

ENGINEERING MATHEMATICS -II

SUBJECT CODE: 71102002002

UNIT-IV

LAPLACE TRANSFORM

UNIT-V

LAPLACE TRANSFORM

5.1 INTRODUCTION

A transformation is an operation which converts a mathematical expression to a different but equivalent form. The well known transformation logarithms reduce multiplication and division to a simpler process of addition subtraction.

The Laplace transform is a powerful mathematical technique which solves linear equations with given initial conditions by using algebra methods. The Laplace transform can also be used to solve systems of differential equations, Partial differential equations and integral equations. In this chapter, we will discuss about the definition, properties of Laplace transform and derive the transforms of some functions which usually occur in the solution of linear differential equations.

5.2 LAPLACE TRANSFORM

Let $f(t)$ be a function of t defined for all $t \geq 0$. then the Laplace transform of $f(t)$, denoted by $L[f(t)]$ is defined by

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

Provided that the integral exists, “s” is a parameter which may be real or complex. Clearly $L[f(t)]$ is a function of s and is briefly written as $F(s)$ (i. e.) $L[f(t)] = F(s)$

Piecewise continuous function

A function $f(t)$ is said to be piecewise continuous is an interval $a \leq t \leq b$, if the interval can be sub divided into a finite number of intervals in each of which the function is continuous and has finite right and left hand limits.

Exponential order

A function $f(t)$ is said to be exponential order if $\lim_{t \rightarrow \infty} e^{-st} f(t)$ is a finite quantity, where $s > 0$ (exists).

Example: 5. 1 Show that the function $f(t) = e^{t^3}$ is not of exponential order.

Solution:

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{-st} e^{t^3} &= \lim_{t \rightarrow \infty} e^{-st+t^3} = \lim_{t \rightarrow \infty} e^{t^3-st} \\ &= e^{\infty} = \infty, \text{ not a finite quantity.} \end{aligned}$$

Hence $f(t) = e^{t^3}$ is not of exponential order.

Sufficient conditions for the existence of the Laplace transform

The Laplace transform of $f(t)$ exists if

- i) $f(t)$ is piecewise continuous in the interval $a \leq t \leq b$
- ii) $f(t)$ is of exponential order.

Note: The above conditions are only sufficient conditions and not a necessary condition.

Example: 5.2 Prove that Laplace transform of e^{t^2} does not exist.

Solution:

$$\begin{aligned}\lim_{t \rightarrow \infty} e^{-st} e^{t^2} &= \lim_{t \rightarrow \infty} e^{-st+t^2} = \lim_{t \rightarrow \infty} e^{t^2-st} \\ &= e^{\infty} = \infty, \text{ not a finite quantity.}\end{aligned}$$

$\therefore e^{t^2}$ is not of exponential order.

Hence Laplace transform of e^{t^2} does not exist.

5.3 PROPERTIES OF LAPLACE TRANSFORM

Property: 1 Linear property

$L[af(t) \pm bg(t)] = aL[f(t)] \pm bL[g(t)]$, where a and b are constants.

Proof:

$$\begin{aligned}L[af(t) \pm bg(t)] &= \int_0^{\infty} [af(t) \pm bg(t)] e^{-st} dt \\ &= a \int_0^{\infty} f(t) e^{-st} dt \pm b \int_0^{\infty} g(t) e^{-st} dt\end{aligned}$$

$$\boxed{L[af(t) \pm bg(t)] = aL[f(t)] \pm bL[g(t)]}$$

Property: 2 Change of scale property.

If $L[f(t)] = F(s)$, then $L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$; $a > 0$

Proof:

$$\begin{aligned}\text{Given } L[f(t)] &= F(s) \\ \therefore \int_0^{\infty} e^{-st} f(t) dt &= F(s) \dots \dots (1)\end{aligned}$$

By the definition of Laplace transform, we have

$$L[f(at)] = \int_0^{\infty} e^{-st} f(at) dt \dots \dots (2)$$

$$\text{Put } at = x \text{ i.e., } t = \frac{x}{a} \Rightarrow dt = \frac{dx}{a}$$

$$\begin{aligned}(2) \Rightarrow L[f(at)] &= \int_0^{\infty} e^{-\frac{sx}{a}} f(x) \frac{dx}{a} \\ &= \frac{1}{a} \int_0^{\infty} e^{-\frac{sx}{a}} f(x) dx\end{aligned}$$

$$\text{Replace } x \text{ by } t, \quad L[f(at)] = \frac{1}{a} \int_0^{\infty} e^{-\frac{st}{a}} f(t) dt$$

$$\boxed{L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right); a > 0}$$

Property: 3 First shifting property.

If $L[f(t)] = F(s)$, then i) $L[e^{-at}f(t)] = F(s+a)$

ii) $L[e^{at}f(t)] = F(s-a)$

Proof:

$$(i) L[e^{-at}f(t)] = F(s+a)$$

Given $L[f(t)] = F(s)$

$$\therefore \int_0^{\infty} e^{-st} f(t) dt = F(s) \dots (1)$$

By the definition of Laplace transform, we have

$$\begin{aligned} L[e^{-at}f(at)] &= \int_0^{\infty} e^{-st} e^{-at} f(t) dt \\ &= \int_0^{\infty} e^{-(s+a)t} f(t) dt \\ &= F(s+a) \quad \text{by (1)} \end{aligned}$$

$$\begin{aligned} \text{(ii) } L[e^{at}f(at)] &= \int_0^{\infty} e^{-st} e^{at} f(t) dt \\ &= \int_0^{\infty} e^{-(s-a)t} f(t) dt \\ &= F(s-a) \quad \text{by (1)} \end{aligned}$$

Property: 4 Laplace transforms of derivatives $L[f'(t)] = sL[f(t)] - f(0)$

Proof:

$$\begin{aligned} L[f'(t)] &= \int_0^{\infty} e^{-st} f'(t) dt = \int_0^{\infty} u dv \\ &= [uv]_0^{\infty} - \int u dv \\ &= [e^{-st} f(t)]_0^{\infty} - \int_0^{\infty} f(t) (-s)e^{-st} dt \\ &= 0 - f(0) + sL[f(t)] \\ &= sL[f(t)] - f(0) \end{aligned}$$

$$L[f'(t)] = sL[f(t)] - f(0)$$

$\begin{aligned} u &= e^{-st} \\ \therefore du &= -se^{-st} dt \\ dv &= f'(t) dt \\ \therefore v &= \int f'(t) dt \\ &= f(t) \end{aligned}$

Property: 5 Laplace transform of derivative of order n

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) \dots - s^{n-3} f''(0) - \dots f^{n-1}(0)$$

Proof:

We know that $L[f'(t)] = sL[f(t)] - f(0) \dots \dots (1)$

$$\begin{aligned} L[f^n(t)] &= L[[f'(t)]'] \\ &= sL[f'(t)] - f'(0) \\ &= s[sL[f(t)] - f(0)] - f'(0) \\ &= s^2 L[f(t)] - sf(0) - f'(0) \end{aligned}$$

$$\text{Similarly, } L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$$

$$\text{In general, } L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) \dots - s^{n-3} f''(0) - \dots f^{n-1}(0)$$

Laplace transform of integrals

Theorem: 1 If $L[f(t)] = F(s)$, then $L\left[\int_0^t f(t) dt\right] = \frac{F(s)}{s}$

Proof:

$$\text{Let } g(t) = \int_0^t f(t) dt$$

$$\therefore g'(t) = f(t)$$

$$\text{And } g(0) = \int_0^0 f(t) dt = 0$$

$$\text{Now } L[g'(t)] = L[f(t)]$$

$$sL[g(t)] - g(0) = L[f(t)]$$

$$sL[g(t)] = L[f(t)] \quad \therefore g(0) = 0$$

$$L[g(t)] = \frac{L[f(t)]}{s}$$

$$\therefore L\left[\int_0^t f(t) dt\right] = \frac{F(s)}{s}$$

Theorem: 2 If $L[f(t)] = F(s)$, then $L[tf(t)] = -\frac{d}{ds}F(s)$

Proof:

$$\text{Given } L[f(t)] = F(s)$$

$$\therefore \int_0^\infty e^{-st} f(t) dt = F(s) \dots \dots (1)$$

Differentiating (1) with respect to s, we get

$$\frac{d}{ds} \int_0^\infty e^{-st} f(t) dt = \frac{d}{ds} F(s)$$

$$\int_0^\infty \frac{\partial}{\partial s} (e^{-st}) f(t) dt = \frac{d}{ds} F(s)$$

$$\int_0^\infty (-t) e^{-st} f(t) dt = \frac{d}{ds} F(s)$$

$$-\int_0^\infty t e^{-st} f(t) dt = \frac{d}{ds} F(s)$$

$$-L[tf(t)] = \frac{d}{ds} F(s)$$

$$\therefore L[tf(t)] = -\frac{d}{ds} F(s)$$

Note: In general $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$

Example: 5.3 If $L[f(t)] = \frac{s^2-s+1}{(2s+1)^2(s-1)}$ then find $L[f(2t)]$.

Solution:

$$\text{Given } L[f(t)] = \frac{s^2-s+1}{(2s+1)^2(s-1)} = F(s)$$

$$\begin{aligned} L[f(2t)] &= \frac{1}{2} F\left(\frac{s}{2}\right) \\ &= \frac{1}{2} \frac{\left(\frac{s}{2}\right)^2 - \frac{s}{2} + 1}{\left(2\frac{s}{2} + 1\right)^2 \left(\frac{s}{2} - 1\right)} \\ &= \frac{1}{2} \frac{\left[\frac{s^2}{4} - \frac{s}{2} + 4\right]}{(s+1)^2 \left(\frac{s-2}{2}\right)} \\ &= \frac{s^2-2s+1}{4(s+1)^2(s-2)} \end{aligned}$$

Laplace transform of some Standard functions

Result: 1 Prove that $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t) dt$

$$L[t^n] = \int_0^\infty e^{-st} t^n dt$$

$$L[t^n] = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$= \int_0^\infty e^{-u} \frac{u^n}{s^{n+1}} du$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n du$$

$$\therefore L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} \quad \because \int_0^\infty e^{-u} u^n du$$

Let $st = u \dots \dots (1)$

$$t = \frac{u}{s}$$

$$dt = \frac{du}{s}$$

When $t \rightarrow 0(1) \Rightarrow u \rightarrow 0$

,

$t \rightarrow \infty, (1) \Rightarrow u \rightarrow \infty$

Note: If n is an integer, then $\Gamma(n+1) = n!$

$$\therefore L[t^n] = \frac{n!}{s^{n+1}} \quad \text{if } n \text{ is an integer}$$

$$\text{If } n = 0, \text{ then } L[1] = \frac{1}{s}$$

$$\text{If } n = 1, \text{ then } L[t] = \frac{1}{s^2}$$

$$\text{Similarly } L[t^2] = \frac{2!}{s^3}$$

$$L[t^3] = \frac{3!}{s^4}$$

Result: 2 Prove that $L(e^{at}) = \frac{1}{s-a}, s > a$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t) dt$

$$\therefore L(e^{at}) = \int_0^\infty e^{-st} e^{at} dt$$

$$= \int_0^\infty e^{-t(s-a)} f(t) dt$$

$$= \left[\frac{e^{-t(s-a)}}{-(s-a)} \right]_0^\infty$$

$$= - \left[0 - \left(\frac{1}{s-a} \right) \right]$$

$$\therefore L(e^{at}) = \frac{1}{s-a}$$

Result: 3 Prove that $L(e^{-at}) = \frac{1}{s+a}, s > a$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t) dt$

$$\therefore L(e^{-at}) = \int_0^\infty e^{-st} e^{-at} dt$$

$$= \int_0^\infty e^{-t(s+a)} f(t) dt$$

$$= \left[\frac{e^{-t(s+a)}}{-(s+a)} \right]_0^\infty$$

$$= - \left[0 - \left(\frac{1}{s+a} \right) \right]$$

$$\therefore L(e^{-at}) = \frac{1}{s+a}$$

Result: 4 Prove that $L[\sin at] = \frac{a}{s^2 + a^2}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t) dt$

$$L[\sin at] = \int_0^\infty e^{-st} \sin at dt$$

$$\therefore L[\sin at] = \frac{a}{s^2 + a^2}, s > |a| \quad \left[\because \int_0^\infty e^{-at} \sin bt dt = \frac{b}{a^2 + b^2} \right]$$

Result: 5 Prove that $L[\cos at] = \frac{s}{s^2 + a^2}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t) dt$

$$L[\cos at] = \int_0^\infty e^{-st} \cos at dt$$

$$\therefore L[\cos at] = \frac{s}{s^2 + a^2}, s > |a| \quad \because \int_0^\infty e^{-at} \cos bt dt = \frac{a}{a^2 + b^2}$$

Result: 6 Prove that $L[\sinh at] = \frac{a}{s^2 - a^2}, s > |a|$

Proof:

$$\text{We have } L[\sinh at] = L\left[\frac{e^{at} - e^{-at}}{2}\right]$$

$$= \frac{1}{2} [L(e^{at}) - L(e^{-at})]$$

$$= \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left[\frac{s+a-s+a}{s^2-a^2} \right]$$

$$= \frac{1}{2} \left[\frac{2a}{s^2-a^2} \right]$$

$$\therefore L[\sinh at] = \frac{a}{s^2 - a^2}, s > |a|$$

Result: 7 Prove that $L[\cosh at] = \frac{s}{s^2 - a^2}, s > |a|$

Proof:

$$\text{We have } L[\cosh at] = L\left[\frac{e^{at} + e^{-at}}{2}\right]$$

$$= \frac{1}{2} [L(e^{at}) + L(e^{-at})]$$

$$= \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left[\frac{s+a+s-a}{s^2-a^2} \right]$$

$$= \frac{1}{2} \left[\frac{2s}{s^2-a^2} \right]$$

$$\therefore L[\cosh at] = \frac{s}{s^2 - a^2}, s > |a|$$

Example: 5.4 Find $L\left[t^{\frac{1}{2}}\right]$

Solution:

$$\text{We have } L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$\text{Put } n = \frac{1}{2}$$

$$\therefore L\left[t^{\frac{1}{2}}\right] = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}}$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}+1}}$$

$$= \frac{\frac{\sqrt{\pi}}{2}}{2s^{\frac{3}{2}}}$$

$$\therefore L\left[t^{\frac{1}{2}}\right] = \frac{\sqrt{\pi}}{2s\sqrt{s}}$$

$$\therefore \Gamma(n+1) = n\Gamma n$$

$$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Example: 5.5 Find the Laplace transform of $t^{-\frac{1}{2}}$ or $\frac{1}{\sqrt{t}}$

Solution:

$$\text{We have } L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$\text{Put } n = -\frac{1}{2}$$

$$\therefore L\left[t^{-\frac{1}{2}}\right] = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$\therefore L\left[\frac{1}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}}$$

$$\therefore \Gamma(n+1) = n\Gamma n$$

$$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

FORMULA

$L[f(t)] = F(s)$	$L[f(t)] = F(s)$
$L[1] = \frac{1}{s}$	$L[\sin at] = \frac{a}{s^2 + a^2}$
$L[t] = \frac{1}{s^2}$	$L[\cos at] = \frac{s}{s^2 + a^2}$
$L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$ if n is not an integer	$L[\cosh at] = \frac{s}{s^2 - a^2}$
$L[t^n] = \frac{n!}{s^{n+1}}$ if n is an integer	$L[\sinh at] = \frac{a}{s^2 - a^2}$
$L(e^{at}) = \frac{1}{s-a}$	
$L(e^{-at}) = \frac{1}{s+a}$	

Problems using Linear property

Example: 5.6 Find the Laplace transform for the following

i. $3t^2 + 2t + 1$	v. $\sin\sqrt{2} t$	ix. $\sin^2 t$
ii. $(t+2)^3$	vi. $\sin(at+b)$	x. $\cos^2 2t$
iii. a^t	vii. $\cos^3 2t$	xi. $\cos 5t \cos 4t$

iv. e^{2t+3}	viii. $\sin^3 t$	
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Solution:

(i) Given $f(t) = 3t^2 + 2t + 1$

$$\begin{aligned}
 L[f(t)] &= L[3t^2 + 2t + 1] \\
 &= L[3t^2] + L[2t] + L[1] \\
 &= L[3t^2] + L[2t] + L[1] \\
 &= 3L[t^2] + 2L[t] + L[1] \\
 &= 3 \frac{2}{s^3} + 2 \frac{1}{s^2} + \frac{1}{s}
 \end{aligned}$$

$$\therefore L[3t^2 + 2t + 1] = \frac{6}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

(ii) Given $f(t) = (t + 2)^3 = t^3 + 3t^2(2) + 3t2^2 + 2^3$

$$\begin{aligned}
 L[f(t)] &= L[t^3 + 3t^2(2) + 3t2^2 + 2^3] \\
 &= L[t^3] + L[6t^2] + L[12t] + L[8] \\
 &= L[t^3] + 6L[t^2] + 12L[t] + 8L[1] \\
 &= \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{12}{s}
 \end{aligned}$$

(iii) Given $f(t) = a^t$

$$\begin{aligned}
 L[f(t)] &= L[a^t] = L[e^{t \log a}] \\
 L[a^t] &= \frac{1}{s - \log a}
 \end{aligned}$$

(iv) Given $f(t) = e^{2t+3}$

$$\begin{aligned}
 L[f(t)] &= L[e^{2t+3}] = L[e^{2t} \cdot e^3] \\
 &= e^3 L[e^{2t}] \\
 &= e^3 \left[\frac{1}{s-2} \right] \\
 \therefore L[e^{2t+3}] &= e^3 \left[\frac{1}{s-2} \right]
 \end{aligned}$$

(v) $L[\sin \sqrt{2}t] = \frac{\sqrt{2}}{s^2+2}$

(vi) Given $f(t) = \sin(at + b) = \sin at \cos b + \cos at \sin b$

$$\begin{aligned}
 L[f(t)] &= L[\sin(at + b)] \\
 &= L[\sin at \cos b + \cos at \sin b] \\
 &= \cos b L[\sin at] + \sin b L[\cos at] \\
 L[\sin(at + b)] &= \cos b \frac{s}{s^2+a^2} + \sin b \frac{s}{s^2+a^2}
 \end{aligned}$$

(vii) Given $f(t) = \cos^3 2t = \frac{1}{4}[3\cos 2t + \cos 6t]$

$$\begin{aligned}
 L[f(t)] &= \frac{1}{4} L[3\cos 2t + \cos 6t] \\
 &= \frac{1}{4} [3L(\cos 2t) + L(\cos 6t)] \\
 &= \frac{1}{4} \left[3 \frac{s}{s^2+4} + \frac{s}{s^2+36} \right]
 \end{aligned}$$

$$\therefore \cos^3 \theta = \frac{3\cos \theta + \cos 3\theta}{4}$$

$$L[\cos^3 2t] = \frac{1}{4} \left[3 \frac{s}{s^2+4} + \frac{s}{s^2+36} \right]$$

(viii) Given $f(t) = \sin^3 t = \frac{1}{4}[3\sin t - \sin 3t]$

$$L[f(t)] = \frac{1}{4} L[3\sin t - \sin 3t]$$

$$= \frac{1}{4} [3L(\sin t) - L(\sin 3t)]$$

$$= \frac{1}{4} \left[3 \frac{1}{s^2+1} - \frac{3}{s^2+9} \right]$$

$$L[\sin^3 t] = \frac{3}{4} \left[\frac{1}{s^2+1} - \frac{1}{s^2+9} \right]$$

(ix) Given $f(t) = \sin^2 t = \frac{1-\cos 2t}{2}$

$$L[f(t)] = L \left[\frac{1-\cos 2t}{2} \right]$$

$$= \frac{1}{2} [L(1) - L(\cos 2t)]$$

$$= \frac{1}{2} \left[\frac{1}{s} - \frac{s}{s^2+4} \right]$$

$$L[\cos^2 2t] = \frac{1}{2} \left[\frac{1}{s} - \frac{s}{s^2+4} \right]$$

(x) Given $f(t) = \cos^2 2t = \frac{1+\cos 4t}{2}$

$$L[f(t)] = L \left[\frac{1+\cos 4t}{2} \right]$$

$$= \frac{1}{2} [L(1) + L(\cos 4t)]$$

$$= \frac{1}{2} \left[\frac{1}{s} + \frac{s}{s^2+16} \right]$$

$$L[\cos^2 2t] = \frac{1}{2} \left[\frac{1}{s} + \frac{s}{s^2+16} \right]$$

(xi) Given $f(t) = \cos 5t \cos 4t$

$$L[f(t)] = L[\cos 5t \cos 4t]$$

$$= \frac{1}{2} [L(\cos 9t) + L(\cos t)]$$

$$= \frac{1}{2} \left[\frac{s}{s^2+81} + \frac{s}{s^2+1} \right]$$

Problems using First Shifting theorem

$$L[e^{-at}f(t)] = L[f(t)]_{s \rightarrow s+a}$$

$$L[e^{at}f(t)] = L[f(t)]_{s \rightarrow s-a}$$

Example: 5.7 Find the Laplace transform for the following:

i. te^{-3t}	vii. $t^2 2^t$
ii. $t^3 e^{2t}$	viii. $t^3 2^{-t}$
iii. $e^{4t} \sin 2t$	ix. $e^{-2t} \sin 3t \cos 2t$
iv. $e^{-5t} \cos 3t$	x. $e^{-3t} \cos 4t \cos 2t$
v. $\sinh 2t \cos 3t$	xi. $e^{4t} \cos 3t \sin 2t$

vi. $\cosh 3t \sin 2t$ (i) te^{-3t}

$$\begin{aligned}
 L[te^{-3t}] &= L[t]_{s \rightarrow s+3} \\
 &= \left(\frac{1}{s^2}\right)_{s \rightarrow s+3} \quad \because L(t) = \frac{1}{s^2} \\
 \therefore L[te^{-3t}] &= \frac{1}{(s+3)^2}
 \end{aligned}$$

(ii) $t^3 e^{2t}$

$$\begin{aligned}
 L[t^3 e^{2t}] &= L[t^3]_{s \rightarrow s-2} \\
 &= \left(\frac{3!}{s^4}\right)_{s \rightarrow s-2} \quad \because L(t) = \frac{3!}{s^{3+1}} \\
 \therefore L[t^3 e^{2t}] &= \frac{6}{(s-2)^4}
 \end{aligned}$$

(iii) $e^{4t} \sin 2t$

$$\begin{aligned}
 L[e^{4t} \sin 2t] &= L[\sin 2t]_{s \rightarrow s-4} \\
 &= \left(\frac{2}{s^2+2^2}\right)_{s \rightarrow s-4} \\
 &= \frac{2}{(s-4)^2+4} \\
 &= \frac{2}{s^2-8s+16+4} \\
 \therefore L[e^{4t} \sin 2t] &= \frac{2}{s^2-8s+20}
 \end{aligned}$$

(iv) $L[e^{-5t} \cos 3t]$

$$\begin{aligned}
 L[e^{-5t} \cos 3t] &= L[\cos 3t]_{s \rightarrow s+5} \\
 &= \left(\frac{s}{s^2+3^2}\right)_{s \rightarrow s+5} \\
 &= \frac{s+5}{(s+5)^2+9} \\
 &= \frac{s+5}{s^2+10s+25+9} \\
 \therefore L[e^{-5t} \cos 3t] &= \frac{s+5}{s^2+10s+34}
 \end{aligned}$$

(v) $L[\sinh 2t \cos 3t]$

$$\begin{aligned}
 L[\sinh 2t \cos 3t] &= L\left[\left(\frac{e^{2t}-e^{-2t}}{2}\right) \cos 3t\right] \\
 &= \frac{1}{2}[L(e^{2t} \cos 3t) - L(e^{-2t} \cos 3t)] \\
 &= \frac{1}{2}[L(\cos 3t)_{s \rightarrow s-2} - L(\cos 3t)_{s \rightarrow s+2}] \\
 &= \frac{1}{2}\left[\left(\frac{s}{s^2+3^2}\right)_{s \rightarrow s-2} - \left(\frac{s}{s^2+3^2}\right)_{s \rightarrow s+2}\right] \\
 \therefore L[\sinh 2t \cos 3t] &= \frac{1}{2}\left[\frac{s-2}{(s-2)^2+9} - \frac{s+2}{(s+2)^2+9}\right]
 \end{aligned}$$

