

Laplace Transform

(is an operator) *

* Let $f(t)$ be a function of $t \geq 0$, then Laplace of $f(t)$ is

$$L[f(t)] = \int_0^\infty e^{-st} \cdot f(t) dt$$

$$= \phi(s) = \bar{f}(s)$$

$t = \text{time (time domain)}$

$s = \text{frequency (freq. domain)}$

t_0, t_1, t_2, \dots

$f(t_0), f(t_1), f(t_2), \dots$

Q. Find Laplace transform of $f(t) = \begin{cases} \cos(t) & \text{for } 0 < t < \pi \\ \sin(t) & \text{for } t > \pi \end{cases}$

Ans. $L[f(t)] = \int_0^\infty e^{-st} \cdot f(t) dt$

$$= \int_0^\pi e^{-st} \cos(t) dt + \int_\pi^\infty e^{-st} \sin(t) dt$$

We know,

$$\int e^{ax} \sin(bx) dx = \frac{e^{ax}}{a^2+b^2} (a\sin(bx) - b\cos(bx))$$

$$\int e^{ax} \cos(bx) dx = \frac{e^{ax}}{a^2+b^2} (a\cos(bx) + b\sin(bx))$$

$$\therefore L[f(t)] = \left[\frac{e^{-st}}{s^2+1} (-s\cos(t) + s\sin(t)) \right]_0^\pi + \left[\frac{e^{-st}}{s^2+1} (-s\sin(t) - \cos(t)) \right]_\pi^\infty$$

$$= \left[\frac{e^{-\pi s}}{s^2+1} (-s(-1)) - \left[1 - (-s) \right] \right] + \left[0 - \frac{e^{-\pi s}}{s^2+1} (0 - (-1)) \right]$$

$$= \left[(t_0) \frac{e^{-\pi s}}{s^2+1} \right] + \left[(t_0) \frac{e^{-\pi s}}{s^2+1} \right] = \frac{e^{-\pi s}}{s^2+1}$$

$$= \frac{s \cdot e^{-\pi s}}{s^2+1} + s - \frac{e^{-\pi s}}{s^2+1}$$

homework problem
(using properties)

[Even if not defined at interval endpoints, integration works (Riemann).]

Laplace for standard functions \rightarrow To evaluate $L(f(t))$

$$\text{Final} \rightarrow L[e^{at}] = \int_0^\infty e^{-st} \cdot e^{at} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt$$

$$= -\left[\frac{e^{-(s-a)t}}{s-a} \right]_0^\infty$$

$$= \frac{1}{s-a} \quad (\text{for } s > a)$$

Similarly,

$$\rightarrow L[e^{-at}] = \frac{1}{s+a} \quad (\text{for } s > -a)$$

Linearity Property:

$$L[a \cdot f(t) + b \cdot g(t)] = a \cdot L[f(t)] + b \cdot L[g(t)]$$

$$\rightarrow L[\cos(at)] = ?$$

$$L[\sin(at)] = ?$$

\therefore we have $e^{iat} = \cos(at) + i\sin(at)$

$$\therefore L[e^{iat}] = L[\cos(at) + i\sin(at)]$$

$$\therefore \frac{e^{iat}}{s-ia} = L[\cos(at)] + i \cdot L[\sin(at)]$$

$$\therefore \frac{s+ia}{s^2+a^2} = L[\cos(at)] + i \cdot L[\sin(at)] \quad (\text{Comparing real and imaginary parts})$$

$$\therefore L[\cos(at)] = \frac{\sin(at)}{s^2+a^2}$$

$$L[\sin(at)] = \frac{-\cos(at)}{s^2+a^2}$$

$$\rightarrow L[\cosh(at)] = ?$$

$$L[\sinh(at)] = ?$$

$$\therefore L[\cosh(at)] = L\left[\frac{e^{at} + e^{-at}}{2}\right]$$

$$= \frac{1}{2} \left(L[e^{at}] + L[e^{-at}] \right)$$

$$= \frac{1}{2} \left(\begin{matrix} [(+s) \cos \theta] + [(-s) \sin \theta] \\ [(-s) \cos \theta] + [(+s) \sin \theta] \end{matrix} \right) = \begin{pmatrix} s \cos \theta - s \sin \theta \\ s \sin \theta + s \cos \theta \end{pmatrix} = s \begin{pmatrix} \cos \theta - \sin \theta \\ \sin \theta + \cos \theta \end{pmatrix} = s \begin{pmatrix} \cos \theta - \sin \theta \\ \sin \theta + \cos \theta \end{pmatrix} = s \begin{pmatrix} \cos \theta - \sin \theta \\ \sin \theta + \cos \theta \end{pmatrix}$$

