2.12 Addendum: No-Deleting Theorem

Corresponding to the just described No-Cloning Theorem, which states that in general an unknown quantum state cannot be cloned, the *No-Deleting Theorem* (Pati and Braunstein, 2000) proves that for two qubits in a random but identical state $|\psi\rangle$, there cannot be a unitary operator to *delete* or *reset* one of the two qubits back to state $|0\rangle$.

THEOREM 2.1 *Given a general quantum state $|\psi\rangle|\psi\rangle|A\rangle$, with two qubits in the identical state $|\psi\rangle$ and an ancilla $|A\rangle$, there cannot be a unitary operator U, such that $U|\psi\rangle|\psi\rangle|A\rangle = |\psi\rangle|0\rangle|A'\rangle$, where A' is the ancilla's state after application of U.*

Proof Assume we had an operator U that is capable of performing the deletion operation:

$$\begin{aligned} U|0\rangle|0\rangle A &= |0\rangle|0\rangle A' \\ U|1\rangle|1\rangle A &= |1\rangle|0\rangle A' \end{aligned}$$

As before we compute the application of U to state $|\psi\rangle|\psi\rangle|A\rangle$ in two different ways. First, for an individual qubit in state $\alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$ and with the operator U as defined above, we get:

$$\begin{aligned} U|\psi\rangle|\psi\rangle A &= |\psi\rangle|0\rangle A' \\ &= \big(\alpha|0\rangle|0\rangle + \beta|1\rangle|0\rangle\big)A' \end{aligned} \tag{2.1}$$

Now let's compute the state as the tensor product of the qubits and apply the hypothetical operator U:

$$\begin{aligned} U|\psi\rangle|\psi\rangle|A\rangle &= U\big((\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)A\big) \\ &= U\big(\alpha^2|0\rangle|0\rangle + \alpha\beta|0\rangle|1\rangle + \beta\alpha|1\rangle|0\rangle + \beta^2|1\rangle|1\rangle\big)A \\ &= \alpha^2 U|0\rangle|0\rangle A + \beta^2 U|1\rangle|1\rangle A + \alpha\beta U|0\rangle|1\rangle A + \beta\alpha U|1\rangle|0\rangle A \\ &= \alpha^2|0\rangle|0\rangle A' + \beta^2|1\rangle|0\rangle A' + \alpha\beta U\big(|0\rangle|1\rangle + |1\rangle|0\rangle\big)A \end{aligned}$$

This form is different from Equation 2.1 above. It has an additional (entangled) component $(|0\rangle|1\rangle + |1\rangle|0\rangle)$, which is typically defined as Φ. With this, the final form becomes:

$$= \big(\alpha^2|0\rangle|0\rangle + \beta^2|1\rangle|0\rangle\big)A' + \alpha\beta U\Phi A \tag{2.2}$$

In general, Equation 2.1 is different from Equation 2.2, a contradiction which proves that no such operator U can exist. □

Note that if either $\alpha = 0$ or $\beta = 0$, we again deal with the equivalent of classical bits. The final term in Equation 2.2 would disappear and thus an operator U for those special states would indeed be feasible.

Bibliography

A. Pati and S. Braunstein. Impossibility of deleting an unknown quantum state. *Nature*, 404(6774):164–165, Mar. 2000. ISSN 0028-0836.