1 \lambda_1 u + c_1 \lambda_1 v + \mu_1 w \\ n_3 &= s_1 \mu_1 u - c_1 \mu_1 v + \lambda_1 w \end{aligned}\right\} \quad (40)$$

Notice that g_1, g_2, g_3, m_1, m_2, and m_3 are functions of θ_5, and h_1, h_2, h_3, n_1, n_2, and n_3 are functions of θ_1. For the reasons of brevity, p_{x1}, p_{y1}, and p_{z1} are replaced by p, q, and r; and u_{x1}, u_{y1} and u_{z1} are replaced by u, v, and w, respectively. Equating the x, y, and z components of $\mathbf{p}_3$ and $\mathbf{p}_3'$, and $\mathbf{u}_3$ and $\mathbf{u}_3'$, yields the following:

$$p_{x3} = p_{x3}', \quad (41)$$

$$p_{y3} = p_{y3}', \quad (42)$$

$$p_{z3} = p_{z3}', \quad (43)$$

$$u_{x3} = u_{x3}', \quad (44)$$

$$u_{y3} = u_{y3}', \quad (45)$$

and

$$u_{z3} = u_{z3}', \quad (46)$$

Equations (41)-(46) are six nonlinear equations free from the variable θ_6. However, only two of the last three equations are independent since they are related by the condition

$$u_{x3}^2 + u_{y3}^2 + u_{z3}^2 = u_{x3}'^2 + u_{y3}'^2 + u_{z3}'^2 = 1 \quad (47)$$

Therefore, there are only five independent equations in five unknowns (θ_1-θ_5). For simplicity, however, we may consider $c_i(\cos\theta_i)$ and $s_i(\sin\theta_i)$ as two independent variables and add

$$c_i^2 + s_i^2 = 1, \text{ for } i = 1, 2, \ldots, 5 \quad (48)$$

as supplementary equations of constraint. This results in ten independent equations in ten unknowns. An inspection of this system of equations reveals that equations (41), (42), and (44) are polynomials of the third degree and equations (43), (46), and (48) with i = 1-5 are of the second degree. The total degree[2] of this system of equations is so large that it is not practical to apply continuation methods. Therefore, it is necessary to further simplify the system of equations. The next section describes how to eliminate θ_3 from the foregoing equations and at the same time obtain a new system of much lower total degree.

Elimination of θ_3. First, we notice that equations (43) and (46) are already free from the angle θ_3. For the convenience of analysis, they have been rewritten in the expanded forms as shown below:

$$\mu_2 h_1 s_2 - \mu_2 h_2 c_2 - \mu_3 g_1 s_4 + \mu_3 g_2 c_4 = -\lambda_2 h_3 + \lambda_2 d_2 + \lambda_3 g_3 + d_3 \quad (49)$$

[2]The total degree of a system of polynomial equations is the product of the degrees of the independent equations.

$$\mu_2 n_1 s_2 - \mu_2 n_2 c_2 - \mu_3 m_1 s_4 + \mu_3 m_2 c_4 = -\lambda_2 n_3 + \lambda_3 m_3 \quad (50)$$

The third equation, free from θ_3, can be obtained by equating the sum of products of equations (41) and (44), (42) and (45), and (43) and (46), i.e.

$$p_{x3} u_{x3} + p_{y3} u_{y3} + p_{z3} u_{z3} = p'_{x3} u'_{x3} + p'_{y3} u'_{y3} + p'_{z3} u'_{z3} \quad (51)$$

Physically, the left-hand side of equation (51) represents the z-component of the position vector, $\mathbf{p}_3$, and the right-hand side represents the z-compoment of the position vector, $\mathbf{p}'_3$, on the sixth coordinate system. Upon substitution of equations (25)–(36) into (51), and after simplification, we obtain

$$a_2 n_2 s_2 + a_2 n_1 c_2 + (a_3 m_2 + d_3 \mu_3 m_1) s_4 + (a_3 m_1 - d_3 \mu_3 m_2) c_4$$
$$= -a_1 n_1 - d_2 n_3 - a_4 m_1 - (d_3 \lambda_3 + d_4) m_3 + k_1 \quad (52)$$

where

$$k_1 = -d_5 \lambda_5 + pu + qv + (r - d_1) w \quad (53)$$

The fourth equation is obtained by equating the sum of the squares of equations (41), (42), and (43), i.e.

$$p_{x3}^2 + p_{y3}^2 + p_{z3}^2 = p'_{x3}{}^2 + p'_{y3}{}^2 + p'_{z3}{}^2 \quad (54)$$

Physically, the left-hand side of equation (54) represents the square of the length of the position vector, $\mathbf{p}_3$, and the right-hand side represents the square of the length of the position vector, $\mathbf{p}'_3$. Substituting equations (25)–(30) into (54) and simplifying yields

$$a_2 h_2 s_2 + a_2 h_1 c_2 + (a_3 g_2 + d_3 \mu_3 g_1) s_4 + (a_3 g_1 - d_3 \mu_3 g_2) c_4$$
$$= -a_1 h_1 - d_2 h_3 - a_4 g_1 - (d_3 \lambda_3 + d_4) g_3 + k_2 \quad (55)$$

where

$$k_2 = 0.5[p^2 + q^2 + (r - d_1)^2 - a_1^2 + a_2^2 + d_2^2$$
$$- a_3^2 - d_3^2 + a_4^2 + d_4^2 - a_5^2 - d_5^2] \quad (56)$$

Notice that although each of the products $p_{x3}\ u_{x3}$ etc. are sixth degree in nature, the resulting equations, equations (52) and (55), are polynomials of second degree. Therefore, many extraneous solutions have been eliminated during the processes of simplification.

Equations (49), (50), (52), and (55) combined with (48) for $i = 1,2,4$, and 5 are a set of eight equations in eight unknowns. Here, the unknowns are c_i and s_i for $i = 1,2,4$, and 5. Since each equation is of second degree, the system can have at most $2^8 = 256$ solutions, unless it has an infinite number of solutions. More exactly, a system of 8 second-degree equations in eight unknowns may have any number of solutions from 0 to 256 or it may have an infinite number. No other possibilities exist. The SYMPOL algorithm begins with 256 generic start points and will generate from these all the solutions, unless there are an infinite number. There will, in fact, be less than 256 solutions to the kinematics problem, but this is irrelevant to SYMPOL, which computes all solutions by a generic procedure which does not depend on the exact number (unless it is infinite).