(vi) $L[\cosh 3t \sin 2t]$

$$L[\cosh 3t \sin 2t] = L\left[\left(\frac{e^{3t}+e^{-3t}}{2}\right) \sin 2t\right]$$

$$\begin{aligned}&= \frac{1}{2}[L(e^{3t}\sin 2t) + L(e^{-3t}\sin 2t)] \\&= \frac{1}{2}[L(\sin 2t)_{s \rightarrow s-3} + L(\sin 2t)_{s \rightarrow s+3}] \\&= \frac{1}{2}\left[\left(\frac{2}{s^2+2^2}\right)_{s \rightarrow s-3} + \left(\frac{2}{s^2+2^2}\right)_{s \rightarrow s+3}\right]\end{aligned}$$

$$\therefore L[\cosh 3t \sin 2t] = \frac{1}{2}\left[\frac{2}{(s-3)^2+4} + \frac{2}{(s+3)^2+4}\right]$$

(vii) $t^2 2^t$

$$\begin{aligned}L[t^2 2^t] &= L[t^2 e^{t \log 2}] \\&= L[t^2 e^{t \log 2}] = L[t^2]_{s \rightarrow s - \log 2} \\&= \left(\frac{2!}{s^3}\right)_{s \rightarrow s - \log 2} \\&= \frac{2}{(s - \log 2)^3} \\ \therefore L[t^2 2^t] &= \frac{2}{(s - \log 2)^3}\end{aligned}$$

(viii) $t^3 2^{-t}$

$$\begin{aligned}L[t^3 2^{-t}] &= L[t^3 e^{\log 2^{-t}}] \\&= L[t^3 e^{-t \log 2}] = L[t^3]_{s \rightarrow s + \log 2} \\&= \left(\frac{3!}{s^4}\right)_{s \rightarrow s + \log 2} \\&= \frac{6}{(s + \log 2)^4} \\ \therefore L[t^3 2^{-t}] &= \frac{6}{(s + \log 2)^4}\end{aligned}$$

(ix) $L[e^{-2t} \sin 3t \cos 2t]$

$$\begin{aligned}L[e^{-2t} \sin 3t \cos 2t] &= L[\sin 3t \cos 2t]_{s \rightarrow s+2} \\&= \frac{1}{2}L[\sin(3t + 2t) + \sin(3t - 2t)]_{s \rightarrow s+2} \\&= \frac{1}{2}L[\sin 5t + \sin t]_{s \rightarrow s+2} \\&= \frac{1}{2}[L(\sin 5t) + L(\sin t)]_{s \rightarrow s+2} \\&= \frac{1}{2}\left[\frac{5}{s^2+5^2} + \frac{1}{s^2+1^2}\right]_{s \rightarrow s+2} \\&= \frac{1}{2}\left[\frac{5}{(s+2)^2+25} + \frac{1}{(s+2)^2+1}\right] \\ \therefore L[e^{-2t} \sin 3t \cos 2t] &= \frac{1}{2}\left[\frac{5}{(s+2)^2+25} + \frac{1}{(s+2)^2+1}\right]\end{aligned}$$

(x) $L[e^{-3t} \cos 4t \cos 2t]$

$$\begin{aligned}L[e^{-3t} \cos 4t \cos 2t] &= L[\cos 4t \cos 2t]_{s \rightarrow s+3} \\&= \frac{1}{2}L[\cos(4t + 2t) + \cos(4t - 2t)]_{s \rightarrow s+3} \\&= \frac{1}{2}L[\cos 6t + \cos 2t]_{s \rightarrow s+3}\end{aligned}$$

$$= \frac{1}{2} [L(\cos 6t) + L(\cos 2t)]_{s \rightarrow s+3}$$

$$= \frac{1}{2} \left[\frac{s}{s^2+6^2} + \frac{s}{s^2+2^2} \right]_{s \rightarrow s+3}$$

$$= \frac{1}{2} \left[\frac{s+3}{(s+3)^2+36} + \frac{s+3}{(s+3)^2+4} \right]$$

$$\therefore L[e^{-3t} \cos 4t \cos 2t] = \frac{1}{2} \left[\frac{s+3}{(s+3)^2+36} + \frac{s+3}{(s+3)^2+4} \right]$$

(xi) $L[e^{4t} \cos 3t \sin 2t]$

$$L[e^{4t} \cos 3t \sin 2t] = L[\cos 3t \sin 2t]_{s \rightarrow s-4}$$

$$= \frac{1}{2} L[\sin(3t + 2t) - \sin(3t - 2t)]_{s \rightarrow s-4}$$

$$= \frac{1}{2} L[\sin 5t - \sin t]_{s \rightarrow s-4}$$

$$= \frac{1}{2} [L(\sin 5t) - L(\sin t)]_{s \rightarrow s-4}$$

$$= \frac{1}{2} \left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1^2} \right]_{s \rightarrow s-4}$$

$$= \frac{1}{2} \left[\frac{5}{(s-4)^2+25} + \frac{1}{(s-4)^2+1} \right]$$

$$\therefore L[e^{4t} \cos 3t \sin 2t] = \frac{1}{2} \left[\frac{5}{(s-4)^2+25} + \frac{1}{(s-4)^2+1} \right]$$

Exercise: 5.1

Find the Laplace transform for the following

1. $\cos^2 3t$ **Ans:** $\frac{1}{4} \left[\frac{3s}{s^2+9} + \frac{s}{s^2+81} \right]$

2. $\sin 3t \cos 4t$ **Ans:** $\frac{1}{4} \left[\frac{7}{s^2+49} - \frac{1}{s^2+1} \right]$

3. $t e^{2t}$ **Ans:** $\frac{1}{(s-2)^2}$

4. $t^4 e^{-3t}$ **Ans:** $\frac{4!}{(s-3)^5}$

5. $e^{4t} \sin 2t$ **Ans:** $\frac{2}{(s-4)^2+4}$

6. $e^{-5t} \cos 3t$ **Ans:** $\frac{s+5}{(s+5)^2+9}$

7. $t^3 3^t$ **Ans:** $\frac{3!}{(s-\log 3)^4}$

8. $t^5 4^{-t}$ **Ans:** $\frac{5!}{(s+\log 4)^6}$

9. $e^{-2t} \sin 3t \cos 2t$ **Ans:** $\frac{5}{(s+2)^2+25} + \frac{1}{(s+2)^2+1}$

10. $e^{-3t} \cos 4t \cos 2t$ **Ans:** $\frac{s+3}{(s+3)^2+36} + \frac{s+3}{(s+3)^2+4}$

11. $\sinh t \sin 4t$ **Ans:** $\frac{4}{(s-1)^2+16} - \frac{4}{(s+1)^2+16}$

12. $\cosh 2t \cos 2t$ **Ans:** $\frac{1}{2} \left[\frac{s-2}{(s-2)^2+4} - \frac{s+2}{(s+2)^2+4} \right]$

5.4 LAPLACE TRANSFORM OF DERIVATIVES AND INTEGRALS

Problems using the formula

$$L[tf(t)] = \frac{-d}{ds} L[f(t)]$$

Example: 5.8 Find the Laplace transform for $t\sin 4t$

Solution:

$$\begin{aligned} L[t\sin 4t] &= \frac{-d}{ds} L[\sin 4t] \\ &= \frac{-d}{ds} \left[\frac{4}{s^2+4} \right] \\ &= \frac{-[(s^2+16)0-4(2s)]}{(s^2+16)^2} \end{aligned}$$

$$\therefore L[t\sin 4t] = \frac{8s}{(s^2+16)^2}$$

Example: 5.9 Find $L[t\sin^2 t]$

Solution:

$$\begin{aligned} L[t\sin^2 t] &= \frac{-d}{ds} L[\sin^2 t] = \frac{-d}{ds} L\left[\frac{(1-\cos 2t)}{2}\right] \\ &= -\frac{1}{2} \frac{d}{ds} [L(1) - L(\cos 2t)] \\ &= -\frac{1}{2} \frac{d}{ds} \left[\frac{1}{s} - \frac{s}{s^2+4} \right] \\ &= -\frac{1}{2} \frac{d}{ds} \left[\frac{s^2+4-s^2}{s(s^2+4)} \right] \\ &= -\frac{1}{2} \frac{d}{ds} \left[\frac{4}{s(s^2+4)} \right] \\ &= -\frac{4}{2} \frac{d}{ds} \left[\frac{1}{s(s^2+4)} \right] \\ &= -2 \left[\frac{0-(3s^2+4)}{(s^3+4s)^2} \right] \\ \therefore L[t\sin^2 t] &= \frac{2(3s^2+4)}{(s^3+4s)^2} \end{aligned}$$

Example: 5.10 Find $L[t\cos^2 2t]$

Solution:

$$\begin{aligned} L[t\cos^2 2t] &= \frac{-d}{ds} L[\cos^2 2t] = \frac{-d}{ds} L\left[\frac{(1+\cos 4t)}{2}\right] \\ &= -\frac{1}{2} \frac{d}{ds} [L(1) + L(\cos 4t)] \\ &= -\frac{1}{2} \frac{d}{ds} \left[\frac{1}{s} + \frac{s}{s^2+16} \right] \\ &= -\frac{1}{2} \left[-\frac{1}{s^2} + \frac{(s^2+16)1-s \cdot 2s}{(s^2+16)^2} \right] \\ &= -\frac{1}{2} \left[-\frac{1}{s^2} + \frac{s^2+16-2s^2}{(s^2+16)^2} \right] \end{aligned}$$

$$\therefore L[t\cos^2 2t] = \frac{1}{2} \left[\frac{1}{s^2} - \frac{16-s^2}{(s^2+16)^2} \right]$$

Example: 5.11 Find the Laplace transform for $t \sinh 2t$ **Solution:**

$$\begin{aligned}
 L[\sinh 2t] &= \frac{-d}{ds} L[\sinh 2t] \\
 &= \frac{-d}{ds} \left[\frac{2}{s^2 - 4} \right] \\
 &= \frac{-[(s^2 - 4)0 - 2(2s)]}{(s^2 - 4)^2}
 \end{aligned}$$

$$\therefore L[t \sinh 2t] = \frac{4s}{(s^2 - 4)^2}$$

Example: 5.12 Find the Laplace transform for $f(t) = \sin at - at \cos at$ **Solution:**

$$\begin{aligned}
 L[\sin at - at \cos at] &= L(\sin at) - a L(t \cos at) \\
 &= \frac{a}{s^2 + a^2} - a \left(\frac{-d}{ds} L[\cos at] \right) \\
 &= \frac{a}{s^2 + a^2} + a \frac{d}{ds} \left[\frac{s}{s^2 + a^2} \right] \\
 &= \frac{a}{s^2 + a^2} + a \left[\frac{(s^2 + a^2)1 - s(2s)}{(s^2 + a^2)^2} \right] \\
 &= \frac{a}{s^2 + a^2} + a \left[\frac{s^2 + a^2 - s^2}{(s^2 + a^2)^2} \right] \\
 &= \frac{a}{s^2 + a^2} + a \left[\frac{a^2 - s^2}{(s^2 + a^2)^2} \right] \\
 &= \frac{a(s^2 + a^2) + a(a^2 - s^2)}{(s^2 + a^2)^2} \\
 &= \frac{as^2 + a^3 + a^3 - as^2}{(s^2 + a^2)^2}
 \end{aligned}$$

$$\therefore L[\sin at - at \cos at] = \frac{2a^3}{(s^2 + a^2)^2}$$

Example: 5.13 Find the Laplace transform for the following**(i) $te^{-3t} \sin 2t$ (ii) $te^{-t} \cos at$ (iii) $t \sinh t \cos 2t$** **Solution:**

$$\begin{aligned}
 \text{(i) } L[te^{-3t} \sin 2t] &= L[t \sin 2t]_{s \rightarrow s+3} = \frac{-d}{ds} L[\sin 2t]_{s \rightarrow s+3} \\
 &= \frac{-d}{ds} \left(\frac{2}{s^2 + 2^2} \right)_{s \rightarrow s+3} \\
 &= \left[\frac{(s^2 + 4)0 - 2(2s)}{(s^2 + 4)^2} \right]_{s \rightarrow s+3} \\
 &= \left[\frac{4s}{(s^2 + 4)^2} \right]_{s \rightarrow s+3} \\
 \therefore L[te^{-3t} \sin 2t] &= \frac{4(s+3)}{((s+3)^2 + 4)^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) } L[te^{-t} \cos at] &= L[t \cos at]_{s \rightarrow s+1} = \frac{-d}{ds} L[\cos at]_{s \rightarrow s+1} \\
 &= \frac{-d}{ds} \left(\frac{s}{s^2 + a^2} \right)_{s \rightarrow s+1}
 \end{aligned}$$

$$\begin{aligned}
&= - \left[\frac{(s^2+a^2)1-s(2s)}{(s^2+a^2)^2} \right]_{s \rightarrow s+1} \\
&= - \left[\frac{a^2-s^2}{(s^2+a^2)^2} \right]_{s \rightarrow s+1} \\
&= \left[\frac{s^2-a^2}{(s^2+a^2)^2} \right]_{s \rightarrow s+1}
\end{aligned}$$

$$\therefore L[te^{-t}\cos at] = \frac{(s+1)^2-a^2}{((s+1)^2+a^2)^2}$$

(iii) $L[t\sinh t \cos 2t]$

$$\begin{aligned}
L[t\sinh t \cos 2t] &= L \left[t \left(\frac{e^t - e^{-t}}{2} \right) \cos 2t \right] \\
&= \frac{1}{2} [L(te^t \cos 2t) - L(te^{-t} \cos 2t)] \\
&= \frac{1}{2} \left[\frac{-d}{ds} L[\cos 2t]_{s \rightarrow s-1} + \frac{d}{ds} L[\cos 2t]_{s \rightarrow s+1} \right] \\
&= \frac{1}{2} \left[\frac{-d}{ds} \left(\frac{s}{s^2+4} \right)_{s \rightarrow s-1} + \frac{d}{ds} \left(\frac{s}{s^2+4} \right)_{s \rightarrow s+1} \right] \\
&= \frac{1}{2} \left[- \left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2} \right]_{s \rightarrow s-1} + \left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2} \right]_{s \rightarrow s+1} \right] \\
&= \frac{1}{2} \left[- \left[\frac{4-s^2}{(s^2+4)^2} \right]_{s \rightarrow s-1} + \left[\frac{4-s^2}{(s^2+4)^2} \right]_{s \rightarrow s+1} \right]
\end{aligned}$$

$$\therefore L[t\sinh t \cos 2t] = \frac{1}{2} \left[\frac{(s-1)^2-4}{((s-1)^2+4)^2} + \frac{4-(s+1)^2}{((s+1)^2+4)^2} \right]$$

Problems using the formula

$$L[t^2 f(t)] = \frac{d^2}{ds^2} L[f(t)]$$

Example: 5.14 Find the Laplace transform for (i) $t^2 \sin t$ (ii) $t^2 \cos 2t$

Solution:

$$\begin{aligned}
\text{(i) } L[t^2 \sin t] &= \frac{d^2}{ds^2} L[\sin t] \\
&= \frac{d^2}{ds^2} \left[\frac{1}{s^2+1} \right] \\
&= \frac{d}{ds} \left(\frac{[(s^2+1)0-1(2s)]}{(s^2+1)^2} \right) \\
&= \frac{d}{ds} \left(\frac{-2s}{(s^2+1)^2} \right) \\
&= -2 \frac{d}{ds} \left(\frac{s}{(s^2+1)^2} \right) \\
&= \frac{-2[(s^2+1)^2(1)-s(2)(s^2+1)(2s)]}{(s^2+1)^4} \\
&= \frac{-2(s^2+1)[(s^2+1)-4s^2]}{(s^2+1)^4} \\
&= \frac{-2[1-3s^2]}{(s^2+1)^3}
\end{aligned}$$

$$\therefore L[t^2 \sin t] = \frac{6s^2-2}{(s^2+1)^3}$$

$$\begin{aligned}
\text{(ii) } L[t^2 \cos 2t] &= \frac{d^2}{ds^2} L[\cos 2t] \\
&= \frac{d^2}{ds^2} \left[\frac{s}{s^2+4} \right] \\
&= \frac{d}{ds} \left(\frac{[(s^2+4)1-s(2s)]}{(s^2+4)^2} \right) \\
&= \frac{d}{ds} \left(\frac{4-s^2}{(s^2+4)^2} \right) \\
&= \frac{[(s^2+4)^2(-2s) - (4-s^2)2(s^2+4)(2s)]}{(s^2+4)^4} \\
&= \frac{2s(s^2+4)[(s^2+4)(-1) - (4-s^2)2]}{(s^2+4)^4} \\
&= \frac{2s[s^2-12]}{(s^2+4)^3} \\
\therefore L[t^2 \cos 2t] &= \frac{2s[s^2-12]}{(s^2+4)^3}
\end{aligned}$$

Example: 5.15 Find the Laplace transform for (i) $t^2 e^{-2t} \cos t$ (ii) $t^2 e^{4t} \sin 3t$

Solution:

$$\begin{aligned}
\text{(i) } L[t^2 e^{-2t} \cos t] &= L[t^2 \cos t]_{s \rightarrow s+2} = \frac{d^2}{ds^2} L[\cos t]_{s \rightarrow s+2} \\
&= \frac{d^2}{ds^2} \left(\frac{s}{s^2+1} \right)_{s \rightarrow s+2} \\
&= \frac{d}{ds} \left[\frac{(s^2+1)1-s(2s)}{(s^2+1)^2} \right]_{s \rightarrow s+2} \\
&= \frac{d}{ds} \left[\frac{1-s^2}{(s^2+1)^2} \right]_{s \rightarrow s+2} \\
&= \left[\frac{[(s^2+1)^2(-2s) - (1-s^2)2(s^2+1)(2s)]}{(s^2+1)^4} \right]_{s \rightarrow s+2} \\
&= (s^2+1) \left[\frac{[(s^2+1)(-2s) - 4s(1-s^2)]}{(s^2+1)^4} \right]_{s \rightarrow s+2} \\
&= \left[\frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3} \right]_{s \rightarrow s+2} \\
&= \left[\frac{2s^3-6s}{(s^2+1)^3} \right]_{s \rightarrow s+2} \\
\therefore L[t^2 e^{-2t} \cos t] &= \frac{2(s+2)^3-6(s+2)}{((s+2)^2+1)^3}
\end{aligned}$$

$$\begin{aligned}
\text{(ii) } L[t^2 e^{4t} \sin 3t] &= L[t^2 \sin 3t]_{s \rightarrow s-4} = \frac{d^2}{ds^2} L[\sin 3t]_{s \rightarrow s-4} \\
&= \frac{d^2}{ds^2} \left(\frac{3}{s^2+9} \right)_{s \rightarrow s-4} \\
&= \frac{d}{ds} \left[\frac{(s^2+9)0-3(2s)}{(s^2+9)^2} \right]_{s \rightarrow s-4} \\
&= \frac{d}{ds} \left[\frac{-6s}{(s^2+9)^2} \right]_{s \rightarrow s-4} = -6 \frac{d}{ds} \left[\frac{s}{(s^2+9)^2} \right]_{s \rightarrow s-4} \\
&= -6 \left[\frac{[(s^2+9)^2(1)-(s)2(s^2+9)(2s)]}{(s^2+9)^4} \right]_{s \rightarrow s-4}
\end{aligned}$$

$$= -6(s^2 + 9) \left[\frac{[(s^2+9)-4s^2]}{(s^2+9)^4} \right]_{s \rightarrow s-4}$$

$$= -6 \left[\frac{9-3s^2}{(s^2+9)^3} \right]_{s \rightarrow s-4}$$

$$= \left[\frac{18s^2-54}{(s^2+9)^3} \right]_{s \rightarrow s-4}$$

$$\therefore L[t^2 e^{4t} \sin 3t] = \frac{18(s-4)^2-54}{((s-4)^2+9)^3}$$

Exercise: 5.2

Find the Laplace transform for the following

1. $t \sin at$ **Ans:** $\frac{2as}{(s^2+a^2)^2}$

2. $t \cos at$ **Ans:** $\frac{s^2-a^2}{(s^2+a^2)^2}$

3. $t e^{-4t} \sin 3t$ **Ans:** $\frac{6(s+4)}{(s+4)^2+9}$

4. $t \cos 2t \sin 6t$ **Ans:** $\frac{8s}{(s^2+64)^2} - \frac{4s}{(s^2+16)^2}$

5. $t e^{-2t} \cos 2t$ **Ans:** $\frac{(s-2)^2-4}{((s+4)^2+4)^2}$

Problems using the formula

$$L \left[\frac{f(t)}{t} \right] = \int_s^\infty L[f(t)] ds$$

This formula is valid if $\lim_{t \rightarrow 0} \frac{f(t)}{t}$ is finite.

The following formula is very useful in this section

$$\int \frac{ds}{s} = \log s$$

$$\int \frac{ds}{s+a} = \log(s+a)$$

$$\int \frac{s ds}{s^2+a^2} = \frac{1}{2} \log(s^2+a^2)$$

$$\int \frac{a ds}{s^2+a^2} = \tan^{-1} \frac{s}{a}$$

Example: 5.16 Find $L \left[\frac{\cos at}{t} \right]$

Solution:

$$\lim_{t \rightarrow 0} \frac{\cos at}{t} = \frac{\cos a(0)}{0} = \frac{1}{0} = \infty$$

\therefore Laplace transform does not exist.

