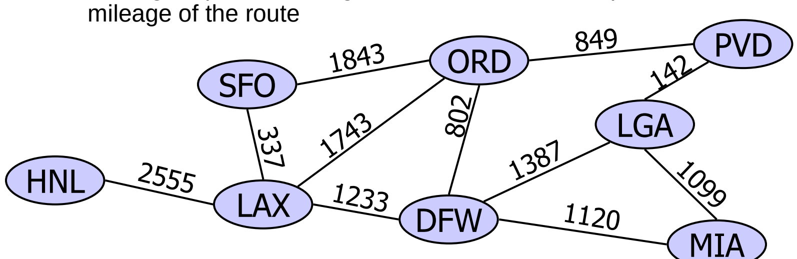
Graphs

CSE 2011

Winter 2007

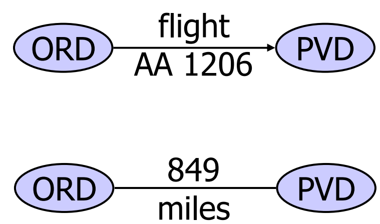
Graph

- A graph is a pair (V, E), where
  - V is a set of nodes, called vertices
  - E is a collection of pairs of vertices, called edges
  - Vertices and edges are objects and store elements
- Example:
  - A vertex represents an airport and stores the three-letter airport code
  - An edge represents a flight route between two airports and stores the mileage of the route



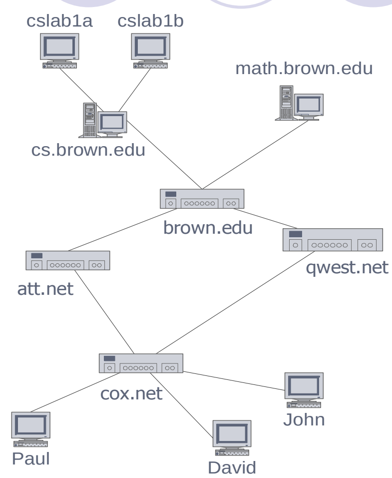
Edge Types

- Directed edge
  - ordered pair of vertices (u,v)
  - first vertex u is the origin
  - second vertex v is the destination
  - e.g., a flight
- Undirected edge
  - unordered pair of vertices (u,v)
  - e.g., a flight route
- Directed graph (digraph)
  - all the edges are directed



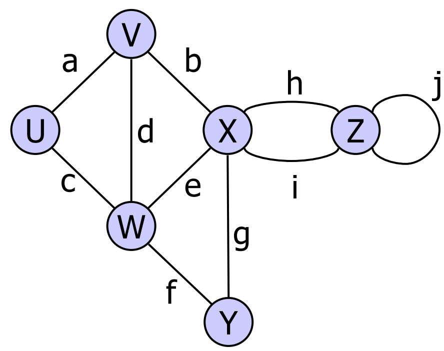
Applications

- Electronic circuits
  - Printed circuit board
  - Integrated circuit
- Transportation networks
  - Highway network
  - Flight network
- Computer networks
  - Local area network
  - Internet
  - Web
- Databases
  - Entity-relationship diagram



Terminology

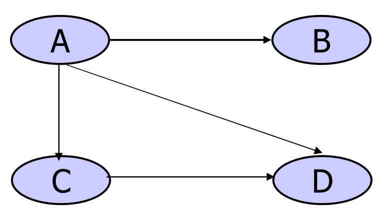
- End vertices (or endpoints) of an edge
  - U and V are the endpoints of a
- Edges incident on a vertex
  - a, d, and b are incident on V
- Adjacent vertices
  - U and V are adjacent
- Degree of a vertex
  - W has degree 4
- Loop
  - j is a loop
    (we will consider only loopless graphs)



Terminology (cont'd)

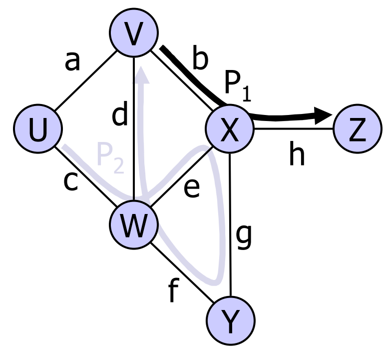
For directed graphs:

- Origin, destination of an edge
- Outgoing edge
- Incoming edge
- Out-degree of vertex v: number of outgoing edges of v
- In-degree of vertex v: number of incoming edges of v



Paths

- Path
  - sequence of alternating vertices and edges
  - begins with a vertex
  - ends with a vertex
  - each edge is preceded and followed by its endpoints
- Path length
  - the total number of edges on the path
- Simple path
  - path such that all vertices are distinct (except that the first and last could be the same)
- Examples
  - P_1=(V,b,X,h,Z) is a simple path
  - P_2=(U,c,W,e,X,g,Y,f,W,d,V) is a path that is not simple



Properties – Undirected Graphs

Property 1

$\Sigma_v \deg(v) = 2E$

Proof: each edge is counted twice

Property 2

In an undirected graph with no loops

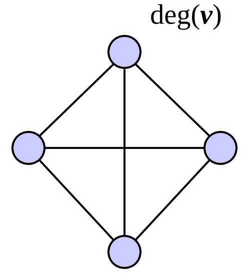
$\mathbf{E} \leq \mathbf{V}\ (\mathbf{V} - 1)/2$

Proof: each vertex has degree at most $(\mathbf{V} - 1)$

What is the bound for a directed graph?

Notation

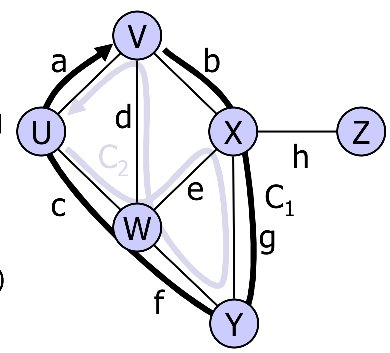
V	number of vertices
E	number of edges
$\deg(v)$	degree of vertex v



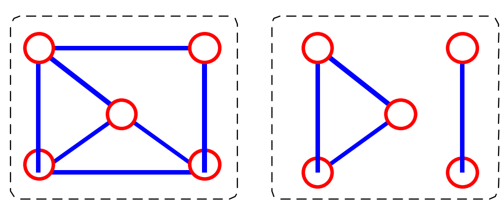
Example
- $V = 4$
- $E = 6$
- $\deg(v) = 3$

Cycles

- Cycle
  - circular sequence of alternating vertices and edges
  - each edge is preceded and followed by its endpoints
- Simple cycle
  - cycle such that all its vertices are distinct (except the first and the last)
- Examples
  - C_1=(V,b,X,g,Y,f,W,c,U,a,V) is a simple cycle
  - C_2=(U,c,W,e,X,g,Y,f,W,d,V,a,U) is a cycle that is not simple
- A directed graph is *acyclic* if it has no cycles ⇒ called DAG (directed acyclic graph)



Connectivity

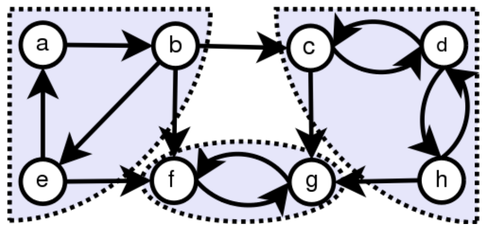


connected not connected

- An undirected graph is *connected* if there is a path from every vertex to every other vertex.

Connectivity (cont'd)

- A directed graph with this property is called *strongly connected*.
- If a directed graph is not strongly connected, but the corresponding undirected graph is connected, then the directed graph is said to be *weakly connected*.



Data Structures

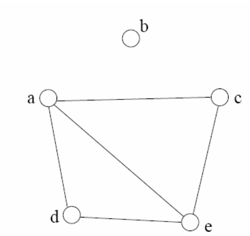
Representation of Graphs

- Two popular computer representations of a graph: Both represent the vertex set and the edge set, but in different ways.