It appears that further reduction in the number of unknowns is possible. For example, since equations (49), (50), (52), and (55) are linear in s_2, c_2, s_4, and c_4, we may solve them in terms of the others, and substitute them into the two equations $c_2^2 + s_2^2 = 1$ and $c_4^2 + s_4^2 = 1$. The resulting equations combined with equation (48) written for $i = 1$ and 5 yield a system of four equations in four unknowns (c_1, s_1, c_5, and s_5). However, because of the complexity involved we did not elect to reduce the number of unknowns in this manner. Instead, we shall use equations (49), (50), (52), and (55) along with equations (48) for $i = 1,2,4$, and 5 as the system of equations for analysis.

Finding θ_3 and θ_6. Once θ_1, θ_2, θ_4, and θ_5 are known, equations (41) and (42) can be used to solve for θ_3. Since both equations (41) and (42) are linear in c_3 and s_3, each solution of θ_1, θ_2, θ_4, and θ_5 yields a unique value of c_3 and s_3 and, therefore, θ_3. The solutions obtained should then be substituted into equations (44) and (45) to check for compatibility. The solutions that do not satisfy equations (44) and (45) are the extraneous solutions and should be disregarded.

Multiplying both sides of equation (10) by $(A_1 A_2 A_3 A_4 A_5)^{-1}$, we obtain

$$A_6 = A_5^{-1} A_4^{-1} A_3{}^{-1} A_2^{-1} A_1^{-1} A_{\text{eq}} \quad (57)$$

Hence, the angle θ_6 can be solved by equating the (1 × 1) and (2 × 1) elements of equation (57), i.e.

$$\cos\theta_6 = (1 \times 1) \text{ element of } (A_5^{-1} A_4^{-1} A_3{}^{-1} A_2^{-1} A_1^{-1} A_{\text{eq}}) \quad (58)$$

$$\sin\theta_6 = (2 \times 1) \text{ element of } (A_5^{-1} A_4^{-1} A_3{}^{-1} A_2^{-1} A_1^{-1} A_{\text{eq}}) \quad (59)$$

Special Cases

The system of equations will decouple when either any three adjacent joint axes intersect at a common point or any three adjacent joint axes are parallel to each other. In these cases, closed-form solutions can be obtained. To demonstrate the decoupling, the following cases are discussed. Fortunately, the continuation methods can handle the decoupled system of equations without special consideration. Therefore, unless computing time is a concern, it is not necessary to have special programs to handle these special cases.

Case (1): Last Three Axes Intersecting. When the last three joint axes intersect at a common point, $a_4 = a_5 = d_5 = 0$ identically. Equations (49) and (55) reduce to

$$\mu_2 (h_1 s_2 - h_2 c_2) = f_1, \quad (60)$$

$$a_2 (h_2 s_2 + h_1 c_2) = f_2, \quad (61)$$

where

$$f_1 = -\lambda_2 h_3 + \lambda_2 d_2 + d_3 + \lambda_3 d_4 \quad (62)$$

$$f_2 = -a_1 h_1 - d_2 h_3 - \lambda_3 d_3 d_4 + [p^2 + q^2 + (r - d_1)^2$$
$$- a_1^2 + a_2^2 + d_2^2 - a_3^2 - d_3^2 - d_4^2]/2 \quad (63)$$

Summing the squares of equations (60)/μ_2 and (61)/a_2, we obtain

$$h_1{}^2 + h_2{}^2 = \left(\frac{f_1}{\mu_2}\right)^2 + \left(\frac{f_2}{a_2}\right)^2,$$

$$\text{provided } \mu_2 \neq 0 \text{ and } a_2 \neq 0. \quad (64)$$

Equation (64) is a second degree polynomial in c_1 and s_1. If we replace c_1 by $(1 - t_1{}^2)/(1 + t_1{}^2)$ and s_1 by $2\,t_1/(1 + t_1{}^2)$, where $t_1 = \tan(\theta_1/2)$, we obtain a fourth degree polynomial in t_1. Therefore, there are at most four solutions of θ_1 for every given position of the hand. Once θ_1 is known, equations (60) and (61) yield a unique solution for θ_2. Since equations (41) and (42) are linear in s_3 and c_3, θ_3 can be solved uniquely once θ_1 and θ_2 are known. Notice that once θ_1, θ_2, and θ_3 are solved, equations (50) and (52) can be simplified, in a similar manner, to a quartic in $\tan(\theta_5/2)$. Therefore, corresponding to each solution of (θ_1, θ_2, and θ_3) there are four possible solutions for θ_5 and θ_4. This results in 16 possible solutions for each given manipulator hand position. In what follows, we show a different approach to avoid eight extraneous solutions.

Subtracting equation (45) × c_3 from equation (44) × s_3 and simplifying, yields

$$-\lambda_3 m_1 s_4 + \lambda_3 m_2 c_4 + \mu_3 m_3 = f_3 \quad (65)$$

where

$$f_3 = (c_2 n_1 + s_2 n_2) s_3 - (-\lambda_2 s_2 n_1 + \lambda_2 c_2 n_2 + \mu_2 n_3) c_3 \quad (66)$$