Example: 5.17 Find $L \left[\frac{\sin at}{t} \right]$

Solution:

$$\lim_{t \rightarrow 0} \frac{\sin at}{t} = \frac{\sin a(0)}{0} = \frac{0}{0}$$

$$= \lim_{t \rightarrow 0} a \cos at$$

(by applying L-Hospital rule)

$$\lim_{t \rightarrow 0} a \cos at = a \cos 0 = a, \text{ finite quantity.}$$

Hence Laplace transform exists

$$\begin{aligned} L\left[\frac{\sin at}{t}\right] &= \int_s^\infty L[(\sin at)] ds \\ &= \int_s^\infty \frac{a}{s^2 + a^2} ds \\ &= \left[\tan^{-1} \frac{s}{a}\right]_s^\infty \\ &= \left[\tan^{-1} \infty - \tan^{-1} \frac{s}{a}\right] \\ &= \left[\frac{\pi}{2} - \tan^{-1} \frac{s}{a}\right] \end{aligned}$$

$$\therefore L\left[\frac{\sin at}{t}\right] = \cot^{-1} \frac{s}{a}$$

Example: 5.18 Find $L\left[\frac{\sin^3 t}{t}\right]$

Solution:

$$\begin{aligned} \frac{\sin^3 t}{t} &= \frac{3 \sin t - \sin 3t}{4t} \\ \lim_{t \rightarrow 0} \frac{\sin^3 t}{t} &= \lim_{t \rightarrow 0} \frac{3 \sin t - \sin 3t}{4t} \\ &= \frac{0-0}{0} = \frac{0}{0} \quad (\text{by applying L-Hospital rule}) \\ &= \lim_{t \rightarrow 0} \frac{3 \sin t - \sin 3t}{4t} = 0 \end{aligned}$$

Hence Laplace transform exists

$$\begin{aligned} L\left[\frac{\sin^3 t}{t}\right] &= L\left[\frac{3 \sin t - \sin 3t}{4t}\right] \\ &= \frac{1}{4} \int_s^\infty L[(3 \sin t - \sin 3t)] ds \\ &= \frac{1}{4} \int_s^\infty \left(3 \frac{1}{s^2 + 1} - \frac{3}{s^2 + 9}\right) ds \\ &= \frac{1}{4} \left[3 \tan^{-1} s - \tan^{-1} \frac{s}{3}\right]_s^\infty \\ &= \frac{1}{4} \left[3(\tan^{-1} \infty - \tan^{-1} s) - \left(\tan^{-1} \infty - \tan^{-1} \frac{s}{3}\right)\right] \\ &= \frac{1}{4} \left[\left(\frac{\pi}{2} - \tan^{-1} s\right) - \left(\frac{\pi}{2} - \tan^{-1} \frac{s}{3}\right)\right] \\ &= \frac{1}{4} \left[\cot^{-1} s - \cot^{-1} \frac{s}{3}\right] \end{aligned}$$

Example: 5.19 Find $L\left[e^{-2t} \frac{\sin 2t \cos 3t}{t}\right]$

Solution:

$$\begin{aligned} L\left[e^{-2t} \frac{\sin 2t \cos 3t}{t}\right] &= L\left[\frac{\sin 2t \cos 3t}{t}\right]_{s \rightarrow s+2} \\ &= \frac{1}{2} \left[\int_s^\infty L(\sin(3t + 2t) - \sin(3t - 2t)) ds\right]_{s \rightarrow s+2} \\ &= \frac{1}{2} \left[\int_s^\infty L((\sin 5t) - L(\sin t)) ds\right]_{s \rightarrow s+2} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left[\int_s^\infty \left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1^2} \right] ds \right]_{s \rightarrow s+2} \\
&= \frac{1}{2} \left[\left[\tan^{-1} \frac{s}{5} - \tan^{-1} s \right]_s^\infty \right]_{s \rightarrow s+2} \\
&= \frac{1}{2} \left[\left[\left(\tan^{-1} \infty - \tan^{-1} \frac{s}{5} \right) - \left(\tan^{-1} \infty - \tan^{-1} s \right) \right] \right]_{s \rightarrow s+2} \\
&= \frac{1}{2} \left[\left(\frac{\pi}{2} - \tan^{-1} \frac{s}{5} \right) - \left(\frac{\pi}{2} - \tan^{-1} s \right) \right]_{s \rightarrow s+2} \\
&= \frac{1}{2} \left[\cot^{-1} \frac{s}{5} - \cot^{-1} s \right]_{s \rightarrow s+2} \\
&= \frac{1}{2} \left[\cot^{-1} \frac{(s+2)}{5} - \cot^{-1}(s+2) \right]
\end{aligned}$$

Example: 5.20 Find the Laplace transform for $\frac{e^{-at}-e^{-bt}}{t}$

Solution:

$$\begin{aligned}
\lim_{t \rightarrow 0} \frac{e^{-at}-e^{-bt}}{t} &= \lim_{t \rightarrow 0} \frac{e^0-e^0}{0} = \frac{1-1}{0} = \frac{0}{0} && \text{(use L- Hospital rule)} \\
&= \lim_{t \rightarrow 0} \frac{-ae^{-at}+be^{-bt}}{1} \\
&= -a + b = b - a = \text{a finite quantity}
\end{aligned}$$

Hence Laplace transform exists.

$$\begin{aligned}
L \left[\frac{e^{-at}-e^{-bt}}{t} \right] &= \int_s^\infty L[e^{-at} - e^{-bt}] ds \\
&= \int_s^\infty [L(e^{-at}) - L(e^{-bt})] ds \\
&= \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b} \right) ds \\
&= [\log(s+a) - \log(s+b)]_s^\infty \\
&= \left[\log \frac{s+a}{s+b} \right]_s^\infty \\
&= \left[\log \frac{s(1+\frac{a}{s})}{s(1+\frac{b}{s})} \right]_s^\infty \\
&= \log 1 - \log \frac{s+a}{s+b} = 0 - \log \frac{s+a}{s+b} && \because \log 1 = 0 \\
&= \log \frac{s+b}{s+a}
\end{aligned}$$

Example: 5.21 Find the Laplace transform of $\frac{1-\cos t}{t}$

Solution:

$$\lim_{t \rightarrow 0} \frac{1-\cos t}{t} = \frac{0}{0} \quad \lim_{t \rightarrow 0} \frac{\sin t}{1} = \frac{0}{1} = 0 \quad \text{(use L- Hospital rule)}$$

$$L \left[\frac{1-\cos t}{t} \right] \text{ exists.}$$

$$\begin{aligned}
L \left[\frac{1-\cos t}{t} \right] &= \int_s^\infty L[(1 - \cos t)] ds \\
&= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1} \right) ds
\end{aligned}$$

$$\begin{aligned}&= \left[\log s - \frac{1}{2} \log(s^2 + 1) \right]_s^\infty \\&= \left[\log s - \log \sqrt{s^2 + 1} \right]_s^\infty \\&= \left[\log \frac{s}{\sqrt{s^2 + 1}} \right]_s^\infty \\&= 0 - \log \frac{s}{\sqrt{s^2 + 1}} \\&= \log \frac{\sqrt{s^2 + 1}}{s}\end{aligned}$$

Example: 5.22 Find the Laplace transform for $\frac{\cos at - \cos bt}{t}$

Solution:

$$\begin{aligned}\lim_{t \rightarrow 0} \frac{\cos at - \cos bt}{t} &= \frac{1-1}{0} = \frac{0}{0} \quad (\text{use L- Hospital rule}) \\&= \lim_{t \rightarrow 0} \frac{-a \sin at + b \sin bt}{1} = 0 = \text{a finite quantity}\end{aligned}$$

Hence Laplace transform exists.

$$\begin{aligned}L \left[\frac{\cos at - \cos bt}{t} \right] &= \int_s^\infty L[\cos at - \cos bt] ds \\&= \int_s^\infty [L(\cos at) - L(\cos bt)] ds \\&= \int_s^\infty \left(\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \right) ds \\&= \left[\frac{1}{2} \log(s^2 + a^2) - \frac{1}{2} \log(s^2 + b^2) \right]_s^\infty \\&= \frac{1}{2} \left[\log \frac{s^2 + a^2}{s^2 + b^2} \right]_s^\infty \\&= \frac{1}{2} \left[\log \frac{s^2 \left(1 + \frac{a^2}{s^2} \right)}{s^2 \left(1 + \frac{b^2}{s^2} \right)} \right]_s^\infty \\&= \frac{1}{2} \left[\log \frac{\left(1 + \frac{a^2}{s^2} \right)}{\left(1 + \frac{b^2}{s^2} \right)} \right]_s^\infty \\&= \frac{1}{2} \left[\log 1 - \log \frac{s^2 + a^2}{s^2 + b^2} \right] = -\frac{1}{2} \left[\log \frac{s^2 + a^2}{s^2 + b^2} \right] \quad [\because \log 1 = 0] \\&= \frac{1}{2} \left[\log \frac{s^2 + b^2}{s^2 + a^2} \right]\end{aligned}$$

Example: 5.23 Find the Laplace transform of $\frac{\sin^2 t}{t}$

Solution:

$$\begin{aligned}\frac{\sin^2 t}{t} &= \frac{1 - \cos 2t}{2t} \\ \lim_{t \rightarrow 0} \frac{1 - \cos 2t}{2t} &= \frac{0}{0} \\ \lim_{t \rightarrow 0} \frac{2 \sin 2t}{2} &= \frac{0}{1} = 0 \quad (\text{use L- Hospital rule})\end{aligned}$$

Laplace transform exists.

$$\begin{aligned}
L\left[\frac{\sin^2 t}{t}\right] &= L\left[\frac{1-\cos 2t}{2t}\right] = \frac{1}{2} \int_s^\infty L[(1-\cos 2t)] ds \\
&= \frac{1}{2} \int_s^\infty [L(1) - L(\cos 2t)] ds \\
&= \frac{1}{2} \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds \\
&= \frac{1}{2} \left[\log s - \frac{1}{2} \log(s^2+4) \right]_s^\infty \\
&= \frac{1}{2} [\log s - \log \sqrt{s^2+4}]_s^\infty \\
&= \frac{1}{2} \left[\log \frac{s}{\sqrt{s^2+4}} \right]_s^\infty \\
&= \frac{1}{2} \left[0 - \log \frac{s}{\sqrt{s^2+4}} \right] \\
&= \frac{1}{2} \log \frac{\sqrt{s^2+4}}{s}
\end{aligned}$$

Example: 5.24 Find the Laplace transform for $\frac{\sin 2t \sin 5t}{t}$

Solution:

$$\begin{aligned}
L\left[\frac{\sin 2t \sin 5t}{t}\right] &= \int_s^\infty L[\sin 2t \sin 5t] ds \\
&= \int_s^\infty \frac{1}{2} [L(\cos(-3t)) - L(\cos 7t)] ds \\
&= \frac{1}{2} \int_s^\infty [L(\cos(3t)) - L(\cos 7t)] ds \quad [\because \cos(-\theta) = \cos \theta] \\
&= \frac{1}{2} \int_s^\infty \left(\frac{s}{s^2+9} - \frac{s}{s^2+49}\right) ds \\
&= \frac{1}{2} \left[\frac{1}{2} \log(s^2+9) - \frac{1}{2} \log(s^2+49) \right]_s^\infty \\
&= \frac{1}{4} \left[\log \frac{s^2+9}{s^2+49} \right]_s^\infty \\
&= \frac{1}{4} \left[\log \frac{s^2 \left(1+\frac{9}{s^2}\right)}{s^2 \left(1+\frac{49}{s^2}\right)} \right]_s^\infty \\
&= \frac{1}{4} \left[\log \frac{\left(1+\frac{9}{s^2}\right)}{\left(1+\frac{49}{s^2}\right)} \right]_s^\infty \\
&= \frac{1}{4} \left[\log 1 - \log \frac{s^2+9}{s^2+49} \right] = -\frac{1}{4} \left[\log \frac{s^2+9}{s^2+49} \right] \quad [\because \log 1 = 0] \\
&= \frac{1}{4} \left[\log \frac{s^2+49}{s^2+9} \right]
\end{aligned}$$

Problems using $L\left[\int_0^t f(t) dt\right] = \frac{1}{s} L[f(t)]$

Example: 5.25 Find the Laplace transform for (i) $\int_0^t e^{-2t} dt$ (ii) $\int_0^t \cos 2t dt$

(iii) $\int_0^t t \sin 3t dt$ (iv) $\int_0^t t \cos t dt$

Solution:

$$(i) \quad L\left[\int_0^t e^{-2t} dt\right] = \frac{1}{s} L[e^{-2t}] = \frac{1}{s} \left(\frac{1}{s+2}\right)$$

$$\therefore L \left[\int_0^t e^{-2t} dt \right] = \frac{1}{s(s+2)}$$

$$(ii) L \left[\int_0^t \cos 2t dt \right] = \frac{1}{s} L[\cos 2t] = \frac{1}{s} \left(\frac{s}{s^2+4} \right)$$

$$\therefore L \left[\int_0^t \cos 2t dt \right] = \frac{1}{s^2+4}$$

$$(iii) L \left[\int_0^t t \sin 3t dt \right] = \frac{1}{s} L[t \sin 3t]$$

$$= \frac{1}{s} \left[\frac{-d}{ds} [L[\sin 3t]] \right]$$

$$= \frac{-1}{s} \left[\frac{d}{ds} \left[\frac{3}{s^2+9} \right] \right]$$

$$= \frac{-1}{s} \left[\frac{-6s}{(s^2+9)^2} \right]$$

$$\therefore L \left[\int_0^t t \sin 3t dt \right] = \frac{6}{(s^2+9)^2}$$

$$(iv) L \left[t \int_0^t \cos t dt \right] = \frac{-d}{ds} L \left[\int_0^t \cos t dt \right]$$

$$= \frac{-d}{ds} \left[\frac{1}{s} \left(\frac{s}{s^2+1} \right) \right]$$

$$= -\frac{d}{ds} \left[\frac{1}{s^2+1} \right]$$

$$= -\left[\frac{-2s}{(s^2+1)^2} \right]$$

$$\therefore L \left[\int_0^t t \sin 3t dt \right] = \frac{2s}{(s^2+1)^2}$$

Example: 5.26 Find the Laplace transform for $e^{-t} \int_0^t t \cos 4t dt$

Solution:

$$L \left[e^{-t} \int_0^t t \cos 4t dt \right] = L \left[\int_0^t t \cos 4t dt \right]_{s \rightarrow s+1} = \left[\frac{-1}{s} \frac{d}{ds} L(\cos 4t) \right]_{s \rightarrow s+1}$$

$$= -\left(\frac{1}{s} \frac{d}{ds} \frac{s}{s^2+16} \right)_{s \rightarrow s+1}$$

$$= \left[\frac{-1}{s} \frac{(s^2+16)1 - s(2s)}{(s^2+16)^2} \right]_{s \rightarrow s+1}$$

$$= \left[\frac{-1}{s} \frac{(s^2+16-2s^2)}{(s^2+16)^2} \right]_{s \rightarrow s+1}$$

$$= \left[\frac{-1}{s} \frac{(-s^2+16)}{(s^2+16)^2} \right]_{s \rightarrow s+1}$$

$$= \left[\frac{1}{s} \frac{(s^2-16)}{(s^2+16)^2} \right]_{s \rightarrow s+1}$$

$$\therefore L \left[e^{-t} \int_0^t t \cos 4t dt \right] = \frac{1}{s+1} \left[\frac{(s+1)^2-16}{((s+1)^2+16)^2} \right]$$

Example: 5.27 Find the Laplace transform of $e^{-t} \int_0^t \frac{\sin t}{t} dt$

Solution:

$$L \left[e^{-t} \int_0^t \frac{\sin t}{t} dt \right] = L \left[\int_0^t \frac{\sin t}{t} dt \right]_{s \rightarrow s+1}$$

$$\begin{aligned}
&= \left[\frac{1}{s} L \left(\frac{\sin t}{t} \right) \right]_{s \rightarrow s+1} \\
&= \left[\frac{1}{s} \int_s^\infty L(\sin t) ds \right]_{s \rightarrow s+1} \\
&= \left[\frac{1}{s} \int_s^\infty \frac{1}{s^2+1} \right]_{s \rightarrow s+1} \\
&= \left[\frac{1}{s} [\tan^{-1} s]_s^\infty \right]_{s \rightarrow s+1} \\
&= \left[\frac{1}{s} (\tan^{-1} \infty - \tan^{-1} s) \right]_{s \rightarrow s+1} \\
&= \left[\frac{1}{s} \left(\frac{\pi}{2} - \tan^{-1} s \right) \right]_{s \rightarrow s+1} \\
&= \left[\frac{1}{s} \cot^{-1} s \right]_{s \rightarrow s+1}
\end{aligned}$$

$$\therefore L \left[e^{-t} \int_0^t \frac{\sin t}{t} dt \right] = \frac{1}{s+1} \cot^{-1}(s+1)$$

Exercise: 5.3

Find the Laplace transform of

1. $\frac{\sin t}{t}$ **Ans:** $\cot^{-1} \frac{s}{2}$
2. $e^{-2t} \frac{\sin t}{t}$ **Ans:** $\cot^{-1}(s+2)$
3. $\frac{\sin at - \sin bt}{t}$ **Ans:** $\cot^{-1} \frac{s}{a} - \cot^{-1} \frac{s}{b}$
4. $\frac{e^{-at} - \cos bt}{t}$ **Ans:** $\log \frac{\sqrt{s^2+b^2}}{s+a}$
5. $\frac{1-e^{-t}}{t}$ **Ans:** $\log \frac{s+1}{s}$
6. $e^{-t} \int_0^t \frac{\sin t}{t} dt$ **Ans:** $\frac{1}{s+1} \cot^{-1}(s+1)$
7. $e^{-t} \int_0^t t \cos t dt$ **Ans:** $\frac{1}{s+1} \left[\frac{s^2+2s}{(s^2+2s+2)^2} \right]$
8. $e^{-t} \int_0^t t e^{-t} \sin t dt$ **Ans:** $\frac{1}{s} \left[\frac{2(s+1)}{s^2+2s+2} \right]$

Evaluation of integrals using Laplace transform

Note: (i) $\int_0^\infty f(t) e^{-st} dt = L[f(t)]$

(ii) $\int_0^\infty f(t) e^{-at} dt = [L[f(t)]]_{s=a}$

(iii) $\int_0^\infty f(t) dt = [L[f(t)]]_{s=0}$

Example: 5.28 If $L[f(t)] = \frac{s+2}{s^2+4}$, then find the value of $\int_0^\infty f(t) dt$

Solution:

$$\text{Given } L[f(t)] = \frac{s+2}{s^2+4}$$

We know that $\int_0^\infty f(t) dt = [L[f(t)]]_{s=0}$

$$= \left[\frac{s+2}{s^2+4} \right]_{s=0} = \frac{2}{4}$$

$$\int_0^{\infty} f(t) dt = \frac{1}{2}$$

Example: 5.29 If $L[f(t)] = \frac{5s+4}{s^2-9}$, then find the value of $\int_0^{\infty} e^{-2t} f(t) dt$

Solution:

$$\text{Given } L[f(t)] = \frac{5s+4}{s^2-9}$$

We know that $\int_0^{\infty} e^{-2t} f(t) dt = [L[f(t)]]_{s=2}$

$$= \left[\frac{5s+4}{s^2-9} \right]_{s=2} = \frac{14}{-5}$$

$$\therefore \int_0^{\infty} e^{-2t} f(t) dt = \frac{-14}{5}$$

Example: 5.30 Find the values of the following integrals using Laplace transforms:

$$(i) \int_0^{\infty} t e^{-2t} \cos 2t dt \quad (ii) \int_0^{\infty} t^2 e^{-t} \sin t dt \quad (iii) \int_0^{\infty} \left(\frac{e^{-t} - e^{-2t}}{t} \right) dt$$

$$(iv) \int_0^{\infty} \left(\frac{1 - \cos t}{t} \right) e^{-t} dt \quad (v) \int_0^{\infty} \left(\frac{e^{-at} - \cos bt}{t} \right) dt$$

Solution:

$$(i) \int_0^{\infty} t e^{-2t} \cos 2t dt = L[t \cos 2t]_{s=2} = \left[\frac{-d}{ds} L(\cos 2t) \right]_{s=2}$$

$$= \frac{-d}{ds} \left(\frac{s}{s^2+4} \right)_{s=2}$$

$$= - \left[\frac{(s^2+4)1 - s(2s)}{(s^2+4)^2} \right]_{s=2}$$

$$= - \left[\frac{(4-s^2)}{(s^2+4)^2} \right]_{s=2}$$

$$= - \frac{(4-4)}{(4+4)^2} = 0$$

$$(ii) \int_0^{\infty} t^2 e^{-t} \sin t dt = L[t^2 \sin t]_{s=1} = \frac{d^2}{ds^2} L[\sin t]_{s=1}$$

$$= \frac{d^2}{ds^2} \left(\frac{1}{s^2+1} \right)_{s=1}$$

$$= \frac{d}{ds} \left[\frac{-1(2s)}{(s^2+1)^2} \right]_{s=1}$$

$$= -2 \frac{d}{ds} \left[\frac{s}{(s^2+1)^2} \right]_{s=1}$$

$$= -2 \left[\frac{[(s^2+1)^2(1) - s \cdot 2(s^2+1)(2s)]}{(s^2+1)^4} \right]_{s=1}$$

$$= -2 \left[\frac{[(s^2+1)[(s^2+1) - 4s^2]]}{(s^2+1)^4} \right]_{s=1}$$

$$= -2 \left[\frac{(1-3s^2)}{(s^2+1)^3} \right]_{s=1}$$

$$= \left[\frac{6s^3-2}{(s^2+1)^3} \right]_{s=1} = \frac{4}{8} = \frac{1}{2}$$

$$(iii) \int_0^{\infty} \left(\frac{e^{-t} - e^{-2t}}{t} \right) dt = L \left[\frac{e^{-t} - e^{-2t}}{t} \right]_{s=0} = \int_s^{\infty} [L[e^{-t} - e^{-2t}]] ds_{s=0}$$

$$\begin{aligned}
&= \int_s^\infty [L(e^{-t}) - L(e^{-2t})] \square s \Big|_{s=0} \\
&= \int_s^\infty \left[\left(\frac{1}{s+1} - \frac{1}{s+2} \right) ds \right]_{s=0} \\
&= \{ [\log(s+1) - \log(s+2)]_s^\infty \}_{s=0} \\
&= \left\{ \left[\log \frac{s+1}{s+2} \right]_s^\infty \right\}_{s=0} \\
&= \left\{ \log \frac{s(1+\frac{1}{s})}{s(1+\frac{2}{s})} \right\}_{s=0} \\
&= \left[0 - \log \frac{s+1}{s+2} \right]_{s=0} \quad \because \log 1 = 0 \\
&= \left[\log \frac{s+2}{s+1} \right]_{s=0} = \log 2
\end{aligned}$$

$$(iv) \int_0^\infty \left(\frac{1-\cos t}{t} \right) e^{-t} dt$$

$$\begin{aligned}
\int_0^\infty \left(\frac{1-\cos t}{t} \right) e^{-t} dt &= L \left[\frac{1-\cos t}{t} \right]_{s=1} = \int_s^\infty [L[(1-\cos t)]] ds \Big|_{s=1} \\
&= \int_s^\infty [L(1) - L(\cos t)] ds \Big|_{s=1} \\
&= \int_s^\infty \left[\left(\frac{1}{s} - \frac{s}{s^2+1} \right) ds \right]_{s=1} \\
&= \left\{ \left[\log s - \frac{1}{2} \log(s^2+1) \right]_s^\infty \right\}_{s=1} \\
&= \left\{ [\log s - \log \sqrt{s^2+1}]_s^\infty \right\}_{s=1} \\
&= \left\{ \left[\log \frac{s}{\sqrt{s^2+1}} \right]_s^\infty \right\}_{s=1} \\
&= \left[0 - \log \frac{s}{\sqrt{s^2+1}} \right]_{s=1} \\
&= \left[\log \frac{\sqrt{s^2+1}}{s} \right]_{s=1} \\
&= \log \sqrt{2}
\end{aligned}$$

$$(v) \int_0^\infty \left(\frac{e^{-at} - \cos bt}{t} \right) dt$$

$$\begin{aligned}
\int_0^\infty \left(\frac{e^{-at} - \cos bt}{t} \right) dt &= L \left[\frac{e^{-at} - \cos bt}{t} \right]_{s=0} = \int_s^\infty [L[(e^{-at} - \cos bt)]] ds \Big|_{s=0} \\
&= \int_s^\infty [L(e^{-at}) - L(\cos bt)] ds \Big|_{s=0} \\
&= \int_s^\infty \left[\left(\frac{1}{s+a} - \frac{s}{s^2+b^2} \right) ds \right]_{s=0} \\
&= \left\{ \left[\log(s+a) - \frac{1}{2} \log(s^2+b^2) \right]_s^\infty \right\}_{s=0} \\
&= \left\{ [\log(s+a) - \log \sqrt{s^2+b^2}]_s^\infty \right\}_{s=0} \\
&= \left\{ \left[\log \frac{s+a}{\sqrt{s^2+b^2}} \right]_s^\infty \right\}_{s=0}
\end{aligned}$$

$$\begin{aligned}
&= \left[0 - \log \frac{s+a}{\sqrt{s^2+b^2}} \right]_{s=0} \\
&= \left[\log \frac{\sqrt{s^2+b^2}}{s+a} \right]_{s=0} \\
&= \log \frac{\sqrt{b^2}}{a} \\
&= \log \frac{b}{a}
\end{aligned}$$

Exercise: 5.4

Find the values of the following integrals using Laplace transforms

- | | |
|--|---------------------------|
| 1. $\int_0^\infty t e^{-2t} \cos t dt$ | Ans: $\frac{3}{25}$ |
| 2. $\int_0^\infty t e^{-3t} \sin t dt$ | Ans: $\frac{13}{250}$ |
| 3. $\int_0^\infty \left(\frac{e^{-at} - e^{-bt}}{t} \right) dt$ | Ans: $\log \frac{b}{a}$ |
| 4. $\int_0^\infty e^{-2t} \frac{\sin^2 t}{t} dt$ | Ans: $\frac{1}{4} \log 2$ |
| 5. $\int_0^\infty \left(\frac{\cos at - \cos bt}{t} \right) dt$ | Ans: $\log \frac{a}{b}$ |

Laplace transform of Piecewise continuous functions

$$\int_0^\infty f(t) e^{-st} dt = L[f(t)]$$

Example: 5.31 Find the Laplace transform of $f(t) = \begin{cases} e^{-t}; & 0 < t < \pi \\ 0; & t > \pi \end{cases}$

Solution:

$$\begin{aligned}
L[f(t)] &= \int_0^\infty f(t) e^{-st} dt \\
&= \int_0^\pi e^{-st} e^{-t} dt + \int_\pi^\infty e^{-st} 0 dt \\
&= \int_0^\pi e^{-(s+1)t} dt \\
&= \left[\frac{e^{-(s+1)t}}{-(s+1)} \right]_0^\pi = \frac{e^{-(s+1)\pi} - e^0}{-(s+1)} \\
\therefore L[f(t)] &= \frac{1 - e^{-(s+1)\pi}}{-(s+1)}
\end{aligned}$$

Example: 5.32 Find the Laplace transform of $f(t) = \begin{cases} \sin t; & 0 < t < \pi \\ 0; & t > \pi \end{cases}$

Solution:

$$\begin{aligned}
L[f(t)] &= \int_0^\infty f(t) e^{-st} dt \\
&= \int_0^\pi e^{-st} \sin t dt + \int_\pi^\infty e^{-st} 0 dt \\
&= \int_0^\pi e^{-st} \sin t dt \\
&= \left[\frac{e^{-st}}{(-s)^2 + 1} (-s \sin t - \cos t) \right]_0^\pi = \frac{e^{-s\pi}}{s^2 + 1} [-s \sin \pi - \cos \pi] - \frac{e^0}{s^2 + 1} [-s \sin 0 - \cos 0] \\
&= \frac{e^{-s\pi}}{s^2 + 1} (0 + 1) - \frac{1}{s^2 + 1} (-1) = \frac{e^{-s\pi} + 1}{s^2 + 1}
\end{aligned}$$

$$\therefore L[f(t)] = \frac{e^{-s\pi} + 1}{s^2 + 1}$$

Example: 5.33 Find the Laplace transform of $f(t) = \begin{cases} t; & 0 < t < 1 \\ 0; & t > 1 \end{cases}$

Solution:

$$\begin{aligned} L[f(t)] &= \int_0^{\infty} f(t)e^{-st} dt \\ &= \int_0^1 e^{-st} t dt + \int_1^{\infty} e^{-st} 0 dt \\ &= \int_0^1 t e^{-st} dt \\ &= \left[t \frac{e^{-st}}{-s} - (1) \frac{e^{-st}}{(-s)^2} \right]_0^1 = \frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} - 0 + \frac{1}{s^2} \\ \therefore L[f(t)] &= -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} \end{aligned}$$

Exercise: 5.5

1. Find the Laplace transform of $f(t) = \begin{cases} 0; & 0 < t < 2 \\ 3; & t > 2 \end{cases}$ **Ans:** $\frac{3e^{-2s}}{s}$
2. Find the Laplace transform of $f(t) = \begin{cases} e^t; & 0 < t < 1 \\ 0; & t > 1 \end{cases}$ **Ans:** $\frac{1-e^{-(s-1)}}{s-1}$
3. Find the Laplace transform of $f(t) = \begin{cases} 1; & 0 < t < 1 \\ 0; & t > 1 \end{cases}$ **Ans:** $\frac{1-e^{-s}}{s}$

Unit step function

The unit step function $U(t - a)$ is defined as $U(t - a) = \begin{cases} 0; & t < a \\ 1; & t > a \end{cases}$

Example: 5.34 Find the Laplace transform of unit step functions.