  1. Adjacency Matrices
     Use a 2D matrix to represent the graph
  2. Adjacency Lists
     Use a set of linked lists, one list per vertex

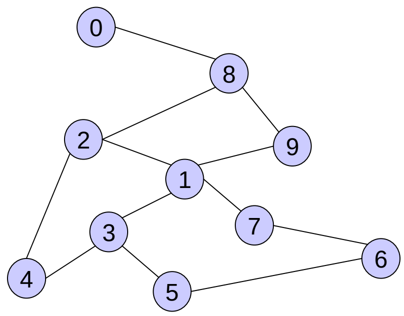
Adjacency Matrix Representation

- 2D array of size ***n*** x ***n*** where ***n*** is the number of vertices in the graph
- A[i][j]=1 if there is an edge connecting vertices i and j; otherwise, A[i][j]=0



	a	b	c	d	e
a	0	0	1	1	1
b	0	0	0	0	0
c	1	0	0	0	1
d	1	0	0	0	1
e	1	0	1	1	0

Adjacency Matrix Example



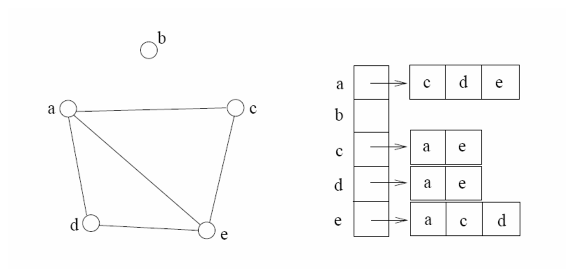
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	0	0	0	0	0	0	0	1	0
1	0	0	1	1	0	0	0	1	0	1
2	0	1	0	0	1	0	0	0	1	0
3	0	1	0	0	1	1	0	0	0	0
4	0	0	1	1	0	0	0	0	0	0
5	0	0	0	1	0	0	1	0	0	0
6	0	0	0	0	0	1	0	1	0	0
7	0	1	0	0	0	0	1	0	0	0
8	1	0	1	0	0	0	0	0	0	1
9	0	1	0	0	0	0	0	0	1	0

Adjacency Matrices (cont'd)

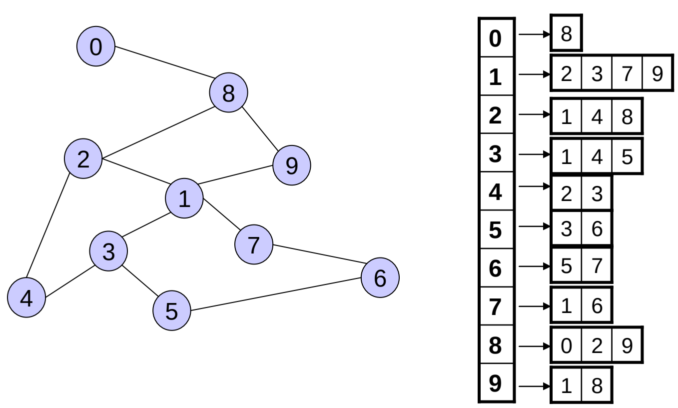
- The storage requirement is $\Theta(V^2)$.
  - not efficient if the graph has few edges.
  - appropriate if the graph is dense; that is E= $\Theta(V^2)$
- If the graph is undirected, the matrix is symmetric. There exist methods to store a symmetric matrix using only half of the space. But the space requirement is still $\Theta(V^2)$.
- We can detect in O(1) time whether two vertices are connected.

Adjacency Lists

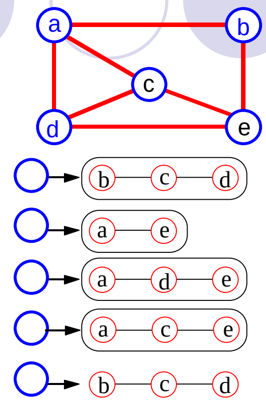
- If the graph is sparse, a better solution is an adjacency list representation.
- For each vertex ***v*** in the graph, we keep a list of vertices adjacent to ***v***.



Adjacency List Example



Adjacency Lists (cont'd)



Space =
Θ (V + Σ_v deg(v)) = Θ (V + E)

- Testing whether u is adjacency to v takes time O(deg(v)) or O(deg(u)).

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Adjacency Lists vs. Adjacency Matrices

- An adjacency list takes Θ(V + E).
  - If E = O(V^2) (dense graph), both use Θ(V^2) space.
  - If E = O(V) (sparse graph), adjacency lists are more space efficient.
- Adjacency lists
  - More compact than adjacency matrices if graph has few edges
  - Requires more time to find if an edge exists
- Adjacency matrices
  - Always require Θ(V^2) space
    - This can waste lots of space if the number of edges is small
  - Can quickly find if an edge exists

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Next time …

- Graph traversal
  - Depth first search
  - Breadth first search

- Topological sort