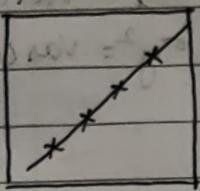


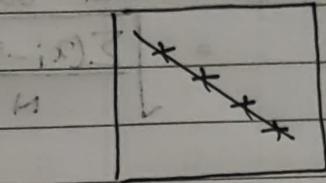
MODULE - 2

Correlation: $\mu = \text{eff of correlation}$ value of r : $-1 \leq r \leq 1$

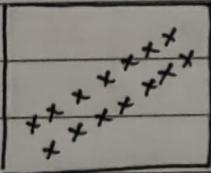
Scatter diagram:



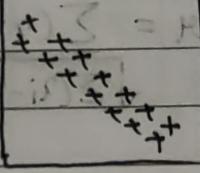
~~Positive~~ Perfect +ve Correlation
 $r = 1$



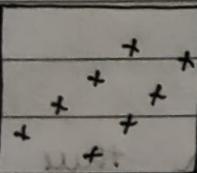
Perfect -ve correlation
 $r = -1$



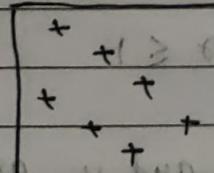
High degree +ve correlation,
 $r = +\text{ve}$



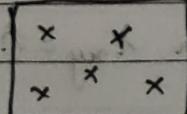
High degree -ve correlation,
 $r = -\text{ve}$



low degree +ve corre



low degree -ve corre

 $r = +\text{ve}$ $r = -\text{ve}$ 

No correlation

 $r = 0$

$\rightarrow r = \text{Karl-Pearson's co-eff of correlation}$:
 $= \frac{\text{cov}(x, y)}{\sigma_x \cdot \sigma_y}$ cov: co-variance

$$= \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{N \cdot \sigma_x \cdot \sigma_y}$$

where σ_x = std deviation of x $\sigma_x^2 = \text{Var}(x)$
 $= \sqrt{\frac{\sum (x_i - \bar{x})^2}{N}}$ $\sigma_y^2 = \text{Var}(y)$

σ_y = std deviation of y

$$= \sqrt{\frac{\sum (y_i - \bar{y})^2}{N}}$$
 no. of observations

OR $r = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2} \sqrt{\sum (y_i - \bar{y})^2}}$

$$OK \quad r = \frac{\sum x_i y_i - N \bar{x} \bar{y}}{\sqrt{\sum x_i^2 - N \bar{x}^2} \sqrt{\sum y_i^2 - N \bar{y}^2}}$$

Properties of r :

1. $-1 \leq r \leq 1$

2. If x and y are independent then there is no correlation between x and y .

3. Correlation co-eff is independent of change of scale and change of origin.

i.e; $u_i = \frac{x_i - a}{h}, v_i = \frac{y_i - b}{h}$ then $r_{xy} = r_{uv}$

Q Calculate correlation co-eff from the given data:

X	23	27	28	29	30	31	33	35	36	39
Y	18	22	23	24	25	26	28	29	30	32

Sol:

X	Y	$U = X - 30$	$V = Y - 25$	U^2	V^2	UV
23	18	-7	-7	49	49	49
27	22	-3	-3	9	9	9
28	23	-2	-2	4	4	4
29	24	-1	-1	1	1	1
30	25	0	0	0	0	0
31	26	1	1	1	1	1
33	28	3	3	9	9	9
35	29	5	4	25	16	20
36	30	6	5	36	25	30
39	32	9	7	81	49	63
		= 11	= 7	= 215	= 163	$\sum UV = 186$

$$N = 10,$$

$$\bar{U} = \frac{\sum U}{N} = 1.1 \quad \text{and } N = 10 \quad \bar{V} = \frac{\sum V}{N} = 0.7 \quad \text{and } N = 10$$

$$\bar{V} = \frac{\sum V}{N} = 0.7 \quad \text{and } N = 10$$

$$r = \frac{\sum UV - N \bar{U} \bar{V}}{\sqrt{\sum U^2 - N \bar{U}^2} \sqrt{\sum V^2 - N \bar{V}^2}}$$

$$= \frac{186 - (10 \times 1.1 \times 0.7)}{\sqrt{215 - 10(1.1)^2} \sqrt{163 - 10(0.7)^2}} = 0.996$$

Answer of question 29 will be in next page

→ Spearman's Rank Correlation co-eff:

$$R = 1 - \frac{6 \sum d^2}{n^3 - n}, \quad -1 \leq R \leq 1$$

where $d = \text{diff between ranks}$

Stud No	Rank in Eng (R_1)	Rank in Maths (R_2)	$d^2 = (R_1 - R_2)^2$
1	10	8	4
2	1	1	0
3	7	4	9
4	5	5	0
5	4	6	4
6	6	9	9
7	2	2	0
8	10	8	4
9	9	10	1
10	8	2	36

$$\sum d^2 = 96, \quad N = 10$$

$$\therefore R = 1 - \frac{6 \sum d^2}{N^3 - N}$$

$$= 1 - \frac{6 \times 96}{(10^3 - 10)} = 0.4181$$

→ For equal ranks:

$$R = 1 - \frac{6}{n^3 - n} \left[\sum d^2 + \frac{1}{12} (m_1^3 - m_1) + \frac{1}{12} (m_2^3 - m_2) + \dots \right]$$

where $m_i = \text{no. of times any item is repeated.}$

Q. Find R for the following data :

$$X \quad R_1 \quad Y \quad R_2 \quad d^2 = (R_1 - R_2)^2$$

$$32 \quad 3 \quad 40 \quad 5 \quad 4$$

$$55 \quad 9 \quad 30 \quad 3.5 \quad 30.25 \quad m_1 = 2,$$

$$49 \quad 7.5 \quad 70 \quad 9 \quad 2.25 \quad m_2 = 2,$$

$$60 \quad 10 \quad 20 \quad 1 \quad 81 \quad m_3 = 2$$

$$43 \quad 5.5 \quad 30 \quad 3.5 \quad 4$$

$$37 \quad 8 \quad 50 \quad 7 \quad 9$$

$$43 \quad 5.5 \quad 72 \quad 10 \quad 20.25$$

$$49 \quad 7.5 \quad 60 \quad 8 \quad 180.25$$

$$10 \quad 1 \quad 45 \quad 6 \quad 25$$

$$20 \quad 2 \quad 25 \quad 2 \quad 0$$

$$\sum d^2 = 176$$

$$R = 1 - \frac{1}{6} \left[\frac{\sum d^2 + \frac{1}{12} (m_1^3 - m_1) + \frac{1}{12} (m_2^3 - m_2) + \frac{1}{12} (m_3^3 - m_3)}{n^3 - n} \right]$$

$$= 1 - \frac{1}{6} \left[176 + \frac{312}{990} \right] = -0.075$$

$$= -0.075$$

$$\textcircled{1} \quad \textcircled{1} \quad \text{Let } r_{xy} = 0.4 \quad (0.1 - 0.4) \times = 0.3 = 0.75$$

$$\text{Cov}(x, y) = 1.6$$

$$\sigma_y^2 = 25 \quad \text{find } \sigma_x$$

$$\textcircled{2} \quad R_{xy} = 0.143$$

$$\sum d^2 = 48 \quad \text{find } n$$

ans ① $r_{xy} = \frac{\text{cov}(x, y)}{\sigma_x \times \sigma_y} = \frac{1.6}{5 \times 6} = 0.4$

$$\therefore \sigma_x = 0.8$$

② $R_{xy} = 1 - \frac{6 \sum d^2}{n^3 n}$

$$0.143 = 1 - \frac{6 \times 48}{n^3 n}$$

$$n^3 n - 336 = 0$$

$$\therefore n = 7, -3.5 \pm 5.98i$$

~~not valid~~

$$\therefore n = 7.$$

Q Find r for : $N=10$, $\sum x = 140$,
 $\sum y = 150$

$$\sum(x-10)^2 = 180$$

$$\sum(y-15)^2 = 215$$

$$\sum(x-10)(y-15) = 60$$

ans: Let $a = x - 10$

$$v = y - 15$$

$$\bar{u} = \frac{\sum u}{N} = \frac{\sum(x-10)}{N} = \frac{\sum x - \sum 10}{N}$$

$$= \frac{\sum x}{N} - \frac{\sum 10}{N}$$

$$= \bar{x} - 10$$

$$= 14 - 10 = 4$$

$$\bar{V} = \frac{\sum V}{N} = \frac{\sum (y - 15)}{N} = \frac{\sum y - \sum 15}{N} \\ = 15 - 15 = 0$$

$$r_{UV} = \frac{\sum UV - N\bar{U}\bar{V}}{\sqrt{\sum U^2 - N\bar{U}^2} \sqrt{\sum V^2 - N\bar{V}^2}}$$

$$= \frac{60 - 0}{\sqrt{180 - 10 \times 15} \sqrt{215 - 10 \times 0}} \\ = 0.91$$

Q A sample of 25 pairs of values of x and y gives the following results:

$$\begin{aligned} \sum x &= 127 & \sum x^2 &= 760 \\ \sum y &= 100 & \sum y^2 &= 449 \\ \sum xy &= 500 \end{aligned}$$

Later on it was found that 2 pairs were taken as $(8, 14)$ and $(8, 6)$ instead of $(8, 12)$ and $(\cancel{6}, 8)$.

Find correct correlation coefficient (r)

ans: correct $\sum x = \text{wrong } \sum x - (8+8) + (6+8) = 125$

correct $\sum y = \text{wrong } \sum y - (14+6) + (12+8) = 100$

correct $\sum x^2 = \text{wrong } \sum x^2 - (8^2+8^2) + (8^2+6^2) = 732$

correct $\sum y^2 = \text{wrong } \sum y^2 - (14^2+6^2) + (12^2+8^2) = 425$

correct $\sum xy = \text{wrong } \sum xy - [(8 \times 14) + (8 \times 6)] + [(8 \times 12) + (6 \times 8)] \\ = 484$

$$\text{correct } \bar{x} = \frac{\sum x}{N} = \frac{125}{25} = 5$$

$$\text{correct } \bar{y} = \frac{\sum y}{N} = \frac{100}{25} = 4$$

$$r = \frac{\sum xy - N\bar{x}\bar{y}}{\sqrt{\sum x^2 - N\bar{x}^2} \sqrt{\sum y^2 - N\bar{y}^2}}$$

$$= -0.3$$

→ Regression lines:

Reg line of y on x : $y - \bar{y} = b_{yx} \text{ byx}(x - \bar{x})$

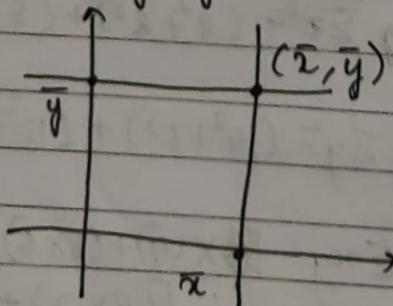
$$\text{where } b_{yx} = \frac{r \sigma_y}{\sigma_x} \quad (\sigma_x^2 = \text{Var}(x))$$

Reg line of x on y : $x - \bar{x} = b_{xy} \text{ byy}(y - \bar{y})$

$$\text{where } b_{xy} = \frac{r \sigma_x}{\sigma_y}$$

b_{xy} & b_{yx} are called regression co-eff which are the slopes of the given 2 lines.

- (i) If $r=0$: $y = \bar{y}$ & $x = \bar{x} \Rightarrow$ these lines are parallel to y & x axis respectively & mutually perpendicular.



(ii) If $r = \pm 1$;

then the regression lines are coincidental or parallel.

$$(iii) b_{xy} = \frac{\sum xy - n\bar{x}\bar{y}}{\sum x^2 - n\bar{x}^2}$$

$$b_{xy} = \frac{\sum xy - n\bar{x}\bar{y}}{\sum y^2 - n\bar{y}^2}$$

$$(iv) r = \pm \sqrt{b_{xy}b_{yx}}$$

(v) If $b_{xy} > 1$ then $b_{yx} < 1$ & vice versa

$$(vi) b_{xy} + b_{yx} > 2$$

(vii) θ is the angle between 2 regression lines then,

$$\tan \theta = \left(\frac{1-r^2}{r} \right) \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

\rightarrow if $r=0$ then $\tan \theta = \infty$ and $\theta = \pi/2$

$$\therefore \theta = \pi/2$$

\rightarrow if $r=\pm 1$ then $\tan \theta = 0$ and $\theta = 0$

$$\therefore \theta = 0$$

(viii) Regression co-eff. are independent of change of origin but not change of scale.

$$\text{i.e., if } u = \frac{x-a}{h}, v = \frac{y-b}{k}$$

$$\text{then } \gamma_{xy} = \gamma_{uv} \text{ but } \sigma_x = h\sigma_u \text{ &}$$

$$\sigma_y = k\sigma_v$$

$$\therefore b_{yx} = \frac{\partial \sigma_y}{\partial x} = \frac{\partial K \sigma_v}{\partial u} = \frac{K}{h} b_{vu}$$

$$\therefore b_{xy} = \frac{h}{K} b_{vu}$$

Q A panel of 2 judges graded the following performances

Perf no.	V(A)	Y(B)	$U = x - \bar{x}$	$V = y - \bar{y}$	UV	U^2
1	36	35	3	2	6	9
2	32	33	-1	0	0	1
3	34	31	1	-2	-2	1
4	31	30	-2	-3	6	4
5	32	34	-1	1	-1	1
6	32	32	-1	-1	1	1
7	34	36	1	3	3	1
	<u>231</u>	<u>231</u>	<u>0</u>	<u>0</u>	<u>13</u>	<u>18</u>

Judge A rewarded 38 marks for the 8th performance while Judge B was absent. How many marks Judge B would have rewarded if he was present?

ans: To find: Regression line Y on x

$$\text{i.e., } y - \bar{y} = b_{yx}(x - \bar{x}) \quad x \leftrightarrow u \quad y \leftrightarrow v$$

but,

$$b_{yx} = \frac{\sum xy - n \bar{x} \bar{y}}{\sum x^2 - n \bar{x}^2} \quad b_{yx} \leftrightarrow b_{vu}$$

$$\bar{x} = \frac{\sum x}{n} = 33 \quad \bar{u} = \frac{\sum u}{n} = 0$$

$$\bar{y} = \frac{\sum y}{n} = 33 \quad \bar{v} = \frac{\sum v}{n} = 0$$

Regnwe have Reg line of V on u :

$$V - \bar{V} = b_{vu} (u - \bar{u})$$

$$\Rightarrow V = 0.72u$$

$$\Rightarrow y - \bar{y} = 0.72(x - \bar{x})$$

$$\Rightarrow y = 33 + 0.72(8x - 33)$$

$$\text{when } x = 38$$

$$\Rightarrow y = 36.6 \approx 37$$

$$\underline{\text{ans: } 37}$$

- Q Estimate marks of student in maths who scored 60 marks in english and marks of student in english who scored 70 marks in maths.

