Lab 1 - Mathematical Modeling: Predator-prey models

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Introduction

In predator-prey models we will consider two dependent populations: the predators and the preys they hunt. We will make three basic assumptions here:

1. The prey species is the only species the predator species hunts: the predator species is thus totally dependent on the preys.
2. The preys are only hunted by the predators
3. The preys have unlimited resources

Because of these assumptions the prey population would grow unbounded without predators and the population $x(t)$ might for example satisfy the following growth model, which causes exponential growth:

$$\frac{dx}{dt} = \epsilon_1 x \tag{1}$$

In the presence of predators the growth of the prey population will be limited, depending on the number of predators $y(t)$, but the chances of a predator to encounter a prey also depend on the number of preys. So we have:

$$\frac{dx}{dt} = \epsilon_1 x - \alpha_1 xy \tag{2}$$

Now we look at the predator population $y(t)$ that will die out without prey:

$$\frac{dy}{dt} = -\epsilon_2 y \tag{3}$$

But it will grow if there are sufficient prey to support reproduction; this depends on both the number of preys and the number of predators:

$$\frac{dy}{dt} = -\epsilon_2 y + \alpha_2 xy \tag{4}$$

Together this leads to the following system of differential equations:

$$\begin{cases} \frac{dx}{dt} = \epsilon_1 x - \alpha_1 xy \\ \frac{dy}{dt} = -\epsilon_2 y + \alpha_2 xy \end{cases} \tag{5}$$

with $\epsilon_1, \epsilon_2, \alpha_1, \alpha_2 \in \mathbb{R}^+$. This model is called the Lotka-Volterra Predator-Prey Model.

Exercise 1

Write a function **simStep** that takes the parameter set $x(t), y(t), \epsilon_1, \alpha_1, \epsilon_2, \alpha_2, \Delta t$ as its input arguments and gives $x(t+\Delta t)$ and $y(t+\Delta t)$ as its output arguments, according to the Lotka-Volterra Predator-Prey Model. This involves discretization of the model; you have to replace the derivatives by appropriate expressions involving the time step. The input will thus be scalar and will contain the current state of the model along with the parameters and the time-step. The function **simStep** should start as:

```
function [x,y]=simStep(x,y,e1,a1,e2,a2,dt)
```

Exercise 2

Write a script file that simulates by iteration the Lotka-Volterra Predator-Prey Model, using **simStep** for 10000 time-steps of size $\Delta t = 0.1$ with the following parameters and initial conditions:

- $\epsilon_1 = 0.04$
- $\alpha_1 = 0.0005$
- $\epsilon_2 = 0.3$
- $\alpha_2 = 0.001$
- $x(0) = 50$
- $y(0) = 5$

Plot the complete trajectory.

Exercise 3

Use the function **plotTrajectory**[1], that can be found on the website, to plot the trajectory that you simulated in the previous exercise with a tail of length 100.
Try various (non-negative) initial conditions that seem to be outside the trajectory or between valid trajectories. What do you notice?
Can you compute the equilibrium point? Or can you find it experimentally by trial-and-error?

Matlab possesses a number of functions to solve ordinary differential equations such as **ode15s**, **ode23s** and **ode45**. These functions solve differential equations of the form:

$$y' = f(t, y) \tag{6}$$

The differential equation from the Lotka-Volterra model is not time-varying and has two variables (x and y). Therefore the y-variable from the Matlab routines is a vector of two variables: $y = \begin{pmatrix} x \\ y \end{pmatrix}$

[1] You can also use the Matlab "comet" function as "comet(trajectory(1,:),trajectory(2,:))", but to my knowledge it does not offer speed control to allow for a nice animation

Exercise 4

Solve the model that has been described earlier with one of the Matlab solvers for the same parameters, initial condition and time steps.

Exercise 5

Plot the solution of the model with `plotTrajectory` and compare it with the results of Exercise 3 and try to explain the differences, if any.

Exercise 6

The Lotka-Volterra model described above involves exponential growth of the preys in absence of predators. It can be made more realistic by including a term which limits the growth rate of the preys depending on the population size. (See for instance the model specification on p.14 of the lecture notes.) This means that an extra quadratic term is included in the first differential equation. Add such a term $-\sigma_1 x^2$ to obtain the model

$$\begin{cases} \frac{dx}{dt} = \epsilon_1 x - \alpha_1 xy - \sigma_1 x^2 \\ \frac{dy}{dt} = -\epsilon_2 y + \alpha_2 xy \end{cases} \tag{7}$$

Adapt your software, simulate and plot the dynamics of this system with parameter value $\sigma_1 = 0.00004$, and find the equilibrium by looking at the simulation and, if needed, changing the initial point.