Breadth First Search

CSE 2011

Winter 2007

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Graph Traversal

- Application examples
  - Given a graph representation and a vertex **s** in the graph
  - Find all paths from **s** to the other vertices
- Two common graph traversal algorithms
  - Breadth-First Search (BFS)
  - Depth-First Search (DFS)

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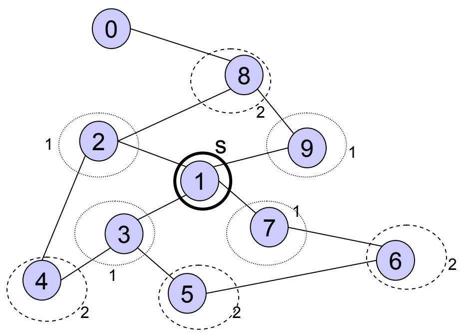
BFS and Shortest Path Problem

- Given any source vertex ***s***, BFS visits the other vertices at increasing distances away from ***s***. In doing so, BFS discovers shortest paths from ***s*** to the other vertices
- What do we mean by "distance"? The number of edges on a path from ***s*** (unweighted graph)



Example

Consider s=vertex 1

Nodes at distance 1?
2, 3, 7, 9

Nodes at distance 2?
8, 6, 5, 4

Nodes at distance 3?
0

How Does BSF Work?

- Similarly to level-order traversal for trees
- Examples:
  - www.student.seas.gwu.edu/~idsv/idsv.html
- Code: similar to code for topological sort (see the next slide)
  - *flag*[*v*] = false: we have not visited *v*
  - *flag*[*v*] = true: we already visited *v*
    - Why does BFS need a flag for each vertex (topological sort does not)?
- The code works for both directed and undirected graphs

Topological Sort

Algorithm *TSort*(G)
Input: a directed acyclic graph G
Output: a topological ordering of vertices

initialize Q to be an empty queue;
for each vertex v
    do if $indegree(v) = 0$
        then *enqueue*(Q, v);
while Q is non-empty
    do $v :=$ *dequeue*(Q);
        output v;
        for each arc (v, w)
            do $indegree(w) = indegree(w) - 1$;
                if $indegree(w) = 0$
                    then *enqueue*(w);

Find all starting points

Decrement indegree(w)

Place new start vertices on the Q

The running time is $O(n + m)$.

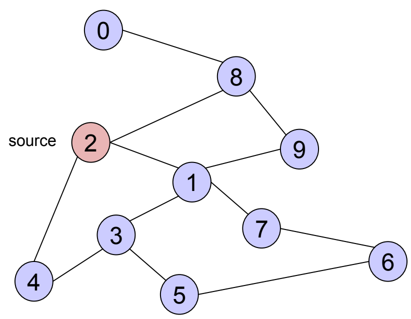
BFS Algorithm

Algorithm *BFS*(s)
Input: s is the source vertex
Output: Mark all vertices that can be visited from s.

for each vertex v
    do $flag[v] :=$ false;
$Q =$ empty queue;
$flag[s] :=$ true;
enqueue(Q, s);
while Q is not empty
    do $v :=$ *dequeue*(Q);
        for each w adjacent to v
            do if $flag[w] =$ false
                then $flag[w] :=$ true;
                    enqueue(Q, w)

flag[ ]: visited or not

BFS Example



Adjacency List

0	8
1	3 7 9 2
2	8 1 4
3	4 5 1
4	2 3
5	3 6
6	7 5
7	1 6
8	2 0 9
9	1 8

Visited Table (T/F)

0	F
1	F
2	F
3	F
4	F
5	F
6	F
7	F
8	F
9	F

Initialize “visited” table (all False)

Q = { }

Initialize **Q** to be empty

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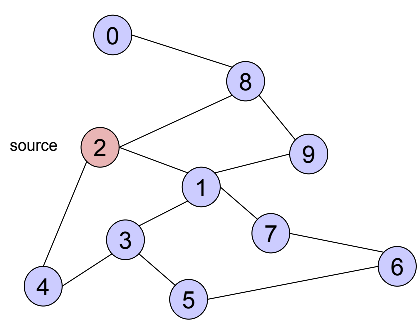

Adjacency List

0	8
1	3 7 9 2
2	8 1 4
3	4 5 1
4	2 3
5	3 6
6	7 5
7	1 6
8	2 0 9
9	1 8

Visited Table (T/F)