<u>avg:</u>	Maths	Eng	x	y
mean:	80	50	60	$0.72 + 50 = 57.2$
S.D:	15	10	70	$0.61 + 70 = 70.61$

$$\lambda = 0.4$$

(57.2)

<u>ans:</u>	Maths	Eng	x	y
	80 (\bar{x})	50 (\bar{y})		$0.72 + 50 = 57.2$
	15 (s_x)	10 (s_y)		

$$b_{xy} = \lambda \frac{s_x}{s_y} = 0.4 \times \frac{15}{10} = 0.6$$

$$by x = \bar{x} + \frac{b_{xy}}{s_x} (y - \bar{y}) = 0.4 \times \frac{10}{15} = 0.27$$

$$x - \bar{x} = b_{xy} (y - \bar{y})$$

$$\Rightarrow x - 80 = 0.6 (y - 50)$$

$$\text{where } y = 60 \quad \therefore x = 86$$

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 50 = 0.27(x - 80)$$

when $x = 70$

$$y - 50 = 0.27 \times (-10)$$

$$\therefore y = 47.3$$

$=$

$$(x - \bar{x}) \cdot 27 \cdot 0 = 0$$

$$(x - \bar{x}) \cdot 27 \cdot 0 = 0$$

Given: $6y = 5x + 90$ - (1) $\Rightarrow x = 6y - 90$

$$15x = 8y + 130$$
 - (2) $\Rightarrow 3 \cdot 5x = 8y + 130$

$$\therefore x^2 = 16$$

find: (i) $x \& \bar{y}$

to solve (ii), x is taken to column 1
in table (iii), y is taken to column 2
and x is taken to column 3

ans: (i) solving eq (1) and (2) simultaneously:

$$6y = 5x + 90 \Rightarrow x = 30 - \bar{x}$$

$$15x = 8y + 130 \therefore y = 40 - \bar{y}$$

(ii) $r = \sqrt{b_{yx} b_{xy}}$

$$① \& ② \Rightarrow y = \frac{5}{6}x + 15$$

$$x = \frac{8}{15}y + \frac{130}{15}$$

$$\therefore b_{yx} = \frac{5}{6} \& b_{xy} = \frac{8}{15}$$

$$\therefore r = \sqrt{\frac{\frac{5}{6} \times \frac{8}{15}}{\frac{4}{3}}} = \frac{2}{3}$$

$$(iii) \sigma y^2 \Rightarrow byz = r \frac{\sigma y}{\sigma z}$$

$$\frac{5}{8} = \frac{2}{3} \times \frac{\sigma y}{4x}$$

$$\sigma y = 5$$

Q If $\sigma z = \sigma y = 6$ and angle between the 2 lines is $\tan^{-1}(3)$. Find r

ans: $\tan \theta = 3$

$$\Rightarrow 3 = \left(\frac{1-r^2}{r} \right) \frac{\sigma z \sigma y}{\sigma z^2 + \sigma y^2}$$

$$6rxz \cancel{+} \cancel{-} : \quad \cancel{r^2} \times \cancel{r} = 3 \cdot 3$$

$$\Rightarrow 3 = \left(\frac{1-r^2}{r} \right) \frac{\sigma z^2}{2\sigma z}$$

$$\Rightarrow 6r = 1-r^2 \quad (5r-1) = 3 \cdot 3 = 9 \cancel{+}$$

$$\Rightarrow r^2 + 6r - 1 = 0$$

$$\Rightarrow r = -3 \pm \sqrt{10}$$

$$\therefore r = -3 + \sqrt{10}$$

Q If $\bar{x}=5$, $\bar{y}=10$ and line of reg. of y on x is parallel to the line: $20y = 9x + 40$. Find y if $x=30$.

ans: Since, Reg line of y on x is parallel to the given line; the slope of Reg line of y on x = slope of $20y = 9x + 40$

$$\Rightarrow byz = \frac{9}{20}$$

$$\therefore y - \bar{y} = byz (x - \bar{x})$$

$$y - 10 = \frac{9}{20} (x - 5)$$

$$\text{when } x = 30, \quad y = \frac{85}{4} = 21.25$$

Q If the tangent of the angle made by the line of reg of y on x is 0.6 and $6y = 25x$. Find r

ans:

$$\tan \theta = 0.6$$

$$by x = 0.6$$

$$r = by x = r \frac{6y}{6x} = \frac{(1+r)}{r}$$

$$0.6 = r \times \frac{25}{6x} \therefore r = 0.3$$

$$\frac{25}{6x} = \frac{(5r-1)}{r} = 8$$

$$\tan \theta = 0.6 = \left(\frac{1-r^2}{r} \right) \frac{6x \cdot 5r-1}{6x^2 + 5r^2} = 8$$

$$0.6 = \frac{1-r^2}{r} \times \frac{25r^2 - 1}{56x^2} = 8 \quad \text{wrong!!}$$

$$r = 0.5, -2$$

$$\frac{0.5}{x} = 8$$

so $x = 0.5$ and $r = 8$

so $x = 0.5$ and $r = 8$

so $x = 0.5$ and $r = 8$

$$P = xy \in$$

$$(8-x) \text{ and } 8-y \therefore$$

PROBABILITY DISTRIBUTION:

* Random variable :

It is a function which assigns real no. to ^{an} outcome of an experiment.

Ex: x : no. of tails, total no. of outcomes = 2^3

$$S: \{(HHH), (HHT), (HTH), (THH), (HTT), (HTTH), (THT), (TTT)\}$$

$$x: 0 : (HHH) : P(x=0) = 1/8$$

$$: 1 : (HHT), (HTH), (THH) : P(x=1) = 3/8 \quad \sum P_i = 1$$

$$: 2 : (HTT), (TTH), (THT) : P(x=2) = 3/8$$

$$: 3 : (TTT) : P(x=3) = 1/8$$

* Probability distribution of a discrete random variable:

Let X be a discrete random variable taking values

x_1, x_2, x_3, \dots and so on with its probabilities

$P(x_1), P(x_2), P(x_3), \dots$ and so on such that $P(x_i) \geq 0$ &

and $\sum P(x_i) = 1$ then $P(x_i)$ is called Probability

mass function or PMF and $(x_i, P(x_i))$ is called

Probability distribution of a random variable x .

Ex: If 2 unbiased dice are thrown then write the PD of the sum of the no's appearing on the toss of these 2 dice.

Total no. of outcomes = $6^2 = 36$

$$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$$

Let X = sum of the numbers appearing on a roll of 2 dice

$x:$	2	3	4	5	6	7	8	9	10	11	12
$P(X=x_i)$:	$1/36$	$2/36$	$3/36$	$4/36$	$(5/36)$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

Q A Random variable X has following PD:

$x:$	0	1	2	3	4	5	6	7
$P(X=x_i)$:	0	K	$2K$	$3K$	$8K$	K^2	$2K^2$	$7K^2 + K$

Find K

$$(i) P\left(\frac{1.5 < x < 4.5}{x > 2}\right)$$

$$(ii) \text{ smallest } \lambda \text{ st } P(X \leq \lambda) > 1/2$$

ans:

$$(i) \text{ Since } \sum P_i = 1, \text{ then } 0 + K + 2K + 3K + 8K + K^2 + 2K^2 + 7K^2 + K = 1$$

$$\Rightarrow 9K + 10K^2 = 1$$

$$\Rightarrow 10K^2 + 9K - 1 = 0$$

$$\therefore K = 1/10, 0.02 - 1 \text{ (not possible)}$$

$$\checkmark \quad X$$

$$\therefore K = 1/10$$

$$(ii) P\left(\frac{1.5 < x < 4.5}{x > 2}\right)$$

$$= P\left[\left(1.5 < x < 4.5\right) \cap (x > 2)\right] / P(x > 2)$$

$$= P\left[\left.2 < x < 4.5\right]\right] / P(x > 2)$$

$$= \frac{P(X=3,4)}{P(X=3,4,5,6,7)} = \frac{\frac{2}{10} + \frac{3}{10}}{\frac{2}{10} + \frac{3}{10} + \frac{8}{10} + \frac{9}{10}} = \frac{5}{7}$$

$$(iii) \text{ If } \lambda = 4 \text{ then } P(X \leq 4) \\ = P(X=1, 2, 3, 4) \\ = 8/10 = 0.8 > 0.5$$

Q A random variable x takes values: $-2, -1, 0, 1, 2$
such that $P(x>0) = P(x=0) = P(x<0)$
 $P(x=-2) = P(x=-1)$
 $P(x=1) = P(x=2)$

Find: (i) $P\left(\frac{-1 \leq x \leq 1}{-2 \leq x \leq 0}\right)$

(ii) $P\left(\frac{x=1}{0 \leq x \leq 2}\right)$

ans:

$$= P(x=-2) + P(x=-1) + P(x=0) + P(x=1) + P(x=2) = 1$$

$$\underbrace{P(x=-2) + P(x=-1)}_{P(x<0)} + \underbrace{P(x=0) + P(x=1) + P(x=2)}_{P(x>0)} = 1$$

$$= 3P(x=0) = 1$$

$$= P(x=0) = 1/3$$

$$P(x=-2) + P(x=-1) = P(x<0) = 1/3$$

$$P(x=-1) = 1/6 = P(x=-2)$$

$$P(x=1) + P(x=2) = 1/3$$

$$P(x=1) = 1/6 = P(x=2)$$

$$(i) P\left(\frac{-1 \leq x \leq 1}{-2 \leq x \leq 0}\right)$$

$$= \frac{P(-1 \leq x \leq 0)}{P(-2 \leq x \leq 0)} = \frac{\frac{1}{6} + \frac{1}{3}}{\frac{1}{6} + \frac{1}{6} + \frac{1}{3}} = \frac{\frac{1}{6} + \frac{2}{6}}{\frac{1}{6} + \frac{1}{6} + \frac{2}{6}} = \frac{\frac{3}{6}}{\frac{4}{6}} = \frac{3}{4}$$

$$(ii) P\left(\frac{x=1}{0 \leq x \leq 2}\right) = \frac{1/6}{\frac{1}{6} + \frac{1}{6} + \frac{1}{3}} = \frac{1}{4}$$

* The Distribution func of discrete random variable
X. Consider $F(x_i) = P(X \leq x_i)$

$$\text{i.e., } F(x_1) = P(x_1)$$

$$F(x_2) = P(x_1) + P(x_2)$$

$$F(x_n) = \sum_{i=1}^n P(x_i) \rightarrow \text{cdf}$$

where, F is called cumulative distribution function abbreviated for CDF (cdf) and x_i , $F(x_i)$ is called cumulative probability distribution.

$$Q. \quad x: 1, 2, 3, 4$$

$$\text{s.t.: } 2P(x=1) = 3P(x=2) = P(x=3) = 5P(x=4)$$

find Pmf (Probability mass function)

$$\text{ans: let } P(x=3) = K$$

$$\Rightarrow x : 1 = (1-x)^3 = 3! - 2x4! + (1-x)^3$$

$$P(x=i) : \frac{K}{2} \quad \frac{K(8-x)}{3} = \frac{2K}{5} = (1-x)^3$$

$$\therefore \sum P(x_i) = 1 \Rightarrow K = (1-x)^3 = 2! = (1-x)^3$$

$$\frac{K}{2} + \frac{K}{3} + \frac{K}{5} = 1$$

$$15K + 10K + 60K + 12K = 60$$

$$K = \frac{60}{126} = \frac{20}{42}$$

$$\therefore x : 1 \quad 2 \quad 3 \quad 4$$

$$P(x=i) : \frac{15}{61} \quad \frac{10}{61} \quad \frac{30}{61} \quad \frac{6}{61} \rightarrow \text{Pmf}$$

~~Exercises:~~

$$P(X=x_i) : \frac{15}{61} + \frac{10}{61} + \frac{30}{61} + \frac{6}{61} \rightarrow \text{Pmf}$$

$$F(x_i) : \frac{15}{61} + \frac{25}{61} + \frac{55}{61} + \frac{61}{61} \rightarrow \text{cdf}$$

* PDF of continuous random variable X :

Consider a continuous random function $y = f(x)$

s.t: (i) f is integrable

(ii) $f(x) \geq 0$

(iii) $\int_a^b f(x) dx = 1, \quad a \in [a, b]$

★ (iv) $P(\alpha \leq X \leq \beta) = \int_{\alpha}^{\beta} f(x) dx \quad \alpha < \alpha < \beta < b$

then f is called probability density function (pdf) of a random variable X .

Q Continuous R.V X has the following pdf $f(x) = Kx^2$. where $0 \leq x \leq 2$.