Equation (50) may be written as

$$\mu_3 (m_1 s_4 - m_2 c_4) + \lambda_3 m_3 = f_4, \quad (67)$$

where

$$f_4 = \mu_2 n_1 s_2 - \mu_2 n_2 c_2 + \lambda_2 n_3 \tag{68}$$

Adding equation (65) × μ_3 to equation (67) × λ_3 and simplifying, yields

$$m_3 = \mu_3 f_3 + \lambda_3 f_4 \tag{69}$$

Hence, we can solve (69) for θ_5

$$\frac{\theta_5}{2} = \pm \tan^{-1}\sqrt{\frac{\mu_3 f_3 + \lambda_3 f_4 + \mu_4\mu_5 - \lambda_4\lambda_5}{-\mu_3 f_3 - \lambda_3 f_4 + \mu_4\mu_5 - \lambda_4\lambda_5}},$$

$$90 \text{ deg} \geq \frac{\theta_5}{2} \geq -90 \text{ deg} \tag{70}$$

Therefore, corresponding to each solution of θ_1, θ_2, and θ_3, there are at most two solutions for θ_5. Once θ_1, θ_2, θ_3, and θ_5 are known, equations (65) and (67) yield θ_4 uniquely. Hence, there are at most eight possible solutions corresponding to each specified position and orientation of the hand. Of the eight possible solutions, there are at most four different configurations defining the geometry of the upper arm (θ_1, θ_2, and θ_3). We have assumed that $a_2 \neq 0$ and $\mu_2 \neq 0$ for the derivations just given. If a_2 or μ_2 equals zero, then equation (60) or (61) further simplifies to $f_2 = 0$ or $f_1 = 0$, and the analysis becomes very straightforward. Note that both a_2 and μ_2 can not be zero simultaneously. For if $a_2 = 0$ and $\mu_2 = 0$, then the second- and third-joint-axes must be coincident and one degree-of-freedom is lost from the manipulator. Once θ_1, θ_2, θ_3, θ_4, and θ_5 are found, θ_6 can be found by the methods outlined in the previous section.

Case (2): Joint Axes 2, 3 and 4 Parallel to Each Other. When the second, third, and fourth joint axes are parallel to each other, $\alpha_2 = \alpha_3 = 0$ identically. Since the common normals between the second and third joint axes and between the third and fourth joint axes are undetermined, we can always define these two normals such that $d_2 = d_3 = 0$. Hence, equations (49) and (50) reduce to

$$a_5\mu_4 s_5 = \mu_1 p\, s_1 - \mu_1 q\, c_1 + \lambda_1 r - \lambda_1 d_1 - \lambda_4 d_5 - d_4 \tag{71}$$

$$-\mu_5\mu_4 c_5 = \mu_1 u\, s_1 - \mu_1 v\, c_1 + \lambda_1 w - \lambda_4\lambda_5 \tag{72}$$

Assuming $a_5 \neq 0$ and $\mu_5 \neq 0$, then s_5 and c_5 can be eliminated by summing the squares of equation (71)/a_5 and equation (72)/μ_5 to yield

$$\begin{aligned}\mu_4{}^2 \;=\; & [(\mu_1 p\, s_1 - \mu_1 q\, c_1 + \lambda_1 r - \lambda_1 d_1 - \lambda_4 d_5 - d_4)/a_5]^2 \\ & + [(\mu_1 u\, s_1 - \mu_1 v\, c_1 + \lambda_1 w - \lambda_4\lambda_5)/\mu_5]^2\end{aligned} \tag{73}$$

Equation (73) is a second degree polynomial in c_1 and s_1. If c_1 is replaced by $(1 - t_1{}^2)/(1 + t_1{}^2)$ and s_1 by $2t_1/(1 + t_1{}^2)$, where $t_1 = \tan(\theta_1/2)$, then equation (73) becomes a fourth degree polynomial in t_1. Therefore, corresponding to each given position and orientation of the manipulator hand, there are at most four solutions for θ_1. Once the solution for θ_1 is obtained, equations (71) and (72) give a unique value of θ_5.

For $\alpha_2 = \alpha_3 = 0$, equations (44) and (45) reduce to

$$c_{34} m_1 + s_{34} m_2 = c_2 n_1 + s_2 n_2 \tag{74}$$

$$s_{34} m_1 - c_{34} m_2 = -s_2 n_1 + c_2 n_2 \tag{75}$$

where $c_{34} = \cos(\theta_3 + \theta_4)$ and $s_{34} = \sin(\theta_3 + \theta_4)$.
Subtracting equation (75) × s_2 from equation (74) × c_2, yields

$$c_{234} m_1 + s_{234} m_2 = n_1 \tag{76}$$

Adding equation (75) × c_2 to equation (74) × s_2, yields

$$s_{234} m_1 - c_{234} m_2 = n_2 \tag{77}$$

where $c_{234} = \cos(\theta_2 + \theta_3 + \theta_4)$, and $s_{234} = \sin(\theta_2 + \theta_3 + \theta_4)$.

Solving equations (76) and (77), yields

$$s_{234} = \frac{m_1 n_2 + m_2 n_1}{m_1^2 + m_2^2} \tag{78}$$

and

$$c_{234} = \frac{m_1 n_1 - m_2 n_2}{m_1^2 + m_2^2} \tag{79}$$

Hence, corresponding to each θ_1 and θ_5, there is a unique solution for $(\theta_2 + \theta_3 + \theta_4)$.

Similarly, equations (41) and (42) reduce to

$$c_{34} g_1 + s_{34} g_2 + c_3 a_3 = c_2 h_1 + s_2 h_2 - a_2 \tag{80}$$

$$s_{34} g_1 - c_{34} g_2 + s_3 a_3 = -s_2 h_1 + c_2 h_2 \tag{81}$$

Subtracting equation (81) × s_2 from equation (80) × c_2, yields

$$c_{234} g_1 + s_{234} g_2 + c_{23} a_3 = h_1 - a_2 c_2 \tag{82}$$

Adding equation (81) × c_2 to equation (80) × s_2, yields

$$s_{234} g_1 - c_{234} g_2 + s_{23} a_3 = h_2 - a_2 s_2 \tag{83}$$

Eliminating c_2 and s_2 from equations (82) and (83), yields

$$2a_3 k_5 c_{23} + 2a_3 k_6 s_{23} - a_2^2 + a_3^2 + k_5^2 + k_6^2 = 0 \tag{84}$$

where

$$c_{23} = \cos(\theta_2 + \theta_3)$$

$$s_{23} = \sin(\theta_2 + \theta_3)$$

$$k_5 = c_{234} g_1 + s_{234} g_2 - h_1$$

and

$$k_6 = s_{234} g_1 - c_{234} g_2 - h_2$$

Equation (84) may be converted into a second degree polynomial in t_{23}, when c_{23} is replaced by $(1 - t_{23}^2)/(1 + t_{23}^2)$ and s_{23} by $2t_{23}/(1 + t_{23}^2)$, where $t_{23} = \tan(\theta_2 + \theta_3)/2$. After $(\theta_2 + \theta_3)$ is solved, equations (82) and (83) yield a unique solution of θ_2. Hence, there are a total of eight possible solutions for θ_1, θ_2, θ_3, θ_4 and θ_5, and the value of θ_6 follows from equations (58) and (59). Note that equations (71) and (72) will decouple if either $a_5 = 0$ or $\mu_5 = 0$. For $a_5 = 0$ or $\mu_5 = 0$, the solution is very straightforward.