0	F
1	F
2	T
3	F
4	F
5	F
6	F
7	F
8	F
9	F

Flag that 2 has been visited

Q = { 2 }

Place source 2 on the queue

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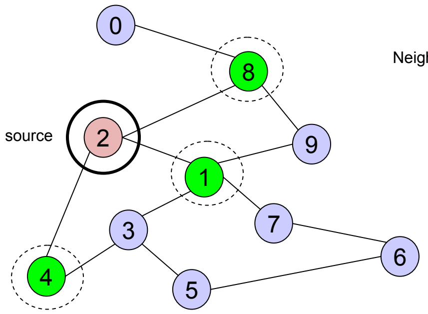


Adjacency List

Neighbors → 2

0	8
1	3 7 9 2
2	8 1 4
3	4 5 1
4	2 3
5	3 6
6	7 5
7	1 6
8	2 0 9
9	1 8

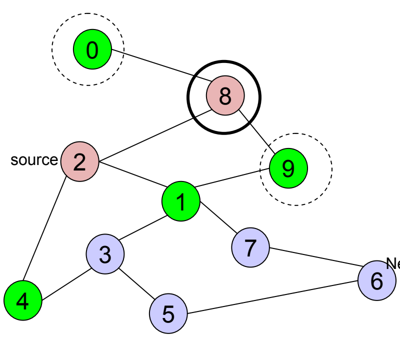
Visited Table (T/F)

0	F
1	T
2	T
3	F
4	T
5	F
6	F
7	F
8	T
9	F

Mark neighbors as visited 1, 4, 8

Q = {2} → { 8, 1, 4 }

Dequeue 2.
Place all unvisited neighbors of 2 on the queue



Adjacency List

Neighbors → 8

0	8
1	3 7 9 2
2	8 1 4
3	4 5 1
4	2 3
5	3 6
6	7 5
7	1 6
8	2 0 9
9	1 8

Visited Table (T/F)

0	T
1	T
2	T
3	F
4	T
5	F
6	F
7	F
8	T
9	T

Mark newly visited neighbors 0, 9

Q = { 8, 1, 4 } → { 1, 4, 0, 9 }

Dequeue 8.
-- Place all unvisited neighbors of 8 on the queue.
-- Notice that 2 is not placed on the queue again, it has been visited!

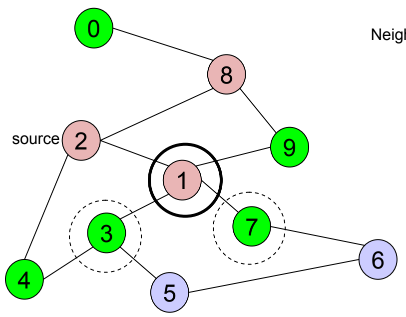
Adjacency List

Neighbors → 1

0	8
1	3 7 9 2
2	8 1 4
3	4 5 1
4	2 3
5	3 6
6	7 5
7	1 6
8	2 0 9
9	1 8

Visited Table (T/F)

0	T
1	T
2	T
3	T
4	T
5	F
6	F
7	T
8	T
9	T



Mark newly visited neighbors 3, 7

Q = { 1, 4, 0, 9 } → { 4, 0, 9, 3, 7 }

Dequeue 1.
-- Place all unvisited neighbors of 1 on the queue.
-- Only nodes 3 and 7 haven't been visited yet.

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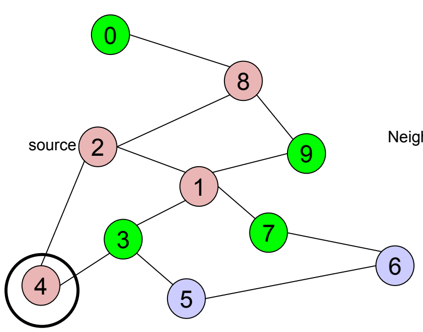
Adjacency List

Neighbors → 4

0	8
1	3 7 9 2
2	8 1 4
3	4 5 1
4	2 3
5	3 6
6	7 5
7	1 6
8	2 0 9
9	1 8

Visited Table (T/F)

0	T
1	T
2	T
3	T
4	T
5	F
6	F
7	T
8	T
9	T



Q = { 4, 0, 9, 3, 7 } → { 0, 9, 3, 7 }

Dequeue 4.
-- 4 has no unvisited neighbors!