$$\frac{[(s^2-a^2)+S]}{s^2-a^2} + \frac{S^2}{s^2-a^2} + \frac{1}{s^2-a^2} =$$

$$[(1)(2)(3)(4)] + [1] = 1 + \varepsilon + 1 - \varepsilon = 2$$

Similarly,

~~Parallelly~~, Similarly,

$$L[\sinh(at)] = \frac{2a}{s^2 - a^2} + \frac{1}{s-a} = \frac{2s}{s^2 - a^2} + \frac{1}{s-a} = \frac{2s}{s+a} + \frac{1}{s-a} = \frac{2s-a}{s^2 - a^2}$$

$$\rightarrow L[t^n] = \left\{ \begin{array}{l} \sqrt{n+1} \\ g^{n+1} \end{array} \right. \quad \left[\left(\frac{1}{2} e_{\text{air}} \right) \right] \cdot b_{\text{air}} \quad . \quad \text{①}$$

$$L(t^n) = \int_0^{\infty} e^{-st} t^n dt$$

$$\text{Let } st = u \quad \therefore t = \frac{u}{s}, \quad dt = \frac{du}{s}$$

$$\therefore L[f^n] = \int_0^\infty e^{-yt} u^n s_{1/2} du + \sum_{j=0}^{n-1} \int_0^\infty e^{-yt} u^j s_{1/2} du$$

$\left[u^j s_{1/2} + \frac{1}{j+1} u^{j+1} s_{1/2} \right]_0^\infty - \left(j+1 \right) s_{1/2}^2$

(Gamma)

$$\therefore L[t^n] = \frac{1}{(s-1)^{n+1}} = \frac{1}{s^n} \cdot \frac{1}{(s-1)^n} = \frac{1}{s^n} \cdot \frac{1}{(s-1)^n} = \frac{1}{s^n} \cdot \frac{1}{(s-1)^n}$$

$$\rightarrow L[i] = \frac{1}{s}$$

$$\rightarrow L[k] = \frac{k}{s}$$

Q. Find $L[t^2 - e^{-2t} + \cosh^2(3t) + \sin(3t)]$

Ans. $= L[t^2] - L[e^{-2t}] + L[\cosh^2(3t)] + L[\sin(3t)]$

$$= \frac{1}{s^3} - \frac{1}{s+2} + \frac{9}{s^2+9} + L\left[\frac{1+\cosh(6t)}{2}\right]$$

$$= \frac{2}{s^3} - \frac{1}{s+2} + \frac{3}{s^2+9} + \frac{1}{2} [L(1) + L(\cosh(6t))]$$

$$= \frac{2}{s^3} - \frac{1}{s+2} + \frac{3}{s^2+9} + \frac{1}{2s} + \frac{9s}{2s^2-72}$$

Q. Find $L[\sin^5(t)]$

Ans. Let $z = \cos(t) + i\sin(t)$ $\therefore \frac{1}{z} = \cos(t) - i\sin(t)$

$$(z - \frac{1}{z}) = 2i\sin(t) \quad \therefore (z - \frac{1}{z})^5 = 2^5 \cdot i \cdot \sin^5(t)$$

$$\therefore 2^5 - 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) - 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) - \frac{1}{z^5} = 2^5 \cdot i \cdot \sin^5(t)$$

$$\therefore = \left(2^5 - 1\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$$\therefore 2i\sin(5t) - 10i\sin(3t) + 20i\sin(t) = 2^5 \cdot i \cdot \sin^5(t)$$

$$\therefore \sin^5(t) = \frac{1}{2^5} \left[\sin(5t) - 5\sin(3t) + 10\sin(t) \right]$$

$$\therefore L[\sin^5(t)] = \frac{1}{2^4} \left[\frac{5}{s^2+25} - 5 \cdot \frac{3}{s^2+9} + 10 \cdot \frac{1}{s^2+1} \right]$$

$$\therefore L[\sin^5(t)] = \frac{1}{2^4} \left[\frac{5}{s^2+25} - \frac{15}{s^2+9} + \frac{10}{s^2+1} \right] = \frac{120}{(s^2+1)(s^2+9)(s^2+25)}$$

(find
out!)

