An attack on Zarankiewicz's problem through SAT solving

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How I came to the problem

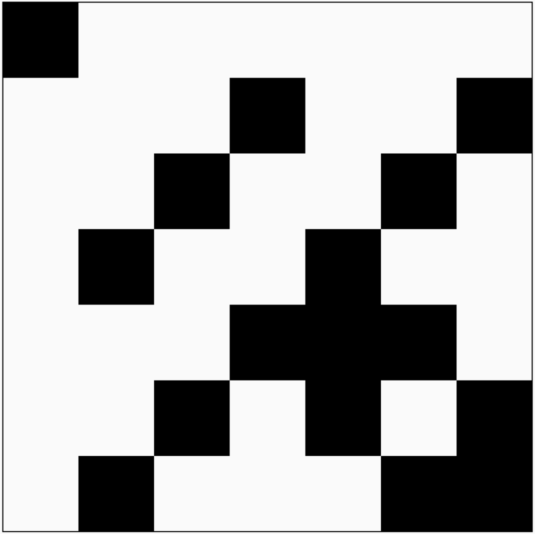
On 20 December 2021 I first saw a question on the Mathematics Stack Exchange (MSE) with an interesting premise:

> *How can 16 squares be shaded in a 7×7 grid so that for* ***any*** *choice of 3 rows and 3 columns, their 9 intersections* ***always*** *contain at least one shaded square (property A)?*

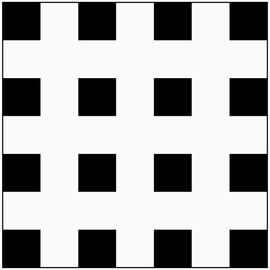
The actual question had lots of fluff and was eventually closed against my wishes. At this point I was unaware that this was a small case of Zarankiewicz's problem, but I had experience with Boolean satisfiability (SAT) solvers from my ATAP at DSO National Laboratories and saw an opportunity to apply them here.

`https://math.stackexchange.com/q/4335395`

Solution



Non-solution



- Here the intersections of 1-indexed rows/columns 2, 4, 6 contain no shaded squares – contiguity is not required.
- It is impossible to shade only 15 (and hence any fewer number of) squares in a 7×7 grid and satisfy property A. In fact the solution on the previous slide is unique up to problem symmetries (more on this later).

How I came to the problem

Using CaDiCaL, the solver I used in my internship, and a (then rudimentary) Python script to generate the necessary files, I computed the fewest number of shaded squares needed to satisfy property A for grid sizes from 3×3 to 11×11:

$$1, 3, 5, 10, 16, 22, 32, 40, 52, \mathbf{64} \ldots$$

This sequence was not in the OEIS *per se*, so I was encouraged to add it, but while my sequence was in the review queue Andrew Howroyd pointed out its inclusion as a column of an existing entry – I saw the link to Zarankiewicz's problem there and realised that the 11×11 term extended yet another sequence for the first time in over 50 years.

How I came to the problem

As soon as my sequence was accepted as A350237 I emailed my supervisors requesting a change of my FYP topic to Zarankiewicz's problem from a rather dry one involving algorithms for medians/centres under permutation distances. I am grateful to them as well as the FYP coordinators for allowing such a change.

Definitions and scope

Zarankiewicz [12] defined his problem in the complementary manner to the MSE post:

> *What is the maximum number $z_{a,b}(m,n)$ of ones an $m \times n$ binary matrix can have if it is admissible, i.e. does not have an all-one $a \times b$ minor?*

As in Guy [2] I suppress indices whenever the matrix or minor is square; in fact my FYP only covers 2×2, 3×3 and 4×4 minors, so for example $\text{A350237}(n) = n^2 - z_3(n)$.

Matrices attaining the maximum number of ones are termed *maximal*. Beyond the values for z I also computed all admissible *square* maximal matrices *up to permuting rows and columns and if applicable transposition*, since clearly an admissible matrix remains admissible under these symmetries.

Arguments

Guy's 1969 paper [2] is still a very valuable reference for Zarankiewicz's problem, complete as it is with hand-computed tables of $z_{a,b}(m,n)$ for $2 \leq a, b \leq 4$ (no $a = b$ restriction) and a few extremely useful "arguments".

Argument A

An admissible matrix with column sums c_i $(1 \leq i \leq n)$ satisfies $\sum_i \binom{c_i}{a} \leq (b-1)\binom{m}{a}$, since otherwise the pigeonhole principle guarantees an all-one $a \times b$ minor.