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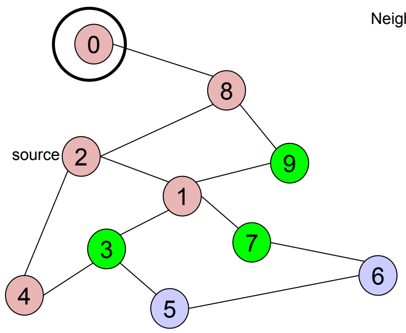

Adjacency List

Neighbors →

0	8
1	3 7 9 2
2	8 1 4
3	4 5 1
4	2 3
5	3 6
6	7 5
7	1 6
8	2 0 9
9	1 8

Visited Table (T/F)

0	T
1	T
2	T
3	T
4	T
5	F
6	F
7	T
8	T
9	T

Q = { 0, 9, 3, 7 } → { 9, 3, 7 }

Dequeue 0.
-- 0 has no unvisited neighbors!

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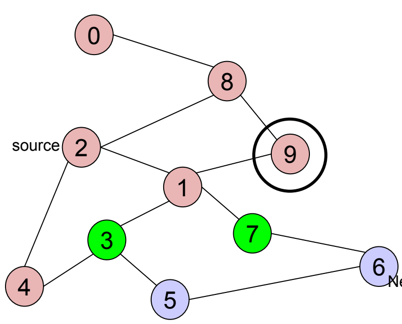

Adjacency List

0	8
1	3 7 9 2
2	8 1 4
3	4 5 1
4	2 3
5	3 6
6	7 5
7	1 6
8	2 0 9
Neighbors → 9	1 8

Visited Table (T/F)

0	T
1	T
2	T
3	T
4	T
5	F
6	F
7	T
8	T
9	T

Q = { 9, 3, 7 } → { 3, 7 }

Dequeue 9.
-- 9 has no unvisited neighbors!

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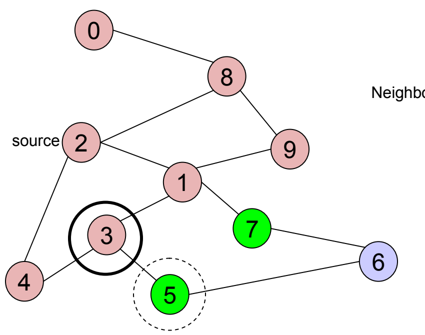

Adjacency List

Neighbors → 3

0	8
1	3 7 9 2
2	8 1 4
3	4 5 1
4	2 3
5	3 6
6	7 5
7	1 6
8	2 0 9
9	1 8

Visited Table (T/F)

0	T
1	T
2	T
3	T
4	T
5	T
6	F
7	T
8	T
9	T

Mark new visited
Vertex 5

Q = { 3, 7 } → { 7, 5 }

Dequeue 3.
-- place neighbor 5 on the queue.

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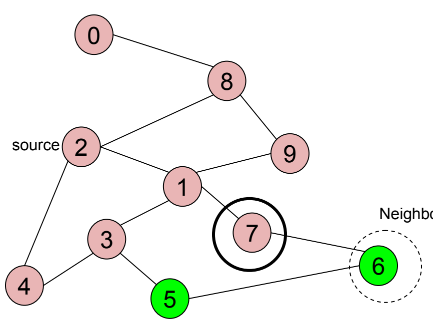

Adjacency List

Neighbors → 7

0	8
1	3 7 9 2
2	8 1 4
3	4 5 1
4	2 3
5	3 6
6	7 5
7	1 6
8	2 0 9
9	1 8

Visited Table (T/F)

0	T
1	T
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T

Mark new visited
Vertex 6

Q = { 7, 5 } → { 5, 6 }

Dequeue 7.
-- place neighbor 6 on the queue

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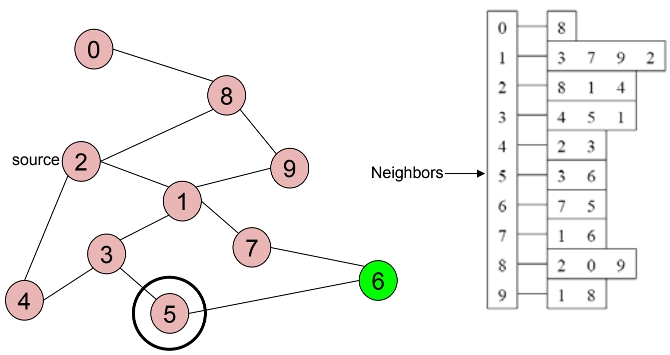
Adjacency List

Visited Table (T/F)



