2.3.3 Addendum: Tensoring States

We stated that in order to build systems of multiple qubits, the individual states of the participating qubits are tensored together. Given our definition of the tensor product in Section 1.3, this was easy to understand when states were expressed as vectors. For example:

$$\begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a \begin{bmatrix} c \\ d \end{bmatrix} \\ b \begin{bmatrix} c \\ d \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}.$$

But what if the states are written as an expression, such as

$$\big(a\,|0\rangle + b\,|1\rangle\big) \otimes \big(c\,|0\rangle + d\,|1\rangle\big)$$

Because of the tensor product's linearity, we can actually "multiply out" the expression, just like a normal product of two terms. Correspondingly, and sometimes confusingly, the product is often written *without* the $\otimes$ operator, such as the following:

$$\big(a\,|0\rangle + b\,|1\rangle\big) \otimes \big(c\,|0\rangle + d\,|1\rangle\big) \equiv \big(a\,|0\rangle + b\,|1\rangle\big)\big(c\,|0\rangle + d\,|1\rangle\big)$$

As we "multiply" the two bracketed terms, the scalar factors turn into simple products and the qubit states are tensored together. For scalar products the order of their operands doesn't matter (mathematically). For qubit states, on the other hand, their ordering must be maintained:

$$\begin{aligned} &\big(a\,|0\rangle + b\,|1\rangle\big)\big(c\,|0\rangle + d\,|1\rangle\big) = \\ &a\,|0\rangle\,\big(c\,|0\rangle + d\,|1\rangle\big) + b\,|1\rangle\,\big(c\,|0\rangle + d\,|1\rangle\big) = \\ &ac\,|00\rangle + ad\,|01\rangle + bc\,|10\rangle + bd\,|11\rangle\,. \end{aligned}$$

Writing this state as a vector results in the same state vector as above: $\begin{bmatrix} ac & ad & bc & bd \end{bmatrix}^T$. To make the required ordering clear, individual qubits sometimes get a subscript, indicating their ordering and who they belong to, such as Alice or Bob:

$$\begin{aligned} &\big(a\,|0_A\rangle + b\,|1_A\rangle\big)\big(c\,|0_B\rangle + d\,|1_B\rangle\big) = \\ &ac\,|0_A 0_B\rangle + ad\,|0_A 1_B\rangle + bc\,|1_A 0_B\rangle + bd\,|1_A 1_B\rangle\,. \end{aligned}$$

Note that the multiplication procedure can be reversed. We can *factor out* individual qubits. This should be no surprise, but it may be helpful to see this at least once:

$$\begin{aligned} &ac\,|00\rangle + ad\,|01\rangle + bc\,|10\rangle + bd\,|11\rangle = \\ &|0\rangle\,\big(ac\,|0\rangle + ad\,|1\rangle\big) + |1\rangle\,\big(bc\,|0\rangle + bd\,|1\rangle\big) = \\ &a\,|0\rangle\,\big(c\,|0\rangle + d\,|1\rangle\big) + b\,|1\rangle\,\big(c\,|0\rangle + d\,|1\rangle\big) = \\ &\big(a\,|0\rangle + b\,|1\rangle\big)\big(c\,|0\rangle + d\,|1\rangle\big). \end{aligned}$$

We will find these types of state manipulations in several places in this book.