Five-Revolute-Joint (5-R) Manipulators

In order to position the hand of a manipulator freely in space, it is necessary that a manipulator has six degrees of freedom. In practice, however, it is sometimes only necessary to specify the position of an axis of the hand, while the orientation of the hand about the axis is not important. A manipulator with five degrees of freedom is then sufficient to perform such a task. Manipulators with five degrees of freedom can often be found in industrial applications such as arc welding and spray painting. The analysis of five-degree-of-freedom robot arms was treated recently by Sugimoto and Duffy [30]. Sugimoto and Duffy introduced a pair of hypothetical joints and links to the five-degree-of-freedom manipulator to form a hypothetical closed-loop 7-R spatial mechanism. The solution was then obtained from the hypothetical closed-loop mechanism. The method works well. However, it is always necessary to calculate the hypothetical pair of links and joints and then solve the hypothetical spatial mechanism, and it is also limited to robots with special geometry. In what follows, we shall show that the method of analysis developed in this paper can be applied directly to the analysis of a five-degree-of-freedom manipulator. There is no need to introduce a hypothetical pair of links and joints.

Figure 3 shows a five-revolute-joint manipulator where the sixth coordinate system is attached to the hand. Only the position of an axis in the hand can be specified freely for a manipulator with five degrees of freedom. Let us assume that the axis of interest is the Z_6-axis. Then, for every specified position of the axis, the origin, O_6, and the direction $\mathbf{u}(u, v, w)$ of the Z_6-axis are known. We observe that the transformation of coordinates of the vectors $\mathbf{p}$ and $\mathbf{u}$ from the sixth

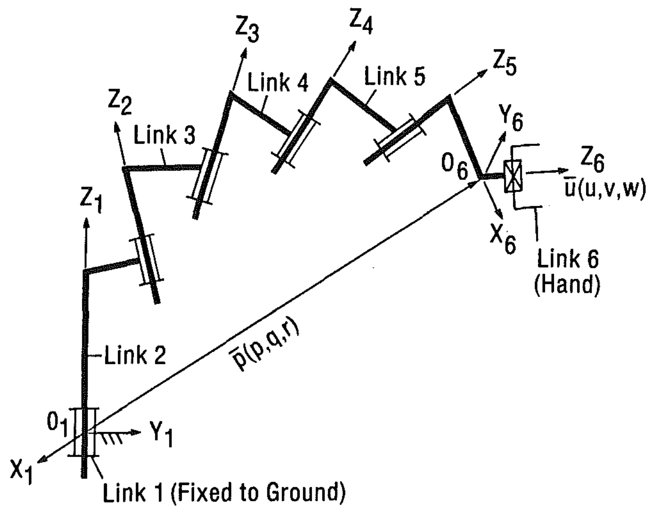

Fig. 3 A five-revolute-joint manipulator

system to the first system is identical to that given for the 6-R manipulator, i.e. equations (17) and (18) work for both types of manipulators. The only difference is that the vectors $\mathbf{p}_1$ and $\mathbf{u}_1$ are directly specified for a 5-R manipulator while they are calculated by equations (15) and (16) for a 6-R manipulator. Hence, the equations (41) through (46) or (49), (50), (52), and (55) can be used for solving θ_1-θ_5. Therefore, solutions of the five-degree-of-freedom manipulator can be considered as a special case of the general six-degree-of-freedom manipulator.

Manipulators with Prismatic Joints

So far we have limited our discussions to the manipulators with revolute joints only. However, the method can also be applied to manipulators with mixed prismatic and revolute joints. For an ith prismatic joint, we simply consider a_i, α_i and θ_i as constants and d_i as a variable. Then equations (41) through (46) along with (25) through (40) can be applied except for the case where the sixth joint is prismatic.

If the third and sixth joints are not prismatic, then it is more convenient to use equations (49), (50), (52), and (55) since θ_3 has already been eliminated from these four equations. If the sixth joint is not prismatic and the third joint is prismatic, then it is more convenient to use equations (41), (42), (44), or (45), and (46) to solve for θ_1, θ_2, θ_4, and θ_5 since these equations are already free from the variable d_3. The variable d_3 can be found from equation (43) once θ_1, θ_2, θ_4, and θ_5 are found.

If the sixth joint is prismatic, then θ_6 is constant and d_6 is variable. Therefore, p, q, and r are now functions of d_6 and an additional equation of constraint is needed. The additional equation can be obtained by equating either the x, y, or z component of the following vector equation:

$$\mathbf{v}_1 = A_1 A_2 A_3 A_4 A_5 \mathbf{v}_6 \tag{85}$$

where $\mathbf{v}_6$ is the unit vector aligned with the X_6-axis and expressed in the sixth system, i.e.,

$$\mathbf{v}_6 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{86}$$

and $\mathbf{v}_1$ is the same unit vector expressed in the first system, which is known when the position and orientation of the hand are specified. Specifically, $\mathbf{v}_1$ is given by

$$\mathbf{v}_1 = \begin{bmatrix} c_6 l_x - s_6 \lambda_6 m_x + s_6 \mu_6 n_x \\ c_6 l_x - s_6 \lambda_6 m_y + s_6 \mu_6 n_y \\ c_6 l_z - s_6 \lambda_6 m_z + s_6 \mu_6 n_z \\ 0 \end{bmatrix} \tag{87}$$

To illustrate the principle, let us consider the "Stanford arm" where the third joint is prismatic [7]. The linkage parameters of the Stanford arm are as follows:

Link i	a_i	d_i	α_i	θ_i
1	0	0	−90 deg	θ_1 (variable)
2	0	d_2 (constant)	90 deg	θ_2 (variable)
3	0	d_3 (variable)	0 deg	0 deg (constant)
4	0	0	−90 deg	θ_4 (variable)
5	0	0	90 deg	θ_5 (variable)
6	0	0	0	θ_6 (variable)

Since $\theta_3 = 0$ deg, we have $s_3 = 0$ and $c_3 = 1$. Substituting the foregoing parameters into equations (41) through (46) yields

$$\left.\begin{aligned} & c_2(c_1 p + s_1 q) - s_2 r = 0 \\ & -s_1 p + c_1 q - d_2 = 0 \\ & s_2(c_1 p + s_1 q) + c_2 r = d_3 \\ & c_2(c_1 u + s_1 v) - s_2 w = c_4 s_5 \\ & -s_1 u + c_1 v = s_4 s_5 \\ & s_2(c_1 u + s_1 v) + c_2 w = c_5 \end{aligned}\right\} \tag{88}$$

Solving equations (88), we obtain

$$\begin{aligned} \theta_1 &= 2\tan^{-1}[(-p \pm \sqrt{p^2+q^2-d_2{}^2})/(d_2+q)], \\ & \qquad -180 \text{ deg} \le \theta_1 \le 180 \text{ deg} \\ \theta_2 &= \tan^{-1}[(c_1 p + s_1 q)/r] \\ d_3 &= s_2(c_1 p + s_1 q) + c_2 r \\ \theta_4 &= \tan^{-1}\{(-s_1 u + c_1 v)/[c_2(c_1 u + s_1 v) - s_2 w]\} \\ \theta_5 &= \cos^{-1}[s_2(c_1 u + s_1 v) + c_2 w] \end{aligned} \tag{89}$$

Hence, θ_1, θ_2, d_3, θ_4, and θ_5 can be found in sequence. Once θ_1, θ_2, d_3, θ_4, and θ_5 are found, θ_6 can be found by solving equations (58) and (59). The solution found is in complete agreement with that given in [7], thereby proving generality of the method of analysis given in this paper.

Continuation Methods

Two continuation methods were used to solve the system of equations (48), (49), (50), (52), and (55). SYMPOL is the generic continuation technique presented by Morgan in [26]. It is not customized to this particular problem; in fact, the method uses only the fact that these equations form a system of eight second-degree polynomials. SYMMAN *is* a special continuation method that is customized to this problem, as described below.

For SYMPOL there is a substantial theory, backed up by much computational experience. This method is slow but reliable. For SYMMAN there is some theory, and experience demonstrates that SYMMAN will usually find a partial solution list quickly. However, neither theory nor experience suggest that SYMMAN can be used reliably to find all solutions. In some very degenerate cases, SYMMAN will not find any solutions at all. For example, if the three joint axes 1, 2, and 3 intersect at a point, then SYMMAN will not work at all. This is because in this case the *initial system* has either an infinite number of solutions or no solutions. However, such a degenerate system can usually be solved in closed form.

Both continuation methods are based on the following idea. Let $F(x) = 0$ denote equations (48), (49), (50), (52), and (55), and let $G(x) = 0$ denote some system of eight equations in eight unknowns which can be solved in closed form. For SYMPOL, $G(x)$ is determined by theory [26]. For SYMMAN, $G(x)$ is a perturbation of the specific simplification of $F(x) = 0$ defined by setting $a_4 = a_5 = d_5 = 0$. Now we define a "continuation equation"

$$H(x, t) = (1-t)G(x) + tF(x) \tag{90}$$

with a new parameter t, called the "continuation parameter," where t goes from 0 to 1. When $t = 0$, $H(x, t) = G(x)$ and, when $t = 1$, $H(x, t) = F(x)$. Thus as t goes from 0 to 1, the solutions of $H(x, t) = 0$ change from solutions of $G(x) = 0$ to solutions of $F(x) = 0$. To implement continuation, we start at a solution to $G(x) = 0$, increment t, solve $H(x, t) = 0$, increment t again, solve $H(x, t) = 0$ again, and so on until $t = 1$. The sequence of solutions to $H(x, t) = 0$ – called a "continuation path" – converges to a solution of $F(x) = 0$. When we do this starting at each solution to $G(x) = 0$, we obtain a collection of solutions to $F(x) = 0$ which we hope includes all solutions. The actual implementation involves defining a differential equation whose solutions are the continuation paths and solving this differential equation using a numerical integration technique. The integration is followed by Newton's method to refine the final solution estimates at the end of each continuation path [25, 26].

Several things can go wrong with this process:

(*a*) $H(x, t) = 0$ might have no solution for some increment of t, or it might have several nearby solutions which are hard to distinguish or find.

(*b*) $G(x) = 0$ might not have enough solutions to match the solutions of $F(x) = 0$, or for some other reason $F(x) = 0$ might have some solutions with no continuation paths converging to them.

(*c*) As t is incremented, the solution to $H(x, t) = 0$ might diverge to infinity.

Each of these types of bad behavior can occur for different choices of F and G, but the theory developed for SYMPOL tells how to avoid (*a*) and (*b*) and how to make (*c*) unimportant. SYMPOL finds all solutions, unless there are an infinite number. A manipulator problem might have an infinite number of solutions when arbitrary wrist rotations of some joint angles do not change the hand position. When there are an infinite number of solutions, SYMPOL will find all *isolated* solutions and tends to find a representative point on each solution *curve*. Using SYMMAN, we can avoid (*a*), but (*b*) and (*c*) may occur. Thus we may miss some solutions when we use SYMMAN.

For SYMPOL we define $G(x)$ by

$$G_j(x) = a_j x_j^{\deg_j} - b_j \quad \text{for } j = 1, 2, \ldots 8 \tag{91}$$

where $\deg_j$ denotes the degree of the polynomial F_j and a_j and b_j are "random" complex numbers. The basic theory of the method [26] shows, for this G, that the only possible bad behavior is (*c*) above. However, each solution of $F(x) = 0$ has a convergent sequence as t goes from 0 to 1, so we still get what we want: all solutions to $F(x) = 0$. The random choice of a_j and b_j is important so that we avoid (*a*) and (*b*) and keep (*c*) under control. (Divergence to infinity can occur only as t approaches 1.)