Find (i) K ,

(ii) $P(0.2 \leq x \leq 0.5)$

(iii) $P[(x \geq 3/4) / (x \geq 1/2)]$

Ans: (i) Since, $f(x)$ is a pdf,

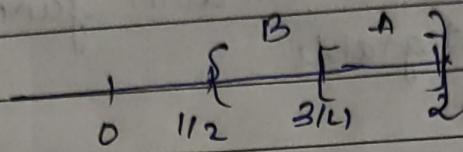
$$\int_0^2 f(x) dx = 1$$

$$\Rightarrow K \left[\frac{x^3}{3} \right]_0^2 = 1 \Rightarrow K = \frac{3}{8}.$$

$$(i) P(0.2 \leq x \leq 0.5) = \frac{3}{8} \int_{0.2}^{0.5} x^2 dx = 0.015$$

(ii) Let $A: x \geq 3/4$
 $B: x \geq 1/2$

$$A = [3/4, 2], B = [1/2, 2]$$



$$\therefore A \subset B \Rightarrow A \cap B = A$$

$$\therefore P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$$

$$\text{but } P(A) = \int_{3/4}^2 \frac{3}{8} x^2 dx = 0.947$$

$$P(B) = \int_{1/2}^2 \frac{3}{8} x^2 dx = 0.984$$

$$\therefore \frac{P(A)}{P(B)} = 0.962$$

* Continuous distribution function (cdf):

X is continuous RV having pdf as $f(x)$ then,
 $F(x)$ is probability of:

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt, \quad -\infty < x < \infty$$

is called cumulative distribution func of RV x .

→ Properties :

- (i) $0 \leq F(x) \leq 1$
- (ii) $x_1 < x_2 \Rightarrow F(x_1) \leq F(x_2)$
- (iii) $F'(x) = f(x)$
- (iv) $P(a \leq x \leq b) = F(b) - F(a)$

Ex: Cdf of continuous RV x is given by,

$$F(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

- find: (i) Pdf
(ii) $P(\frac{1}{2} \leq x \leq 4/5)$

Ans (i) Pdf: $f(x) = \begin{cases} 0, & x < 0 \\ 2x, & 0 \leq x < 1 \\ 0, & x > 1 \end{cases}$

(ii) $P\left(\frac{1}{2} \leq x \leq \frac{4}{5}\right) = F(4/5) - F(1/2)$

$$= \left(\frac{4}{5}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{0.39}{25}$$

or $= \int_{1/2}^{4/5} f(x) dx = \int_{1/2}^{4/5} 2x dx = \frac{0.39}{2}$

Q Find the distribution function corresponding to the following pdf:

$$f(x) = \begin{cases} 1/2x^2e^{-x}, & 0 \leq x < \infty \\ 0, & \text{o.w.} \end{cases}$$

ans:

$$F(x) = \int_0^x f(x) dx$$

$$= \int_0^x \frac{1}{2}x^2e^{-x} dx$$

$$= \frac{1}{2} \left[x^2 \left(\frac{e^{-x}}{-1} \right) - (2x) \left(\frac{e^{-x}}{-1} \right) + (2) \left(\frac{e^{-x}}{-1} \right) \right]_0^x$$

$$= \frac{1}{2} \left[-x^2 e^{-x} - 2x e^{-x} - 2 e^{-x} + 2 \right]_0^x$$

$$F(x) = \begin{cases} 1 - \frac{e^{-x}}{2}(x^2 + 2x + 2), & 0 \leq x < \infty \\ 0, & \text{o.w.} \end{cases}$$

* Expectation and Variance:

Discrete:

(i) If X is a discrete R.V then Expectation is given by

$$\begin{aligned} E(X) &= \sum x_i p_i = \text{mean}(X) \\ &= \bar{x} \end{aligned}$$

(2) Variance of X is :

$$\text{Var}(x) = E(x - \bar{x})^2$$

$$= E(x^2) - [E(x)]^2$$

$$= \sum x_i^2 p_i - [E(x)]^2$$

$$= u'_2 - (u'_1)^2$$

For Continuous Random Variable x' :

$$(1) E(x) = \int_{-\infty}^{+\infty} x \cdot f(x) dx = u'_1$$

$$(2) \text{Var}(x) = \int_{-\infty}^{+\infty} x^2 p dx - [E(x)]^2$$

$$P = f(x)$$

$$= \int_{-\infty}^{+\infty} x^2 f(x) dx - [E(x)]^2$$

Q A fair coin is tossed till head appears. What is the expectation the no. of tosses required.

Ans:

H TH

$x : 1 \quad 2 \quad 3 \quad \dots$

$$P(x=i) : \frac{1}{2} \quad \frac{1 \times \frac{1}{2}}{2} \quad \frac{\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}}{3} \quad \dots$$

$$E(x) \Rightarrow S = \frac{1}{2} + 2 \times \frac{1}{2^2} + 3 \times \frac{1}{2^3} + 4 \times \frac{1}{2^4}$$

$$\frac{1}{2}S = \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} -$$

$$S - \frac{1}{2}S = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$$

$$\frac{1}{2}S = \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \right)$$

$$S = \frac{1}{1 - \frac{1}{2}} = 2$$

$$\therefore E(X) = 2$$

\therefore No. of tosses required $= \underline{\underline{2}}$

Q Find the Expectation of

(i) Sum of

(ii) Product of

the numbers appearing on a throw of n dice

1	2	3	\dots	n
x_1	x_2	x_3	\dots	x_n

Ans:

$$x_1: 1, 2, 3, \dots, n$$

$$(i) E(x_1 + x_2 + x_3 + \dots + x_n) = E(\sum x_i) = \sum E(x_i)$$

$$(ii) E(x_1 \cdot x_2 \cdot x_3 \cdots \cdot x_n) = E(\prod_{i=1}^n x_i) = \prod_{i=1}^n (E(x_i))$$

(i) Let x_i = the number appearing on i th die

$$x_1: 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$$

$$P(x_1 = x_i) \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6}$$

$$\therefore E(X_1) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = \frac{7}{2}$$

$$\therefore E(\sum_{i=1}^n X_i) = \sum_{i=1}^n (E(X_i)) = \sum_{i=1}^n \frac{7}{2} = \frac{n \cdot 7}{2}$$

$$(ii) E\left(\prod_{i=1}^n X_i\right) = \prod_{i=1}^n E(X_i) = \left(\frac{7}{2}\right)^n$$

Q Find the Expectation of the number of failures preceding the first success in an infinite series of independent trials with probabilities P and q of success and failure respectively.

ans:

	p	q/p	$q/q/p$	- - -
$x:$	0	1	2	.

$$P(x=x) : p \quad q/p \quad q^2/p \quad \dots$$

$$\therefore E(x) = q/p + 2q^2/p + 3q^3/p + \dots = q/p(1 + 2q + 3q^2 + \dots)$$

$$= q/p(1-q)^{-2}$$

$$= q/p \times p^{-2} = \frac{q}{p}$$

Q Capital X is a continuous R.V with pdf:
 $f(x) = \begin{cases} Kx^2(1-x^3) & \text{for } 0 \leq x < 1 \\ 0 & \text{o.w.} \end{cases}$

(i) find K

(ii) $P(0 \leq x \leq 1/2)$

(iii) mean & var of x

ans: (i) Since f is a pdf;

$$\int_0^1 f(x) dx = 1$$

$$\Rightarrow K \int_0^1 (x^2 - x^5) dx = 1$$

$$\Rightarrow K \left(\frac{x^3}{3} - \frac{x^6}{6} \right)_0^1 = 1$$

$$\Rightarrow K \left(\frac{1}{3} - \frac{1}{6} \right) = 1$$

$$\Rightarrow K \left(\frac{1}{6} \right) \therefore K = 6 //$$

$$(ii) P(0 \leq x \leq 1/2) = \int_0^{1/2} f(x) dx$$

$$= 6 \int_0^{1/2} (x^2 - x^5) dx$$

$$= 15/64 //$$

(iii) mean = $E(x)$

$$= \int_{-\infty}^{\infty} x f(x) dx$$

$$= 6 \int_0^{\infty} (x^3 - x^6) dx$$

$$= 6 \left(\frac{x^4}{4} - \frac{x^7}{7} \right)_0^{\infty} = 6 \left(\frac{1}{4} - \frac{1}{7} \right) = \frac{9}{14}.$$

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$$\begin{aligned} \text{Var } E(x^2) &= \int_0^1 x^2 f(x) dx \\ &= 6 \int_0^1 (x^4 - x^2) dx \\ &= 6 \left(\frac{x^5}{5} - \frac{x^3}{3} \right)_0^1 = 6 \left(\frac{1}{5} - \frac{1}{8} \right) = \frac{9}{20} \end{aligned}$$

$$\begin{aligned} \text{Var}(x) &= E(x^2) - (E(x))^2 \\ &= \frac{9}{20} - \left(\frac{9}{14} \right)^2 = \frac{9}{245} \text{ //} \end{aligned}$$

Binomial Distribution

Let X be a discrete R.V then, X follows a binomial distribution with parameters n and p .

$$\text{If } P(X=x) = {}^n C_x p^x q^{n-x} \text{ where } q = 1-p$$

→ " X follows a binomial distribution with parameters n and $p" \Rightarrow X \sim B(n, p)$

Mean & Variance

$$\begin{aligned}
 \text{Mean} = E(X) &= \sum_{x=0}^n x_i p_i \\
 &= \sum_{x=0}^n x {}^n C_x p^x q^{n-x} \\
 &= {}^n C_1 p q^{n-1} + 2 {}^n C_2 p^2 q^{n-2} + 3 {}^n C_3 p^3 q^{n-3} \\
 &\quad \dots \dots {}^n C_n p^n q^0 \\
 &= npq^{n-1} + 2 \times \frac{n(n-1)}{2!} p^2 q^{n-2} + \dots \dots np^n \\
 &= np [q^{n-1} + \frac{n(n-1)}{2!} p q^{n-2} + \frac{(n-1)(n-2)}{3!} p^2 q^{n-3} \\
 &\quad + \dots \dots + p^{n-1}] \\
 &= np [{}^n C_0 p^0 q^{n-1} + {}^{n-1} C_1 p^1 q^{(n-1)-1} + \\
 &\quad + {}^{n-1} C_2 p^2 q^{(n-1)-2} + \dots \dots {}^{n-1} C_{n-1} p^{n-1} q^0] \\
 &= np [(p+q)^{n-1}]
 \end{aligned}$$

$$\boxed{E(X) = np}$$

Variance :

$$\text{Var}(x) = E(x^2) - [E(x)]^2 = \sum_{i=1}^n x_i^2 p_i - (\mu)^2 = npq + np^2 - (\mu)^2$$

Now,

$$E(x^2) = \sum_{i=1}^n x_i^2 p_i$$

$$= \sum_{x=0}^n x^2 n c_x p^x q^{n-x} \quad [npq + np^2 - (\mu)^2]$$

$$= \sum_{x=0}^n n c_x p^x q^{n-x} [x + x(x-1)]$$

$$= \sum_{x=0}^n n c_x p^x q^{n-1} + \sum_{x=0}^n x(x-1) n c_x p^x q^{n-2}$$

~~$x = \binom{n}{x}$~~ ~~$x = 0, 1, 2, \dots, n$~~ ~~$x = 0, 1, 2, \dots, n$~~

$$= np + np(n+1) + \dots + x + \dots$$

$$= np + 2 n c_2 p^2 q^{n-2} + 3 n c_3 p^3 q^{n-3} + \dots$$

$$= np + \left[\frac{2(n)(n-1)}{2!} p^2 q^{n-2} + \frac{3(n)(n-1)(n-2)}{3!} p^3 q^{n-3} + \dots \right]$$

$$= np + n(n-1)p^2 [q^{n-2} + (n-2)pq^{n-3} + \dots]$$

$$= np + n(n-1)p^2 [c_{n-2} c_0 p^0 q^{n-2} + c_{n-2} c_1 p^1 q^{(n-2)-1} + \dots]$$

$$= np + n(n-1)p^2 (p+q)^{n-2}$$

$$= np + np(n-1)p^2$$

$$= np + n^2 p^2 - np^2$$

$$= np[1 + np - p] = np[q + np]$$

$$= npq + np^2$$

$$\begin{aligned} \text{Var}(x) &= E(x^2) - [E(x)]^2 \\ &= npq + n^2p^2 - (np)^2 \\ &= npq \end{aligned}$$

$$\boxed{\text{Var}(x) = npq}$$

Properties:

- If $X_1 \sim B(n_1, p_1)$ and $X_2 \sim B(n_2, p_2)$ then:
 $X_1 + X_2 \sim B(n_1 + n_2, p)$

where, $p = p_1 = p_2$

- mode is the value of x which has the maximum probability.

case(i):

If $(n+1)p \in \mathbb{Z}$ say k then there are 2 modes k and $k-1$.

case(ii):

If $(n+1)p \notin \mathbb{Z}$ then mode is an integral part of $(n+1)p$. Ex: $n=16, p=2/3$

$$(n+1)p = 17 \times \frac{2}{3} = \frac{34}{3} \therefore \text{mode} = 11$$

- Q If X is binomially distributed with $E(x)=2$ & $\text{Var}(x)=4/3$. Find the probability distribution of x and p (at least one success).