Solution:

$$\begin{aligned} L[U(t - a)] &= \int_0^{\infty} U(t - a)e^{-st} dt \\ &= \int_0^a 0 dt + \int_a^{\infty} (1)e^{-st} dt = \int_a^{\infty} e^{-st} dt \\ &= \left[\frac{e^{-st}}{-s} \right]_a^{\infty} = 0 - \frac{e^{-sa}}{-s} = \frac{e^{-sa}}{s} \\ L[U(t - a)] &= \frac{e^{-sa}}{s} \end{aligned}$$

Second Shifting theorem

Statement: If $L[f(t)] = F(s)$, then $L[f(t - a)U(t - a)] = e^{-as}F(s)$

Proof:

$$U(t - a)f(t - a) = \begin{cases} 0; & t < a \\ f(t - a); & t > a \end{cases}$$

By the definition of Laplace transform,

$$\begin{aligned} L[U(t - a)f(t - a)] &= \int_0^{\infty} U(t - a)f(t - a)e^{-st} dt \\ &= \int_0^a 0 dt + \int_a^{\infty} f(t - a)e^{-st} dt \end{aligned}$$

$$L[U(t - a)f(t - a)] = \int_0^{\infty} e^{-s(a+x)} f(x) dx$$

$$\begin{aligned}
 &= \int_0^{\infty} e^{-sa} e^{-sx} f(x) dx \\
 &= e^{-sa} \int_0^{\infty} e^{-sx} f(x) dx
 \end{aligned}$$

Replace x by t

$$\begin{aligned}
 L[U(t-a)f(t-a)] &= e^{-sa} \int_0^{\infty} e^{-st} f(t) dt \\
 &= e^{-sa} L[f(t)] = e^{-sa} F(s)
 \end{aligned}$$

$$L[U(t-a)f(t-a)] = e^{-sa} F(s)$$

$$\text{Let } t - a = x \cdots (1)$$

$$t = a + x$$

$$dt = dx$$

$$\text{When } t = a, (1) \Rightarrow x = 0$$

$$\text{When } t = \infty, (1) \Rightarrow x = \infty$$

5.5 PERIODIC FUNCTIONS

Definition: A function $f(t)$ is said to be periodic if $f(t+T) = f(t)$ for all values of t and for certain values of T . The smallest value of T for which $f(t+T) = f(t)$ for all t is called periodic function.

Example:

$$\sin t = \sin(t + 2\pi) = \sin(t + 4\pi) \cdots$$

$\therefore \sin t$ is periodic function with period 2π .

Let $f(t)$ be a periodic function with period T . Then

$$L[f(t)] = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt$$

Problems on Laplace transform of Periodic function

Example: 5.35 Find the Laplace transform of $f(t) = \begin{cases} \sin \omega t; & 0 < t < \frac{\pi}{\omega} \\ 0; & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$ $f\left(t + \frac{2\pi}{\omega}\right) = f(t)$

Solution:

The given function is a periodic function with period $T = \frac{2\pi}{\omega}$

$$\begin{aligned}
 L[f(t)] &= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt \\
 &= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \left[\int_0^{\frac{\pi}{\omega}} \sin \omega t e^{-st} dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st} (0) dt \right] \\
 &= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \int_0^{\frac{\pi}{\omega}} \sin \omega t e^{-st} dt \\
 &= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-st}}{(-s)^2 + \omega^2} (-s \sin \omega t - \omega \cos \omega t) \right]_0^{\frac{\pi}{\omega}} \\
 &= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \left\{ \frac{e^{-\frac{s\pi}{\omega}}}{s^2 + \omega^2} [-s \sin \pi - \omega \cos \pi] + \frac{\omega}{s^2 + \omega^2} \right\} \\
 &= \frac{1}{1 - e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-\frac{s\pi}{\omega}} \omega + \omega}{s^2 + \omega^2} \right] \\
 &= \frac{1}{1^2 - \left(e^{-\frac{\pi s}{\omega}}\right)^2} \left[\frac{\omega \left(e^{-\frac{s\pi}{\omega}} + 1\right)}{s^2 + \omega^2} \right]
 \end{aligned}$$

$$= \frac{1}{\left(1 - e^{-\frac{\pi s}{\omega}}\right)\left(1 + e^{-\frac{\pi s}{\omega}}\right)} \left[\frac{\omega \left(e^{-\frac{s\pi}{\omega}} + 1\right)}{s^2 + \omega^2} \right]$$

$$\therefore L[f(t)] = \frac{\omega}{\left(1 - e^{-\frac{\pi s}{\omega}}\right)(s^2 + \omega^2)}$$

Example: 5.36 Find the Laplace transform of $f(t) = \begin{cases} E; 0 \leq t \leq a \\ -E; a \leq t \leq 2a \end{cases}$ given that $f(t + 2a) = f(t)$.

Solution:

The given function is a periodic function with period $T = 2a$

$$\begin{aligned} L[f(t)] &= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt \\ &= \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t) dt \\ &= \frac{1}{1 - e^{-2as}} \left[\int_0^a E e^{-st} dt + \int_a^{2a} -E e^{-st} dt \right] \\ &= \frac{1}{1 - e^{-2as}} \left[E \int_0^a e^{-st} dt - E \int_a^{2a} e^{-st} dt \right] \\ &= \frac{E}{1 - e^{-2as}} \left[\left[\frac{e^{-st}}{-s} \right]_0^a - \left[\frac{e^{-st}}{-s} \right]_a^{2a} \right] \\ &= \frac{E}{1 - e^{-2as}} \left[\frac{e^{-as}}{-s} + \frac{1}{s} - \frac{e^{-2as}}{s} - \frac{e^{-as}}{s} \right] \\ &= \frac{E}{1 - e^{-2as}} \left[\frac{1 - 2e^{-as} + e^{-2as}}{s} \right] \\ &= \frac{E}{1^2 - (e^{-as})^2} \left[\frac{(1 - e^{-as})^2}{s} \right] \\ &= \frac{E}{(1 - e^{-as})(1 + e^{-as})} \left[\frac{(1 - e^{-as})^2}{s} \right] \\ &= \frac{E (1 - e^{-as})}{s (1 + e^{-as})} \end{aligned}$$

$$\therefore L[f(t)] = \frac{E}{s} \tanh\left(\frac{as}{2}\right)$$

Example: 5.37 Find the Laplace transform of $f(t) = \begin{cases} 1; 0 \leq t \leq \frac{a}{2} \\ -1; \frac{a}{2} \leq t \leq a \end{cases}$ given that $f(t + a) = f(t)$.

Solution:

The given function is a periodic function with period $T = a$

$$\begin{aligned} L[f(t)] &= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt \\ &= \frac{1}{1 - e^{-as}} \int_0^a e^{-st} f(t) dt \\ &= \frac{1}{1 - e^{-as}} \left[\int_0^{\frac{a}{2}} (1) e^{-st} dt + \int_{\frac{a}{2}}^a (-1) e^{-st} dt \right] \\ &= \frac{1}{1 - e^{-as}} \left[\int_0^{\frac{a}{2}} e^{-st} dt - \int_{\frac{a}{2}}^a e^{-st} dt \right] \\ &= \frac{1}{1 - e^{-as}} \left[\left[\frac{e^{-st}}{-s} \right]_0^{\frac{a}{2}} - \left[\frac{e^{-st}}{-s} \right]_{\frac{a}{2}}^a \right] \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{1-e^{-as}} \left[\frac{e^{-\frac{sa}{2}}}{-s} + \frac{1}{s} + \frac{e^{-as}}{s} - \frac{e^{-\frac{sa}{2}}}{s} \right] \\
 &= \frac{1}{1-e^{-as}} \left[\frac{1-2e^{-\frac{sa}{2}}+e^{-as}}{s} \right] \\
 &= \frac{1}{1^2 - \left(e^{-\frac{sa}{2}}\right)^2} \left[\frac{\left(1-e^{-\frac{sa}{2}}\right)^2}{s} \right] \\
 &= \frac{1}{\left(1-e^{-\frac{sa}{2}}\right)\left(1+e^{-\frac{sa}{2}}\right)} \left[\frac{\left(1-e^{-\frac{sa}{2}}\right)^2}{s} \right] \\
 &= \frac{1}{s} \frac{\left(1-e^{-\frac{sa}{2}}\right)}{\left(1+e^{-\frac{sa}{2}}\right)} \quad \left[\because \tanh x = \frac{(1-e^{-2x})}{(1+e^{-2x})} \right]
 \end{aligned}$$

$$\therefore L[f(t)] = \frac{1}{s} \tanh\left(\frac{as}{2}\right)$$

Example: 5.38 Find the Laplace transform of $f(t) = \begin{cases} t; & 0 \leq t \leq a \\ 2a - t; & a \leq t \leq 2a \end{cases}$ given that $f(t + 2a) = f(t)$.

Solution:

The given function is a periodic function with period $T = 2a$

$$\begin{aligned}
 L[f(t)] &= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt \\
 &= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t) dt \\
 &= \frac{1}{1-e^{-2as}} \left[\int_0^a t e^{-st} dt + \int_a^{2a} (2a - t) e^{-st} dt \right] \\
 &= \frac{1}{1-e^{-2as}} \left[\left[t \left(\frac{e^{-st}}{-s} \right) - \left(\frac{e^{-st}}{(-s)^2} \right) \right]_0^a - \left[(2a - t) \left(\frac{e^{-st}}{-s} \right) - (-1) \left(\frac{e^{-st}}{(-s)^2} \right) \right]_a^{2a} \right] \\
 &= \frac{1}{1-e^{-2as}} \left[\frac{-ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} \right] \\
 &= \frac{1}{1-e^{-2as}} \left[\frac{1-2e^{-as}+e^{-2as}}{s^2} \right] \\
 &= \frac{1}{1^2 - (e^{-as})^2} \left[\frac{(1-e^{-as})^2}{s^2} \right] \\
 &= \frac{1}{(1-e^{-as})(1+e^{-as})} \left[\frac{(1-e^{-as})^2}{s^2} \right] \\
 &= \frac{1}{s^2} \frac{(1-e^{-as})}{(1+e^{-as})} \\
 &= \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)
 \end{aligned}$$

Exercise: 5.6

1. Find the Laplace transform of

$$f(t) = \begin{cases} 1; & 0 \leq t \leq \frac{a}{2} \\ -1; & \frac{a}{2} \leq t \leq a \end{cases} \text{ given that } f(t + a) = f(t). \quad \text{Ans: } \frac{k}{s} \tanh\left(\frac{as}{2}\right)$$

2. Find the Laplace transform of

$$f(t) = \begin{cases} t; 0 \leq t \leq a \\ 2a - t; a \leq t \leq 2a \end{cases} \text{ given that } f(t + 2a) = f(t). \quad \text{Ans: } \frac{1}{s^2} \tanh\left(\frac{s\pi}{2}\right)$$

3. Find the Laplace transform of

$$f(t) = \begin{cases} \frac{t}{a}; 0 \leq t \leq a \\ \frac{2a-t}{a}; a \leq t \leq 2a \end{cases} \text{ given that } f(t + 2a) = f(t). \quad \text{Ans: } \frac{1}{as^2} \tanh\left(\frac{sa}{2}\right)$$

4. Find the Laplace transform of

$$f(t) = \begin{cases} \sin t; 0 < t < \pi \\ 0; \pi < t < 2\pi \end{cases} \quad f(t + 2\pi) = f(t) \quad \text{Ans: } \frac{1}{(1 - e^{-\pi s})(s^2 + 1)}$$

5.6 INVERSE LAPLACE TRANSFORM

Definition

If the Laplace transform of a function $f(t)$ is $F(s)$ i.e., $L[f(t)] = F(s)$, then $f(t)$ is called an inverse Laplace transform of $F(s)$ and we write symbolically $f(t) = L^{-1}[F(s)]$, where L^{-1} is called the inverse Laplace transform operator.

Inverse Laplace transform of elementary functions

$L[f(t)] = F(s)$	$L^{-1}[F(s)] = f(t)$
$L[1] = \frac{1}{s}$	$L^{-1}\left[\frac{1}{s}\right] = 1$
$L[t] = \frac{1}{s^2}$	$L^{-1}\left[\frac{1}{s^2}\right] = t$
$L[t^n] = \frac{n!}{s^{n+1}}$ if n is an integer	$L^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$ $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$
$L[e^{at}] = \frac{1}{s - a}$	$L^{-1}\left[\frac{1}{s - a}\right] = e^{at}$
$L[e^{-at}] = \frac{1}{s + a}$	$L^{-1}\left[\frac{1}{s + a}\right] = e^{-at}$
$L[\sin at] = \frac{a}{s^2 + a^2}$	$L^{-1}\left[\frac{1}{s^2 + a^2}\right] = \frac{\sin at}{a}$
$L[\cos at] = \frac{s}{s^2 + a^2}$	$L^{-1}\left[\frac{s}{s^2 + a^2}\right] = \cos at$

$L[\sinh at] = \frac{a}{s^2 - a^2}$	$L^{-1}\left[\frac{1}{s^2 - a^2}\right] = \frac{\sinh at}{a}$
$L[\cosh at] = \frac{s}{s^2 - a^2}$	$L^{-1}\left[\frac{s}{s^2 - a^2}\right] = \cosh at$

Result on inverse Laplace transform**Result: 1 Linear property**

$$L[f(t)] = F(s) \text{ and } L[g(t)] = G(s) \text{ ,then } L^{-1}[aF(s) \pm bG(s)] = aL^{-1}[F(s)] \pm bL^{-1}[G(s)]$$

Where a and b are constants.

Proof:

$$\begin{aligned} \text{We know that } L[aF(s) \pm bG(s)] &= aL[F(s)] \pm bL[G(s)] \\ &= aF(s) \pm bG(s) \end{aligned}$$

$$(i.e.) aF(s) \pm bG(s) = L[af(t) \pm bg(t)]$$

Operating L^{-1} on both sides, we get

$$L^{-1}[aF(s) \pm bG(s)] = af(t) \pm bg(t)$$

$$L^{-1}[aF(s) \pm bG(s)] = aL^{-1}[F(s)] \pm bL^{-1}[G(s)]$$

$$\therefore f(t) = L^{-1}[F(s)]$$

$$\therefore g(t) = L^{-1}[G(s)]$$

Result: 2 First shifting property

$$(i) L^{-1}[F(s + a)] = e^{-at} L^{-1}[F(s)]$$

$$(ii) L^{-1}[F(s - a)] = e^{at} L^{-1}[F(s)]$$

Proof:

$$\text{Let } L[e^{-at}f(t)] = F[s + a]$$

Operating L^{-1} on both sides, we get

$$e^{-at}f(t) = L^{-1}[F[s + a]]$$

$$L^{-1}[F[s + a]] = e^{-at} L^{-1}[F(s)]$$

Result: 3 Multiplication by s.

$$\text{If } L^{-1}[F(s)] = f(t) \text{ and } f(0) = 0, \text{ then } L^{-1}[sF(s)] = \frac{d}{dt} L^{-1}[F(s)]$$

Proof:

$$\text{We know that } L[f'(t)] = sL[f(t)] - f(0) = sF(s)$$

Operating L^{-1} on both sides, we get

$$f'(t) = L^{-1}[sF(s)]$$

$$\frac{d}{dt} f(t) = L^{-1}[sF(s)]$$

$$\frac{d}{dt} L^{-1}[F(s)] = L^{-1}[sF(s)]$$

$$\therefore L^{-1}[sF(s)] = \frac{d}{dt} L^{-1}[F(s)]$$

Result: 4 Division by s.

$$L^{-1} \left[\frac{F(s)}{s} \right] = \int_0^t L^{-1}[F(s)] dt$$

Proof:

We know that $L \left[\int_0^t f(t) dt \right] = \frac{1}{s} L[f(t)] = \frac{1}{s} F(s)$

Operating L^{-1} on both sides, we get

$$\int_0^t f(t) dt = L^{-1} \left[\frac{1}{s} F(s) \right]$$

$$\int_0^t L^{-1}[F(s)] dt = L^{-1} \left[\frac{1}{s} F(s) \right]$$

$$\therefore L^{-1} \left[\frac{F(s)}{s} \right] = \int_0^t L^{-1}[F(s)] dt$$

Result: 5 Inverse Laplace transform of derivative

$$L^{-1}[F(s)] = \frac{-1}{t} L^{-1} \left[\frac{d}{ds} F(s) \right]$$

Proof:

We know that $L[tf(t)] = \frac{-d}{ds} L[f(t)] = \frac{-d}{ds} F(s)$

Operating L^{-1} on both sides, we get

$$tf(t) = -L^{-1} \left[\frac{d}{ds} F(s) \right]$$

$$L^{-1}[F(s)] = \frac{-1}{t} L^{-1} \left[\frac{d}{ds} F(s) \right]$$

$$f(t) = \frac{-1}{t} L^{-1} \left[\frac{d}{ds} F(s) \right]$$

$$L^{-1}[F(s)] = \frac{-1}{t} L^{-1} \left[\frac{d}{ds} F(s) \right]$$

Result: 6 Inverse Laplace transform of integral

$$L^{-1}[F(s)] = t L^{-1} \left[\int_s^\infty F(s) ds \right]$$

Proof:

We know that $L \left[\frac{f(t)}{t} \right] = \int_s^\infty L(f(t)) ds$

$$= \int_s^\infty F(s) ds$$

Operating L^{-1} on both sides, we get

$$\frac{f(t)}{t} = L^{-1} \left[\int_s^\infty F(s) ds \right]$$

$$f(t) = t L^{-1} \left[\int_s^\infty F(s) ds \right]$$

$$L^{-1}[F(s)] = t L^{-1} \left[\int_s^\infty F(s) ds \right]$$

Problems under inverse Laplace transform of elementary functions

Example: 5.39 Find the inverse Laplace for the following

(i) $\frac{1}{2s+3}$ (ii) $\frac{1}{4s^2+9}$ (iii) $\frac{s^3-3s^2+7}{s^4}$ (iv) $\frac{3s+5}{s^2+36}$

Solution:

$$\begin{aligned} \text{(i)} \quad L^{-1} \left[\frac{1}{2s+3} \right] &= L^{-1} \left[\frac{1}{2 \left[s + \frac{3}{2} \right]} \right] \\ &= \frac{1}{2} e^{-\frac{3t}{2}} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad L^{-1} \left[\frac{1}{4s^2+9} \right] &= L^{-1} \left[\frac{1}{4 \left[s^2 + \frac{9}{4} \right]} \right] \\ &= \frac{1}{4} L^{-1} \left[\frac{1}{\left[s^2 + \frac{9}{4} \right]} \right] \\ &= \frac{1}{4} \frac{1}{3/2} \sin \frac{3}{2} t \\ &= \frac{1}{6} \sin \frac{3}{2} t \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad L^{-1} \left[\frac{s^3-3s^2+7}{s^4} \right] &= L^{-1} \left[\frac{s^3}{s^4} - \frac{3s^2}{s^4} + \frac{7}{s^4} \right] \\ &= L^{-1} \left[\frac{1}{s} \right] - 3L^{-1} \left[\frac{1}{s^2} \right] + 7L^{-1} \left[\frac{1}{s^4} \right] \\ L^{-1} \left[\frac{s^3-3s^2+7}{s^4} \right] &= 1 - 3t + \frac{7t^3}{3!} \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad L^{-1} \left[\frac{3s+5}{s^2+36} \right] &= 3L^{-1} \left[\frac{s}{s^2+36} \right] + 5L^{-1} \left[\frac{1}{s^2+36} \right] \\ L^{-1} \left[\frac{3s+5}{s^2+36} \right] &= 3\cos 6t + \frac{5\sin 6t}{6} \end{aligned}$$

Inverse Laplace transform using First shifting theorem

$$L^{-1}[F(s+a)] = e^{-at} L^{-1}[F(s)]$$

Example: 5.40 Find the inverse Laplace transform for the following:

(i) $\frac{1}{(s+2)^2}$	(ii) $\frac{1}{(s-3)^4}$	(iii) $\frac{1}{(s+3)^2+9}$	(iv) $\frac{1}{s^2-2s+2}$
(v) $\frac{1}{s^2-4s+13}$	(vi) $\frac{s+2}{(s+2)^2+25}$	(vii) $\frac{s+2}{s^2+4s+20}$	(viii) $\frac{s}{(s+3)^2}$
(ix) $\frac{s}{(s-4)^3}$	(x) $\frac{s}{s^2-2s+2}$	(xi) $\frac{2s+3}{s^2+6s+25}$	(xii) $\frac{s}{s^2+6s-7}$

