Random Variables, Sampling, Estimation and Inference

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Outline

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Objectives

- Review basic concepts of Random Variables, Sampling, Estimation and Inference

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- Intiutively understand these concepts with examples
- Be able to use them in a practical problem setting

Definition

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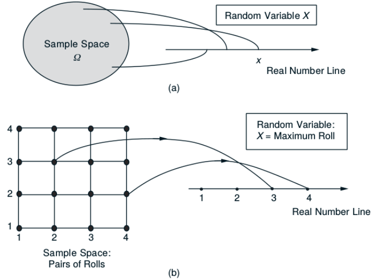

- Intuitively, we can think of rv as a bunch of values from real line
- It is often much easier to deal with summary variables than the actual outcomes

Probability Distribution of RV

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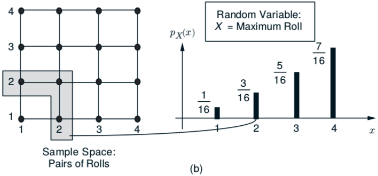

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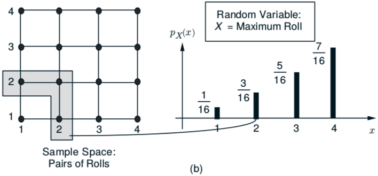

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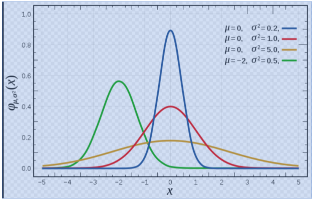
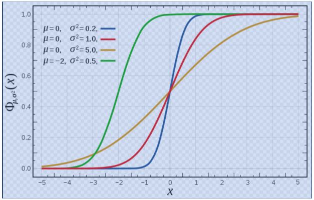
- Example: Distribution of Normal RV





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- Expectation is a linear operation and hence following identities hold
  - $E(g(X)) = \sum_{x \in X} g(x)P(X = x)$
  - $E(aX + bY + c) = aE(X) + bE(Y) + c$

Properties of RVs

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- To quantify extent of linear relation between two RVs X and Y, we define
    - Covariance of X and Y as

    $$Cov(X,Y) = E((X-E(X))(Y-E(Y)))$$

    - Correlation of X and Y as

    $$\rho_{xy} = Cov(X,Y)/\sqrt{Var(X)}\sqrt{Var(Y)}$$

Population, Sample and Realization

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- For example, to understand the effect of education on future employment of individuals in a particular state
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  - data consists of age, education, current employment and other factors of n individuals from the population of interest
- From statistical point of view, we have two sets of RVs in this experiment
  - one corresponding to the whole population called population RVs
  - other corresponds to the sample, sample RVs, because we could have chosen any set of n individuals from population (so sample is itself random)
- After the observation is made, it is called 'realization'of the corresponding RV

Random Sample

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- Alternatively, $X_1, X_2 ... X_n$ are called independent and identically distributed RVs

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- Sample mean can be used as an esimator of population mean,

$$\overline{\mathrm{X}} = (X_1 + X_2 + X_3 ... + X_n)/n$$

- Sample variance is a good estimator of population variance,

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- When we are using a qualifier 'good ', we have to be more precise,
  - this leads us to three properties of estimators - unbiasedness, efficiency and consistency

Unbiasedness and Efficiency

- Mean Square Error (MSE) to evaluate an estimator:
  - MSE of an estimator W of a parameter θ is a function of θ defined as $E_\theta(W-\theta)^2$,

$$E_\theta(W-\theta)^2 = \underbrace{Var_\theta W}_{\text{Variance}} + \underbrace{(E_\theta W-\theta)^2}_{\text{Bias}}$$

  - MSE incorporates to components, one measuring the variability of estimator (precision) and the other its bias (accuracy)
  - An estimator is '**Unbiased**'if its bias term is zero
  - For example, population mean estimator $\overline{X}$ is unbiased since $E\overline{X} = \mu_x$
- An estimator is said to be '**Efficient** 'if its variance component is least among all the estimators with same bias

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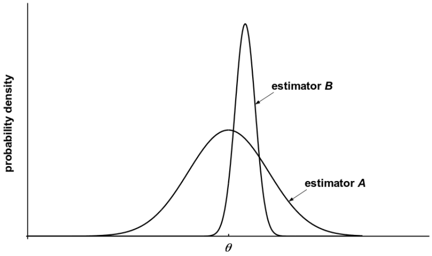

- Unbiasedness and efficiency are finite sample properties of estimators

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  - A sequence of random variables, $X_1, X_2...$, converges in probability to a random variable X if, for every $\epsilon > 0$,

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  - Intuitively, limit implies that for large n 'bulk 'of the distribution of X_n is concentrated near X
- An estimator is said to be consistent if it satisfies these two properties:
  - the estimator possesses probability limit
  - and the limit is the true value of the population parameter

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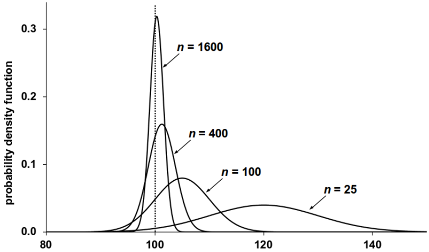

Hypothesis Testing

- It is yet another method of inference like point estimation
- A hypothesis is a statement about the population parameter
- Goal of a hypothesis test is to decide, based on sample from the population, which of two complementary hypothesis (Null and Alternative) is true
- Hypothesis testing procedure is a rule that species for which values of sample do we accept one hypothesis over the other
- Some of the popular Hypothesis testing procedures include LRT, Bayesian tests etc

Hypothesis Testing - Example

Dougherty Slides from Study Guide

Evaluating Hypothesis tests

Dougherty Slides from Study Guide

Hypothesis tests - Confidence Interval approach

Dougherty Slides from Study Guide

"In God we trust, all others bring data."

William Edwards Deming (1900 - 1993).