Q. $y = \cos(t) \cdot \cos(2t) \cdot \cos(3t)$. Find $L[y]$ giving clear steps to answer.

Ans.

$$= \frac{1}{2} [\cos(3t) + \cos(t)] \cdot \cos(3t), (2) \Phi = [(1)H] + (1)E$$

$$= \frac{1}{2} [\cos(3t) \cdot \cos(3t) + \cos(t) \cdot \cos(3t)]$$

$$= \frac{1}{2} \left[\left(\frac{1 + \cos(6t)}{2} \right) + \frac{1}{2} (\cos(4t) + \cos(2t)) \right], \text{ by alignment}$$

$$= \frac{1}{4} [1 + \cos(6t) + \cos(4t) + \cos(2t)], \text{ by } \dots$$

$$\therefore L[\cos(t) \cdot \cos(2t) \cdot \cos(3t)] = \frac{1}{4} \left[\frac{1 + (s^2 + s^4 + s^6 + s^8)}{s^2 + 3s^4 + 3s^6 + s^8} \right],$$

$$(1)(1)(1) = (1)(1)$$

$$(1)(2)(1) = [(1)(2)(1)] \dots$$

Q. Find $L\left[\frac{\cos(\sqrt{t})}{\sqrt{t}}\right]$.

$$\text{Ans. } \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\therefore \cos(\sqrt{t}) = \frac{1}{2!} - \frac{t}{4!} + \frac{t^3}{6!} - \frac{t^5}{8!} + \dots$$

$$\therefore L\left[\frac{\cos(\sqrt{t})}{\sqrt{t}}\right] = \frac{\frac{1}{2}}{s^{1/2}} - \frac{1}{2!} \frac{\frac{1}{4}}{s^{3/2}} + \frac{\frac{1}{24}}{4!} \frac{\frac{1}{16}}{s^{5/2}} - \frac{1}{8!} \frac{\frac{1}{256}}{s^{7/2}} + \dots$$

$$\therefore = \frac{\sqrt{\pi}}{\sqrt{s}} \left[1 - \left(\frac{1}{4s} \right) + \frac{3}{96} \cdot \frac{1}{s^2} - \frac{15}{8 \times 6 \times 5!} \cdot \frac{1}{s^3} + \dots \right]$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}} \left[1 - \left(\frac{1}{4s} \right) + \frac{1}{2!} \left(\frac{1}{4s} \right)^2 - \frac{1}{3!} \left(\frac{1}{4s} \right)^3 + \frac{1}{4!} \left(\frac{1}{4s} \right)^4 - \dots \right]$$

$$\text{Let } \frac{1}{4s} = x$$

$$\therefore L\left[\frac{\cos(\sqrt{t})}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{\sqrt{s}} \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots \right)$$

$$L\left[\frac{\cos(\sqrt{t})}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{\sqrt{s}} \cdot e^{-x}$$

$$\therefore L\left[\frac{\cos(\sqrt{t})}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{\sqrt{s}} \cdot e^{-\frac{1}{4s}}$$

- Change of scale property $\rightarrow L[f(at)] = \frac{1}{a} \phi(s/a)$

If $L[f(t)] = \phi(s)$, then $L[f(at)] = \frac{1}{a} \phi(s/a)$

Example: If $L[\operatorname{erf}(st)] = \frac{1}{s} \phi(1/s)$, then $L[\operatorname{erf}(2st)]$

Find $L[\operatorname{erf}(2st)]$

Solution: $L[\operatorname{erf}(2st)] = L[\operatorname{erf}(\sqrt{4s}t)]$

Let $f(t) = \operatorname{erf}(st)$

$\therefore f(\sqrt{4s}t) = \operatorname{erf}(\sqrt{4s}t)$

$\therefore L[\operatorname{erf}(\sqrt{4s}t)] = \frac{1}{\sqrt{4s}} \cdot \phi(s/\sqrt{4s})$

$$= \frac{1}{2\sqrt{s}} \cdot \frac{\sqrt{4s} + \sqrt{4s^2 - 1}}{\sqrt{4s^2 - 1}} = \frac{\sqrt{4s} + \sqrt{4s^2 - 1}}{2\sqrt{s}\sqrt{4s^2 - 1}} = \frac{\sqrt{4s} + \sqrt{4s^2 - 1}}{2\sqrt{s}\sqrt{4s^2 - 1}} = \frac{\sqrt{4s} + \sqrt{4s^2 - 1}}{2\sqrt{s}\sqrt{4s^2 - 1}}$$