Ans:

$$E(X) = 2 = np$$

$$\text{Var}(X) = \frac{4}{3} = npq$$

$$\text{we have } \frac{4}{3} = 2q \therefore q = \frac{2}{3}$$

$$\therefore p = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\text{but } np = 2 \therefore \frac{1}{3}n = 2 \therefore n = 6$$

$$\therefore P(X=2) = {}^n C_2 p^2 q^{n-2}$$

$$= {}^6 C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{6-2}$$

$$P(X=2) : \frac{64}{729}, \frac{192}{729}, \frac{240}{729}, \frac{160}{729}, \frac{60}{729}, \frac{12}{729}, \frac{1}{729}$$

$$P(\text{at least one success}) = 1 - P(\text{no success})$$

$$= 1 - \frac{64}{729} = 0.9$$

Poisson:

$$X \sim P(m), \quad P(X=x) = \frac{e^{-m} m^x}{x!}$$

but $m = np$ is finite

$$n \rightarrow \infty$$

$$p \rightarrow 0$$

Uniform Distribution

discrete: $P(X=x) = \frac{1}{n}$

$$P(X=x) = \frac{1}{n} = P(X=x)$$

continuous:

$$P(X=x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

Exponential

continuous: $P(X=x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$

Q Probability that man aged 60 will live upto 70 is 0.65
what is the probability out of 10 such men that at least 7 will live upto 70.

$$n = 10$$

$$P = 0.65$$

$$q = 0.35$$

$X = \text{no. of men who will live upto 70}$

$$\text{To find: } P(X \geq 7) = \sum_{x=7}^{10} {}^{10}_2 C_2 (0.65)^x (0.35)^{10-x} = 0.51$$

Q Ratio of Probability of 3 successes in 5 independent trials to the probability of 2 successes in 5 independent trials is $\frac{1}{4}$. Find the probability of 4 successes in 6 independent trials.

ans: $X = \text{no. of successes}$

$$n=5, X=3 \text{ then, } P(X=3) = {}^5C_3 p^3 q^2 \quad (1)$$

$$n=5, X=2 \text{ then, } P(X=2) = {}^5C_2 p^2 q^3 \quad (2)$$

$$(1) : (2) = 1/4 \quad (\text{given})$$

$$\text{So, } \frac{P(X=3)}{P(X=2)} = \frac{{}^5C_3 p^3 q^2}{{}^5C_2 p^2 q^3} = \frac{1}{4} \quad (1-4q^2 - 2q^2 + (2q+q^2-1)q^2) = 1/4$$

$$= \frac{{}^5C_3 p^3 q^2}{{}^5C_2 p^2 q^3} = \frac{1}{4} \quad \therefore 4p = q \quad \text{but } q = 1-p$$

$\therefore \text{For } n=6;$

$$\therefore q = 4/5$$

$$P(X=4) = ?$$

$$= {}^6C_4 p^4 q^2$$

$$= {}^6C_4 \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^2 = \frac{48}{625} = 0.0768$$

Q A communication has n components, each of which functions independently with probability p . The total system will work efficiently if at least $\frac{1}{2}$ of its components are working. For what value of p , 5 component system will work more efficiently than a 3 component system.

ans

$X = \text{no of components}$

5 component system will work more effectively than a 3 component system if P of 5 component sys is greater than P of 3 component sys.

$$P(5 \text{ comp}) > P(3 \text{ comp})$$

$$\text{if } P(X=3, 4, 5) > P(X=2, 3)$$

$$\text{if } \sum_{x=3}^5 {}^5C_x p^x q^{5-x} > \sum_{x=2}^3 {}^3C_x p^x q^{3-x}$$

$$\text{if } -5c_3 p^3 q^2 + 5c_4 p^4 q + 5c_5 p^5 > 3c_2 p^2 q + 3c_3 p^3$$

$$\text{if, } 10p^3(1-p)^2 + 5p^4(1-p) + p^5 - 3p^2(1-p) \bar{p} p^3 > 0$$

(newip) $\bar{p} = (1-x)/2$

$$\text{if, } 10p^3(1-2p+p^2) + 5p^4 - 5p^5 + p^5 - (3p^2 + 3p^3 - p^3) > 0$$

+ (cancel)

$$\text{if, } 12p^3 - 15p^4 + 6p^5 - 3p^2 > 0$$

$$\text{if, } 3p^2(4p - 5p^2 + 2p^3 - 1) > 0$$

$$\text{if, } 3p^2(p-1)^2(2p-1) > 0$$

$$\text{if, } 2p-1 > 0$$

$$\text{if, } p > \frac{1}{2}$$

Now if this equation has odd powers of p then it is not divisible by 2. If all coefficients are even then sum of even powers of p = sum of odd powers of p .

$$2p^3 - 5p^2 + 4p - 1 = 0$$

$$\begin{array}{r|rrrr} 1 & 2 & -5 & 4 & -1 \\ \hline & 2 & -3 & 1 & \\ \hline & 2 & -3 & 1 & 0 \end{array} \rightarrow \text{sum is 0.}$$

$$\begin{array}{r|rrr} & 2 & -1 & \\ \hline 2 & -1 & 0 & \end{array}$$

\rightarrow linear eqn.

$$\therefore (p-1)^2(2p-1) = 0$$

Q 7 dice are thrown 729 times. How many times do you expect at least 4 dice to show 3 or 5.

ans:

$$n = 7 \text{ dice} \times 729 \text{ times} = 5043$$

$$N = 729 \text{ want for at least 4 dice to show 3 or 5}$$

$$x = \text{no. of dice showing 3 or 5}$$

$$P = \frac{1}{6} + \frac{1}{6} = \frac{1}{3} \quad \therefore P = \frac{2}{3}$$

$$\therefore P(x \geq 4) = \sum_{x=4}^{7} {}^7C_x P^x q^{7-x}$$

$$= \sum_{x=4}^{7} \left(\frac{1}{2}\right)^x \left(\frac{2}{3}\right)^{7-x} \frac{379}{3^7}$$

\therefore In 729 times:

$$= 729 \times \frac{379}{3^7} \approx 126$$

Q Let x and y be 2 independent binomial variates with parameters $n_1 = 6$, $p_1 = 1/2$ and $n_2 = 4$, $p_2 = 1/2$ respectively. Evaluate probability of ~~$x+y=3$~~ $x+y=3$ i.e. $P(x+y=3)$.

Ans: Let, $x+y=z$

then $z \sim B(10, 1/2)$

$$\therefore P(z=3) = {}^{10}C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^7$$

$$= 0.117$$

Also,

$$P(x+y \geq 3) = P(z \geq 3)$$

$$= 1 - [P(z=0) + P(z=1) + P(z=2)]$$

$$\text{Ans} = 1 - \left[10C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{10} + 10C_1 \left(\frac{1}{2}\right) \left(\frac{1}{2}\right)^9 + 10C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^8 \right] \\ = 0.945$$

Q 3 fair coins are tossed 3000 times. Find the frequencies of distribution of heads and tails and tabulate the results. Also find mean & S.D. of the result.

$$\frac{1}{2} = 9 \therefore \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 9$$

ans:

$$n = 3 \\ N = 3000 \\ P = 1/2$$

$$\therefore P(X=x) = \binom{3}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{3-x}$$

$$X : 0 \quad 1 \quad 2 \quad 3$$

$$P(X=2) = \frac{3!}{2!1!} \times \frac{1}{2^3} \times \frac{1}{2^1} = \frac{3}{16}$$

$$f = NP$$

Binomial distribution is said to be skewed if

mean = NP , $N = n$, standard deviation \sqrt{NPq}

$$\text{SD} = \sqrt{Npq} \quad (N = 3000, p = 1/2) \\ (N = 3000, q = 1/2) \quad \therefore \text{SD} = \sqrt{3000 \times 1/2 \times 1/2} = 39$$

* Fitting of Binomial distribution:

$$P(X=x) = nC_x p^x q^{n-x}$$

Q Seven coins are tossed and no. of heads obtained are noted. The experiment is repeated 128 times. Fit a binomial distribution if:

(i) coins are unbiased

(ii) nature of the coins is not known

ans. (i) $n = 7$, $N = 128 = 2^7$

$X = \text{no. of heads}$

$P = 1/2 = q$

$P(X=x) = {}^7 C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{7-x} = 88.8 = q^n$

$$P(X=0) = 1/2^7 (2.0)^7 (8.0)^0 = (2-x)q \therefore$$

$$P(X=1) = 7/2^7$$

$$P(X=2) = 21/2^7 \quad \begin{matrix} n \\ 2 \end{matrix} \quad 8 \quad 8 \quad 1 \quad 0 \quad : X$$

$$P(X=3) = 35/2^7$$

$$P(X=4) = 35/2^7 \quad 5.68 \quad 8.28 \quad 2.86 \quad 2.8 \quad 1.1 \quad : q^n$$

$$P(X=5) = 21/2^7 \quad 18 \quad 18 \quad 18 \quad P \quad 1 =$$

$$P(X=6) = 7/2^7$$

$$P(X=7) = 1/2^7$$

The following frequency distribution:

x	0	1	2	3	4	5	6	7	Total
Freq.	7	19	35	30	23	7	1	128.	

$$x: 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$$

Given Freq. $\therefore 1 - 7 = 21 \times 35 = 35, 21, 7, 1 = 5/128$

(ii) the nature of the coin is unknown : p is not 1/2

$$\text{mean} = \bar{x} = \frac{\sum xf_i}{\sum f_i}$$

: if probability is known
basically we use (i)

$$= \frac{0x7 + 1x6 + 2x19 + 3x35 + 4x11 + 5x7}{128}$$

$$= 3.38$$

$$= np$$

$$\Rightarrow p = \frac{3.38}{7} = 0.48$$

$$f_s = f_{S1} = n, F = n \quad (i)$$

$$\text{above } f_{S1} = X$$

$$(p = S1) = 9$$

$$\left(\frac{1}{n}\right)^n \left(\frac{1}{n}\right)^{n-p} : (x=x)9$$

$$\therefore q = 1 - p = 0.52$$

$$\therefore P(X=2) = {}^7C_2 p^2 (0.48)^2 (0.52)^{7-2} = (x=x)9$$

$$F_{S1}F = (1-x)9$$

$$X: 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$$

freq =

$$\begin{aligned} NP &: 1.31 \quad 8.5 \quad 23.5 \quad 36.2 \quad 36.2 \quad 23.5 \quad 8.5 \quad p = 1.31 \\ &= 1 \quad 9 \quad 29 \quad 36 \quad 36 \quad 24 \quad 9 \quad (x=x)9 \\ &= \underline{\underline{140}} \end{aligned}$$

$$F_{S1}F : (2=x)9$$

$$F_{S1}F : (3=x)9$$

Poisson's distribution:

If $n \rightarrow \infty$

$p \rightarrow 0$

but $np = \text{finite}$

$pn: \text{very big}$

$p: \text{very small}$

* Discrete R.V X is said to follow Poisson's distribution with parameter m if $P(X=a) = \frac{e^{-m} m^a}{a!}, a = 0, 1, 2, \dots$

$$\begin{aligned}
 \text{mean}(x) = E(x) &= \sum_{i=0}^{\infty} x_i P(x=i) + (\bar{x}) \cdot 0 = M \text{ a.v.} \\
 &= \sum_{x=0}^{\infty} x \frac{e^{-m} m^x}{x!} \stackrel{m \rightarrow \infty}{\approx} \\
 &= \sum_{x=1}^{\infty} x \frac{e^{-m} m^x}{x(x-1)!} \quad | \quad m = (x) \text{ a.v.} \\
 &= e^{-m} \sum_{x=1}^{\infty} \frac{m^x}{(x-1)!} \quad | \quad \text{soil} \\
 &= e^{-m} m \sum_{x=1}^{\infty} \frac{m^{x-1}}{(x-1)!}
 \end{aligned}$$

mit weiterer analoger Rechnung erhalten $m = \sum x_i P(x=i)$ ist die Mittelwertsregel

oder Mittelwert $\approx e^{-m} m e^m \approx x \dots + x + x = M$

und $x = \sum x_i P(x=i)$ ist die Varianzregel

$E(x) = M$ Mittelwert $\sum x_i P(x=i) = M$ Varianzregel

$$\text{Var}(x) = E(x^2) - [E(x)]^2$$

Jetzt ist der Mittelwertrechnung nach zu zeigen $M = \sum x_i P(x=i)$

$$E(x^2) = \sum_{i=0}^{\infty} x_i^2 P_i \quad (0 \leq x_i \leq n, 0 \leq P_i \leq 1)$$

$$= \sum_{x=0}^{\infty} x^2 \frac{e^{-m} m^x}{x!}$$

$$= \sum_{x=0}^{\infty} (x + x(x-1)) \frac{e^{-m} m^x}{x!} \quad \text{R.V.} = H = M = \text{Mittelw.} \quad \text{und}$$

$$= x! \cdot \frac{e^{-m} m^x}{x!} = \overline{P}_V =$$

$$= \sum_{x=0}^{\infty} \frac{x e^{-m} m^x}{x!} + \sum_{x=0}^{\infty} x(x-1) \frac{e^{-m} m^x}{x!}$$

$$= m + \sum_{x=2}^{\infty} \frac{e^{-m} m^x}{(x-2)!}$$

$$= m + e^{-m} m^2 \sum_{x=2}^{\infty} \frac{m^{x-2}}{(x-2)!}$$

$$= m + e^{-m} m^2 e^m$$

$$\therefore E(x^2) = m + m^2$$

$$\begin{aligned}\therefore \text{Var}(x) &= E(x^2) + [E(x)]^2 \\ &= m + m^2 - m^2 \\ &= m\end{aligned}$$

$$\boxed{\text{Var}(x) = m}$$

Note:

If x_1, x_2, \dots, x_n are poisson variates with parameters m_1, m_2, \dots, m_n , then $y = x_1 + x_2 + \dots + x_n$ is also a poisson variate with parameter $m_1 + m_2 + \dots + m_n$ and $y = x_1 - x_2 - \dots - x_n$ are not poisson variates.