Solution:

$$\text{(i)} \quad L^{-1} \left[\frac{1}{(s+2)^2} \right] = e^{-2t} L^{-1} \left[\frac{1}{s^2} \right] = e^{-2t} t$$

$$\text{(ii)} \quad L^{-1} \left[\frac{1}{(s-3)^4} \right] = e^{3t} L^{-1} \left[\frac{1}{s^4} \right] = e^{-2t} \frac{t^3}{3!}$$

$$\text{(iii)} \quad L^{-1} \left[\frac{1}{(s+3)^2+9} \right] = e^{-3t} L^{-1} \left[\frac{1}{s^2+9} \right] = e^{-3t} \frac{\sin 3t}{3}$$

$$\text{(iv)} \quad L^{-1} \left[\frac{1}{s^2-2s+2} \right] = L^{-1} \left[\frac{1}{(s-1)^2+1} \right] = e^t L^{-1} \left[\frac{1}{s^2+1} \right] = e^t \sin t$$

$$\text{(v)} \quad L^{-1} \left[\frac{1}{s^2-4s+13} \right] = L^{-1} \left[\frac{1}{(s-2)^2+9} \right] = e^{2t} L^{-1} \left[\frac{1}{s^2+9} \right] = e^{2t} \frac{\sin 3t}{3}$$

$$\text{(vi)} \quad L^{-1} \left[\frac{s+2}{(s+2)^2+25} \right] = e^{-2t} L^{-1} \left[\frac{s}{s^2+25} \right] = e^{-2t} \cos 5t$$

$$\text{(vii)} \quad L^{-1} \left[\frac{s+2}{s^2+4s+20} \right] = L^{-1} \left[\frac{s+2}{(s+2)^2+16} \right]$$

$$= e^{-2t} L^{-1} \left[\frac{s}{s^2+16} \right] = e^{-2t} \cos 4t$$

$$\begin{aligned} \text{(viii)} \quad L^{-1} \left[\frac{s}{(s+3)^2} \right] &= L^{-1} \left[\frac{s+3-3}{(s+3)^2} \right] \\ &= L^{-1} \left[\frac{s+3}{(s+3)^2} \right] - L^{-1} \left[\frac{3}{(s+3)^2} \right] \\ &= L^{-1} \left[\frac{1}{s+3} \right] - 3L^{-1} \left[\frac{1}{(s+3)^2} \right] \\ &= e^{-3t} - 3e^{-3t} L^{-1} \left[\frac{1}{s^2} \right] \\ &= e^{-3t} - 3e^{-3t} t \end{aligned}$$

$$\begin{aligned} \text{(ix)} \quad L^{-1} \left[\frac{s}{(s-4)^3} \right] &= L^{-1} \left[\frac{s-4+4}{(s-4)^3} \right] \\ &= L^{-1} \left[\frac{s-4}{(s-4)^3} \right] + L^{-1} \left[\frac{4}{(s-4)^3} \right] \\ &= L^{-1} \left[\frac{1}{(s-4)^2} \right] + 4L^{-1} \left[\frac{1}{(s-4)^3} \right] \\ &= e^{4t} L^{-1} \left[\frac{1}{s^2} \right] + 4e^{4t} L^{-1} \left[\frac{1}{s^3} \right] \\ &= e^{4t} t + 4e^{4t} \frac{t^2}{2!} \\ &= e^{4t} t + 2e^{4t} t^2 \end{aligned}$$

$$\begin{aligned} \text{(x)} \quad L^{-1} \left[\frac{s}{s^2-2s+2} \right] &= L^{-1} \left[\frac{s}{(s-1)^2+1} \right] = L^{-1} \left[\frac{s-1+1}{(s-1)^2+1} \right] \\ &= L^{-1} \left[\frac{s-1}{(s-1)^2+1} \right] + L^{-1} \left[\frac{1}{(s-1)^2+1} \right] \\ &= e^t L^{-1} \left[\frac{s}{s^2+1} \right] + e^t L^{-1} \left[\frac{1}{s^2+1} \right] \end{aligned}$$

$$L^{-1} \left[\frac{s}{s^2-2s+2} \right] = e^t \cos t + e^t \sin t$$

$$\begin{aligned} \text{(xi)} \quad L^{-1} \left[\frac{2s+3}{s^2+6s+25} \right] &= L^{-1} \left[\frac{2s+3}{(s+3)^2+16} \right] = L^{-1} \left[\frac{2(s+3-3)+3}{(s+3)^2+16} \right] \\ &= L^{-1} \left[\frac{2(s+3)-6+3}{(s+3)^2+16} \right] \\ &= e^{-3t} L^{-1} \left[\frac{2s-3}{s^2+16} \right] \\ &= e^{-3t} \left[2L^{-1} \left[\frac{s}{s^2+16} \right] - 3L^{-1} \left[\frac{1}{s^2+16} \right] \right] \end{aligned}$$

$$L^{-1} \left[\frac{2s+3}{s^2+6s+25} \right] = e^{-3t} \left(2\cos 4t - \frac{3\sin 4t}{4} \right)$$

$$\begin{aligned} \text{(xii)} \quad L^{-1} \left[\frac{s}{s^2+6s-7} \right] &= L^{-1} \left[\frac{s}{(s+3)^2-16} \right] = L^{-1} \left[\frac{s+3-3}{(s+3)^2-16} \right] \\ &= e^{-3t} L^{-1} \left[\frac{s-3}{s^2-16} \right] \\ &= e^{-3t} L^{-1} \left[\frac{s}{s^2-16} \right] - 3e^{-3t} L^{-1} \left[\frac{1}{s^2-16} \right] \end{aligned}$$

$$L^{-1} \left[\frac{s}{s^2+6s-7} \right] = e^{-3t} \left[\cosh 4t - \frac{3\sinh 4t}{4} \right]$$

Exercise: 5.7

Find the inverse Laplace transform for the following:

1. $\frac{2s-3}{s^2+5^2}$

Ans: $2\cos 5t - \frac{3\sin 5t}{5}$

2. $\frac{3s+5}{s^2+16}$

Ans: $3\cos 4t + \frac{5\sin 4t}{4}$

3. $\frac{1}{4s^2+9}$

Ans: $\frac{1}{6}\sin \frac{3}{2}t$

4. $\frac{1}{(s+4)^5}$

Ans: $e^{-4t} \frac{t^4}{4!}$

5. $\frac{1}{s^2-4s+13}$

Ans: $\frac{e^{2t}}{3}\sin 3t$

Inverse using the formula

$$L^{-1}[F(s)] = \frac{-1}{t} L^{-1} \left[\frac{d}{ds} F(s) \right]$$

Note: This formula is used when $F(s)$ is $\cot^{-1} \phi(s)$ or $\tan^{-1} \phi(s)$ or $\log \phi(s)$

Example: 5.41 Find the inverse Laplace transform for the following

(i) $\cot^{-1} \left(\frac{s}{a} \right)$ (ii) $\tan^{-1} \left(\frac{a}{s} \right)$ (iii) $\cot^{-1} as$

(iv) $\tan^{-1}(s+a)$ (v) $\log \left(\frac{s+a}{s+b} \right)$ (vi) $\cot^{-1} \left(\frac{2}{s+1} \right)$ (vii) $\tan^{-1} \left(\frac{2}{s^2} \right)$

Solution:

$$\begin{aligned}
 \text{(i) } L^{-1} \left[\cot^{-1} \left(\frac{s}{a} \right) \right] &= \frac{-1}{t} L^{-1} \left[\frac{d}{ds} \left(\cot^{-1} \left(\frac{s}{a} \right) \right) \right] \\
 &= \frac{-1}{t} L^{-1} \left[\frac{-1}{1 + \frac{s^2}{a^2}} \left(\frac{1}{a} \right) \right] = \frac{1}{t} L^{-1} \left[\frac{-1}{\frac{a^2 + s^2}{a^2}} \left(\frac{1}{a} \right) \right] \\
 &= \frac{1}{t} L^{-1} \left[\frac{a}{s^2 + a^2} \right]
 \end{aligned}$$

$$L^{-1} \left[\cot^{-1} \left(\frac{s}{a} \right) \right] = \frac{1}{t} \sin at$$

$$\begin{aligned}
 \text{(ii) } L^{-1} \left[\tan^{-1} \left(\frac{a}{s} \right) \right] &= \frac{-1}{t} L^{-1} \left[\frac{d}{ds} \left(\tan^{-1} \left(\frac{a}{s} \right) \right) \right] \\
 &= \frac{-1}{t} L^{-1} \left[\frac{1}{1 + \left(\frac{a}{s} \right)^2} \left(\frac{-a}{s^2} \right) \right] = \frac{-1}{t} L^{-1} \left[\frac{1}{\frac{s^2 + a^2}{s^2}} \left(\frac{-a}{s^2} \right) \right] \\
 &= \frac{1}{t} L^{-1} \left[\frac{a}{s^2 + a^2} \right]
 \end{aligned}$$

$$L^{-1} \left[\tan^{-1} \left(\frac{a}{s} \right) \right] = \frac{1}{t} \sin at$$

$$\begin{aligned}
 \text{(iii) } L^{-1} [\cot^{-1} as] &= \frac{-1}{t} L^{-1} \left[\frac{d}{ds} (\cot^{-1}(as)) \right] \\
 &= \frac{-1}{t} L^{-1} \left[\frac{-1}{1 + a^2 s^2} (a) \right] = \frac{1}{t} L^{-1} \left[\frac{a}{a^2 (s^2 + \frac{1}{a^2})} \right] \\
 &= \frac{1}{at} L^{-1} \left[\frac{1}{s^2 + \frac{1}{a^2}} \right] = \frac{1}{at} \left[\frac{\sin \frac{1}{a} t}{\frac{1}{a}} \right]
 \end{aligned}$$

$$L^{-1} [\cot^{-1} as] = \frac{1}{t} \sin \frac{t}{a}$$

(iv) $L^{-1} [\tan^{-1}(s+a)] = e^{-at} L^{-1} [\tan^{-1} s]$

$$= e^{-at} \left[\frac{-1}{t} L^{-1} \left[\frac{d}{ds} (\tan^{-1} s) \right] \right]$$

$$= e^{-at} \left(\frac{-1}{t} \right) L^{-1} \left[\frac{1}{1+s^2} \right]$$

$$= \frac{-1}{t} e^{-at} L^{-1} \left[\frac{1}{1+s^2} \right]$$

$$L^{-1} \left[\cot^{-1} \left(\frac{s}{a} \right) \right] = \frac{-e^{-at}}{t} \sin t$$

$$(v) L^{-1} \left[\log \left(\frac{s+a}{s+b} \right) \right] = \frac{-1}{t} L^{-1} \left[\frac{d}{ds} \left(\log \left(\frac{s+a}{s+b} \right) \right) \right]$$

$$= \frac{-1}{t} L^{-1} \left[\frac{d}{ds} (\log(s+a) - \log(s+b)) \right]$$

$$= \frac{-1}{t} L^{-1} \left[\frac{1}{s+a} - \frac{1}{s+b} \right]$$

$$= \frac{-1}{t} [e^{-at} - e^{-bt}]$$

$$L^{-1} \left[\log \left(\frac{s+a}{s+b} \right) \right] = \frac{-1}{t} [e^{-at} - e^{-bt}]$$

$$(vi) L^{-1} \left[\cot^{-1} \left(\frac{2}{s+1} \right) \right] = e^{-t} L^{-1} \left[\cot^{-1} \left(\frac{2}{s} \right) \right]$$

$$= e^{-t} \left(\frac{-1}{t} \right) L^{-1} \left[\frac{d}{ds} \left(\cot^{-1} \left(\frac{2}{s} \right) \right) \right]$$

$$= e^{-t} \left(\frac{-1}{t} \right) L^{-1} \left[\frac{-1}{1+\frac{4}{s^2}} \left(\frac{-2}{s^2} \right) \right] = -\frac{e^{-t}}{t} L^{-1} \left[\frac{1}{\frac{s^2+4}{s^2}} \left(\frac{2}{s^2} \right) \right]$$

$$= -\frac{e^{-t}}{t} L^{-1} \left[\frac{2}{s^2+4} \right]$$

$$L^{-1} \left[\cot^{-1} \left(\frac{2}{s+1} \right) \right] = -\frac{e^{-t}}{t} \sin 2t$$

$$(vii) L^{-1} \left[\tan^{-1} \left(\frac{2}{s^2} \right) \right] = \frac{-1}{t} L^{-1} \left[\frac{d}{ds} \left(\tan^{-1} \left(\frac{2}{s^2} \right) \right) \right]$$

$$= \frac{-1}{t} L^{-1} \left[\frac{1}{1+\left(\frac{2}{s^2}\right)^2} \left(\frac{-4}{s^3} \right) \right] = \frac{4}{t} L^{-1} \left[\frac{1}{\frac{s^4+4}{s^4}} \left(\frac{1}{s^3} \right) \right]$$

$$= \frac{4}{t} L^{-1} \left[\frac{s}{s^4+4} \right]$$

$$= \frac{4}{t} L^{-1} \left[\frac{s}{(s^2)^2+2^2} \right]$$

$$= \frac{4}{t} L^{-1} \left[\frac{s}{(s^2+2)^2-(2s)^2} \right]$$

$$= \frac{4}{t} L^{-1} \left[\frac{s}{(s^2+2+2s)(s^2+2-2s)} \right]$$

$$\because a^2 + b^2 = (a+b)^2 - 2ab$$

$$= \frac{4}{t} L^{-1} \left[\frac{s}{-4s} \left(\frac{1}{s^2+2+2s} - \frac{1}{s^2+2-2s} \right) \right] \quad \because \left\{ \frac{1}{(s^2+ax+b)(s^2+ax+c)} = \frac{1}{c-b} \left[\frac{1}{s^2+ax+b} - \frac{1}{s^2+ax+c} \right] \right\}$$

$$= \frac{-1}{t} L^{-1} \left[\left(\frac{1}{s^2+2s+2} - \frac{1}{s^2-2s+2} \right) \right]$$

$$= \frac{-1}{t} L^{-1} \left[\frac{1}{(s+1)^2+1} - \frac{1}{(s-1)^2+1} \right]$$

$$\begin{aligned}
&= \frac{-1}{t} \left(e^{-t} L^{-1} \left[\frac{1}{s^2+1} \right] - e^t L^{-1} \left[\frac{1}{s^2+1} \right] \right) \\
&= \frac{-1}{t} (e^{-t} \sin t - e^t \sin t) \\
&= \frac{\sin t}{t} (e^{-t} - e^t) \\
&= \frac{\sin t}{t} 2 \sinh t
\end{aligned}$$

$$L^{-1} \left[\tan^{-1} \left(\frac{2}{s^2} \right) \right] = \frac{2 \sin t \sinh t}{t}$$

Inverse using the formula

$$L^{-1}[sF(s)] = \frac{d}{dt} L^{-1}[F(s)]$$

Example: 5.42 Find $L^{-1} \left[s \log \left(\frac{s^2+a^2}{s^2+b^2} \right) \right]$

Solution:

$$\begin{aligned}
L^{-1} \left[s \log \left(\frac{s^2+a^2}{s^2+b^2} \right) \right] &= \frac{d}{dt} L^{-1} \left[s \log \left(\frac{s^2+a^2}{s^2+b^2} \right) \right] \dots (1) \\
L^{-1} \left[\log \left(\frac{s^2+a^2}{s^2+b^2} \right) \right] &= L^{-1} \frac{d}{ds} \left[\log \left(\frac{s^2+a^2}{s^2+b^2} \right) \right] \\
&= \frac{-1}{t} L^{-1} \left[\frac{d}{ds} (\log(s^2 + a^2) - \log(s^2 + b^2)) \right] \\
&= \frac{-1}{t} L^{-1} \left[\frac{1}{s^2+a^2} 2s - \frac{1}{s^2+b^2} 2s \right] \\
&= \frac{-2}{t} L^{-1} \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} \right] \\
&= \frac{-2}{t} [\cos at - \cos bt] \\
&= \frac{2}{t} [\cos bt - \cos at]
\end{aligned}$$

Substituting in (1), we get

$$\begin{aligned}
L^{-1} \left[s \log \left(\frac{s^2+a^2}{s^2+b^2} \right) \right] &= \frac{d}{dt} \left[\frac{2}{t} [\cos bt - \cos at] \right] \\
&= 2 \left[\frac{t(-b \sin bt + a \sin at) - (\cos bt - \cos at)}{t^2} \right] \\
L^{-1} \left[s \log \left(\frac{s^2+a^2}{s^2+b^2} \right) \right] &= 2 \left[\frac{t(-b \sin bt + a \sin at) - (\cos bt - \cos at)}{t^2} \right]
\end{aligned}$$

Inverse using the formula

$$L^{-1} \left[\frac{F(s)}{s} \right] = \int_0^t L^{-1}[F(s)] dt$$

This formula is used when $F(s) = \frac{\text{one term}}{s(\text{another term})}$

Example: 5.43 Find $L^{-1} \left[\frac{1}{s(s^2+a^2)} \right]$

Solution:

$$\begin{aligned}
L^{-1} \left[\frac{1}{s(s^2+a^2)} \right] &= \int_0^t L^{-1} \left[\frac{1}{s^2+a^2} \right] dt \\
&= \int_0^t \left[\frac{\sin at}{a} \right] dt
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{a} \left[\frac{-\cos at}{a} \right]_0^t \\
&= \frac{-1}{a^2} [\cos at]_0^t \\
&= \frac{-1}{a^2} (\cos at - \cos 0) = \frac{-1}{a^2} (\cos at - 1)
\end{aligned}$$

$$\therefore L^{-1} \left[\frac{1}{s(s^2+a^2)} \right] = \frac{1-\cos at}{a^2}$$

Example: 5.44 Find $L^{-1} \left[\frac{1}{s(s^2-a^2)} \right]$

Solution:

$$\begin{aligned}
L^{-1} \left[\frac{1}{s(s^2+a^2)} \right] &= \int_0^t L^{-1} \left[\frac{1}{(s^2-a^2)} \right] dt \\
&= \int_0^t \left[\frac{\sinh at}{a} \right] dt \\
&= \frac{1}{a} \left[\frac{\cosh at}{a} \right]_0^t \\
&= \frac{1}{a^2} [\cosh at]_0^t \\
&= \frac{1}{a^2} (\cosh at - \cosh 0) = \frac{1}{a^2} (\cosh at - 1)
\end{aligned}$$

$$\therefore L^{-1} \left[\frac{1}{s(s^2-a^2)} \right] = \frac{\cosh at - 1}{a^2}$$

Example: 5.45 Find $L^{-1} \left[\frac{1}{s(s+a)} \right]$

Solution:

$$\begin{aligned}
L^{-1} \left[\frac{1}{s(s+a)} \right] &= \int_0^t L^{-1} \left[\frac{1}{(s+a)} \right] dt \\
&= \int_0^t e^{-at} dt \\
&= \left[\frac{e^{-at}}{-a} \right]_0^t \\
&= \frac{-1}{a} (e^{-at} - 1)
\end{aligned}$$

$$\therefore L^{-1} \left[\frac{1}{s(s+a)} \right] = \frac{1-e^{-at}}{a}$$

Inverse using Partial Fraction

Example: 5.46 Find $L^{-1} \left[\frac{s-2}{s(s+2)(s-1)} \right]$

Solution:

$$\begin{aligned}
\frac{s-2}{s(s+2)(s-1)} &= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1} \\
&= \frac{A(s+2)(s-1) + Bs(s-1) + Cs(s+2)}{s(s+2)(s-1)}
\end{aligned}$$

$$A(s+2)(s-1) + Bs(s-1) + Cs(s+2) = s-2 \dots (1)$$

Put $s = 0$ in (1)

$$A(2)(-1) = -2$$

Put $s = -2$ in (1)

$$B(-2)(-3) = -4$$

Put $s = 1$ in (1)

$$3C = -1$$

$$\Rightarrow A = 1$$

$$\Rightarrow B = \frac{-4}{6} = \frac{-2}{3}$$

$$\Rightarrow C = \frac{-1}{3}$$

$$\therefore \frac{s-2}{s(s+2)(s-1)} = \frac{1}{s} - \frac{2}{s+2} - \frac{1}{3(s-1)}$$

$$L^{-1} \left[\frac{s-2}{s(s+2)(s-1)} \right] = L^{-1} \left[\frac{1}{s} \right] - \frac{2}{3} L^{-1} \left[\frac{1}{s+2} \right] - \frac{1}{3} L^{-1} \left[\frac{1}{s-1} \right]$$

$$L^{-1} \left[\frac{s-2}{s(s+2)(s-1)} \right] = 1 - \frac{2}{3} e^{-2t} - \frac{1}{3} e^t$$

Example: 5.47 Find $L^{-1} \left[\frac{2s-3}{(s-1)(s-2)^2} \right]$

Solution:

$$\begin{aligned} \frac{2s-3}{(s-1)(s-2)^2} &= \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} \\ &= \frac{A(s-2)^2 + B(s-1)(s-2) + C(s-1)}{(s-1)(s-2)^2} \end{aligned}$$

$$A(s-2)^2 + B(s-1)(s-2) + C(s-1) = 2s-3 \dots (1)$$

Put $s = 1$ in (1)

$$A = -1$$

Put $s = 2$ in (1)

$$C = 1$$

Equating the coefficient of s^2

$$A + B = 0$$

$$B = -A \Rightarrow B = 1$$

$$\therefore \frac{2s-3}{(s-1)(s-2)^2} = \frac{-1}{s-1} + \frac{1}{s-2} + \frac{1}{(s-2)^2}$$

$$\begin{aligned} L^{-1} \left[\frac{2s-3}{(s-1)(s-2)^2} \right] &= -L^{-1} \left[\frac{1}{s-1} \right] + L^{-1} \left[\frac{1}{s-2} \right] + L^{-1} \left[\frac{1}{(s-2)^2} \right] \\ &= -e^t + e^{2t} + e^{2t} L^{-1} \left[\frac{1}{s^2} \right] \end{aligned}$$

$$\therefore L^{-1} \left[\frac{2s-3}{(s-1)(s-2)^2} \right] = -e^t + e^{2t} + e^{2t} t$$

Example: 5.48 Find the inverse Laplace transform of $\frac{5s^2-15s-11}{(s+1)(s-2)^3}$

Solution:

$$\begin{aligned} \frac{5s^2-15s-11}{(s+1)(s-2)^3} &= \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3} \\ &= \frac{A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1)}{(s+1)(s-2)^3} \end{aligned}$$

$$A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1) = 5s^2 - 15s - 11 \dots (1)$$

Put $s = -1$ in (1)

$$A(-27) = 9$$

$$A = \frac{9}{-27} \Rightarrow A = \frac{-1}{3}$$

Put $s = 2$ in (1)

$$D(3) = -21$$

$$D = \frac{-21}{3} = -7$$

Equating the coefficient of s^3

$$A + B = 0$$

$$B = -A \Rightarrow B = \frac{1}{3}$$

Put $s = 0$ in (1), we get

$$-8A + 4B - 2C + D = -11$$

$$\frac{8}{3} + \frac{4}{3} - 2C - 7 = -11$$

$$4 - 2C = 7 - 11$$

$$-2C = -8 \Rightarrow C = 4$$

$$\therefore \frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} = \frac{-1}{3(s+1)} + \frac{1}{3(s-2)} + \frac{4}{(s-2)^2} - \frac{7}{(s-2)^3}$$

$$\begin{aligned} L^{-1} \left[\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} \right] &= \frac{-1}{3} L^{-1} \left[\frac{1}{s+1} \right] + \frac{1}{3} L^{-1} \left[\frac{1}{s-2} \right] + 4 L^{-1} \left[\frac{1}{(s-2)^2} \right] - 7 L^{-1} \left[\frac{1}{(s-2)^3} \right] \\ &= \frac{-1}{3} e^{-t} + \frac{1}{3} e^{2t} + 4 e^{2t} L^{-1} \left[\frac{1}{s^2} \right] - 7 e^{2t} L^{-1} \left[\frac{1}{s^3} \right] \end{aligned}$$

$$L^{-1} \left[\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} \right] = \frac{-1}{3} e^{-t} + \frac{1}{3} e^{2t} + 4 e^{2t} L^{-1} \left[\frac{1}{s^2} \right] - 7 e^{2t} \frac{t^2}{2}$$

Example: 5.49 Find the inverse Laplace transform of $\frac{4s+5}{(s+1)(s^2+4)}$

Solution:

$$\begin{aligned} \frac{4s+5}{(s+1)(s^2+4)} &= \frac{A}{s+1} + \frac{Bs+c}{s^2+4} \\ &= \frac{A(s^2+4) + (Bs+c)(s+1)}{(s+1)(s^2+4)} \end{aligned}$$

$$A(s^2 + 4) + (Bs + c)(s + 1) = 4s + 5 \dots \dots (1)$$

Put $s = -1$ in (1)	Equating coefficients of s^2 term in (1)	Put $s = 0$ in (1)	
$A(1 + 4) + 0 = 4(-1) + 5$		$A + B = 0$	$A(4) + C = 5$
$A(5) = 1 \Rightarrow A = \frac{1}{5}$		$B = -A \Rightarrow B = -\frac{1}{5}$	$C = 5 - 4A = 5 - \frac{4}{5}$
			$= \frac{25-4}{5} = \frac{21}{5}$

$$\begin{aligned} \therefore \frac{4s+5}{(s+1)(s^2+4)} &= \frac{\frac{1}{5}}{s+1} + \frac{-\frac{1}{5}s + \frac{21}{5}}{s^2+4} \\ &= \frac{1}{5(s+1)} - \frac{s}{5(s^2+4)} + \frac{21}{5} \frac{1}{(s^2+4)} \end{aligned}$$

$$\begin{aligned} L^{-1} \left[\frac{4s+5}{(s+1)(s^2+4)} \right] &= \frac{1}{5} L^{-1} \left[\frac{1}{s+1} \right] - \frac{1}{5} L^{-1} \left[\frac{s}{s^2+4} \right] + \frac{21}{5} L^{-1} \left[\frac{1}{s^2+4} \right] \\ &= \frac{1}{5} e^{-t} - \frac{1}{5} \cos 2t + \frac{21}{5} \frac{\sin 2t}{2} \end{aligned}$$

$$L^{-1} \left[\frac{4s+5}{(s+1)(s^2+4)} \right] = \frac{1}{5} e^{-t} - \frac{1}{5} \cos 2t + \frac{21}{10} \sin 2t$$