As noted above, $G(x)$ for SYMMAN is a perturbation of the simplification of (48)–(50), (52), (55) defined by setting $a_4 = a_5 = d_5 = 0$. (See the Appendix.) This will be discussed in two steps. Forgetting about the perturbation, the logic of the simplification is that in this case $G(x) = 0$ is *solvable in closed form* and *physical*.

By solvable in closed form, we mean that we can write down formulas (using the quartic formula) for the solutions. The cases in which $G(x) = 0$ has an infinite number or no solutions are exceptions. For example, this occurs when the three axes 1, 2, and 3 intersect at a point. In these cases, SYMMAN cannot be used.

By physical, we mean that this case corresponds to an actual manipulator, namely one with the last three joint axes intersecting in a common point. For this simplifed case, there are at most 16 solutions, unless there are an infinite number. These 16 consist of eight significant and eight extraneous solutions. We observe that by proper back substitution we can omit the eight extraneous as described in the previous section.

The SYMMAN continuation equations, $H(x, t)$, are defined by letting the t parameter multiply each of the terms containing a_4, a_5, and d_5 in the system. (See the Appendix.) It turns out that the resulting continuation is subject to all three of the things that can go wrong in the abovementioned list. However, by a slight modification of $H(x, t)$ we can prevent (*a*) from occurring. We have to live with (*b*) and (*c*), although in our experience (*c*) has not occurred.

If (*a*) happens, then the whole continuation procedure will break down. To prevent this, we perturb the system of equations by adding a small random complex constant of the form $(t-1)k$, with a different constant k for each equation. The theory of continuation shows that this "trick" will eliminate (*a*) as a problem. However, practically speaking the magnitude of the k must be chosen with care. The existence of the $(t-1)k$ terms makes the solutions nonphysical. If the k's are too large, then the physical interpretation of the solutions will become unreasonable. If the k's are too small, the continuation may fail because of a singularity. We have found that, after normalizing the system coefficients about unity, k's whose order of magnitude is 10^{-4} seem to work well. In other words, no singularities were encountered, and a large fraction of the total solution list was found in the test examples. (See the next section.) The basic problem with SYMMAN is simply that, even when no singularities are encountered, at most sixteen solutions can be found. Further, no theory guarantees that *any* solutions will be found. The basic justification for SYMMAN rests on physical reasoning, along with the fact that it has worked well in tests.

Numerical Examples

Overview. The key parameters for SYMPOL and SYMMAN are the path-integration accuracy, EPS, and the maximum arc length for each continuation path, MAXARC [25, 26]. Guidelines for setting these parameters are not determined by theory and must be established by experience. For SYMPOL we used EPS = 10^{-6} and MAXARC = 250. For all the given examples and for most of the other tests we made, the path arc length in SYMPOL was less than 250, except for diverging paths. Occasionally, a path required an arc length over 250, but never, in our testing, over 800. For SYMMAN we used EPS = 10^{-4} and MAXARC = 250. For all examples and tests the path arc lengths were, in fact, less than 10.

The system of equations (48)–(50), (52), and (55) can have

Table 1 Hand position and orientation

Given vectors	x-component	y-component	z-component
$\boldsymbol{\rho}$	0.22441776*	0.71549788	0.79551628
$\mathbf{l}$	−0.71511545	0.65150320	0.25328538
$\mathbf{m}$	−0.69899036	−0.66895464	−0.25280857
$\mathbf{n}$	0.00473084	−0.35783135	0.93377425

*We show only eight digits here. However, full double precision (fifteen digits) was used in the computer codes.

Table 2(*a*) **Linkage parameters**

i	a_i (unit)	d_i (unit)	α_i (deg)
1	0.0	0.0	−90.00
2	1.000000	0.345000	0.0
3	−0.047000	0.0	90.00
4	0.0	1.000000	−90.00
5	0.0	0.0	90.00
6	0.0	0.130000	0.0

Table 2(*b*) **Linkage parameters**

i	a_i (unit)	d_i (unit)	α_i (deg)
1	0.450000	0.500000	80.00
2	0.550000	0.600000	93.00
3	0.750000	0.400000	120.00
4	0.750000	1.000000	120.00
5	0.550000	0.400000	93.00
6	0.450000	0.600000	80.00

Table 2(*c*) **Linkage parameters**

i	a_i (unit)	d_i (unit)	α_i (deg)
1	0.500000	0.187500	80.00
2	1.000000	0.375000	15.00
3	0.125000	0.250000	120.00
4	0.625000	0.875000	75.00
5	0.312500	0.500000	100.00
6	0.250000	0.125000	60.00

Table 3(*a*) **Computing time and total number of solutions found**

	Computing CPU time (s)	Number of solutions found		
		Significant		Extraneous
		Real	Complex	
SYMPOL	212.363	8	0	8
SYMMAN	2.012	8	0	8

Table 3(*b*) **Computing time and total number of solutions found**

	Computing CPU time (s)	Number of solutions found		
		Significant		Extraneous
		Real	Complex	
SYMPOL	266.513	6	10	16
SYMMAN	8.028	6	2	8

Table 3(*c*) **Computing time and total number of solutions found**

	Computing CPU time (s)	Number of solutions found		
		Significant		Extraneous
		Real	Complex	
SYMPOL	250.868	12	4	16
SYMMAN	10.618	6	0	10

at most 256 solutions, unless it is singular, in which case it may have an infinite number of solutions. For SYMPOL, there are always 256 generic start points. For each start point, a continuation path is generated. Some of these paths converge to solutions to (48)–(50), (52), and (55). The rest diverge to infinity. In practice, diverging paths are detected by determining that their arc length exceeds MAXARC. Each isolated solution to the system has a path converging to it. All solutions are isolated, unless there are an infinite number of solutions. In this latter case, some solutions will be isolated, and others will form algebraic hypersurfaces, most commonly curves. Thus SYMPOL will find all solutions, unless there are an infinite number. However, solutions to equations (48)–(50), (52), and (55) are not necessarily solutions to equations (41)–(46). Those solutions that satisfy equations (48)–(50), (52), and (55) but not (41)–(46), are the so-called "extraneous" solutions. The others we call "significant." Significant solutions can be real or complex, but only the real solutions have physical meaning. For SYMMAN, there are only 16 start points. Hence, SYMMAN is faster than SYMPOL but can produce at most 16 solutions.