Ex: If mean of a poisson distribution is 4. find $P(m-2\sigma < x < m+2\sigma)$

$$\text{Ans: mean} = m = 4 = \text{Var}$$

$$\sigma = \sqrt{\text{Var}} = \sqrt{4} = 2$$

$$P(m-2\sigma < x < m+2\sigma)$$

$$= P(0 < x < 8)$$

$$\text{But, } P(x=x) = \frac{e^{-4}}{4^x} \frac{x!}{x!}$$

$$\therefore P(0 < x < 8) = \sum_{x=1}^{7} P(x=x)$$

$$= 0.93$$

Q Car Hire has 2 cars. The no. of demands for a car on each day is distributed between +5 as poisson variate with mean 1.5 calculate the proportion that neither car is used and some demand is refused.

ans $X = \text{no. of demands}$

$$m = 1.5$$

$$P(X=x) = \frac{e^{-1.5} (1.5)^x}{x!}$$

$$(i) P(X=0) = e^{-1.5} = 0.22$$

$$\begin{aligned} (ii) P(X > 2) &= 1 - P(X \leq 2) \\ &= 1 - [P(X=0) + P(X=1) + P(X=2)] \\ &= 1 - [0.22 + 0.33 + 0.25] \\ &= 0.19 \approx 0.2 \end{aligned}$$

Q Car Hire has 2 cars. The no. of demands for a car on each day is distributed between +5 as poisson variate with mean 1.5

Calculate the proportion that neither car is used and some demand is refused.

ans $x = \text{no. of demands}$

$$m = 1.5$$

$$P(x=x) = \frac{e^{-1.5}}{x!} (1.5)^x$$

$$(i) P(x=0) = e^{-1.5} = 0.22$$

$$\begin{aligned} (ii) P(x>2) &= 1 - P(x \leq 2) \\ &= 1 - [P(x=0) + P(x=1) + P(x=2)] \\ &= 1 - [0.22 + 0.33 + 0.25] \\ &= 0.19 \approx 0.2 \end{aligned}$$

Q If x_1, x_2, x_3 be three dependent Poisson's variables with parameters $m_1=1, m_2=2, m_3=3$. Find:

$$(i) P[(x_1+x_2+x_3) \geq 3]$$

$$(ii) P[(x_1=1) / (x_1+x_2+x_3)=3]$$

ans: Let $z = x_1 + x_2 + x_3$, with parameter $m_1 + m_2 + m_3 = 6 = m$

$$\therefore P(z=x) = \frac{e^{-m} m^x}{x!} = \frac{e^{-6} 6^x}{x!}$$

$$P(z \geq 3) = 1 - P(z < 3)$$

$$= 1 - P(z=0, 1, 2)$$

$$= 1 - e^{-6} \left[\sum_{z=0}^2 \frac{6^z}{z!} \right]$$

$$= 1 - e^{-6} \left[1 + \frac{6}{1!} + \frac{6^2}{2!} \right] = 0.94$$

$$(ii) P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$P(A \cap B) = P(A)P(B)$

when independent

$$P[(x_1=1) / (x_1+x_2+x_3=3)]$$

$$= \frac{P[(x_1=1) \cap (x_1+x_2+x_3=3)]}{P(x_1+x_2+x_3=3)}$$

$$= \frac{P[(x_1=1) \cap (x_2+x_3=2)]}{P(x_1+x_2+x_3=3)}$$

$$= \frac{P(x_1=1) \times P(x_2+x_3=2)}{P(x_1+x_2+x_3=3)}$$

$$\begin{aligned} & [6-x]9 + [5-x]e^{-1} \times e^{-5} \cdot 5^2 \\ & [28-6x + e^{-1} + 5e^{-5}] = 25 \\ & e^{-6} \cdot 6^3 = 72 \\ & 31 \end{aligned}$$

Q An insurance company found that only ~~10%~~^{0.01} of the population met with an accident every year. If its 1000 policyholders are selected randomly, what is the probability that no more than 2 of its clients meet with an accident next year?

$$X =$$

$$P = 0.01\% = 0.0001$$

$$n = 1000$$

$$m = np$$

$$= 0.1$$

$$P(X=x) = \frac{e^{-m} m^x}{x!} = \frac{e^{-0.1} 0.1^x}{x!}$$

We want $P(X \leq 2) = P(X=0,1,2)$

$$= e^{-0.1} \left[\sum_{x=0}^2 \frac{(0.1)^x}{x!} \right]$$

$$= e^{-0.1} \left[1 + \frac{0.1}{1} + \frac{0.1^2}{2} \right] = 0.999$$

Q Using Poisson's distribution find the approx value of

$$300 C_2 (0.02)^2 (0.98)^{298} + 300 C_3 (0.02)^3 (0.98)^{297}$$

Ans: We observe that they are Probabilities of Binomial distribution.

$$n = 300$$

$$P = 0.02$$

$$q = 0.98$$

$$m = np = 6$$

$$\therefore P(X=x) = \frac{e^{-m} m^x}{x!} = \frac{e^{-6} 6^x}{x!}$$

$$\therefore \text{Given exp} = \frac{e^{-6} 6^2}{2!} + \frac{e^{-6} 6^3}{3!}$$

$$= 0.13438$$

Q In a certain factory turning out blades, there is a chance that 1/500 for a blade to be defective. Blades are supplied in a packet of 10. Calculate approx no. of packets containing

(i) no defective

(ii) 1 defective

(iii) 2 defective blades in a consignment of 10000 packets.

ans: $X = \text{no. of defective blades (in 1 packet)}$

$$P = 1/500$$

$$n = 10$$

$$m = np = 0.02$$

$$P(X=x) = \frac{e^{-m} m^x}{x!} = \frac{e^{-0.02} \times 0.02^x}{x!}$$

$$P(X=0) = e^{-0.02} = 0.98$$

$$P(X=1) = \frac{e^{-0.02}(0.02)^1}{1!} = 0.0196$$

$$P(X=2) = 0.000196$$

$$N = 10,000$$

$$\text{no. of packets having no defective blades} = NP(X=0) = 9800$$

$$\begin{array}{ccccccc} 1 & , & 1 & , & 1 & , & 1 \\ \text{no. of packets having 1 defective blade} & = & NP(X=1) & = & 1960.3 \end{array}$$

$$\begin{array}{ccccccc} 1 & , & 1 & , & 1 & , & 2 \\ \text{no. of packets having 2 defective blades} & = & NP(X=2) & = & 19.6 \end{array}$$

Q Fit a poisson distribution to the following data :

$$x_i: \text{no. of deaths: } 0 \quad 1 \quad 2 \quad = 3 \quad 4$$

$$f_i: \text{Freq: } 123 \quad 59 \quad 14 \quad 3 \quad 1$$

ans:

$$m = \sum x_i f_i = \frac{100}{200} = 0.5$$

$$N = \sum f_i = 200$$

$$P(X=x) = \frac{e^{-0.5} 0.5^x}{x!}$$

$$x: \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$$

$$P(X=x): \quad 0.606 \quad 0.303 \quad 0.075 \quad 0.0012 \quad 0.0001579$$

$$\begin{array}{lllll} \text{Exp freq:} & 121.2 & 60.6 & 15 & 2.4 \\ \text{NP} & & & & 0.3158 \end{array}$$

* Normal Distribution:

Continuous R.V X follows a normal distribution with parameters m (mean) & σ (s.d) if its pdf is given by :

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \times e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \quad -\infty < x < \infty$$

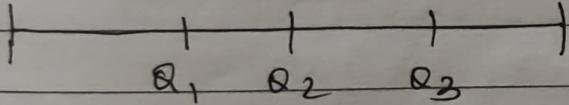
1. mean = mode = median = m

2. If x_1, \dots, x_n are independent normal variables with mean m_1, m_2, \dots, m_n & variance $\sigma_1^2, \sigma_2^2, \sigma_3^2, \dots$ then $y = a_1x_1 + a_2x_2 + \dots + a_nx_n$ is also a normal variable with mean,

$$m = a_1m_1 + a_2m_2 + \dots + a_nm_n$$

$$\text{& } \sigma^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \dots + a_n^2\sigma_n^2$$

3.



Quartiles

$$Q_1 = m - \frac{\sigma}{3}, \quad Q_3 = m + \frac{\sigma}{3}$$

$$Q.D = \frac{Q_3 - Q_1}{2} = \frac{\sigma}{3}$$

4. If $X \sim N(m, \sigma^2)$ then $Z = \frac{X-m}{\sigma}$ is std normal variate (SNV) is also a normal variate with $Z \sim N(0, 1)$.

Ex:

1. Normal distribution w:

$$m = 50, \sigma = 15$$

find: (i) Q_1, Q_2
(ii) Q.D

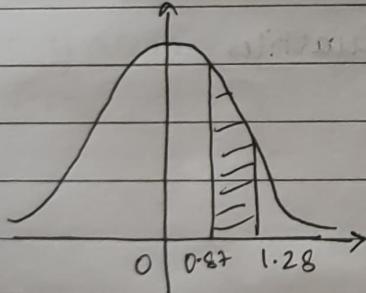
2. If $Q_1 = 36, Q_3 = 44$ find mean, std.

$$\text{f}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}, P(a \leq x \leq b) = ?$$

$$P(a \leq x \leq b) = \int_a^b f(x) dx$$

$$\frac{z-0}{1} = \frac{x-m}{\sigma}$$

$$\underline{\text{Q}} \quad \text{Find } P(0.87 < z < 1.28)$$



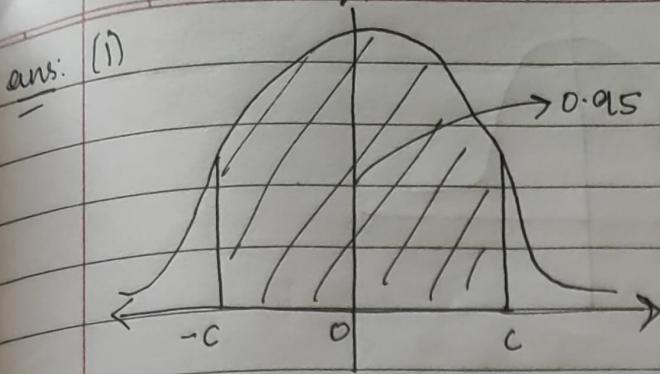
$$\begin{aligned} P(0.87 < z < 1.28) &= P(0 < z < 1.28) - P(0 < z < 0.87) \\ &= 0.3997 - 0.3078 \\ &= 0.0919. \end{aligned}$$

$$\underline{\text{Q}} \quad \text{find const} \quad (i) P(-c < z < c) = 0.95$$

$$(ii) P(|z| < c) = 0.01$$

$$(iii) P(x > c) = 0.02 \quad \left. \begin{array}{l} \text{lim} \\ m=120 \end{array} \right\}$$

$$(iv) P(x < c) = 0.05 \quad \left. \begin{array}{l} \text{lim} \\ \sigma = 10 \end{array} \right\}$$

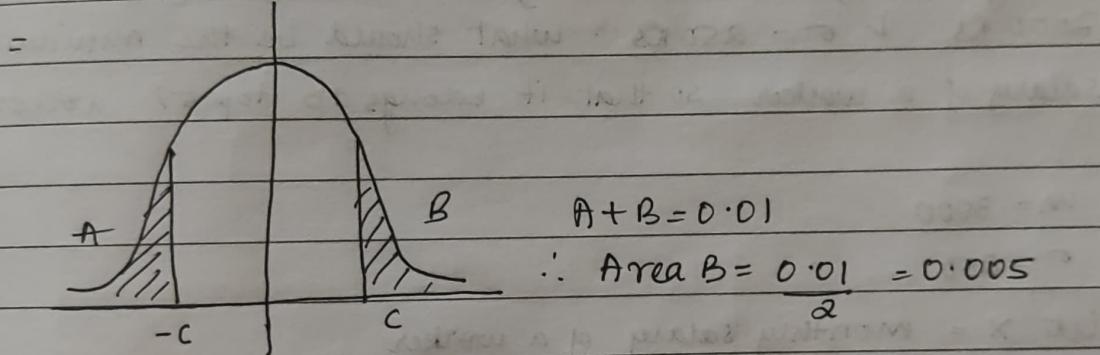


$$P(0 < Z < c) = \frac{0.95}{2} = 0.475$$

$$\therefore c = 1.9 + 0.06 = 1.96$$

(ii) $P(|Z| > c)$

$$= P(Z > c, Z < -c) = 0.01$$



$$\therefore P(0 < Z < c) = 0.5 - 0.005$$

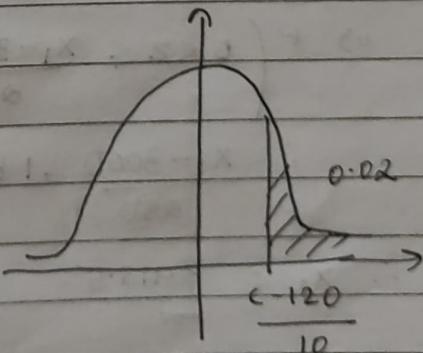
$$= 0.495$$

$$\therefore c = 2.58$$

(iii) $P(X > c) = 0.02$

$$\Rightarrow P\left(\frac{X-m}{\sigma} > \frac{c-m}{\sigma}\right) = 0.02$$

$$\Rightarrow P\left(Z > \frac{c-120}{10}\right) = 0.02$$



$$P\left(0 < Z < \frac{c-120}{10}\right) = 0.5 - 0.02$$

$$= 0.48$$

$$\frac{c-120}{10} = 2.06$$

$$\therefore c = 140.6$$

(iv) $P(X < C) = 0.05$

$$= P\left(\frac{X - \mu}{\sigma} < \frac{C - \mu}{\sigma}\right) = 0.05^{0.05}$$

$$= P\left(Z < \frac{C - 120}{10}\right) = 0.05$$

$$= P\left(0 < Z < \frac{C - 120}{10}\right) = 0.5 - 0.05 = 0.45$$

$$\therefore \frac{C - 120}{10} = 1.65 \quad \therefore C = 136.5$$

Q Monthly salary x is normally distributed between 3 thousand 3000 Rs & $\sigma = 250$ Rs. What should be the minimum salary of a worker so that it belongs to top 5% workers.