Exercise: 5.8

Find the Inverse Laplace transforms using partial fraction for the following

1. $\frac{1}{(s+1)(s+3)}$

Ans: $\frac{1}{2} (e^{-t} - e^{-3t})$

2. $\frac{1}{s(s+1)(s+2)}$

Ans: $\frac{1}{2} (e^{-2t} - 2e^{-t} + 1)$

3. $\frac{54-3s-5}{(s+1)(s^2-3s+2)}$

Ans: $2e^{-t} + 2e^{\frac{3t}{2}} \cosh \frac{t}{2} + 8e^{\frac{3t}{2}} \sinh \frac{t}{2}$

5. 7 INITIAL AND FINAL VALUE THEOREMS

Initial value theorem

Statement: If $L[f(t)] = F(s)$, then $\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$

Proof:

$$\begin{aligned}\text{We know that } L[f'(t)] &= s L[f(t)] - f(0) \\ &= sF(s) - f(0)\end{aligned}$$

$$\begin{aligned}\therefore sF(s) &= L[f'(t)] + f(0) \\ &= \int_0^{\infty} e^{-st} f'(t) dt + f(0)\end{aligned}$$

Taking limit as $s \rightarrow \infty$ on both sides, we have

$$\begin{aligned}\lim_{s \rightarrow \infty} sF(s) &= \lim_{s \rightarrow \infty} \left[\int_0^{\infty} e^{-st} f'(t) dt + f(0) \right] \\ &= \lim_{s \rightarrow \infty} \left[\int_0^{\infty} e^{-st} f'(t) dt \right] + f(0) \\ &= \int_0^{\infty} \lim_{s \rightarrow \infty} [e^{-st} f'(t)] dt + f(0) \\ &= 0 + f(0) \qquad \because e^{-\infty} = 0 \\ &= f(0) \\ &= \lim_{t \rightarrow 0} f(t) \\ \therefore \lim_{s \rightarrow \infty} sF(s) &= \lim_{t \rightarrow 0} f(t)\end{aligned}$$

Final value theorem

Statement: If the Laplace transforms of $f(t)$ and $f'(t)$ exist and $L[f(t)] = F(s)$, then $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$

Proof:

$$\begin{aligned}\text{We know that } L[f'(t)] &= s L[f(t)] - f(0) \\ &= sF(s) - f(0)\end{aligned}$$

$$\begin{aligned}\therefore sF(s) &= L[f'(t)] + f(0) \\ &= \int_0^{\infty} e^{-st} f'(t) dt + f(0)\end{aligned}$$

Taking limit as $s \rightarrow 0$ on both sides, we have

$$\begin{aligned}\lim_{s \rightarrow 0} sF(s) &= \lim_{s \rightarrow 0} \left[\int_0^{\infty} e^{-st} f'(t) dt + f(0) \right] \\ &= \lim_{s \rightarrow 0} \left[\int_0^{\infty} e^{-st} f'(t) dt \right] + f(0) \\ &= \int_0^{\infty} \lim_{s \rightarrow 0} [e^{-st} f'(t)] dt + f(0) \\ &= \int_0^{\infty} f'(t) dt + f(0) \\ &= [f(t)]_0^{\infty} + f(0) \\ &= f(\infty) - f(0) + f(0) \\ &= f(\infty) \\ &= \lim_{t \rightarrow \infty} f(t) \\ \therefore \lim_{t \rightarrow \infty} f(t) &= \lim_{s \rightarrow 0} sF(s)\end{aligned}$$

Example: 5.50 Verify the initial value theorem for the function $f(t) = ae^{-bt}$

Solution:

$$\text{Given } f(t) = ae^{-bt}$$

$$F(s) = L[f(t)]$$

$$= L[ae^{-bt}]$$

$$= a \frac{1}{s+b}$$

$$sF(s) = \frac{as}{s+b}$$

$$\text{Initial value theorem is } \lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

$$\begin{aligned} \lim_{t \rightarrow 0} f(t) &= \lim_{t \rightarrow 0} ae^{-bt} \\ &= a \cdots \cdots \cdots (1) \end{aligned}$$

$$\begin{aligned} \lim_{s \rightarrow \infty} sF(s) &= \lim_{s \rightarrow \infty} \left[\frac{as}{s+b} \right] \\ &= \lim_{s \rightarrow \infty} \left[\frac{as}{s \left(1 + \frac{b}{s} \right)} \right] = \lim_{s \rightarrow \infty} \left[\frac{a}{\left(1 + \frac{b}{s} \right)} \right] \\ &= a \cdots \cdots \cdots (2) \end{aligned}$$

$$\text{From (1) and (2), } \lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

\therefore Initial value theorem is verified

Example: 5.51 Verify the initial value theorem and Final value theorem for the function

$$f(t) = 1 + e^{-t}[\sin t + \cos t].$$

Solution:

$$\text{Given } f(t) = 1 + e^{-t}[\sin t + \cos t]$$

$$F(s) = L[f(t)]$$

$$= L[1 + e^{-t}[\sin t + \cos t]]$$

$$= L[1] + L[e^{-t}[\sin t + \cos t]]$$

$$= L[1] + L[\sin t + \cos t]_{s \rightarrow s+1}$$

$$= \frac{1}{s} + \left[\frac{1}{s^2+1} + \frac{s}{s^2+1} \right]_{s \rightarrow s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$F(s) = \frac{1}{s} + \frac{1}{s^2+2s+2} + \frac{s+1}{s^2+2s+2}$$

$$sF(s) = 1 + \frac{s}{s^2+2s+2} + \frac{s^2+s}{s^2+2s+2}$$

$$\text{Initial value theorem is } \lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

$$\lim_{t \rightarrow 0} f(t) = \lim_{t \rightarrow 0} [1 + e^{-t}[\sin t + \cos t]]$$

$$= 1 + 0 + 1 = 2 \cdots \cdots \cdots (1)$$

$$\lim_{s \rightarrow \infty} sF(s) = \lim_{s \rightarrow \infty} \left[1 + \frac{s}{s^2+2s+2} + \frac{s^2+s}{s^2+2s+2} \right]$$

$$\begin{aligned}
&= 1 + \lim_{s \rightarrow \infty} \left[\frac{1}{s \left(1 + \frac{2}{s} + \frac{2}{s^2}\right)} + \frac{\left(1 + \frac{1}{s}\right)}{\left(1 + \frac{2}{s} + \frac{2}{s^2}\right)} \right] \\
&= 1 + 0 + 1 = 2 \dots \dots \dots (2)
\end{aligned}$$

From (1) and (2), $\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$

\therefore Initial value theorem is verified

Final value theorem is $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$

$$\begin{aligned}
\lim_{t \rightarrow \infty} f(t) &= \lim_{t \rightarrow \infty} (1 + e^{-t} [\sin t + \cos t]) \\
&= 1 + 0 = 1 \dots \dots \dots (3)
\end{aligned}$$

$$\begin{aligned}
\lim_{s \rightarrow 0} sF(s) &= \lim_{s \rightarrow 0} \left[1 + \frac{s}{s^2 + 2s + 2} + \frac{s^2 + s}{s^2 + 2s + 2} \right] \\
&= 1 + 0 + 0 = 1 \dots \dots \dots (4)
\end{aligned}$$

From (3) and (4), $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$

\therefore Final value theorem is verified.

Example: 5.52 Verify the initial value theorem and Final value theorem for the function

$$f(t) = L^{-1} \left[\frac{1}{s(s+2)^2} \right]$$

Solution:

$$\begin{aligned}
\text{Given } f(t) &= L^{-1} \left[\frac{1}{s(s+2)^2} \right] \dots (1) \\
&= \int_0^t L^{-1} \left[\frac{1}{(s+2)^2} \right] dt = \int_0^t e^{-2t} L^{-1} \left[\frac{1}{s^2} \right] dt \\
&= \int_0^t e^{-2t} t dt \\
&= \int_0^t t e^{-2t} dt \\
&= \left[t \left(\frac{e^{-2t}}{-2} \right) - \frac{(1)e^{-2t}}{(-2)^2} \right]_0^t \\
&= -t \frac{e^{-2t}}{2} - \frac{e^{-2t}}{4} - 0 + \frac{1}{4}
\end{aligned}$$

$$\therefore f(t) = \frac{1}{4} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}$$

From (1), $F(s) = \frac{1}{s(s+2)^2}$

$$sF(s) = \frac{1}{(s+2)^2}$$

Initial value theorem is $\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$

$$\begin{aligned}
\lim_{t \rightarrow 0} f(t) &= \lim_{t \rightarrow 0} \left[\frac{1}{4} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4} \right] \\
&= \frac{1}{4} - 0 - \frac{1}{4} = 0
\end{aligned}$$

$$\therefore \lim_{t \rightarrow 0} f(t) = 0 \dots (2)$$

$$\lim_{s \rightarrow \infty} sF(s) = \lim_{s \rightarrow \infty} \frac{1}{(s+2)^2} = 0$$

$$\therefore \lim_{s \rightarrow \infty} sF(s) = 0 \dots (3)$$

From (2) and (3), $\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$

\therefore Initial value theorem is verified

Final value theorem is $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$

$$\begin{aligned}\lim_{t \rightarrow \infty} f(t) &= \lim_{t \rightarrow \infty} \left[\frac{1}{4} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4} \right] \\ &= \frac{1}{4} - 0 - 0 = \frac{1}{4} \dots (4)\end{aligned}$$

$$\begin{aligned}\lim_{s \rightarrow 0} sF(s) &= \lim_{s \rightarrow 0} \left[\frac{1}{(s+2)^2} \right] \\ &= \frac{1}{4} \dots (5)\end{aligned}$$

From (4) and (5), $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$

\therefore Final value theorem is verified

Example: 5.53 Verify the initial value theorem and Final value theorem for the function

$$f(t) = e^{-t}(t+2)^2$$

Solution:

$$\begin{aligned}\text{Given } f(t) &= e^{-t}(t+2)^2 \\ &= e^{-t}(t^2 + 4t + 4)\end{aligned}$$

$$\begin{aligned}F(s) &= L[f(t)] \\ &= L[e^{-t}(t^2 + 4t + 4)] \\ &= L[t^2 + 4t + 4]_{s \rightarrow s+1} \\ &= [L(t^2) + 4L(t) + 4L(1)]_{s \rightarrow s+1} \\ &= \left[\frac{2!}{s^3} + 4 \frac{1}{s^2} + 4 \frac{1}{s} \right]_{s \rightarrow s+1} \\ &= \frac{2}{(s+1)^3} + 4 \frac{1}{(s+1)^2} + 4 \frac{1}{s+1}\end{aligned}$$

$$sF(s) = \frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1}$$

Initial value theorem is $\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$

$$\begin{aligned}\lim_{t \rightarrow 0} f(t) &= \lim_{t \rightarrow 0} [e^{-t}(t^2 + 4t + 4)] \\ &= 4 \dots (1)\end{aligned}$$

$$\begin{aligned}\lim_{s \rightarrow \infty} sF(s) &= \lim_{s \rightarrow \infty} \left[\frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1} \right] \\ &= \lim_{s \rightarrow \infty} \left[\frac{2s}{s^3 \left(1 + \frac{1}{s}\right)^3} + \frac{4s}{s^2 \left(1 + \frac{1}{s}\right)^2} + \frac{4s}{s \left(1 + \frac{1}{s}\right)} \right] \\ &= \lim_{s \rightarrow \infty} \left[\frac{2}{s^2 \left(1 + \frac{1}{s}\right)^3} + \frac{4}{s \left(1 + \frac{1}{s}\right)^2} + \frac{4}{\left(1 + \frac{1}{s}\right)} \right] \\ &= 0 + 0 + 4\end{aligned}$$

$$= 4 \cdots (2)$$

$$\text{From (1) and (2), } \lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

\therefore Initial value theorem is verified

$$\text{Final value theorem is } \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

$$\begin{aligned} \lim_{t \rightarrow \infty} f(t) &= \lim_{t \rightarrow \infty} [e^{-t}(t^2 + 4t + 4)] \\ &= 0 \cdots (3) \end{aligned}$$

$$\begin{aligned} \lim_{s \rightarrow 0} sF(s) &= \lim_{s \rightarrow 0} \left[\frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1} \right] \\ &= 0 \cdots (4) \end{aligned}$$

$$\text{From (3) and (4), } \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

\therefore Final value theorem is verified.

Example: 5.54 If $L[f(t)] = \frac{1}{s(s+1)}$, find the $\lim_{t \rightarrow 0} f(t)$ and $\lim_{t \rightarrow \infty} f(t)$ using initial and final value theorems.

Solution:

$$\text{Given } L[f(t)] = \frac{1}{s(s+1)} \cdots (1)$$

$$\text{i.e., } F(s) = \frac{1}{s(s+1)} \Rightarrow sF(s) = \frac{1}{(s+1)}$$

$$\begin{aligned} \text{Initial value theorem is } \lim_{t \rightarrow 0} f(t) &= \lim_{s \rightarrow \infty} sF(s) \\ &= \lim_{s \rightarrow \infty} \frac{1}{(s+1)} = 0 \end{aligned}$$

$$\begin{aligned} \text{Final value theorem is } \lim_{t \rightarrow \infty} f(t) &= \lim_{s \rightarrow 0} sF(s) \\ &= \lim_{s \rightarrow 0} \frac{1}{(s+1)} = 1 \end{aligned}$$

Exercise: 5.9

1. Verify the initial value theorem for the function $f(t) = e^{-t} \sin t$
2. Verify the initial value theorem for the function $f(t) = \sin^2 t$
3. Verify the initial value theorem for the function $f(t) = 1 + e^{-t} + t^2$
4. Verify the Final value theorem for the function $f(t) = 1 - e^{-at}$
5. Verify the Final value theorem for the function $f(t) = t^2 e^{-3t}$

5.8 CONVOLUTION THEOREM

Definition: Convolution of two functions

The convolution of two functions $f(t)$ and $g(t)$ is denoted by $f(t) * g(t)$ and defined by

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du.$$

State and prove Convolution theorem

Statement: If $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$, then $L[f(t)] * L[g(t)] = F(s)G(s)$

Proof:

We have $f(t) * g(t) = \int_0^t f(u)g(t-u)du$

$$\begin{aligned} L[f(t) * g(t)] &= \int_0^\infty [f(t) * g(t)] e^{-st} dt \\ &= \int_0^\infty \int_0^t f(u)g(t-u)du e^{-st} dt \\ &= \int_0^\infty \int_0^t f(u)g(t-u)e^{-st} du dt \dots (1) \end{aligned}$$

Now we have no change the order of integration.

$$u = 0, u = t; t = 0, t = \infty$$

Change of order is . Draw horizontal strip PQ

At P, $t = u$, At A $u = \infty$

$$\begin{aligned} L[f(t) * g(t)] &= \int_0^\infty \int_u^\infty f(u)g(t-u)e^{-st} dt du \\ &= \int_0^\infty f(u) \left[\int_u^\infty g(t-u)e^{-st} dt \right] du \dots (2) \end{aligned}$$

Put $t - u = x \dots (3)$

$$t = u + x \Rightarrow dt = dx$$

When $t = u$; $(3) \Rightarrow x = 0$

When $t = \infty$; $(3) \Rightarrow x = \infty$

$$\begin{aligned} (2) \Rightarrow L[f(t) * g(t)] &= \int_0^\infty f(u) \left[\int_0^\infty g(x)e^{-s(u+x)} dx \right] du \\ &= \int_0^\infty f(u) \left[\int_0^\infty g(x)e^{-su}e^{-sx} dx \right] du \\ &= \int_0^\infty f(u)e^{-su} du \int_0^\infty g(x)e^{-sx} dx \\ &= L[f(u)]L[g(x)] \end{aligned}$$

$$\therefore L[f(t) * g(t)] = F(s)G(s)$$

Note: Convolution theorem is very useful to compute inverse Laplace transform of product of two terms

Convolution theorem is $L[f(t) * g(t)] = F(s)G(s)$

$$L^{-1}[F(s)G(s)] = f(t) * g(t)$$

$$L^{-1}[F(s)G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

Problems under Convolution theorem

Example: 5.55 Find $L^{-1} \left[\frac{1}{(s+a)(s+b)} \right]$ using convolution theorem.

Solution:

$$\begin{aligned} L^{-1} \left[\frac{1}{(s+a)(s+b)} \right] &= L^{-1} \left[\frac{1}{(s+a)} \right] * L^{-1} \left[\frac{1}{(s+b)} \right] \\ &= e^{-at} * e^{-bt} \\ &= \int_0^t e^{-au} e^{-b(t-u)} du \\ &= e^{-bt} \int_0^t e^{-au} e^{bu} du \\ &= e^{-bt} \int_0^t e^{(b-a)u} du \end{aligned}$$

$$\begin{aligned}
&= e^{-bt} \left[\frac{e^{(b-a)u}}{b-a} \right]_0^t \\
&= \frac{e^{-bt}}{b-a} [e^{(b-a)t} - 1] \\
&= \frac{e^{-bt}}{b-a} [e^{bt-at} - 1] \\
&= \frac{1}{b-a} [e^{-bt+bt-at} - e^{-bt}]
\end{aligned}$$

$$\therefore L^{-1} \left[\frac{1}{(s+a)(s+b)} \right] = \frac{1}{b-a} [e^{-at} - e^{-bt}]$$

Example: 5.56 Find the inverse Laplace transform $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ by using convolution theorem.

Solution:

$$\begin{aligned}
L^{-1} \left[\frac{s^2}{(s^2+a^2)(s^2+b^2)} \right] &= L^{-1} \left[\frac{s}{(s^2+a^2)} \frac{s}{(s^2+b^2)} \right] \\
&= L^{-1} \left[\frac{s}{(s^2+a^2)} \right] * L^{-1} \left[\frac{s}{(s^2+b^2)} \right] \\
&= \cos at * \cos bt \\
&= \int_0^t \cos au \cos b(t-u) du \\
&= \int_0^t \frac{\cos(au+bt-bu) + \cos(au-bt+bu)}{2} du \\
&= \frac{1}{2} \int_0^t (\cos(au+bt-bu) + \cos(au-bt+bu)) du \\
&= \frac{1}{2} \int_0^t [\cos(a-b)u + bt + \cos(a+b)u - bt] du \\
&= \frac{1}{2} \left[\frac{\sin[(a-b)u+bt]}{a-b} + \frac{\sin[(a+b)u+bt]}{a+b} \right]_0^t \\
&= \frac{1}{2} \left[\frac{\sin(at-bt+bt)}{a-b} + \frac{\sin(at-bt+bt)}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b} \right] \\
&= \frac{1}{2} \left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b} \right] \\
&= \frac{1}{2} \left[\frac{(a+b)\sin at + (a-b)\sin at - (a+b)\sin bt + (a-b)\sin bt}{a^2-b^2} \right] \\
&= \frac{1}{2} \left[\frac{2a\sin at - 2b\sin bt}{a^2-b^2} \right] \\
&= \frac{1}{2} \left[\frac{2(a\sin at - b\sin bt)}{a^2-b^2} \right]
\end{aligned}$$

$$\therefore L^{-1} \left[\frac{s^2}{(s^2+a^2)(s^2+b^2)} \right] = \frac{a\sin at - b\sin bt}{a^2-b^2}$$

Example: 5.57 Find the inverse Laplace transform $\frac{1}{(s^2+a^2)(s^2+b^2)}$ by using convolution theorem.

Solution:

$$\begin{aligned}
L^{-1} \left[\frac{1}{(s^2+a^2)(s^2+b^2)} \right] &= L^{-1} \left[\frac{1}{(s^2+a^2)} \frac{1}{(s^2+b^2)} \right] \\
&= L^{-1} \left[\frac{1}{(s^2+a^2)} \right] * L^{-1} \left[\frac{1}{(s^2+b^2)} \right] \\
&= \frac{1}{a} \sin at * \frac{1}{b} \sin bt
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{ab} \int_0^t \sin au \sin b(t-u) du \\
&= \frac{1}{ab} \int_0^t \frac{\cos(au-bt+bu) - \cos(au+bt-bu)}{2} du \\
&= \frac{1}{2ab} \int_0^t (\cos(au-bt+bu) - \cos(au+bt-bu)) du \\
&= \frac{1}{2} \int_0^t [\cos((a+b)u-bt) - \cos((a-b)u+bt)] du \\
&= \frac{1}{2ab} \left[\frac{\sin((a+b)u-bt)}{a+b} - \frac{\sin((a-b)u+bt)}{a-b} \right]_0^t \\
&= \frac{1}{2ab} \left[\frac{\sin(at+bt-bt)}{a+b} - \frac{\sin(at-bt+bt)}{a-b} + \frac{\sin bt}{a+b} + \frac{\sin bt}{a-b} \right] \\
&= \frac{1}{2ab} \left[\frac{\sin at}{a+b} - \frac{\sin at}{a-b} - \frac{\sin bt}{a+b} + \frac{\sin bt}{a-b} \right] \\
&= \frac{1}{2ab} \left[\frac{(a-b)\sin at - (a+b)\sin at + (a-b)\sin bt + (a+b)\sin bt}{a^2-b^2} \right] \\
&= \frac{1}{2ab} \left[\frac{-2b\sin at + 2a\sin bt}{a^2-b^2} \right] \\
&= \frac{1}{2ab} \left[\frac{2(a\sin bt - b\sin at)}{a^2-b^2} \right] \\
\therefore L^{-1} \left[\frac{1}{(s^2+a^2)(s^2+b^2)} \right] &= \frac{a\sin bt - b\sin at}{ab(a^2-b^2)}
\end{aligned}$$

Example: 5.58 Find the inverse Laplace transform $\frac{s}{(s^2+4)(s^2+9)}$ by using convolution theorem.

Solution:

$$\begin{aligned}
L^{-1} \left[\frac{s}{(s^2+4)(s^2+9)} \right] &= L^{-1} \left[\frac{1}{(s^2+4)} \cdot \frac{s}{(s^2+9)} \right] \\
&= L^{-1} \left[\frac{1}{(s^2+4)} \right] * L^{-1} \left[\frac{s}{(s^2+9)} \right] \\
&= \frac{1}{2} \sin 2t * \cos 3t \\
&= \frac{1}{2} \int_0^t \sin 2u \cos 3(t-u) du \\
&= \frac{1}{2} \int_0^t \frac{\sin(2u+3t-3u) + \sin(2u-3t+3u)}{2} du \\
&= \frac{1}{4} \int_0^t [\sin(3t-u) + \sin(5u-3t)] du \\
&= \frac{1}{4} \left[\frac{-\cos(3t-u)}{-1} - \frac{\cos(5u-3t)}{5} \right]_0^t \\
&= \frac{1}{4} \left[\frac{\cos(3t-t)}{1} - \frac{\cos(5t-3t)}{5} - \frac{\cos 3t}{1} + \frac{\cos 3t}{5} \right] \\
&= \frac{1}{4} \left[\cos 2t - \frac{\cos 2t}{5} - \cos 3t + \frac{\cos 3t}{5} \right] \\
&= \frac{1}{4} \left[\frac{5\cos 2t - \cos 2t - 5\cos 3t + \cos 3t}{5} \right] \\
&= \frac{1}{20} [4\cos 2t - 4\cos 3t] \\
\therefore L^{-1} \left[\frac{s}{(s^2+4)(s^2+9)} \right] &= \frac{\cos 2t - \cos 3t}{5}
\end{aligned}$$

Example: 5.59 Find $L^{-1} \left[\frac{s}{(s^2+a^2)^2} \right]$ by using convolution theorem.