Argument I

Every minor of every admissible matrix is itself admissible.

Arguments

My SAT solving approach first assumes a value for $z_{a,b}(m, n)$; because of symmetries I only need to consider unordered partitions of the assumed number of ones across rows and across columns (row and column partitions). Arguments A and I usually eliminate a great many partitions from consideration, resulting in far less time needed to generate them through a simple lexicographic algorithm.

Roman's bound

Theorem [7]

For all integers $p \geq a - 1$

$$z_{a,b}(m,n) \leq \left\lfloor \frac{b-1}{\binom{p}{a-1}} \binom{m}{a} + \frac{(p+1)(a-1)}{a} n \right\rfloor$$

and equality holds with $p = a$ or $p = a - 1$ when $\ell(m,a,b) \leq n$, where $\ell(m,a,b) \approx \frac{b-1}{a+1}\binom{m}{a}$ is related to the hypergraph packing formulation of Zarankiewicz's problem.

Roman's bound

My tables for z rely on the equality cases of this theorem – and nothing else! Why?

- Guy's tables are known to contain errors (cf. [4]). One goal of this FYP is to independently verify those tables' entries as far as they extend, and then some.
- The structure of maximal matrices in this part of (m, n)-space can be summarised as "column sums as equal as possible" – there are typically a huge number of solutions even after accounting for symmetries.

SAT formulation

As explained in some other FYP presentations the norm for expressing SAT problems is the conjunctive normal form (CNF), a conjunction (AND) of clauses or disjunctions (ORs) of Boolean variables and their negations. For example

$$(\neg x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_3)$$

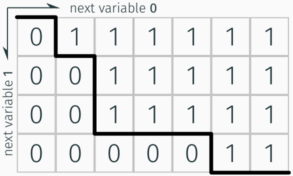
Formulating Zarankiewicz's problem for given parameters $(a, b, m, n$ and assumed $z_{a,b}(m, n))$ as a CNF instance is very simple – the variables are the matrix entries and each minor engenders a clause stating that it has at least one zero entry:

$$\bigwedge_{\text{(all minors)}} \bigvee_{(v \text{ in minor})} \neg v$$

SAT formulation

With these clauses as a base I now follow Marijn Heule's cube-and-conquer paradigm [3], solving possibly several instances* for one parameter set where each instance enforces a row and column partition pair not ruled out by the arguments.

I force a row or column to have exactly its specified number of ones using the equality variation of Sinz's cardinality constraint encoding [9] discussed in Wynn [10], an encoding deemed in the latter reference as fastest and most efficient for general use.

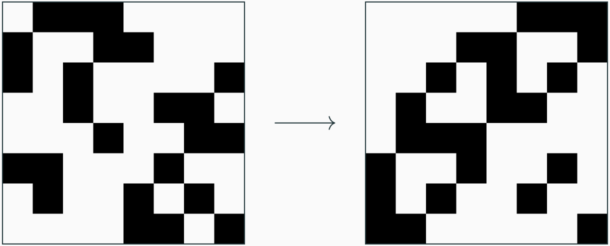


Sinz's encoding enforcing that exactly 4 of 11 variables are true.

*There may be none, which implies that the assumed *z* value is too high without any need for SAT solving.

SAT formulation

To further break the symmetries of Zarankiewicz's problem – reducing the total number of satisfying assignments for each instance while still allowing all non-isomorphic solutions – within the SAT framework I require all columns and all rows with the same sum to be *simultaneously* lexicographically sorted.



SAT formulation

Theorem

Every binary matrix A can be made to satisfy the above property by alternately sorting same-sum rows and same-sum columns a finite number of times. In other words, no generality is lost here.

Proof

The integral resource function $f(A) = \sum_{i=1}^{m} \sum_{j=1}^{n} 2^{i+j-2} a_{ij}$ is clearly bounded and strictly increases or strictly decreases (depending on the sort order) every time two out-of-order rows or columns are swapped.

Software used

This lexicographic constraint eliminates most but not all isomorphic solutions; removing all isomorphs is equivalent to solving the graph isomorphism problem, which is better handled by a dedicated program such as Nauty's ***`shortg`*** [5] than a SAT solver. Nevertheless, the constraint is very cheap to implement in CNF.

For the maximal square matrices, beyond finding all non-isomorphic solutions using ***`shortg`*** and noting their row and column partitions, I used GAP to compute their full automorphism groups with abstract descriptions. Finally, the SAT solver I used throughout my FYP proper was CaDiCaL's successor Kissat.