ans: $m = 3000$

$\sigma = 250$

Let x = monthly salary of a worker

x_1 = minimum salary of top 5% workers

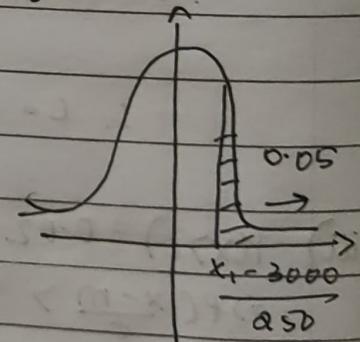
Given: $P(x > x_1) = 0.05$

$$\Rightarrow P\left(z > \frac{x_1 - 3000}{250}\right) = 0.05$$

$$\Rightarrow P\left(0 < z \leq \frac{x_1 - 3000}{250}\right) = 0.5 - 0.05 = 0.45$$

$$\therefore \frac{x_1 - 3000}{250} = 1.65$$

$$\therefore x_1 = 3412.5$$



Q The diameter of tops of cans is normally distributed with standard deviation 0.05. At what mean diameter the machine should be set so that not more than 5% of the tops have dia exceeding 3 cm.

ans:

X = diameter of the top

$$P(X > 3) = 5\% = 0.05$$

$$P\left(\frac{X-m}{0.05} > \frac{3-m}{0.05}\right) = 0.05$$

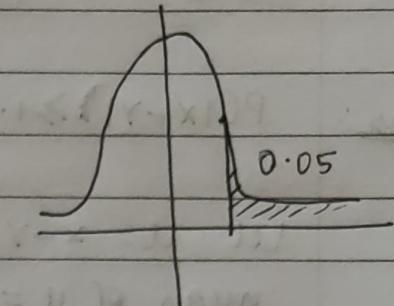
$$P\left(Z > \frac{3-m}{0.05}\right) = 0.05$$

∴

$$P\left(0 < Z < \frac{3-m}{0.05}\right) = 0.5 - 0.05 = 0.45$$

$$\Rightarrow \frac{3-m}{0.05} = 1.65$$

$$\therefore m = 2.9175$$

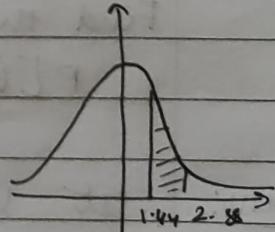


Q If X_1, X_2 are two independent normal variables with mean 30 & 25 and var = 16 & 12. If $Y = 3x_1 - 2x_2$. Find prob of : $P(60 \leq Y \leq 80)$

ans: mean of $3x_1 - 2x_2$ is = $3 \times 30 - 2 \times 25 = 40 = m$

var of $3x_1 - 2x_2$ is = $9 \times 16 + 4 \times 12 = 192 = \sigma^2$

$$\begin{aligned} \therefore P(60 \leq Y \leq 80) &= P\left(\frac{60-m}{\sigma} \leq Z \leq \frac{80-m}{\sigma}\right) \\ &= P(1.44 \leq Z \leq 2.88) \end{aligned}$$



$$= P(0 \leq Z \leq 2.88) - P(0 \leq Z \leq 1.44)$$

$$= 0.988 - 0.557$$

$$= 0.4980 - 0.4251$$

$$= 0.0729$$

Q If X & Y are normally dist with $m=8$ - std as $(52, 3)$ and $(50, 2)$ respectively. Find the prob. that randomly chosen values pairs of X & Y differ by 1.7 or more.

ans: $P(|X-Y| \geq 1.7) \Rightarrow$ to find

Let $U = X-Y$

mean of $U = 52 - 50 = 2$

var of $U = 9 + 4 = 13$

$$P(|U| \geq 1.7) = 1 - P(|U| < 1.7)$$

$$= 1 - P(-1.7 < U < 1.7)$$

$$= 1 - P\left(\frac{-1.7 - 2}{\sqrt{13}} < Z < \frac{1.7 - 2}{\sqrt{13}}\right)$$

$$= 1 - P(-1.026 < Z < -0.083)$$

$$= 1 - [P(0 < Z \leq 1.026) - P(0 < Z \leq 0.083)]$$

$$= 1 - \left[\frac{0.3485 - 0.0319}{0.3485} \right]$$

$$= 0.6834$$

Q If X & Y are independent N.V with mean $m_1 = 8$, $m_2 = 12$ & $\sigma_1 = 2$, $\sigma_2 = 4\sqrt{3}$.

Find the value of alpha α such that:

$$P[(2X-Y) \leq 2\alpha] = P[(X+2Y) \geq 3\alpha]$$

ans: Let $U = 2X-Y$

$$V = X+2Y$$

mean of $U = 4$

mean of $V = 32$

var of $U = 64$

var of $V = 196$

$$\begin{aligned} P(U \leq 2\alpha) &= P(V \geq 3\alpha) \\ \Rightarrow P\left(Z \leq \frac{2\alpha - 4}{8}\right) &= P\left(Z \geq \frac{3\alpha - 32}{14}\right) \\ \Rightarrow P\left(Z \leq \frac{2\alpha - 4}{8}\right) &= P\left(Z \leq -\left(\frac{3\alpha - 32}{14}\right)\right) \\ \Rightarrow \frac{2\alpha - 4}{8} &= -\left(\frac{3\alpha - 32}{14}\right) \end{aligned}$$

$$\Rightarrow 14\alpha - 28 = -12\alpha + 128$$

$$\Rightarrow 26\alpha = 128 + 28$$

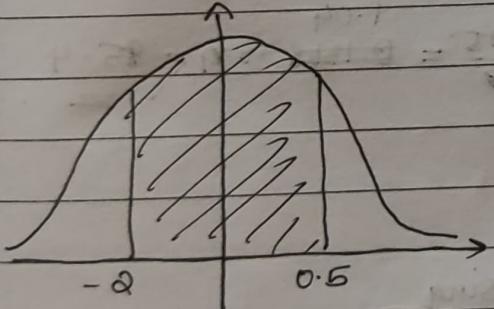
$$\Rightarrow 26\alpha = 156$$

$$\therefore \alpha = \frac{156}{26} = 6$$

Q Income of grp of 10,000 persons are normally distributed with mean 520 & $s.d = 60$. Find :

- no. of persons having income between 400 & 550.
- lowest income of richest 500.

ans:



$$m = 520$$

$$\sigma = 60$$

$$\begin{aligned} \text{find : (i) } P(400 < X < 550) &= P(-2 < Z < 0.5) \\ &= P(0 < Z < 0.5) + P(0 < Z < 2) \\ &= 0.6687 \end{aligned}$$

$$\begin{aligned} \text{no. of persons having income between 400 & 550} &= NP \\ &= 10,000 \times 0.6687 \\ &= 6687 \end{aligned}$$

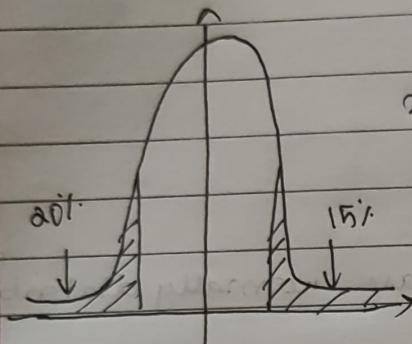
$$\begin{aligned} \text{P(Person selected at random among 500} \\ \text{from 10,000 persons)} &= \frac{500}{10,000} = 0.05 \end{aligned}$$

x_1 = lowest income

$$P(X > x_1) = 0.05$$

- Q In a competitive exam top 15% of students will get grade A, while bottom 20% will be declared failed. The grades are normally distributed with mean % of marks = 75 and $SD = 10$. Determine lowest % of marks to receive grade A and lowest % of marks that passes.

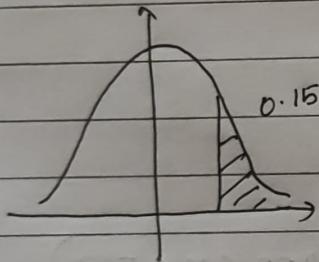
ans:



$$x_1 = \text{marks req for grade A}$$

$$\Rightarrow P(X > x_1) = 0.15$$

$$\Rightarrow P\left(Z > \frac{x_1 - 75}{10}\right) = 0.15$$



$$P(0 \leq Z \leq \frac{x_1 - 75}{10}) = 0.5 - 0.15$$

$$= 0.35$$

$$\therefore \frac{x_1 - 75}{10} = 1.04 \Rightarrow x_1 = 85.4$$

let x_2 = lowest marks for passing

$$P(X < x_2) = 0.2$$

$$x_2 = 66.6$$

$$P\left(Y < \frac{x_2 - 75}{10}\right) = 0.2$$

$$P\left(0 < Z < \frac{x_2 - 75}{10}\right) = 0.5 - 0.2$$

$$= 0.3$$

$$\frac{x_2 - 75}{10} \neq$$

$$\frac{-75 + x_2}{10} = -0.84$$

$$P\left(\frac{75 - x_2}{10} > Z > 0\right) = 0.3 \quad \therefore x_2 = 66.6$$

Q If the actual amount of coffee which a machine puts into 6 ounce jars is a random variable having normal distribution & $\sigma = 0.05$ and if only 3% contains less than 6 ounces of coffee, what must be the mean fill of these jars?

ans: x = amount of coffee.

$$P(x < 6) = 3\%$$

$$P\left(\frac{x-m}{\sigma} < \frac{6-m}{\sigma}\right) = 0.03$$

$$\text{i.e } P\left(\frac{z-m}{\sigma} < \frac{6-m}{\sigma}\right) = 0.03$$

$$\Rightarrow P\left(0 \leq z \leq \frac{6-m}{\sigma}\right) = 0.5 - 0.03 \\ = 0.47$$

$$\frac{6-m}{\sigma} = -1.88$$

$$\therefore m = 6.094$$

6/xx/

Q If the actual amount of coffee which a machine puts into 6 ounce jars is a random variable having normal distribution & $\sigma = 0.05$ and if only 3% contain less than 6 ounces of coffee, what must be the mean fill of these jars.

ans: x = amount of coffee.

$$P(x < 6) = 3\%$$

$$P\left(\frac{x-m}{\sigma} < \frac{6-m}{\sigma}\right) = 0.03$$

$$\text{i.e. } P\left(\frac{z < 6-m}{\sigma}\right) = 0.03$$

$$\Rightarrow P\left(0 \leq z \leq \frac{6-m}{\sigma}\right) = 0.5 - 0.03 \\ = 0.47$$

$$\frac{6-m}{\sigma} = -1.88$$

$$\therefore m = 6.094$$

Q Find the mean & s.d. of the normal distribution of marks where 58% obtained marks below 75, 4% obtained above 80 and rest between 75 & 80.

ans: x = marks obtained

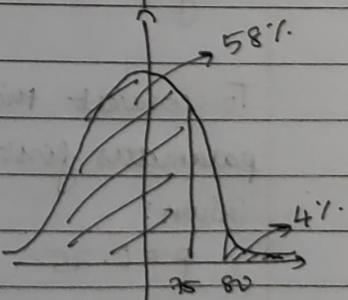
Let m and σ be mean & S.D.

$$\Rightarrow P(m \leq x \leq 75) = 8\% = 0.08$$

$$\therefore P\left(\frac{m-m}{\sigma} \leq z \leq \frac{75-m}{\sigma}\right) = 0.08$$

$$\therefore P\left(0 \leq z \leq \frac{75-m}{\sigma}\right) = 0.08$$

$$\therefore \frac{75-m}{\sigma} = 0.2 \Rightarrow m + 0.2\sigma = 75 \quad -(1)$$



Also,

$$P(X > 80) = 0.1 = 0.04$$

$$\Rightarrow P\left(Z > \frac{80-m}{\sigma}\right) = 0.04$$

$$\Rightarrow P\left(0 < Z < \frac{80-m}{\sigma}\right) = 0.5 - 0.04 = 0.46$$

$$\therefore \frac{80-m}{\sigma} = 1.75$$

$$\therefore 80 = m + 1.75\sigma \quad \text{--- (2)}$$

\therefore From (1) and (2) we get,

$$m = 74.36$$

$$\sigma = 3.225$$

Q

The prob that electronic component will fail ⁱⁿ less than 1200 hours of continuous use is 0.25. Using normal approximation to Binomial dist find the prob that amongst 200 such components fewer than 45 will fail in less than 1200 hrs. of continuous use.

ans:

X = no. of components which will fail in less than 1200 hrs

To find: $P(X < 45)$

To convert this into normal dist we need to find the parameters first : m, σ

Given:

$$P = 0.25$$

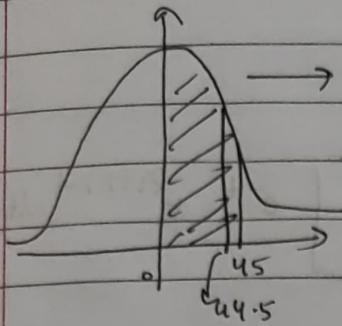
$$q = 1 - p = 0.75$$

$$n = 200$$

$$m = np = 200 \times 0.25 = 50$$

$$S.d (\sigma) = \sqrt{npq} = \sqrt{50 \times 0.25 \times 0.75} = 4.123$$

$$Z = \frac{x - m}{\sigma} = \frac{x - 50}{6.123}$$



→ AE Rule: subtract 0.5

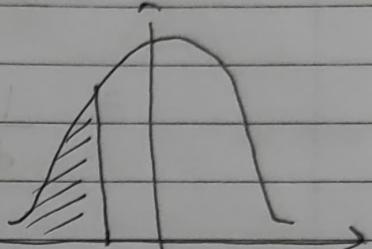
∴ Req Prob is $P(x \leq 44.5)$

$$= P\left(Z \leq \frac{44.5 - 50}{6.123}\right)$$

$$= P(Z \leq -0.898) = 0.5 - P(0 < Z < 0.898)$$

$$= 0.5 - 0.313$$

$$= 0.1867$$



* Exponential distⁿ:

Continuous R.V X is said to follow exponential dist with parameter $\lambda > 0$ if its pdf is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{N.B.: } \int_0^{\infty} f(x) dx = 1$$

Mean & Variance

$$E(X^r) = \int_0^{\infty} x^r f(x) dx$$

$$= \int_0^{\infty} x^r \lambda e^{-\lambda x} dx$$

$$\text{put } \lambda x = t \quad \therefore dx = \frac{dt}{\lambda}$$

$$\therefore X \int_0^{\infty} \left(\frac{t}{\lambda}\right)^x e^{-t} dt$$

$$= \frac{1}{\lambda^x} \int_0^{\infty} t^x e^{-t} dt = \frac{1}{\lambda^x} \int_0^{\infty} e^{-t} t^{(x+1)-1} dt$$

$$= \frac{1}{\lambda^x} \Gamma(x+1)$$

$$\therefore E(X^x) = \frac{x!}{\lambda^x}$$

$$\therefore \text{mean} = E(X) = \frac{1}{\lambda}$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$P(X > k) = \int_k^{\infty} \lambda e^{-\lambda x} dx = e^{-\lambda k}$$

Q Mileage which car owners get is a R.V having an exponential distribution with mean 40,000 Km. Find the probability that

One of these tires will last at least 20,000 Km & atmost 30,000 Km.