Solution:

$$\begin{aligned}
L^{-1} \left[\frac{s}{(s^2+a^2)^2} \right] &= L^{-1} \left[\frac{1}{(s^2+a^2)} \frac{s}{(s^2+a^2)} \right] \\
&= L^{-1} \left[\frac{1}{(s^2+a^2)} \right] * L^{-1} \left[\frac{s}{(s^2+a^2)} \right] \\
&= \frac{1}{a} \sin at * \cos at \\
&= \frac{1}{a} \int_0^t \sin au \cos a(t-u) du \\
&= \frac{1}{a} \int_0^t \frac{\sin(au+at-au) + \sin(au-at+au)}{2} du \\
&= \frac{1}{2a} \int_0^t [\sin at + \sin(2au-at)] du \\
&= \frac{1}{2a} \left[\int_0^t \sin at du + \int_0^t \sin(2au-at) du \right] \\
&= \frac{1}{2a} \left[\sin at \int_0^t du + \int_0^t \sin(2au-at) du \right] \\
&= \frac{1}{2a} \left[\sin at (u)_0^t - \left(\frac{\cos(2au-at)}{2a} \right)_0^t \right] \\
&= \frac{1}{2a} \left[t \sin at - \frac{\cos(2at-at)}{2a} + \frac{\cos at}{2a} \right] \\
&= \frac{1}{2a} \left[t \sin at - \frac{\cos at}{2a} + \frac{\cos at}{2a} \right] \\
&= \frac{1}{2a} t \sin at
\end{aligned}$$

$$\therefore L^{-1} \left[\frac{s}{(s^2+a^2)^2} \right] = \frac{t \sin at}{2a}$$

Example: 5.60 Find $L^{-1} \left[\frac{1}{(s^2+a^2)^2} \right]$ by using convolution theorem.**Solution:**

$$\begin{aligned}
L^{-1} \left[\frac{1}{(s^2+a^2)^2} \right] &= L^{-1} \left[\frac{1}{(s^2+a^2)} \frac{1}{(s^2+a^2)} \right] \\
&= L^{-1} \left[\frac{1}{(s^2+a^2)} \right] * L^{-1} \left[\frac{1}{(s^2+a^2)} \right] \\
&= \frac{1}{a} \sin at * \frac{1}{a} \sin at \\
&= \frac{1}{a^2} \int_0^t \sin au \sin a(t-u) du \\
&= \frac{1}{a^2} \int_0^t \frac{\cos(au-at+au) - \cos(au+at-au)}{2} du \\
&= \frac{1}{2a^2} \int_0^t [\cos(2au-at) - \cos at] du \\
&= \frac{1}{2a^2} \left[\int_0^t \cos(2au-at) du - \int_0^t \cos at du \right] \\
&= \frac{1}{2a^2} \left[\int_0^t \cos(2au-at) du - \cos at \int_0^t du \right] \\
&= \frac{1}{2a^2} \left[\left(\frac{\sin(2au-at)}{2a} \right)_0^t - \cos at (u)_0^t \right] \\
&= \frac{1}{2a^2} \left[\frac{\sin(2at-at)}{2a} - \frac{\sin(-at)}{2a} - t \cos at \right]
\end{aligned}$$

$$= \frac{1}{2a^2} \left[\frac{\sin at}{2a} + \frac{\sin at}{2a} - t \cos at \right]$$

$$= \frac{1}{2a^2} \left[\frac{2 \sin at}{2a} - t \cos at \right]$$

$$\therefore L^{-1} \left[\frac{1}{(s^2+a^2)^2} \right] = \frac{1}{2a^2} \left[\frac{\sin at}{a} - t \cos at \right]$$

Example: 5.61 Find $L^{-1} \left[\frac{s^2}{(s^2+a^2)^2} \right]$ by using convolution theorem.

Solution:

$$L^{-1} \left[\frac{s^2}{(s^2+a^2)^2} \right] = L^{-1} \left[\frac{s}{(s^2+a^2)} \frac{s}{(s^2+a^2)} \right]$$

$$= L^{-1} \left[\frac{s}{(s^2+a^2)} \right] * L^{-1} \left[\frac{s}{(s^2+a^2)} \right]$$

$$= \cos at * \cos at$$

$$= \int_0^t \cos au \cos a(t-u) du$$

$$= \int_0^t \frac{\cos(au+at-au) + \cos(au-at+au)}{2} du$$

$$= \frac{1}{2} \int_0^t [\cos at + \cos(2au-at)] du$$

$$= \frac{1}{2} \left[\int_0^t \cos at du + \int_0^t \cos(2au-at) du \right]$$

$$= \frac{1}{2} \left[\cos at \int_0^t du + \int_0^t \cos(2au-at) du \right]$$

$$= \frac{1}{2} \left[\cos at (u)_0^t + \left(\frac{\sin(2au-at)}{2a} \right)_0^t \right]$$

$$= \frac{1}{2} \left[t \cos at + \frac{\sin(2at-at)}{2a} + \frac{\sin at}{2a} \right]$$

$$= \frac{1}{2} \left[t \cos at + \frac{\sin at}{2a} + \frac{\sin at}{2a} \right]$$

$$= \frac{1}{2} \left[t \cos at + \frac{2 \sin at}{2a} \right]$$

$$\therefore L^{-1} \left[\frac{s^2}{(s^2+a^2)^2} \right] = \frac{1}{2} \left[t \cos at + \frac{\sin at}{a} \right]$$

Example: 5.62 Find $L^{-1} \left[\frac{s^2}{(s^2+4)^2} \right]$ by using convolution theorem.

Solution:

$$L^{-1} \left[\frac{s^2}{(s^2+2^2)^2} \right] = L^{-1} \left[\frac{s}{(s^2+2^2)} \frac{s}{(s^2+2^2)} \right]$$

$$= L^{-1} \left[\frac{s}{(s^2+2^2)} \right] * L^{-1} \left[\frac{s}{(s^2+2^2)} \right]$$

$$= \cos 2t * \cos 2t$$

$$= \int_0^t \cos 2u \cos 2(t-u) du$$

$$= \int_0^t \frac{\cos(2u+2t-2u) + \cos(2u-2t+2u)}{2} du$$

$$= \frac{1}{2} \int_0^t [\cos 2t + \cos(4u-2t)] du$$

$$= \frac{1}{2} \left[\int_0^t \cos 2t du + \int_0^t \cos(4u-2t) du \right]$$

$$\begin{aligned}
&= \frac{1}{2} \left[\cos 2t \int_0^t du + \int_0^t \cos(4u - 2t) du \right] \\
&= \frac{1}{2} \left[\cos 2t (u)_0^t + \left(\frac{\sin(4u - 2t)}{4} \right)_0^t \right] \\
&= \frac{1}{2} \left[t \cos 2t + \frac{\sin(4t - 2t)}{4} - \frac{\sin(-2t)}{4} \right] \\
&= \frac{1}{2} \left[t \cos 2t + \frac{\sin 2t}{4} + \frac{\sin 2t}{4} \right] \\
&= \frac{1}{2} \left[t \cos 2t + \frac{2 \sin 2t}{4} \right] \\
\therefore L^{-1} \left[\frac{s^2}{(s^2 + a^2)^2} \right] &= \frac{1}{2} \left[t \cos 2t + \frac{\sin 2t}{2} \right]
\end{aligned}$$

Example: 5.63 Find $L^{-1} \left[\frac{1}{s(s^2 + 4)} \right]$ by using convolution theorem.

Solution:

$$\begin{aligned}
L^{-1} \left[\frac{1}{s(s^2 + 4)} \right] &= L^{-1} \left[\frac{1}{s} \cdot \frac{1}{s^2 + 4} \right] \\
&= L^{-1} \left[\frac{1}{s} \right] * L^{-1} \left[\frac{1}{s^2 + 4} \right] \\
&= 1 * \frac{\sin 2t}{2} \\
&= \frac{\sin 2t}{2} * 1 \\
&= \int_0^t \frac{\sin 2u}{2} (1) du \\
&= \left[\frac{-\cos 2u}{4} \right]_0^t = \frac{1}{4} (-\cos 2t + 1) \\
&= \frac{1}{4} (1 - \cos 2t)
\end{aligned}$$

Example: 5.64 Find the inverse Laplace transform $\frac{s+2}{(s^2 + 4s + 13)^2}$ by using convolution theorem.

Solution:

$$\begin{aligned}
L^{-1} \left[\frac{s+2}{(s^2 + 4s + 13)^2} \right] &= L^{-1} \left[\frac{s+2}{s^2 + 4s + 13} \cdot \frac{1}{s^2 + 4s + 13} \right] \\
&= L^{-1} \left[\frac{s+2}{s^2 + 4s + 13} \right] * L^{-1} \left[\frac{1}{s^2 + 4s + 13} \right] \\
&= L^{-1} \left[\frac{s+2}{(s+2)^2 + 9} \right] * L^{-1} \left[\frac{1}{(s+2)^2 + 9} \right] \\
&= e^{-2t} L^{-1} \left[\frac{s}{s^2 + 9} \right] * e^{-2t} L^{-1} \left[\frac{1}{s^2 + 9} \right] \\
&= e^{-2t} \cos 3t * \frac{e^{-2t} \sin 3t}{3} \\
&= \int_0^t e^{-2u} \cos 3u e^{-2(t-u)} \frac{\sin 3(t-u)}{3} du \\
&= \int_0^t e^{-2u} \cos 3u e^{-2t+2u} \frac{\sin(3t-3u)}{3} du \\
&= \frac{1}{3} \int_0^t e^{-2u-2t+2u} \cos 3u \sin(3t-3u) du \\
&= \frac{e^{-2t}}{3} \int_0^t \frac{\sin(3u+3t-3u) - \sin(3u-3t+3u)}{2} du
\end{aligned}$$

$$\begin{aligned}
&= \frac{e^{-2t}}{6} \int_0^t [\sin 3t - \sin(6u - 3t)] du \\
&= \frac{e^{-2t}}{6} \left[\int_0^t \sin 3t du - \int_0^t \sin(6u - 3t) du \right] \\
&= \frac{e^{-2t}}{6} \left[\sin 3t \int_0^t du - \int_0^t \sin(6u - 3t) du \right] \\
&= \frac{e^{-2t}}{6} \left[\sin 3t (u)_0^t + \left(\frac{\cos(6u - 3t)}{6} \right)_0^t \right] \\
&= \frac{e^{-2t}}{6} \left[t \sin 3t + \frac{\cos(6t - 3t)}{6} - \frac{\cos(-3t)}{6} \right] \\
&= \frac{e^{-2t}}{6} \left[t \sin 3t + \frac{\cos 3t}{6} - \frac{\cos 3t}{6} \right] \\
&= \frac{e^{-2t}}{6} t \sin 3t
\end{aligned}$$

$$\therefore L^{-1} \left[\frac{s+2}{(s^2+4s+13)^2} \right] = \frac{e^{-2t}}{6} t \sin 3t$$

Example: 5.65 Find the inverse Laplace transform $\frac{1}{(s+1)(s^2+4)}$ by using convolution theorem.

Solution:

$$\begin{aligned}
L^{-1} \left[\frac{1}{(s^2+4)(s+1)} \right] &= L^{-1} \left[\frac{1}{s+1} \cdot \frac{1}{s^2+4} \right] \\
&= L^{-1} \left[\frac{1}{s+1} \right] * L^{-1} \left[\frac{1}{s^2+4} \right] \\
&= e^{-t} * \cos 2t \\
&= \int_0^t e^{-(t-u)} \cos 2u du \\
&= e^{-t} \int_0^t e^u \cos 2u du \\
&= e^{-t} \left[\frac{e^u}{1^2+2^2} (\cos 2u + 2 \sin 2u) \right]_0^t \\
&= \frac{e^{-t}}{5} [e^t (\cos 2t + 2 \sin 2t) - e^0 (\cos 0 - 0)] \\
&= \frac{e^{-t}}{5} [e^t (\cos 2t + 2 \sin 2t) - 1]
\end{aligned}$$

$$\therefore \int e^{at} \cos bt dt = \frac{e^{at}}{a^2 + b^2} (a \cos bt + b \sin bt)$$

$$\therefore L^{-1} \left[\frac{1}{(s^2+4)(s+1)} \right] = \frac{e^{-t}}{5} [e^t (\cos 2t + 2 \sin 2t) - 1]$$

Exercise: 5.10

Find the inverse Laplace transforms using convolution theorem for the following

1. $\frac{1}{s(s^2+1)}$

Ans: $1 - \cos t$

2. $\frac{s}{(s^2+4)^2}$

Ans: $\frac{1}{8} \left[\frac{\sin 2t}{2} - t \cos 2t \right]$

3. $\frac{s^2}{(s^2+4)^2}$

Ans: $\frac{1}{2} \left[t \cos 2t + \frac{\sin 2t}{2} \right]$

4. $\frac{1}{(s+1)(s^2+1)}$

Ans: $\frac{1}{2} [e^{-t} + \sin t - \cos t]$

5. $\frac{1}{(s+1)(s^2+4)}$

Ans: $-\frac{1}{5} e^{-t} + \frac{1}{5} \cos 2t - \frac{1}{10} \sin 2t$

5.9 SOLUTION OF DIFFERENTIAL EQUATION BY LAPLACE TRANSFORM TECHNIQUE

There are so many methods to solve a linear differential equation. If the initial conditions are known, then Laplace transform technique is easier to solve the differential equation. The Laplace transform transforms the differential equation into an algebraic equation.

$$L[y'(t)] = sL[y(t)] - y(0)$$

$$L[y''(t)] = s^2L[y(t)] - sy(0) - y'(0)$$

Problems using Partial Fraction

Example: 5.66 Solve $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2$, given $x = 0$ and $\frac{dx}{dt} = 5$ for $t = 0$ using Laplace transform method.

Solution:

$$\text{Given } x'' - 3x' + 2x = 2; x(0) = 0; x'(0) = 5$$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] - 3L[x'(t)] + 2L[x(t)] = 2L(1)$$

$$[s^2L[x(t)] - sx(0) - x'(0)] - 3[sL[x(t)] - x(0)] + 2L[x(t)] = \frac{2}{s}$$

Substituting $x(0) = 0; x'(0) = 5$

$$[s^2L[x(t)] - 0 - 5] - 3[sL[x(t)] - 0] + 2L[x(t)] = \frac{2}{s}$$

$$s^2L[x(t)] - 3sL[x(t)] + 2L[x(t)] = \frac{2}{s} + 5$$

$$s^2L[x(t)] - 3sL[x(t)] + 2L[x(t)] = \frac{2}{s} + 5$$

$$\text{Put } L[x(t)] = \bar{x}$$

$$s^2\bar{x} - 3s\bar{x} + 2\bar{x} = \frac{2}{s} + 5$$

$$[s^2 - 3s + 2]\bar{x} = \frac{2}{s} + 5$$

$$(s - 1)(s - 2)\bar{x} = \frac{2}{s} + 5$$

$$\bar{x} = \frac{2+5s}{s(s-1)(s-2)}$$

$$\text{Consider } \frac{2+5s}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$\frac{2+5s}{s(s-1)(s-2)} = \frac{A(s-1)(s-2) + Bs(s-2) + Cs(s-1)}{s(s-1)(s-2)}$$

$$A(s-1)(s-2) + Bs(s-2) + Cs(s-1) = 2 + 5s \cdots (1)$$

$$\text{Put } s = 0 \text{ in (1)}$$

$$A(-1)(-2) = 2$$

$$A = 1$$

$$\text{Put } s = 1 \text{ in (1)}$$

$$B(1)(-1) = 7$$

$$B = -7$$

$$\text{Put } s = 2 \text{ in (1)}$$

$$C(2)(1) = 2 + 10$$

$$C = 6$$

$$\frac{2+5s}{s(s-1)(s-2)} = \frac{1}{s} - \frac{7}{s-1} + \frac{6}{s-2}$$

$$\therefore \bar{x} = \frac{1}{s} - 7 \frac{1}{s-1} + 6 \frac{1}{s-2}$$

$$x(t) = L^{-1} \left[\frac{1}{s} \right] - 7L^{-1} \left[\frac{1}{s-1} \right] + 6L^{-1} \left[\frac{1}{s-2} \right]$$

$$x(t) = 1 - 7e^t + 6e^{2t}$$

Example: 5.67 Using Laplace transform solve the differential equation $y'' - 3y' - 4y = 2e^{-t}$, with $y(0) = 1 = y'(0)$.

Solution:

$$\text{Given } y'' - 3y' - 4y = 2e^{-t}; \text{ with } y(0) = 1 = y'(0).$$

Taking Laplace transform on both sides, we get,

$$L[y''(t)] - 3L[y'(t)] - 4L[y(t)] = 2L(e^{-t})$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] - 4L[y(t)] = 2 \frac{1}{s+1}$$

Substituting $y(0) = 1 = y'(0)$.

$$[s^2L[y(t)] - s - 1] - 3[sL[y(t)] - 1] - 4L[y(t)] = \frac{2}{s+1}$$

$$s^2L[y(t)] - s - 1 - 3sL[y(t)] + 3 - 4L[y(t)] = \frac{2}{s+1}$$

$$s^2L[y(t)] - 3sL[y(t)] - 4L[y(t)] = \frac{2}{s+1} + s - 2$$

$$\text{Put } L[y(t)] = \bar{y}$$

$$s^2\bar{y} - 3s\bar{y} - 4\bar{y} = \frac{2}{s+1} + s - 2$$

$$[s^2 - 3s - 4]\bar{y} = \frac{2}{s+1} + s - 2$$

$$[s^2 - 3s - 4]\bar{y} = \frac{2+s(s+1)-2(s+1)}{s+1}$$

$$= \frac{2+s^2+s-2s-2}{s+1}$$

$$(s+1)(s-4)\bar{y} = \frac{s^2-s}{s+1}$$

$$\bar{y} = \frac{s^2-s}{(s+1)(s+1)(s-4)}$$

$$\bar{y} = \frac{s^2-s}{(s+1)^2(s-4)}$$

$$\text{Consider } \frac{s^2-s}{(s+1)^2(s-4)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-4}$$

$$\frac{s^2-s}{(s+1)^2(s-4)} = \frac{A(s+1)(s-4)+B(s-4)+C(s+1)^2}{(s+1)^2(s-4)}$$

$$A(s+1)(s-4) + B(s-4) + C(s+1)^2 = s^2 - s \dots (1)$$

Put $s = -1$ in (1) | Put $s = 4$ in (1) | equating the coefficients of s^2 , we get

$$-5B = 1 + 1 \quad 25C = 16 - 4 \quad A + C = 1 \Rightarrow A = 1 - C \Rightarrow 1 - \frac{12}{25}$$

$$B = \frac{-2}{5} \quad C = \frac{12}{25} \quad A = \frac{13}{25}$$

$$\frac{s^2-s}{(s+1)^2(s-4)} = \frac{25}{25(s+1)} - \frac{2}{5(s+1)^2} + \frac{12}{25(s-4)}$$

$$\therefore \bar{y} = \frac{13}{25(s+1)} - \frac{2}{5(s+1)^2} + \frac{12}{25(s-4)}$$

$$y(t) = \frac{13}{25} L^{-1} \left[\frac{1}{(s+1)} \right] - \frac{2}{5} L^{-1} \left[\frac{1}{(s+1)^2} \right] + \frac{12}{25} L^{-1} \left[\frac{1}{s-4} \right]$$

$$y(t) = \frac{13}{25} e^{-t} - \frac{2}{5} t e^{-t} + \frac{12}{25} e^{4t}$$

Example: 5.68 Solve the differential equation $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{-t}$, with $y(0) = 1$ and $y'(0) = 0$ using

Laplace transform.

Solution:

Given $y'' - 3y' + 2y = e^{-t}$; with $y(0) = 1$ and $y'(0) = 0$.

Taking Laplace transform on both sides, we get,

$$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L(e^{-t})$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = \frac{1}{s+1}$$

Substituting $y(0) = 1$ and $y'(0) = 0$.

$$[s^2L[y(t)] - s - 0] - 3[sL[y(t)] - 1] + 2L[y(t)] = \frac{1}{s+1}$$

$$s^2L[y(t)] - s - 3sL[y(t)] + 3 + 2L[y(t)] = \frac{1}{s+1}$$

$$s^2L[y(t)] - 3sL[y(t)] + 2L[y(t)] = \frac{1}{s+1} + s - 3$$

$$\text{Put } L[y(t)] = \bar{y}$$

$$s^2\bar{y} - 3s\bar{y} + 2\bar{y} = \frac{1}{s+1} + s - 3$$

$$[s^2 - 3s + 2]\bar{y} = \frac{1}{s+1} + s - 3$$

$$[s^2 - 3s + 2]\bar{y} = \frac{1+s(s+1)-3(s+1)}{s+1}$$

$$= \frac{1+s^2+s-3s-3}{s+1}$$

$$(s-1)(s-2)\bar{y} = \frac{s^2-2s-2}{s+1}$$

$$\bar{y} = \frac{s^2-2s-2}{(s+1)(s-1)(s-2)}$$

$$\text{Consider } \frac{s^2-2s-2}{(s+1)(s-1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$\frac{s^2-2s-2}{(s+1)(s-1)(s-2)} = \frac{A(s-1)(s-2)+B(s+1)(s-2)+C(s+1)(s-1)}{(s+1)(s-1)(s-2)}$$

$$A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1) = s^2 - 2s - 2 \dots (1)$$

$$\text{Put } s = -1 \text{ in (1)} \quad \left| \quad \text{put } s = 1 \text{ in (1)} \quad \left| \quad \text{put } s = 2 \text{ in (1)} \right. \right.$$

$$6A = 1 + 2 - 2 \quad \left| \quad -2B = 1 - 4 \quad \left| \quad 3C = 4 - 4 - 2 \right. \right.$$

$$A = \frac{1}{6} \quad \left| \quad B = \frac{3}{2} \quad \left| \quad C = \frac{-2}{3} \right. \right.$$

$$\therefore \frac{s^2-2s-2}{(s+1)(s-1)(s-2)} = \frac{1}{6(s+1)} + \frac{3}{2(s-1)} - \frac{2}{3(s-2)}$$

$$\bar{y} = \frac{1}{6(s+1)} + \frac{3}{2(s-1)} - \frac{2}{3(s-2)}$$

$$y(t) = \frac{1}{6}L^{-1}\left[\frac{1}{(s+1)}\right] + \frac{3}{2}L^{-1}\left[\frac{1}{s-1}\right] - \frac{2}{3}L^{-1}\left[\frac{1}{s-2}\right]$$

$$y(t) = \frac{1}{6}e^{-t} + \frac{3}{2}e^t - \frac{2}{3}e^{2t}$$

Example: 5.69 Using Laplace transform solve the differential equation $y'' + 2y' - 3y = \sin t$, with $y(0) = y'(0) = 0$.

Solution:

Given $y'' + 2y' - 3y = \sin t$ with $y(0) = 0 = y'(0)$.

Taking Laplace transform on both sides, we get,

$$L[y''(t)] + 2L[y'(t)] - 3L[y(t)] = L(\sin t)$$

$$[s^2L[y(t)] - sy(0) - y'(0)] + 2[sL[y(t)] - y(0)] - 3L[y(t)] = \frac{1}{s^2+1}$$

Substituting $y(0) = 0 = y'(0)$.

$$[s^2L[y(t)] - 0 - 0] + 2[sL[y(t)] - 0] - 3L[y(t)] = \frac{1}{s^2+1}$$

$$s^2L[y(t)] + 2sL[y(t)] - 3L[y(t)] = \frac{1}{s^2+1}$$

$$s^2L[y(t)] + 2sL[y(t)] - 3L[y(t)] = \frac{1}{s^2+1}$$

$$\text{Put } L[y(t)] = \bar{y}$$

$$s^2\bar{y} + 2s\bar{y} - 3\bar{y} = \frac{1}{s^2+1}$$

$$[s^2 + 2s - 3]\bar{y} = \frac{1}{s^2+1}$$

$$(s-1)(s+3)\bar{y} = \frac{1}{s^2+1}$$

$$\bar{y} = \frac{1}{(s-1)(s+3)(s^2+1)}$$

$$\text{Consider } \frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$$

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A(s^2+1)(s+3) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3)}{(s-1)(s+3)(s^2+1)}$$

$$A(s^2+1)(s+3) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3) = 1 \dots (1)$$

Put $s = 1$ in (1) Put $s = -3$ in (1) equating the coefficients of s^2 , we get

$$8A = 0 + 1 \quad B(-4)(10) = 1 \quad A + B + C = 0 \Rightarrow C = -A - B = \frac{-1}{8} + \frac{1}{40}$$

$$A = \frac{1}{8} \quad B = \frac{-1}{40} \quad C = \frac{-1}{10}$$

Put $s = 0$ in (1), we get

$$3A - B - 3D = 1 \Rightarrow \frac{3}{8} + \frac{1}{40} - 3D = 1$$

$$3D = \frac{3}{8} + \frac{1}{40} - 1$$

$$3D = \frac{15+1-40}{40} \Rightarrow D = \frac{-24}{40 \times 3} \Rightarrow D = \frac{-1}{5}$$

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{1}{8(s-1)} - \frac{1}{40(s+3)} + \frac{\left(\frac{-1}{10}\right)s - \frac{1}{5}}{s^2+1}$$

$$\therefore \bar{y} = \frac{1}{8(s-1)} - \frac{1}{40(s+3)} - \frac{s}{10(s^2+1)} - \frac{1}{5(s^2+1)}$$

$$y(t) = \frac{1}{8}L^{-1}\left[\frac{1}{(s-1)}\right] - \frac{1}{40}L^{-1}\left[\frac{1}{(s+3)}\right] - \frac{1}{10}L^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{5}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$y(t) = \frac{1}{8}e^t - \frac{1}{40}e^{-3t} - \frac{1}{10}(\cos t - 2\sin t)$$

Example: 5.70 Using Laplace transform solve the differential equation $y'' - 3y' + 2y = 4e^{2t}$, with $y(0) = -3$ and $y'(0) = 5$.