Results

Table 1: $z_2(n)$ (OEIS A072567). Italics denote new values; subscripts indicate the number of non-isomorphic solutions for that size if greater than 1.

n	1	2	3	4	5	6	7	8
$z_2(n)$	1	3	6	9	12_2	16	21	24_3
n	9	10	11	12	13	14	15	16
$z_2(n)$	29	34	39	45	52	56	*61*	$\mathit{67}_4$
n	17	18	19	20	21	22	23	24
$z_2(n)$	*74*	*81*	*88*	*96*	*105*	$\mathit{108}_{10}$	*115*	*122*

Results

Table 2: $z_3(n)$ (OEIS A350304).

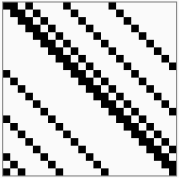
n	1	2	3	4	5	6	7	8
$z_3(n)$	1	4	8	13	20	26	33	42
n	9	10	11	12	13	14	15	16
$z_3(n)$	49_7	60	*69*	$\mathit{80}_2$	*92*	*105*	*120*	*128*

Table 3: $z_4(n)$.

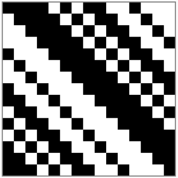
n	4	5	6	7	8	9	10	11	12	13
$z_4(n)$	15	22	31	42	51	$\mathit{61}_9$	*74*	$\mathit{86}_4$	$\mathit{100}_2$	*117*

Results

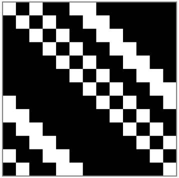
Circulant maximal matrices feature most prominently in the 2×2 minor case, since it is easy to prove that $z_2(q^2+q+1) = (q+1)(q^2+q+1)$ where q is a prime power using a projective plane construction [6], which itself can be rearranged into a circulant matrix [8]. But the motif also appears elsewhere:



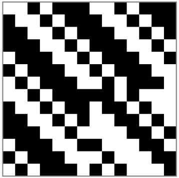
$z_2(23)$



$z_3(15)$



$z_4(13)$



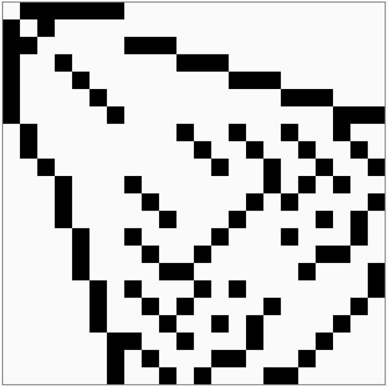
$z_3(14)$

Results

Proving uniqueness of the maximal matrix for $z_2(23)$ required more than just SAT solving; a count using Marc Thurley's sharpSAT revealed exactly $6^6 = 46656$ instance solutions even after imposing the cardinality and lexicographic constraints. For this case I therefore took one maximal matrix and successfully generated 6^6 distinct instance solutions by randomly permuting, then alternately sorting its rows and columns (the arguments had already ruled out all row and column partitions except the most level one, (5^{23}) (5^{23})), thereby showing that all instance solutions were isomorphic to one another.

Results

Maximal matrices for $z_a(m)$ where $m \leq 2a$ are simply described and unique by a paper of Yang [11], but they get extremely complicated as soon as $m > 2a$. *Every* such matrix found, however, is not totally asymmetric, even those without a transpose symmetry (example on right), suggesting that a purely random search like that done in [1] is unlikely to yield maximal matrices.



$z_2(22)$ – has S_4 of order 24 as its automorphism group

Limitations and future work

I did all of the SAT solving on a single laptop computer (my own) using Kissat, which is a single-processor solver. There was thus a "natural" limit of $z \approx 100$ to the range of the table I could complete within reasonable time. My ideas for future work include:

- Taking cube-and-conquer to its fullest potential for Zarankiewicz's problem by further splitting into instances with *partially assigned matrices* and distributing the larger number of instances across many processors (e.g. Charity Engine).
- Exploring other ways to obtain upper and lower bounds on the z-function, such as the "neighbouring theorem" of Collins [1].
- Applying SAT solving to cases with non-square minors – no part of my approach requires square minors.

Links

My full thesis, with tables for $z_{\{2,3,4\}}(m, n)$ and a complete listing of maximal square matrices, has been published on the arXiv at

https://arxiv.org/abs/2203.02283

Supporting code can be found at

https://github.com/Parcly-Taxel/Kyoto

Questions?

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