$\therefore L[\operatorname{erf}(\sqrt{4s}t)] = \frac{2}{\sqrt{s}\sqrt{4s+1}}$

$$= \frac{2}{\sqrt{s}\sqrt{4s+1}} = \frac{2}{\sqrt{s}\sqrt{4s+1}} = \frac{2}{\sqrt{s}\sqrt{4s+1}} = \frac{2}{\sqrt{s}\sqrt{4s+1}}$$

$$= \frac{2}{\sqrt{s}\sqrt{4s+1}} = \frac{2}{\sqrt{s}\sqrt{4s+1}} = \frac{2}{\sqrt{s}\sqrt{4s+1}} = \frac{2}{\sqrt{s}\sqrt{4s+1}}$$

$$= \frac{2}{\sqrt{s}\sqrt{4s+1}} = \frac{2}{\sqrt{s}\sqrt{4s+1}} = \frac{2}{\sqrt{s}\sqrt{4s+1}} = \frac{2}{\sqrt{s}\sqrt{4s+1}}$$

$$\therefore L[\operatorname{erf}(2st)] = \frac{2}{\sqrt{s}\sqrt{4s+1}} = \frac{2}{\sqrt{s}\sqrt{4s+1}} = \frac{2}{\sqrt{s}\sqrt{4s+1}}$$

$$= \frac{2}{\sqrt{s}\sqrt{4s+1}} = \frac{2}{\sqrt{s}\sqrt{4s+1}} = \frac{2}{\sqrt{s}\sqrt{4s+1}}$$

- first Shifting Property \rightarrow

\rightarrow shifting shifting formula

If $L[f(t)] = \phi(s)$,

$$(2) \phi = [(t)] \perp \text{FT}$$

$L[(t)]$ then $L[e^{at} \cdot f(t)] = (\phi(s-a)) \perp \Rightarrow (3) e$ bno

also, $L[e^{-at} \cdot f(t)] = \phi(s+a)$

$$(2) \phi \stackrel{2.0}{=} [(t)e] \perp \text{next}$$

$$\text{Example: Prove that } L[\sinh(t/2) \cdot \sin(\sqrt{3}t/2)] = \frac{\sqrt{3}}{2} \left[\frac{s}{s^4 + s^2 + 1} \right]$$

Solution:

$$\text{LHS} = \sinh(t/2) \cdot \sin(\sqrt{3}t/2)$$

$$= (e^{t/2} - e^{-t/2}) \cdot \sin(\sqrt{3}t/2) = [(t)] \perp \text{FT}$$

$$(2) \phi \stackrel{2.0}{=} [(t)] \perp \text{next}$$

$$= \frac{1}{2} [e^{t/2} \cdot \sin(\sqrt{3}t/2) - e^{-t/2} \cdot \sin(\sqrt{3}t/2)]$$

\perp

$$\text{Let } \sin(\sqrt{3}t/2) = f(t) \perp \Rightarrow [(t)] \perp : s = n$$

$$\therefore L[\sinh(t/2) \cdot \sin(\sqrt{3}t/2)] = \frac{1}{2} (L[e^{t/2} \cdot f(t)] - L[e^{-t/2} \cdot f(t)])$$

$$(2) \phi = (2) \phi \stackrel{2.0}{=} [(2)] \perp : s = n$$

$$= \frac{1}{2} (\phi(s - \gamma_2) - \phi(s + \gamma_2))$$

$$= \frac{1}{2} \left[\frac{\sqrt{3}/2}{(s - \gamma_2)^2 + 3/4} - \frac{\sqrt{3}/2}{(s + \gamma_2)^2 + 3/4} \right]$$

$$= \frac{\sqrt{3}}{4} \left[\frac{2s}{(s^2 + 1)^2 - s^2} \right]$$

$$= \frac{\sqrt{3}}{2} \left[\frac{s}{s^4 + s^2 + 1} \right] //$$

Hence Proved.