The hand position and orientation for all the numerical examples tested are given in Table 1, where $\boldsymbol{\rho}$ is the position vector and **l**, **m**, and **n** are the orientation vectors. For each of the three examples, there are three associated tables. Thus for Example 1, Table 2(*a*) gives the (input) linkage parameters that define the manipulator. Table 3(*a*) summarizes the SYMPOL and SYMMAN output in terms of CPU time and how many solutions were found in each of three categories: real-significant, complex-significant, and extraneous. Table 4(*a*) lists the real-significant solutions found by SYMPOL and indicates which were also found by SYMMAN. Note that θ_3, and θ_6 were found by back substitution after θ_1, θ_2, θ_4, and θ_5 were computed.

Example 1: Last Three Axes Intersecting. In this case the system partially decouples. We notice there are four different configurations of the upper arm (θ_1, θ_2, and θ_3), and there are two different configurations for the wrist joints (θ_4, θ_5, and θ_6) corresponding to each configuration of the upper arm.

Example 2: Symmetrical Robot. This robot is taken from the first example of [16] except for a scale factor of 2. It is symmetric about the fourth joint axis.

Example 3: General Robot. The linkage parameters given in Table 2(*c*) represents one of the many general robots with arbitrary linkage proportions that we tested. Note that SYMMAN missed six of the real significant solutions.

Summary. Equations of constraint were derived for the kinematic analysis of the most general six-revolute-joint manipulators. Solutions can be obtained by simultaneously solving a system of eight second-degree polynomial equations in eight unknowns using continuation methods.

We found that both SYMPOL and SYMMAN can be used to solve the indirect-position problem of the general 6-R manipulator. SYMPOL can also be used to solve the decoupled cases, without any special considerations. Although SYMMAN did solve the decoupled case presented in Example 1, it will not find any solutions at all for some other decoupled cases, such as when the three joint axes (1, 2, and 3) or (3, 4, and 5) intersect at a point, or the two axes (4 and 5) or (5 and 6) are parallel to one another. SYMPOL consistently found 16 significant solutions (except for the decoupled cases), taking about 4 minutes per run on the IBM 370-3033, while SYMMAN found up to eight significant solutions in only a few seconds. We notice that although the number of real-significant solutions changes from case to case, the total number of significant solutions (real and complex) is 16 for all the 6-R manipulators tested except for

Table 4(*a*) Real solutions found by SYMPOL

No.	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	Also found by SYMMAN
	Joint displacements (deg)						
1	−80.62	−76.06	−28.47	176.23	−125.23	34.71	yes
2	−80.62	−76.06	−28.47	−3.77	125.23	−145.29	yes
3	−80.62	162.66	−146.15	−36.03	5.23	−107.20	yes
4	−80.62	162.66	−146.15	143.97	−5.23	72.80	yes
5	47.89	−103.94	−146.15	−162.84	124.38	−84.24	yes
6	47.89	−103.94	−146.15	17.16	−124.38	95.76	yes
7	47.89	17.34	−28.47	−107.55	14.80	−166.02	yes
8	47.89	17.34	−28.47	72.45	−14.80	13.98	yes

Table 4(*b*) Real solutions found by SYMPOL

No.	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	Also found by SYMMAN
	Joint displacements (deg)						
1	−146.88	170.87	−11.22	−25.99	−108.51	60.82	yes
2	−167.72	−173.52	128.00	−179.64	−3.12	179.99	yes
3	21.50	135.15	−104.31	64.39	−89.40	77.38	yes
4	63.74	−47.27	−172.43	−114.49	−50.04	−11.94	yes
5	17.31	19.31	42.89	−164.02	29.10	−17.23	yes
6	26.20	6.88	−62.10	−45.96	−130.25	−129.34	yes

Table 4(*c*) Real solutions found by SYMPOL

No.	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	Also found by SYMMAN
	Joint displacements (deg)						
1	167.68	83.55	168.07	65.84	−88.67	−44.77	no
2	−143.00	100.07	131.85	18.46	−59.49	−71.52	no
3	115.86	−168.65	−66.22	157.17	−111.41	156.71	no
4	107.56	2.00	−111.47	166.77	−173.54	−105.56	no
5	−106.07	−140.86	22.07	−161.28	35.54	134.45	yes
6	−65.37	142.24	56.06	−70.90	−51.63	−116.13	yes
7	120.52	31.27	−143.03	114.15	−143.62	−64.39	no
8	7.75	103.87	−113.21	−21.37	−79.90	82.26	yes
9	−16.69	97.90	−25.97	−80.98	−25.72	−3.44	yes
10	47.26	163.44	−119.49	28.32	−41.13	81.08	no
11	20.93	58.74	−125.17	−27.07	−125.66	106.21	yes
12	38.93	−56.45	−149.20	12.28	72.23	67.43	yes

the decoupled cases where there are eight significant solutions. Since we know that SYMPOL found all the solutions for these examples, this suggests that there are only 16 possible significant solutions for the most general 6-R manipulator, even though Roth et al. [3] predicted up to 32 solutions.

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APPENDIX

SYMMAN Continuation Function Definition

The following eight equations, $H_1(x, t) = 0, \ldots, H_8(x, t) = 0$, define the SYMMAN continuation equations.