Ans: X = mileage of a car

$$M = 40,000 = \frac{1}{\lambda} \quad \therefore \lambda = 2.5 \times 10^{-5}$$

$$\begin{aligned}
 \text{(i)} \quad P(X > 20,000) &= \int_{20,000}^{\infty} \frac{1}{40,000} \times e^{-\frac{x}{40,000}} dx \\
 &\stackrel{\text{QDK}}{=} -e^{-\frac{x}{40,000}} \Big|_{20,000}^{\infty} \quad (\text{using note}) \\
 &= e^{-\frac{20,000}{40,000}} \\
 &= e^{-0.5} = 0.606 \\
 \text{(ii)} \quad P(X \leq 30,000) &= \int_0^{30,000} \frac{1}{40,000} e^{-\frac{x}{40,000}} dx \\
 &= \frac{1}{40,000} \left[e^{-\frac{x}{40,000}} \right]_0^{30,000} \\
 &= - \left[e^{-\frac{30,000}{40,000}} - 1 \right] = 1 - e^{-0.75} \\
 &= .
 \end{aligned}$$

Note:

~~Exponential distribution~~

If X is exponentially distributed then $P(X \geq s+t | X \geq s) = P(X \geq t)$ for $s, t > 0$.

The length of a shower has exponential distribution with parameter $\lambda = 2$ time being measured in mins then what is the prob. that shower will last more than 3 mins. If shower has lasted 2 mins then find the prob that it will last 1 more min.

Ans: (i) $P(X > 3) = \int_3^{\infty} 2 e^{-2z} dz = e^{-6}$

X = length of shower

Given $\lambda = 2$, $f(x) = 2e^{-2x}$

$$(ii) \text{ we want } P(X > 2 + 1) = P(X > 3) \\ = \int_1^\infty e^{-2x} dx = e^{-2}$$

* uniform distribution:

Discrete R.V X is said to follow uniform distribution if $P(X = x_i) = \frac{1}{n} \quad \forall x_i = 1, 2, \dots, n$

Ex: when coin is tossed $P=1/2$.

$$\text{Mean: } E(X) = \sum p_i x_i$$

$$= \left[\sum x_i \frac{1}{n} \right] = \frac{1}{n} + \frac{2}{n} + \dots + \frac{n}{n}$$

$$= \frac{1}{n} (1+2+\dots+n) = \frac{\cancel{n}(n+1)}{\cancel{n}} \times \frac{1}{2}$$

Variance: E

$$E(X^2) = \sum x_i^2 p_i$$

$$= 1^2 \frac{1}{n} + 2^2 \frac{1}{n} + \dots + n^2 \frac{1}{n}$$

$$= \frac{1}{n} (1^2 + 2^2 + \dots + n^2) =$$

$$= \frac{1}{n} \times \frac{n(n+1)(2n+1)}{6} =$$

$$= \frac{(n+1)(2n+1)}{6}$$

$$\text{Variance: } E(x^2) - [E(x)]^2$$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{(n+1)}{12} (4n+2 - 3n-3) = \frac{n^2-1}{12}$$

continuous Uniform Distribution:

A continuous R.V x follows uniform dist over the interval $[a, b]$ if:

$$f(x) = \begin{cases} K, & a \leq x \leq b \\ 0, & \text{o.w.} \end{cases}$$

$$\therefore \int_a^b f(x) dx = 1$$

$$= \int_a^b K dx = 1 \Rightarrow K(b-a) = 1 \quad K = \frac{1}{b-a}$$

$$\therefore f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{o.w.} \end{cases}$$

$$\text{mean} = \int_a^b x f(x) dx$$

$$= \int_a^b x \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \left(\frac{x^2}{2} \right)_a^b = \frac{1}{2(b-a)} (b^2 - a^2)$$

$$= \frac{b+a}{2}$$

$$\begin{aligned}
 E(X^2) &= \int_a^b x^2 f(x) dx \\
 &= \int_a^b x^2 \frac{1}{b-a} dx \\
 &= \frac{1}{b-a} \left(\frac{x^3}{3} \right)_a^b = \frac{1}{3(b-a)} (b^3 - a^3) \\
 &= \frac{b^3 - a^3}{3(b-a)} = \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} \\
 &= \frac{b^2 + ab + a^2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{Variance} &= E(X^2) - (E(X))^2 \\
 &= \frac{b^2 + ab + a^2}{3} - \left(\frac{b+a}{2} \right)^2 \\
 &= \frac{b^2 + ab + a^2}{3} - \frac{(b^2 + a^2 + 2ab)}{4} \\
 &= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}.
 \end{aligned}$$

$$\begin{aligned}
 E(x^2) &= \int_a^b x^2 f(x) dx \\
 &= \int_a^b x^2 \frac{1}{b-a} dx \\
 &= \frac{1}{b-a} \left(\frac{x^3}{3} \right)_a^b = \frac{1}{3(b-a)} (b^3 - a^3) \\
 &= \frac{b^3 - a^3}{3(b-a)} = \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)}
 \end{aligned}$$

$$\begin{aligned}
 \text{Variance} &= E(x^2) - (E(x))^2 \\
 &= \frac{b^3 - a^3}{3(b-a)} - \left(\frac{b+a}{2} \right)^2 \\
 &= \frac{b^3 - a^3}{3(b-a)} - \left(b^2 + a^2 + 2ab \right) \\
 &= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}.
 \end{aligned}$$

* Baye's Theorem:

\Rightarrow let E_1, E_2, \dots, E_m be mutually disjoint events with prob of $P(E_i) \neq 0$. let A be any arbitrary event such that $A \subseteq \bigcup_{i=1}^m E_i$, $P(A) > 0$

$$\text{then, } P(E_i/A) = \frac{P(E_i) P(A/E_i)}{\sum_{i=1}^m P(E_i) P(A/E_i)}$$

$$P(E_i/A) = \frac{P(E_i) P(A/E_i)}{P(A)}$$

$$(we have, A = (A \cap E_1) \cup (A \cap E_2) \cup \dots \cup (A \cap E_n))$$

$$(A \cap E_1) \cup (A \cap E_2) \cup \dots \cup (A \cap E_n) = A$$

$$(A \cap E_1) \cup (A \cap E_2) \cup \dots \cup (A \cap E_n) = A \cap (E_1 \cup E_2 \cup \dots \cup E_n)$$

$$(A \cap E_1) \cup (A \cap E_2) \cup \dots \cup (A \cap E_n) = P(A \cap E_1) + P(A \cap E_2) + \dots + P(A \cap E_n)$$

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \dots + P(A \cap E_n)$$

$$= \sum_{i=1}^n P(A \cap E_i)$$

$$= \sum_{i=1}^n P(A|E_i) P(E_i)$$

$$* P(A \cup B) = P(A) + P(B)$$

$$P(A \cap B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A)P(B)$$

Ex: A product is produced in 3 factories X, Y, Z. X produces three times as many items as Y. Y and Z produce same no. of items. Assume that 3% of items produced by each factory X and Z are defective. While 5% produced by Y are defective. An item is selected at random. (i) What is the prob that this item is defective? (ii) If the item is already known to be defective then what is the prob that it is produced by X, Y or Z.

at,

ans: A = item is defective.

E₁ = Item produced by X, Y and Z respectively.

Events E_i	Prior prob $P(E_i)$	Cond Prob $P(A E_i)$	Joint prob $P(A \cap E_i)$ $= P(E_i)P(A E_i)$	Posterior prob $P(E_i A)$ $(5) = (4) \div P(A)$
E_1	$1/5$	0.03	$(4) = (2) \times (3)$ $(E_1) = 0.018$	$(A) = 0.529$
E_2	$1/5$	0.05	0.01	0.294
E_3	$1/5$	0.03	0.006 $(E_3) = 0.006$	0.176

$$P(A) = 0.0349$$

A factory produces certain types of outputs by 3 types of machines. Daily productions are as follows:

$M_1 = 3000$ units
 $M_2 = 2500$ units
 $M_3 = 4500$ units

It is already known that % of outputs by M_1 are defective, $M_2 = 1.2\%$, $M_3 = 2\%$. An item drawn is already found to be defective. What is the prob that it is from M_1, M_2 or M_3 .

ans: E_i = item produced by M_i

E_i	$P(E_i)$	$P(A E_i)$	$P(G \cap E_i)$	$P(E_i A)$
E_1	0.01	0.01	0.003	(0.01) 0.2
E_2	0.012	0.012	0.003	0.2
E_3	0.02	0.02	0.009	0.6
			<u>0.015</u>	

$$P(G|A) = \frac{P(E_i)P(A|E_i)}{\sum P(E_i)P(A|E_i)}$$

$\underbrace{P(A)}_{\sum P(E_i)P(A|E_i)}$

Q $A \rightarrow n$ white + 2 black

$B \rightarrow 2$ white + n black

1 bag is selected at random & 2 balls are drawn without replacement. If both the balls drawn are white & prob that A was used is $6/7$. Find the value of n .

ans:

E_1 : bag A selected

E_2 : bag B selected

A: Two balls drawn are white.

E_i	$P(E_i)$	$P(G E_i)$	$P(G \cap E_i)$	$P(E_i A)$
E_1	$1/2$	$\frac{nC_2}{n+2C_2}$		
E_2	$1/2$	$\frac{2C_2}{n+2C_2}$		

$$E_1 \quad 1/2 \quad \frac{nC_2}{n+2C_2}$$

$$E_2 \quad 1/2 \quad \frac{2C_2}{n+2C_2}$$

By Bayes's theorem,
prob that 2 balls are drawn from A:

$$P(E_2/A) = \frac{P(E_2)P(A/E_2)}{(P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}) \stackrel{E_1}{=} \frac{6}{7}$$

$$\Rightarrow \frac{\frac{1}{2} \times \frac{nC_2}{n+2C_2}}{\frac{1}{2} \frac{nC_2}{n+2C_2} + \frac{1}{2} \frac{2C_2}{n+2C_2}} = \frac{6}{7}$$

$$\Rightarrow \frac{\frac{n(n-1)}{2}}{\frac{n(n-1)}{2} + 2} = 6/7$$

Answers are also obtained by inspection and
no answers other than 6C2 / 8C2 remain
 $\Rightarrow \frac{n(n-1)}{n(n-1)+2} = 6/7$ solving this

$$\Rightarrow 7n(n-1) = 6(n(n-1)+2)$$

$$\Rightarrow 7n^2 - 7n = 6n^2 - 6n + 12$$

$$\Rightarrow n^2 - n - 12 = 0$$

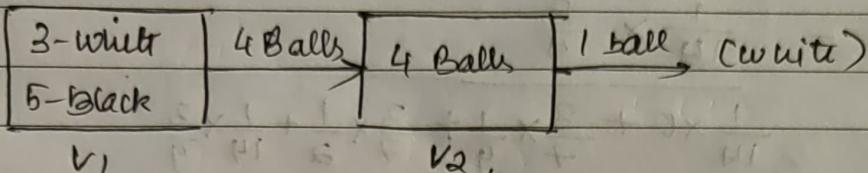
$$\Rightarrow n = 4, -3$$

$$\therefore n = 4$$

Q A vessel containing 3-white, 5-black. 4 balls are transferred to an empty vessel. From this vessel a ball is drawn at random & is found to be white. What is the prob that out of 4 balls, 3 are white and 1 black.

3 white + 1 black.

ans:



Define: $E_1 = 0 \text{ white and } 4 \text{ black}$

$E_2 = 1 \text{ white and } 3 \text{ black}$

$E_3 = 2 \text{ white and } 2 \text{ black}$

$E_4 = 3 \text{ white and } 1 \text{ black}$

$\rightarrow (x,y) \leftarrow (V, v)$

A = a ball drawn is white

$$P(E_1) = \frac{3C_0 \times 5C_4}{8C_4} = \frac{1}{14}$$

$$P(E_2) = \frac{3C_1 \times 5C_3}{8C_4} = \frac{3}{14}$$

$$P(E_3) = \frac{3C_2 \times 5C_2}{8C_4} = \frac{3}{14}$$

$$P(E_4) = \frac{3C_3 \times 5C_1}{8C_4} = \frac{3}{14}$$

$$P(E/E_1) = 0 \quad \text{To find: } P(E_4/E)$$

$$P(E/E_2) = \frac{1}{4}$$

$$P(E/E_3) = \frac{1}{2}$$

$$P(E/E_4) = \frac{3}{4}$$

$$P(E_4|E) = P(E_4) P(E|E_4)$$

$$= \sum_{i=1}^4 P(E_i) P(E|E_i)$$

$$\frac{\frac{1}{14} \times \frac{3}{4}}{\frac{1}{14} \times 0 + \frac{3}{7} \times \frac{1}{4} + \frac{3}{7} \times \frac{1}{2} + \frac{1}{14} \times \frac{3}{4}}$$

$$= \frac{1}{14} = 0.14$$

2D or Joint Conditional Prob.

$$(x, y) \rightarrow (x, y) \in \mathbb{R}^2$$

Let x and y be 2 R.V defined on the example space S . Then, a function (x, y) which assigns a point in \mathbb{R}^2 is called 2 Dimensional R.V.