Solution:

Given $y'' - 3y' + 2y = 4e^{2t}$; with $y(0) = -3$ and $y'(0) = 5$.

Taking Laplace transform on both sides, we get,

$$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = 4L(e^{2t})$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = 4\frac{1}{s-2}$$

Substituting $y(0) = -3$ and $y'(0) = 5$.

$$[s^2L[y(t)] + 3s - 5] - 3[sL[y(t)] + 3] + 2L[y(t)] = \frac{4}{s-2}$$

$$s^2L[y(t)] + 3s - 5 - 3sL[y(t)] - 9 + 2L[y(t)] = \frac{4}{s-2}$$

$$s^2L[y(t)] - 3sL[y(t)] + 2L[y(t)] = \frac{4}{s-2} - 3s + 14$$

$$\text{Put } L[y(t)] = \bar{y}$$

$$s^2\bar{y} - 3s\bar{y} + 2\bar{y} = \frac{4}{s-2} - 3s + 14$$

$$[s^2 - 3s + 2]\bar{y} = \frac{4}{s-2} + 14 - 3s$$

$$[s^2 - 3s + 2]\bar{y} = \frac{4 + (14-3s)(s-2)}{s-2}$$

$$(s-1)(s-2)\bar{y} = \frac{4 + (14-3s)(s-2)}{s-2}$$

$$\bar{y} = \frac{4 + (14-3s)(s-2)}{(s-1)(s-2)^2}$$

$$\text{Consider } \frac{4 + (14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$\frac{4 + (14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{A(s-2)^2 + B(s-1)(s-2) + C(s-1)}{(s-1)(s-2)^2}$$

$$A(s-2)^2 + B(s-1)(s-2) + C(s-1) = 4 + (14-3s)(s-2) \dots (1)$$

Put $s = 1$ in (1)

$$A = 4 - 11$$

$$A = -7$$

Put $s = 2$ in (1)

$$C = 4 + 0$$

$$C = 4$$

equating the coefficients of s^2 , we get

$$A + B = -3 \Rightarrow -7 + B = -3$$

$$B = 4$$

$$\frac{4 + (14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

$$\therefore \bar{y} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

$$y(t) = -7L^{-1}\left[\frac{1}{(s-1)}\right] + 4L^{-1}\left[\frac{1}{(s-2)}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2}\right]$$

$$= -7e^t + 4e^{2t} + 4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$y(t) = -7e^t + 4e^{2t} + 4e^{2t}t$$

Example: 5.71 Using Laplace transform solve the differential equation $y'' - 4y' + 8y = e^{2t}$, with $y(0) = 2$ and $y'(0) = -2$.

Solution:

Given $y'' - 4y' + 8y = e^{2t}$; with $y(0) = 2$ and $y'(0) = -2$.

Taking Laplace transform on both sides, we get,

$$L[y''(t)] - 4L[y'(t)] + 8L[y(t)] = L(e^{2t})$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 4[sL[y(t)] - y(0)] + 8L[y(t)] = \frac{1}{s-2}$$

Substituting $y(0) = 2$ and $y'(0) = -2$.

$$[s^2L[y(t)] - 2s + 2] - 4[sL[y(t)] - 2] + 8L[y(t)] = \frac{1}{s-2}$$

$$s^2L[y(t)] - 2s + 2 - 4sL[y(t)] + 8 + 8L[y(t)] = \frac{1}{s-2}$$

$$s^2L[y(t)] - 4sL[y(t)] + 8L[y(t)] = \frac{1}{s-2} + 2s - 10$$

$$\text{Put } L[y(t)] = \bar{y}$$

$$s^2\bar{y} - 4s\bar{y} + 8\bar{y} = \frac{1}{s-2} + 2s - 10$$

$$[s^2 - 4s + 8]\bar{y} = \frac{1}{s-2} + 2s - 10$$

$$[s^2 - 4s + 8]\bar{y} = \frac{1+(2s-10)(s-2)}{s-2}$$

$$\bar{y} = \frac{1+(2s-10)(s-2)}{(s-2)(s^2-4s+8)}$$

$$= \frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]}$$

$$\text{Consider } \frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]} = \frac{A}{s-2} + \frac{B(s-2)+C}{(s-2)^2+4}$$

$$= \frac{A[(s-2)^2+4] + B[(s-2)+C](s-2)}{[s-2][(s-2)^2+4]}$$

$$A[(s-2)^2+4] + B[(s-2)+C](s-2) = 1 + (2s-10)(s-2) \dots (1)$$

Put $s = 2$ in (1) Put $s = 0$ in (1) equating the coefficients of s^2 , we get

$$4A = 1 + 0 \quad 8A + 4B - 2C = 21 \quad A + B = 2 \Rightarrow \frac{1}{4} + B = 2$$

$$A = \frac{1}{4} \quad C = -6 \quad B = \frac{7}{4}$$

$$\frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]} = \frac{\frac{1}{4}}{s-2} + \frac{\frac{7}{4}(s-2)-6}{(s-2)^2+4}$$

$$\therefore \bar{y} = \frac{1}{4(s-2)} + \frac{7}{4} \frac{(s-2)}{(s-2)^2+4} - 6 \frac{1}{(s-2)^2+4}$$

$$y(t) = \frac{1}{4}L^{-1}\left[\frac{1}{(s-2)}\right] + \frac{7}{4}L^{-1}\left[\frac{(s-2)}{(s-2)^2+4}\right] - 6L^{-1}\left[\frac{1}{(s-2)^2+4}\right]$$

$$= \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}L^{-1}\left[\frac{s}{s^2+4}\right] - 6e^{2t}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}\cos 2t - 6e^{2t}\frac{\sin 2t}{2}$$

$$y(t) = \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}\cos 2t - 3e^{2t}\sin 2t$$

Problems without using Partial Fraction

Example: 5.72 Solve using Laplace transform $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$, with $x = 2, \frac{dx}{dt} = -1$ at $t = 0$

Solution:

$$\text{Given } x'' - 2x' + x = e^t; x(0) = 2; x'(0) = -1$$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] - 2L[x'(t)] + L[x(t)] = L(e^t)$$

$$[s^2L[x(t)] - sx(0) - x'(0)] - 2[sL[x(t)] - x(0)] + L[x(t)] = \frac{1}{s-1}$$

Substituting $x(0) = 2; x'(0) = -1$

$$[s^2L[x(t)] - 2s + 1] - 2[sL[x(t)] - 2] + L[x(t)] = \frac{1}{s-1}$$

$$s^2L[x(t)] - 2sL[x(t)] + L[x(t)] = \frac{1}{s-1} + 2s - 5$$

$$s^2L[x(t)] - 2sL[x(t)] + L[x(t)] = \frac{1}{s-1} + 2s - 5$$

$$\text{Put } L[x(t)] = \bar{x}$$

$$s^2\bar{x} - 2s\bar{x} + \bar{x} = \frac{1}{s-1} + 2s - 5$$

$$[s^2 - 2s + 1]\bar{x} = \frac{1}{s-1} + 2s - 5$$

$$(s-1)^2\bar{x} = \frac{1}{s-1} + 2s - 5$$

$$\bar{x} = \frac{1}{(s-1)(s-1)^2} + \frac{2s}{(s-1)^2} - \frac{5}{(s-1)^2}$$

$$x(t) = L^{-1}\left[\frac{1}{(s-1)^3}\right] + 2L^{-1}\left[\frac{s}{(s-1)^2}\right] - 5L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= e^t L^{-1}\left[\frac{1}{s^3}\right] + 2L^{-1}\left[\frac{s-1+1}{(s-1)^2}\right] - 5e^t L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^t \frac{t^2}{2!} + 2L^{-1}\left[\frac{s-1}{(s-1)^2} + \frac{1}{(s-1)^2}\right] - 5e^t t$$

$$= e^t \frac{t^2}{2!} + 2L^{-1}\left[\frac{1}{s-1}\right] + 2L^{-1}\left[\frac{1}{(s-1)^2}\right] - 5e^t t$$

$$= e^t \frac{t^2}{2!} + 2e^t + 2e^t L^{-1}\left[\frac{1}{s^2}\right] - 5e^t t$$

$$= e^t \frac{t^2}{2} + 2e^t + 2e^t t - 5e^t t$$

$$\therefore x = \frac{t^2 e^t}{2} + 2e^t - 3e^t t$$

Example: 5.73 Solve the following differential equation using Laplace transform

$$(D^2 - 2D + 1)y = t^2 e^t \text{ Given } y(0) = 2 \text{ and } Dy(0) = 3$$

Solution:

$$\text{Given } (D^2 - 2D + 1)y = t^2 e^t \text{ with } y(0) = 2 \text{ and } Dy(0) = 3$$

$$\text{ie., } D^2y - 2Dy + y = t^2e^t$$

$$y'' - 2y' + y = t^2e^t \text{ With } y(0) = 2 \text{ and } y'(0) = 3$$

Apply Laplace transform on both sides, we get

$$L[y''(t)] - 2L[y'(t)] + L[y(t)] = L[t^2e^t]$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 2[sL[y(t)] - y(0)] + L[y(t)] = L[t^2]_{s \rightarrow s-1}$$

Substituting $y(0) = 2$ and $y'(0) = 3$.

$$[s^2L[y(t)] - 2s - 3] - 2[sL[y(t)] - 2] + L[y(t)] = \left[\frac{2!}{s^3}\right]_{s \rightarrow s-1}$$

$$s^2L[y(t)] - 2s - 3 - 2sL[y(t)] + 4 + L[y(t)] = \frac{2}{(s-1)^3}$$

$$s^2L[y(t)] - 2sL[y(t)] + L[y(t)] = \frac{2}{(s-1)^3} + 2s - 1$$

$$\text{Put } L[y(t)] = \bar{y}$$

$$s^2\bar{y} - 2s\bar{y} + \bar{y} = \frac{2}{(s-1)^3} + 2s - 1$$

$$[s^2 - 2s + 1]\bar{y} = \frac{2}{(s-1)^3} + 2s - 1$$

$$(s-1)^2\bar{y} = \frac{2}{(s-1)^3} + 2s - 1$$

$$\bar{y} = \frac{2}{(s-1)^5} + \frac{2s}{(s-1)^2} - \frac{1}{(s-1)^2}$$

$$\begin{aligned} y(t) &= L^{-1} \left[\frac{2}{(s-1)^5} \right] + 2L^{-1} \left[\frac{s}{(s-1)^2} \right] - L^{-1} \left[\frac{1}{(s-1)^2} \right] \\ &= 2e^t L^{-1} \left[\frac{1}{s^5} \right] + 2L^{-1} \left[\frac{s-1+1}{(s-1)^2} \right] - e^t L^{-1} \left[\frac{1}{s^2} \right] \\ &= 2e^t \frac{t^4}{4!} + 2L^{-1} \left[\frac{s-1}{(s-1)^2} + \frac{1}{(s-1)^2} \right] - e^t t \\ &= 2e^t \frac{t^4}{24} + 2L^{-1} \left[\frac{1}{s-1} \right] + 2L^{-1} \left[\frac{1}{(s-1)^2} \right] - e^t t \\ &= e^t \frac{t^4}{12} + 2e^t + 2e^t L^{-1} \left[\frac{1}{s^2} \right] - e^t t \\ &= e^t \frac{t^4}{12} + 2e^t + 2e^t t - e^t t \end{aligned}$$

$$\therefore x = \frac{t^4 e^t}{12} + 2e^t + e^t t$$

Example: 5.74 Solve using Laplace transform $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 6t^2e^{-3t}$, given that $y(0) = 0$ and $y'(0) = 0$

Solution:

$$\text{Given } \frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 6t^2e^{-3t} \text{ with } y(0) = 0 \text{ and } y'(0) = 0$$

$$y'' + 6y' + 9y = 6t^2e^{-3t} \text{ With } y(0) = 0 \text{ and } y'(0) = 0$$

Apply Laplace transform on both sides, we get

$$L[y''(t)] + 6L[y'(t)] + 9L[y(t)] = 6L[t^2e^{-3t}]$$

$$[s^2L[y(t)] - sy(0) - y'(0)] + 6[sL[y(t)] - y(0)] + 9L[y(t)] = 6L[t^2]_{s \rightarrow s+3}$$

Substituting $y(0) = 0$ and $y'(0) = 0$.

$$[s^2L[y(t)] - 0 - 0] + 6[sL[y(t)] - 0] + 9L[y(t)] = 6 \left[\frac{2!}{s^3} \right]_{s \rightarrow s+3}$$

$$s^2L[y(t)] + 6sL[y(t)] + 9L[y(t)] = \frac{12}{(s+3)^3}$$

$$s^2L[y(t)] + 6sL[y(t)] + 9L[y(t)] = \frac{12}{(s+3)^3}$$

$$\text{Put } L[y(t)] = \bar{y}$$

$$s^2\bar{y} + 6s\bar{y} + 9\bar{y} = \frac{12}{(s+3)^3}$$

$$[s^2 + 6s + 9]\bar{y} = \frac{12}{(s+3)^3}$$

$$(s+3)^2\bar{y} = \frac{12}{(s+3)^3}$$

$$\bar{y} = \frac{12}{(s+3)^5}$$

$$y(t) = L^{-1} \left[\frac{12}{(s+3)^5} \right] = 12e^{-3t} L^{-1} \left[\frac{1}{s^5} \right]$$

$$= 12e^{-3t} \frac{t^4}{4!}$$

$$\therefore y = \frac{t^4 e^{-3t}}{2}$$

Example: 5.75 Solve $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = e^{-t} \sin t$; $x(0) = 0$ and $x'(0) = 1$

Solution:

$$\text{Given } x'' + 2x' + 5x = e^{-t} \sin t; x(0) = 0; x'(0) = 1$$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] + 2L[x'(t)] + 5L[x(t)] = L[e^{-t} \sin t]$$

$$[s^2L[x(t)] - sx(0) - x'(0)] + 2[sL[x(t)] - x(0)] + 5L[x(t)] = L[\sin t]_{s \rightarrow s+1}$$

Substituting $x(0) = 0$; $x'(0) = 1$

$$[s^2L[x(t)] - 0 - 1] + 2[sL[x(t)] - 0] + 5L[x(t)] = \left[\frac{1}{s^2+1} \right]_{s \rightarrow s+1}$$

$$s^2L[x(t)] + 2sL[x(t)] + 5L[x(t)] - 1 = \frac{1}{(s+1)^2+1}$$

$$s^2L[x(t)] + 2sL[x(t)] + 5L[x(t)] = \frac{1}{(s+1)^2+1} + 1$$

$$\text{Put } L[x(t)] = \bar{x}$$

$$s^2\bar{x} + 2s\bar{x} + 5\bar{x} = \frac{1}{(s+1)^2+1} + 1$$

$$[s^2 + 2s + 5]\bar{x} = \frac{1}{(s+1)^2+1} + 1$$

$$[s^2 + 2s + 5]\bar{x} = \frac{1}{s^2+2s+2} + 1$$

$$\bar{x} = \frac{1}{(s^2+2s+2)(s^2+2s+5)} + \frac{1}{s^2+2s+5}$$

$$= \frac{1}{5-2} \left[\frac{1}{s^2+2s+2} - \frac{1}{s^2+2s+5} \right] + \frac{1}{s^2+2s+5}$$

$$\begin{aligned} & \frac{1}{(s^2 + ax + b)(s^2 + ax + c)} \\ &= \frac{1}{c-b} \left[\frac{1}{s^2 + ax + b} - \frac{1}{s^2 + ax + c} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3} \left[\frac{1}{s^2+2s+2} - \frac{1}{s^2+2s+5} \right] + \frac{1}{s^2+2s+5} \\
&= \frac{1}{3(s^2+2s+2)} - \frac{1}{3(s^2+2s+5)} + \frac{1}{s^2+2s+5} \\
\bar{x} &= \frac{1}{3(s^2+2s+2)} + \frac{2}{3(s^2+2s+5)} \\
x(t) &= \frac{1}{3} L^{-1} \left[\frac{1}{(s^2+2s+2)} \right] + \frac{2}{3} L^{-1} \left[\frac{1}{(s^2+2s+5)} \right] \\
&= \frac{1}{3} L^{-1} \left[\frac{1}{(s+1)^2+1} \right] + \frac{2}{3} L^{-1} \left[\frac{1}{(s+1)^2+4} \right] \\
&= \frac{1}{3} e^{-t} L^{-1} \left[\frac{1}{s^2+1} \right] + \frac{2}{3} e^{-t} L^{-1} \left[\frac{1}{s^2+4} \right] \\
&= \frac{1}{3} e^{-t} \sin t + \frac{2}{3} e^{-t} \frac{\sin 2t}{2} \\
\therefore x &= \frac{1}{3} e^{-t} [\sin t + \sin 2t]
\end{aligned}$$

Example: 5.76 Solve using Laplace transform $\frac{d^2y}{dt^2} + \frac{dy}{dt} = t^2 + 2t$, given that $y = 4, y' = -2$ when $t = 0$

Solution:

Given $\frac{d^2y}{dt^2} + \frac{dy}{dt} = t^2 + 2t$ with $y(0) = 4$ and $y'(0) = -2$

$y'' + y' = t^2 + 2t$ with $y(0) = 4$ and $y'(0) = -2$

Apply Laplace transform on both sides, we get

$$L[y''(t)] + L[y'(t)] = L(t^2) + L(2t)$$

$$[s^2L[y(t)] - sy(0) - y'(0)] + [sL[y(t)] - y(0)] = \frac{2}{s^3} + 2\frac{1}{s^2}$$

Substituting $y(0) = 4$ and $y'(0) = -2$.

$$[s^2L[y(t)] - 4s + 2] + [sL[y(t)] - 4] = \frac{2}{s^3} + \frac{2}{s^2}$$

$$s^2L[y(t)] + sL[y(t)] - 4s + 2 - 4 = \frac{2+2s}{s^3}$$

$$s^2L[y(t)] + sL[y(t)] = \frac{2(1+s)}{s^3} + 4s + 2$$

$$\text{Put } L[y(t)] = \bar{y}$$

$$s^2\bar{y} + s\bar{y} = \frac{2(1+s)}{s^3} + 2(2s + 1)$$

$$s(s^2 + s)\bar{y} = \frac{2(s+1)}{s^3} + 2(2s + 1)$$

$$s(s + 1)\bar{y} = \frac{2(s+1)}{s^3} + 2(2s + 1)$$

$$\begin{aligned}
\bar{y} &= \frac{2(s+1)}{s^4(s+1)} + \frac{2(2s+1)}{s(s+1)} \\
&= \frac{2}{s^4} + 2 \left[\frac{s+(s+1)}{s(s+1)} \right] \\
&= \frac{2}{s^4} + 2 \left[\frac{s}{s(s+1)} + \frac{s+1}{s(s+1)} \right] \\
&= \frac{2}{s^4} + 2 \left[\frac{1}{s+1} + \frac{1}{s} \right] \\
\bar{y} &= \frac{2}{s^4} + \frac{2}{s+1} + \frac{2}{s}
\end{aligned}$$

$$\begin{aligned}
 y(t) &= 2L^{-1}\left[\frac{2}{s^4}\right] + 2L^{-1}\left[\frac{1}{s+1}\right] + 2L^{-1}\left[\frac{1}{s}\right] \\
 &= 2\frac{t^3}{3!} + 2e^{-t} + 2(1) \\
 \therefore y &= \frac{t^3}{3} + 2e^{-t} + 2
 \end{aligned}$$

Example: 5.77 Solve using Laplace transform $\frac{d^2x}{dt^2} + 9x = \cos 2t$, if $x(0) = 1$; $x\left(\frac{\pi}{2}\right) = -1$

Solution:

$$\text{Given } x'' + 9x = \cos 2t; x(0) = 1; x\left(\frac{\pi}{2}\right) = -1$$

Since $x'(0)$ is not given assume $x'(0) = k$

Taking Laplace transform on both sides, we get,

$$\begin{aligned}
 L[x''(t)] + L[9x(t)] &= L[\cos 2t] \\
 [s^2L[x(t)] - sx(0) - x'(0)] + 9L[x(t)] &= L[\cos 2t]
 \end{aligned}$$

Substituting $x(0) = 1$; $x\left(\frac{\pi}{2}\right) = -1$

$$[s^2L[x(t)] - s - k] + 9L[x(t)] = \frac{s}{s^2+4}$$

$$s^2L[x(t)] + 9L[x(t)] = \frac{s}{s^2+4} + s + k$$

$$[s^2 + 9]L[x(t)] = \frac{s}{s^2+4} + s + k$$

$$\text{Put } L[x(t)] = \bar{x}$$

$$[s^2 + 9]\bar{x} = \frac{s}{s^2+4} + s + k$$

$$\begin{aligned}
 \bar{x} &= \frac{s}{(s^2+9)(s^2+4)} + \frac{s}{s^2+9} + \frac{k}{s^2+9} \\
 &= \frac{s}{9-4} \left[\frac{1}{s^2+4} - \frac{1}{s^2+9} \right] + \frac{s}{s^2+9} + \frac{k}{s^2+9} \\
 &= \frac{s}{5} \left[\frac{1}{s^2+4} - \frac{1}{s^2+9} \right] + \frac{s}{s^2+9} + \frac{k}{s^2+9} \\
 &= \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)} + \frac{s}{s^2+9} + \frac{k}{s^2+9}
 \end{aligned}$$

$$\begin{aligned}
 \bar{x} &= \frac{s}{5(s^2+4)} + \frac{(5s-s)}{5(s^2+9)} + \frac{k}{s^2+9} \\
 &= \frac{1}{5} \frac{s}{s^2+4} + \frac{4}{5} \frac{s}{s^2+9} + \frac{k}{s^2+9} \\
 x(t) &= \frac{1}{5} L^{-1} \left[\frac{s}{s^2+4} \right] + \frac{4}{5} L^{-1} \left[\frac{s}{s^2+9} \right] + k L^{-1} \left[\frac{1}{s^2+9} \right] \\
 &= \frac{1}{5} \cos 2t + \frac{4}{5} \cos 3t + k \frac{\sin 3t}{3} \dots (1)
 \end{aligned}$$

$$\text{Given } x\left(\frac{\pi}{2}\right) = -1$$

$$\text{Put } t = \frac{\pi}{2} \text{ in (1)}$$

$$\begin{aligned}
 (1) \Rightarrow x\left(\frac{\pi}{2}\right) &= \frac{1}{5} \cos \frac{2\pi}{2} + \frac{4}{5} \cos \frac{3\pi}{2} + k \frac{\sin \frac{3\pi}{2}}{3} \\
 -1 &= \frac{1}{5}(-1) + 0 + \frac{k}{3}(-1)
 \end{aligned}$$

$$\begin{aligned}-\frac{k}{3} &= \frac{1}{5} - 1 \Rightarrow -\frac{k}{3} = \frac{-4}{5} \Rightarrow k = \frac{12}{5} \\ &= \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{12}{5}\frac{\sin 3t}{3} \\ \therefore x(t) &= \frac{1}{5}[\cos 2t + 4\cos 3t + 4\sin 3t]\end{aligned}$$

Exercise: 5.11

1. Solve using Laplace transform $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 5y = 5$, given that $y = 0, \frac{dy}{dt} = 2$ when $t = 0$

Ans: $-1 - \frac{1}{6}e^{-5t} + \frac{5}{6}e^t$

2. Using Laplace transform solve the differential equation $y'' + 5y' + 6y = 2$, with

$y(0) = 0 = y'(0)$. Where $y' = \frac{dy}{dt}$ **Ans:** $y(t) = \frac{1}{3} - e^{-2t} + \frac{2}{3}e^{-3t}$

3. Using Laplace transform solve the differential equation $y'' + 4y' + 3y = e^{-t}$, with

$y(0) = 1; y'(0) = 0$. **Ans:** $y(t) = \frac{-1}{4}e^{-3t} - \frac{5}{4}e^{-t} + \frac{1}{2}te^{-t}$

4. Solve using Laplace transform $\frac{d^2y}{dt^2} + y = \sin t$ given $y = 1, \frac{dy}{dt} = 0$ when $t = 0$

Ans: $y(t) = \sin t - t \cos t$

5. Solve using Laplace transform $\frac{d^2y}{dt^2} + 9y = \cos 2t$, if $y(0) = 1; y\left(\frac{\pi}{2}\right) = -1$

Ans: $y(t) = \frac{1}{5}[\cos 2t + 4\cos 3t + 4\sin 3t]$