Let $c_1, s_1, c_2, s_2, c_4, s_4, c_5$, and s_5 be denoted by $x_1, x_2, x_3, x_4, x_5, x_6, x_7$, and x_8 respectively. We take

$$\text{factor} = (|a_1| + |a_2| + |a_3| + |a_4| + |a_5| + |d_1| + |d_2| + |d_3| + |d_4| + |d_5|)/10$$

for the normalization factor. Then we have

$$\begin{aligned} H_1(x, t) = \; & [-x_1 x_3 \lambda_1 \mu_2 q \\ & + x_1 x_4 \mu_2 p \\ & + x_2 x_3 \lambda_1 \mu_2 p \\ & + x_2 x_4 \mu_2 q \\ & - x_5 x_8 \mu_3 \lambda_4 a_5 t \\ & - x_6 x_7 \mu_3 a_5 t \\ & - x_1 \mu_1 \lambda_2 q \\ & + x_2 \mu_1 \lambda_2 p \\ & - x_3 \mu_1 \mu_2 (r - d_1) \\ & - x_4 \mu_2 a_1 \\ & + x_5 \mu_3 \mu_4 d_5 t \\ & - x_6 \mu_3 a_4 t \\ & - x_8 \lambda_3 \mu_4 a_5 t \\ & + \lambda_1 \lambda_2 r - \lambda_1 \lambda_2 d_1 - \lambda_2 d_2 - d_3 - \lambda_3 d_4 \\ & \quad - \lambda_3 \lambda_4 d_5 t]/\text{factor} \\ & + k_1 (t-1) = 0 \end{aligned} \tag{A1}$$

$$\begin{aligned} H_2(x, t) = \; & -x_1 x_3 \lambda_1 \mu_2 v \\ & + x_1 x_4 \mu_2 u \\ & + x_2 x_3 \lambda_1 \mu_2 u \\ & + x_2 x_4 \mu_2 v \\ & + x_5 x_7 \mu_3 \lambda_4 \mu_5 \\ & - x_6 x_8 \mu_3 \mu_5 \\ & - x_1 \mu_1 \lambda_2 v \\ & + x_2 \mu_1 \lambda_2 u \\ & - x_3 \mu_1 \mu_2 w \\ & + x_5 \mu_3 \mu_4 \lambda_5 \\ & + x_7 \lambda_3 \mu_4 \mu_5 \\ & + \lambda_1 \lambda_2 w - \lambda_3 \lambda_4 \lambda_5 + (t-1) k_2 = 0 \end{aligned} \tag{A2}$$

$$\begin{aligned} H_3(x, t) = \; & [x_1 x_3 a_2 u \\ & + x_1 x_4 \lambda_1 a_2 v \\ & + x_2 x_3 a_2 v \\ & - x_2 x_4 \lambda_1 a_2 u \\ & - x_5 x_7 \mu_3 \lambda_4 \mu_5 d_3 \\ & + x_5 x_8 \mu_5 a_3 \\ & + x_6 x_7 \lambda_4 \mu_5 a_3 \\ & + x_6 x_8 \mu_3 \mu_5 d_3 \\ & - x_1 (-a_1 u + \mu_1 d_2 v) \\ & + x_2 (a_1 v + \mu_1 d_2 u) \\ & + x_4 \mu_1 a_2 w \\ & - x_5 \mu_3 \mu_4 \lambda_5 d_3 \\ & + x_6 \mu_4 \lambda_5 a_3 \\ & - x_7 (\mu_4 \mu_5 d_4 + \lambda_3 \mu_4 \mu_5 d_3) \\ & + x_8 \mu_5 a_4 t \\ & + d_1 w + \lambda_1 d_2 w + \lambda_3 \lambda_4 \lambda_5 d_3 + \lambda_4 \lambda_5 d_4 + \\ & \quad + \lambda_5 d_5 t - p u - q v - r w]/\text{factor} \\ & + k_3 (t-1) = 0 \end{aligned}$$

$$\begin{aligned} H_4(x, t) = \; & [x_1 x_3 a_2 p \\ & + x_1 x_4 \lambda_1 a_2 q \\ & + x_2 x_3 a_2 q \\ & - x_2 x_4 \lambda_1 a_2 p \\ & + x_5 x_7 a_3 a_5 t \\ & + x_5 x_8 \mu_3 \lambda_4 a_5 d_3 t \\ & + x_6 x_7 \mu_3 a_5 d_3 t \\ & - x_6 x_8 \lambda_4 a_3 a_5 t \\ & + x_1 (a_1 p - \mu_1 d_2 q) \\ & + x_2 (a_1 q + \mu_1 d_2 p) \\ & - x_3 a_1 a_2 \\ & + x_4 (-\mu_1 a_2 d_1 + \mu_1 a_2 r) \\ & + x_5 (a_3 a_4 - \mu_3 \mu_4 d_3 d_5) t \\ & + x_6 (\mu_4 a_3 d_5 + \mu_3 a_4 d_3) t \\ & + x_7 a_4 a_5 t \\ & + x_8 (\mu_4 a_5 d_4 + \lambda_3 \mu_4 a_5 d_3) t \\ & + (-a_1^2 - d_1^2 - a_2^2 - d_2^2 + a_3^2 + d_3^2 + a_4^2 t \\ & \quad + d_4^2 + a_5^2 t + d_5^2 t - p^2 - q^2 - r^2)/2 \end{aligned} \tag{A4}$$

$$+d_1\, r+\lambda_1\, d_2\, r-\lambda_1\, d_1\, d_2+\lambda_3\, d_3\, d_4 + \lambda_3\, \lambda_4\, d_3\, d_5\, t+\lambda_4\, d_4\, d_5\, t]/(\text{factor}^2) + k_4(t-1)=0$$

$$H_5(x,t)=x_1^2+x_2^2-1=0 \qquad (A5)$$

$$H_6(x,t)=x_3^2+x_4^2-1=0 \qquad (A6)$$

$$H_7(x,t)=x_5^2+x_6^2-1=0 \qquad (A7)$$

$$H_8(x,t)=x_7^2+x_8^2-1=0 \qquad (A8)$$

where

$$k_1=1.9656\text{E}-04+i\ 2.4712\text{E-}04$$
$$k_2=2.7202\text{E}-04+i\ 8.4575\text{E-}04$$
$$k_3=3.8144\text{E}-04+i\ 4.8048\text{E-}04$$
$$k_4=-4.9656\text{E}-04-i\ 2.4712\text{E-}04$$

(with $i = \sqrt{-1}$). These k_1, k_2, k_3,and k_4 define the "perturbation" of the SYMMAN system referred to above. Note that when $t = 0$ and $k_1 = k_2 = k_3 = k_4 = 0$, (A1)–(A8) define the equations of constraint for a manipulator having the last three axes intersecting at a common point. When $t = 1$, (A1)–(A8) define the equations of constraint for the given general manipulator.