- Second Shifting Property \rightarrow \leftarrow subsequent property (contd)

If $L[f(t)] = \phi(s)$

and $g(t) = \begin{cases} f(t-a) & \text{for } t > a \\ 0 & \text{for } t \leq a \end{cases}$

then $L[g(t)] = e^{-as} \phi(s)$

$$\text{Given } L[f(t)] = \int_{0}^{\infty} f(t) e^{-st} dt = \int_{0}^{\infty} g(t) e^{-st} dt = \int_{a}^{\infty} f(t-a) e^{-st} dt = \int_{a}^{\infty} g(t) e^{-st} dt$$

- Effect of Multiplication by t \rightarrow

$$(s^2 \phi(s))' = s^2 \phi(s) + 2s \phi'(s) = 2s \phi(s)$$

If $L[f(t)] = \phi(s)$

then $L[t^n \cdot f(t)] = (-1)^n \cdot \frac{d^n}{ds^n} [\phi(s)]$

$$[(s^2 \phi(s))' = s^2 \phi(s) + 2s \phi'(s)] \Rightarrow$$

If:

$$n=1 : L[t \cdot f(t)] = -\frac{d}{ds} \phi(s) = -\phi'(s)$$

$$n=2 : L[t^2 \cdot f(t)] = \frac{d^2}{ds^2} \phi(s) = \phi''(s)$$

$$\frac{d}{ds} [s^2 \phi(s)] = s^2 \phi(s) + 2s \phi'(s)$$

$$\frac{d^2}{ds^2} [s^2 \phi(s)] = 2s \phi(s) + s^2 \phi'(s)$$

$$\frac{d^3}{ds^3} [s^2 \phi(s)] = 2s^2 \phi(s) + 2s \phi'(s)$$

(repeat process)

Q. Find $L[t \cdot e^{3t} \cdot \sin(ut)]$ $\epsilon=2$ to $[f_{200}, \phi]$ b/w \therefore

$$\text{Ans. } L[e^{3t} \cdot (t \cdot \sin(ut))]_{200} \downarrow \epsilon_{b-} = [f_{200}, \phi] \downarrow \therefore \text{LWA}$$

$$= L[e^{3t} \cdot f(t)] = \phi(s-3)$$

$$\therefore \phi(s) = L[f(t)] \left(\frac{2}{1+s^2} \right) \epsilon_{b-} =$$

$$\left[2s \cdot (L[t \cdot \sin(ut)]) - (s-2) \cdot \epsilon_{(1+s^2)} \right] \downarrow =$$

$$= -\frac{d}{ds} (L[\sin(ut)])$$

$$= -\frac{d}{ds} \left(\frac{4}{s^2+16} \right)$$

$$= \frac{8s}{(s^2+16)^2}$$

$$\left[(s^2+16)^2 \cdot \epsilon_{\infty} - 8s \cdot \epsilon_{01} \right] \downarrow = (\epsilon) \phi$$

$$\therefore \phi(s-3) = \frac{8(s-3)}{(s-3)^2+16^2}$$

$$12 = (\epsilon) \phi$$

Q. Find $L[e^{3t} \cdot t \cdot \sqrt{1+\sin t}]$

$$\text{Ans. } \therefore L[e^{3t} \cdot f(t)] = \phi(s-3)$$

$$\therefore \phi(s) = L[f(t)] = L[t \cdot \sqrt{1+\sin t}] \text{ b/w to 200}$$

$$= -\frac{d}{ds} (L[\sqrt{1+\sin t}])$$

$$= -\frac{d}{ds} \left(L[\sin(\frac{t}{2}) + \cos(\frac{t}{2})] \right) \downarrow \text{next}$$

$$\therefore \frac{d}{ds} \left(\frac{\phi^{1/2}}{s^2+1/4} \right) \downarrow = \left[s(\frac{1}{2}) \right] \downarrow \text{b/w}$$

(complete!!)

$$= \frac{1}{2} \left(\frac{2s}{(s^2+1/4)^2} \right) + \frac{1}{s^2+1/4}$$

S 8 PS

Q. Find $L[t^3 \cdot \text{cost}]$ at $s=3$

$$\begin{aligned} \text{Ans. } \therefore L[t^3 \cdot \text{cost}] &= -\frac{d^3}{ds^3} (L[\text{cost}]) \\ &= -\frac{d^3}{ds^3} \left(\frac{s}{s^2+1} \right) \\ &= -\left[(s^2+1)^3 \cdot (6s^2-6) - (2s^3-6s) \cdot 3 \cdot (s^2+1) \cdot 2s \right] \\ &= \phi(s) \end{aligned}$$

 \therefore At $s=3$,

$$\phi(3) = -\left[\frac{10^3 \cdot 48 - (36 \cdot 3 \cdot 10 \cdot 2 \cdot 3)}{10^6} \right]$$

$$\phi(3) = \frac{21}{1250}$$

• Effect of Division by t^n If $L[f(t)] = \phi(s)$,

$$\text{then } L\left[\frac{f(t)}{t}\right] = \int_s^\infty \phi(s) \cdot ds$$

$$\text{also, } L\left[\frac{f(t)^2}{t^2}\right] = \int_s^\infty \int_s^\infty (\phi(s) \cdot ds) \cdot ds$$