0	T
1	T
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T

Q = { 5, 6} → { 6 }

Dequeue 5.
-- no unvisited neighbors of 5

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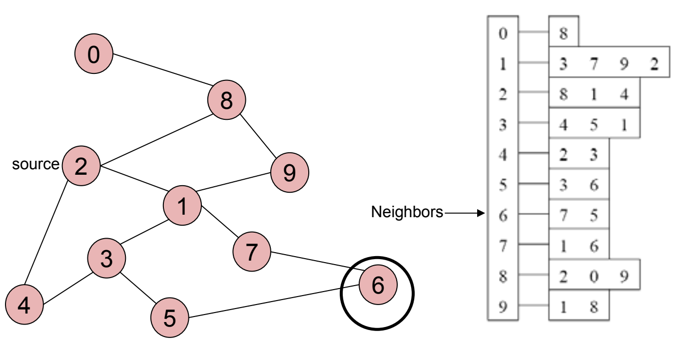
Adjacency List

Visited Table (T/F)



0	T
1	T
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T

Q = { 6 } → { }

Dequeue 6.
-- no unvisited neighbors of 6

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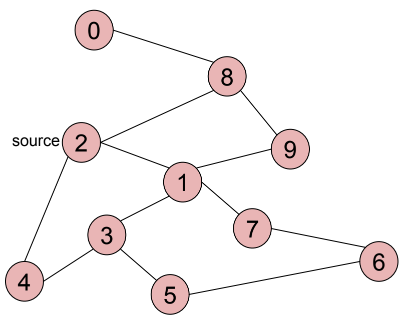

Q = { } **STOP!!! Q is empty!!!**

Adjacency List

0	8			
1	3	7	9	2
2	8	1	4	
3	4	5	1	
4	2	3		
5	3	6		
6	7	5		
7	1	6		
8	2	0	9	
9	1	8		

Visited Table (T/F)

0	T
1	T
2	T
3	T
4	T
5	T
6	T
7	T
8	T
9	T

What did we discover?

Look at “visited” tables.

There exists a path from source vertex 2 to all vertices in the graph

Applications of BFS

What can we do with the BFS code we just discussed?

- Is there a path from source *s* to a vertex *v*?
  - Check *flag*[*v*].
- Is an undirected graph connected?
  - Scan array *flag*[ ].
  - If there exists *flag*[*u*] = false then …
- Is a directed graph strongly connected?
  - Scan array *flag*[ ].
  - If there exists *flag*[*u*] = false then …
- To output the contents (e.g., the vertices) of a connected (strongly connected) graph
  - What if the graph is not connected (weakly connected)? Add just a little bit of code and invoke method BFS( ) ⇒ discussed later.

Other Applications of BFS

- To find the shortest path from a vertex *s* to a vertex *v* in an unweighted graph
- To find the length of such a path
- To find out if a graph contains cycles
- To find the connected components of a graph that is not connected
- To construct a BSF tree/forest from a graph

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Running Time of BFS

- Assume adjacency list
  - V = number of vertices;   E = number of edges

```
Algorithm BFS(s)
Input: s is the source vertex
Output: Mark all vertices that can be visited from s.
for each vertex v
    do flag[v] := false;
Q = empty queue;
flag[s] := true;
enqueue(Q, s);
while Q is not empty
   do v := dequeue(Q);
      for each w adjacent to v
          do if flag[w] = false
                then flag[w] := true;
                     enqueue(Q, w)
```

Each vertex will enter *Q* at most once. dequeue is O(1).

The for loop takes time proportional to deg(*v*).

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Running Time of BFS (cont'd)

- Recall: Given a graph with E edges

$$\Sigma_{\text{vertex } v} \ \deg(v) = 2E$$

- The total running time of the while loop is:

$$O(\Sigma_{\text{vertex } v} (1 + \deg(v))) = O(V+E)$$

- This is the sum over all the iterations of the while loop!
- Homework: What is the running time of BFS if we use an adjacency matrix?

Applications of BFS

- Is there a path from source *s* to a vertex *v*?
- Is an undirected graph connected?
- Is a directed graph strongly connected?
- To output the contents (e.g., the vertices) of a connected (strongly connected) graph
  - Discuss non-connected (weakly connected) graphs later
- To find the shortest path from a vertex *s* to a vertex *v* in an unweighted graph
- To find the length of such a path
- To find out if a graph contains cycles
- To find the connected components of a graph that is not connected
- To construct a BSF tree/forest from a graph