Joint prob mass function:

If x, y is 2 Dimensional discrete RV with each possible outcome (x_i, y_j) with the prob P_{ij} satisfying the conditions;

- (i) $P(x_i, y_j) \geq 0 \quad \forall i, j$
- (ii) $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} P(x_i, y_j) = 1$

The P is called joint pmf and joint prob distribution is given as follows:

$x \backslash y$	y_1	y_2	\dots	y_m	Total
x_1	P_{11}	P_{12}	\dots	P_{1m}	P_1
x_2	P_{21}	P_{22}	\dots	P_{2m}	P_2
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
x_n	P_{n1}	P_{n2}	\dots	P_{nm}	P_n
Total	P_1'	P_2'	\dots	P_m'	

Marginal prob distribution of x is given by:

$$x : x_1 \ x_2 \ \dots \ x_n$$

$$P(x) : P_1 \ P_2 \ \dots \ P_n$$

where $P_i' = \sum_{j=1}^m P_{ij}$

and, $P_n = \sum_{j=1}^m P_{nj}$

$$y : y_1 \ y_2 \ \dots \ y_m$$

$$P(y) : P_1' \ P_2' \ \dots \ P_m'$$

where, $P_1' = \sum_{i=1}^n P_{i1}$

$$P_m' = \sum_{i=1}^n P_{im}$$

$$P(X=x_i, Y=y_i) = \frac{P(X=x_i, Y=y_i)}{P(Y=y_i)}$$

Q Joint Prob dist of X and Y is given by:

$$P(X=x, Y=y) = \frac{x+3y}{24} ; x, y = 1, 2.$$

Find pmf and marginal prob dist of X and Y

		X		Total
		1	2	
Y	1	$\frac{1}{6}$	$\frac{5}{24}$	$\frac{3}{8}$
	2	$\frac{7}{24}$	$\frac{13}{24}$	$\frac{5}{8}$
Total		$\frac{11}{24}$	$\frac{13}{24}$	$\frac{1}{2}$
X	1		2	Total

$$P(X) : \frac{11}{24} \quad \frac{13}{24}$$

$$P(Y) : \frac{3}{8} \quad \frac{5}{8}$$

Q Joint prob dist of X_1 & X_2 is given by,

$$P(X_1=x_1, X_2=x_2) = \frac{1}{27}(x_1+2x_2)$$

$$\Rightarrow x_1, x_2 = 0, 1, 2$$

Find pmf of X_1 & X_2 indep in terms of x_1 & x_2

Ans: $P(X_1=x_1) = \sum_{x_2=0}^n P(x_1, x_2)$

$$= \sum_{x_2=0}^2 \frac{1}{27}(x_1+2x_2)$$

$$= \frac{1}{27} [x_1 + (x_1+2) + (x_1+4)]$$

$$= \frac{1}{27} [3x_1 + 6] = \frac{x_1+2}{9}$$

$$P(X=x_2) = \sum_{x_1=0}^2 P(x_1, x_2)$$

$$= \sum_{x_1=0}^{x_2} \frac{1}{27} (x_1+2x_2)$$

$$= \frac{1}{27} [2x_2 + (1+2x_2) + (2+2x_2)]$$

$$= \frac{1}{27} [6x_2 + 3] = \frac{2x_2 + 1}{9}$$

Q 3 balls are drawn at random from a box containing 4 black & 3 red. If X denotes no. of white balls, Y = no. of Red balls. Find Joint P.d.f. of X & Y

Ans:

$X \backslash Y$	0	1	2	3	Total
0	$1/21$	$3/14$	$4/7$	$11/84$	$35/84$
1	$1/7$	$2/7$	$1/14$	0	$42/84$
2	$1/21$	$1/28$	0	0	$7/84$
Total:	$5/21$	$15/28$	$3/14$	$11/84$	

$$P(X=0, Y=0) = \frac{4C_3}{9C_3} = \frac{4C_1}{9C_6} = \frac{1}{21}$$

$$P(X=0, Y=1) = \frac{3C_1 \times 4C_2}{9C_3} = \frac{3}{14}$$

X = no. of white balls

Y = no. of red balls

$$\begin{aligned}
 \text{(ii)} \quad P(X \leq 1) &= \frac{35}{84} + \frac{42}{84} = \frac{77}{84} \\
 P(X \leq 1, Y \leq 2) &= P(X=0, Y \leq 2) + P(X=1, Y \leq 2) = \frac{76}{84} \\
 P(Y \leq 2 | X \leq 1) &= \frac{P(X \leq 1, Y \leq 2)}{P(X \leq 1)} = \frac{76}{77}
 \end{aligned}$$

$$P(X+Y \leq 2) = \sum_{j=0}^2 P(X=0, Y=j) + \sum_{j=0}^1 P(X=1, Y=j)$$

$$+ P(X=2, Y=0)$$

$$= \frac{37}{48}$$

(iii) check whether X & Y are independent,

$$P(X, Y) = P(X)P(Y)$$

$$P(0, 1) = \frac{3}{14} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{not equal}$$

$$P(0) \times P(1) = \frac{35}{84} \times \frac{15}{28}$$

hence, not independent.

* Continuous prob distribution:

Let (x, y) be 2-D continuous R.V and let $f_{xy}^{(x,y)}$ be a function of (x, y) such that:

$$(i) \quad f_{xy}^{(x,y)} \geq 0$$

$$(ii) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{xy}^{(x,y)} dx dy = 1$$

$$(iii) \quad \int_a^b \int_c^d f_{xy}^{(x,y)} dx dy = P(a \leq x \leq b, c \leq y \leq d)$$

* Marginal pdf of x-axis : $f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x, y) dy$

$$f_x(x) \geq 0 \text{ and } \int_{-\infty}^{\infty} f_x(x) dx = 1$$

* Marginal pdf of y-axis : $f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x, y) dx$

$$f_y(y) \geq 0 \Rightarrow \int_{-\infty}^{\infty} f_y(y) dy = 1$$

Q 2-D R.V has joint pdf $f_{xy}(xy) = \begin{cases} 15e^{-3x-5y} & x>0, y>0 \\ 0 & \text{o.w} \end{cases}$

Find : (i) $P(1 < x < 2, 0.2 < y < 0.3)$

(ii) $P(x < 2, y > 0.2)$

(iii) Marginal prob distribution of x & y

$$\underline{\text{any (i)}} \quad P(1 < x < 2, 0.2 < y < 0.3) = \int_{0.2}^{0.3} \int_{1}^{2} 15 e^{-3x-5y} dx dy$$

$$= 15 \int_{0.2}^{0.3} e^{-5y} dy \int_{1}^{2} e^{-3x} dx$$

$$= 15 \left(\frac{e^{-5y}}{-5} \right)_{0.2}^{0.3} \times \left(\frac{e^{-3x}}{-3} \right)_{1}^{2}$$

$$= (e^{-5 \times 0.3} - e^{-5 \times 0.2}) \times (e^{-6} - e^{-3})$$

$$= 0.00684$$

$$\underline{\text{(ii)}} \quad P(x < 2, y > 0.2) = \int_{0.2}^{\infty} \int_{0}^{2} 15 e^{-3x-5y} dx dy = 0.367$$

$$(iii) f_x(x) = \int_{y=0}^{\infty} 15e^{-3x-5y} dy = 15e^{-3x} \left(\frac{e^{-5y}}{-5} \right) \Big|_0^{\infty}$$

$$= -3e^{-3x}(-1)$$

$$f_x(x) = \begin{cases} 3e^{-3x} & x > 0 \\ 0 & 0 \leq x \leq 0 \end{cases}$$

$$f_y(y) = 15e^{-5y} \int_0^{\infty} e^{-3x} dx = \frac{15}{3} e^{-5y} \text{ (F1)}$$

$$f_y(y) = \begin{cases} 5e^{-5y} & y > 0 \\ 0 & 0 \leq y \leq 0 \end{cases}$$

$$\text{Q} f_{xy}(x, y) = \begin{cases} x(x-y) & 0 < x < 2, -x < y < x \\ 0 & \text{o.w.} \end{cases}$$

find : (i) C

$$(ii) f_x(x)$$

$$(iii) f_{y/x}(y/x)$$

$$\text{ans: we have } \int_{y=-x}^2 \int_0^2 f_{xy}(x, y) dx dy = 1$$

$$= c \int_{y=-x}^2 \int_{x=0}^1 x(x-y) dx dy = 1$$

$$= c \int_{x=0}^1 x \left(xy - \frac{y^2}{2} \right) \Big|_{y=-x}^2 dx = 1$$

$$= c \int_0^2 x \left(x^2 - \frac{x^2}{2} + x^2 + \frac{x^2}{2} \right) dx = 1$$

$$= 2c \left(\frac{24}{4} \right)_0^2 = 1 \quad \Rightarrow \quad \frac{c(16)}{2} = 1 \quad \therefore c = \frac{1}{8}$$

$$(ii) f_x(x) = \int_{y=-x}^2 f_{xy}(x, y) dy \\ = \frac{1}{8} \int_{y=-x}^x x(x-y) dy = \frac{x^3}{4}, \quad 0 < x < 2$$

$$(iii) f_{y|x}(y|x) = \frac{f_{xy}(x, y)}{f_x(x)}$$

$$= \frac{\frac{1}{8}x(x-y)}{x^3/4} = \frac{x-y}{2x^2}$$

$$\text{Q} \quad f_{xy}(x, y) = xy^2 + \frac{x^2}{8}, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 1$$

find:

$$(i) P(X>1)$$

$$(iv) P(Y<1/2 | X>1)$$

$$(ii) P(Y<1/2)$$

$$(v) P(X>Y)$$

$$(iii) P(X>1, Y<1/2)$$

$$(vi) P(X+Y \leq 1)$$

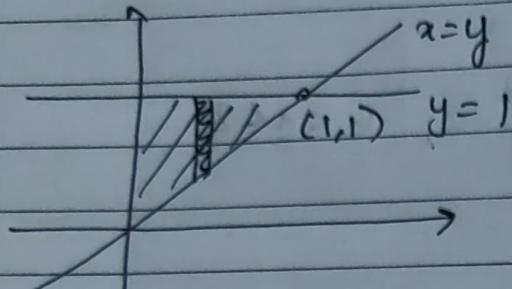
$$\underline{\text{Ans:}} \quad (i) P(X>1) = \int_{y=0}^{1/2} \int_{x=1}^2 \left(xy^2 + \frac{x^2}{8} \right) dx dy = \frac{19}{24}$$

$$(ii) P(Y<1/2) = \int_{y=0}^{1/2} \int_{x=0}^2 \left(xy^2 + \frac{x^2}{8} \right) dx dy = \frac{1}{4}$$

$$(iii) \int_{y=0}^{1/2} \int_{x=0}^2 \left(xy^2 + \frac{x^2}{8} \right) dx dy = \frac{1}{4}$$

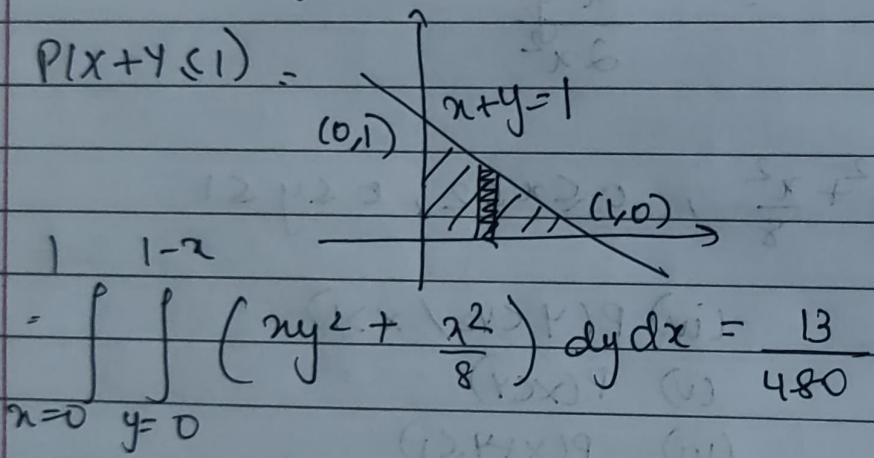
$$(iii) P(X>1, Y<1/2) = \int_{y=0}^{1/2} \int_{x=1}^2 \left(xy^2 + \frac{x^2}{8} \right) dx dy = \frac{5}{24}$$

$$(iv) P(Y < 1/2 | X > 1) = \frac{P(X > 1, Y < 1/2)}{P(X > 1)} = \frac{5/24}{1/4} = \frac{5}{6}$$



$$(v) P(X < Y) = \int_{x=0}^1 \int_{y=x}^1 \left(xy^2 + \frac{x^2}{8} \right) dy dx = \frac{53}{480}$$

$$(vi) P(X+Y \leq 1)$$



$$= \int_{x=0}^1 \int_{y=0}^{1-x} \left(xy^2 + \frac{x^2}{8} \right) dy dx = \frac{13}{480}$$