Q. Find $L \left[\frac{e^{-2t} \cdot \sin(2t) \cosh(t)}{t} \right]$

$\left[\frac{f(s)}{s} \right] \text{ if } f(t) = \dots$

Ans. Let $f(t) = e^{-2t} \cdot \sin(2t) \cdot \cosh(t)$, $t > 0$ very small
 $= e^{-2t} \cdot \left(\frac{e^t + e^{-t}}{2} \right) \cdot \sin(2t)$
 $= \frac{1}{2} (e^{-t} + e^{-3t}) \cdot \sin(2t)$

$(\text{H.S. of } e^{-2t} = (2t - t_0) \cdot 2^n)$

$\therefore L[f(t)] = \frac{1}{2} (L[e^{-t} \cdot \sin(2t)] + L[e^{-3t} \cdot \sin(2t)])$

~~$\frac{1}{2} (\phi(s+1) + \phi(s+3))$~~ \leftarrow [Using first-shifting property and $L[\sin(st)]$]

$$= \frac{1}{2} \left(\frac{\pi}{(s+1)^2 + 4} + \frac{3/2 \cdot \pi}{(s+3)^2 + 4} \right)$$

$$= \phi(s) \quad \left[\text{ab. } (\phi) \right] = \left[(s) \right] \text{ if } s > 0$$

$\therefore L[f(t)] = \int_s^\infty \phi(s) \cdot ds$

(Taking S.O.R.)

(using property)

$$= \int_s^\infty \left[\left(\frac{\pi}{(s+1)^2 + 4} + \frac{3/2 \cdot \pi}{(s+3)^2 + 4} \right) \right] ds$$

$$\therefore L[f(t)] = \left[\frac{1}{2} \tan^{-1}\left(\frac{s+1}{2}\right) + \frac{1}{2} \tan^{-1}\left(\frac{s+3}{2}\right) \right]_s^\infty$$

$$= \frac{1}{2} \left[\left(\frac{\pi}{2} + \frac{\pi}{2} \right) - \left(\tan^{-1}\left(\frac{s+1}{2}\right) + \tan^{-1}\left(\frac{s+3}{2}\right) \right) \right]$$

$$= \frac{1}{2} \left[\cot^{-1}\left(\frac{s+1}{2}\right) + \cot^{-1}\left(\frac{s+3}{2}\right) \right]$$

$\left[\text{ab. } (\phi) \right] = \left[\text{ab. } (\phi) \right]$

(using property)

$$(2)_{\text{pol. }} = (n!)_{\text{pol. }} \quad (n \geq 2 \text{ even})$$

Q. Find $L\left[\frac{\sin^2 t}{t}\right]$

Hence prove that $\int_0^\infty \frac{e^{-st} \cdot \sin^2 t}{t} dt = \frac{1}{4} \log_e(5)$

Ans. Let $f(t) = \sin^2(t)$

$$= \frac{1}{2} (1 - \cos(2t))$$

$$\therefore L[f(t)] = \frac{1}{2} (L[1] - L[\cos(2t)])$$

$$= \frac{1}{2} \left(\frac{1}{s} - \frac{s}{s^2 + 4} \right)$$

$$= \phi(s)$$

$$\therefore L\left[\frac{f(t)}{t}\right] = \int_s^\infty \phi(s) \cdot ds$$

$$= \frac{1}{2} \left[\log(s) - \frac{1}{2} \log(s^2 + 4) \right]_s^\infty$$

$$= \frac{1}{2} \left[\frac{1}{2} \log(s^2) - \frac{1}{2} \log(s^2 + 4) \right]_s^\infty$$

(since cannot
directly put ∞)

$$= \frac{1}{4} \left(\log\left(\frac{s^2}{s^2 + 4}\right) \right)_s^\infty = \frac{1}{4} \left(\log\left(\frac{1}{1 + 4/s^2}\right) \right)_s^\infty$$

$$\therefore L\left[\frac{f(t)}{t}\right] = \frac{1}{4} \left(0 - \log\left(\frac{1}{1 + 4/s^2}\right) \right) = \frac{1}{4} \log\left(\frac{s^2 + 4}{s^2}\right) = L\left[\frac{\sin^2 t}{t}\right]$$

By first definition, $L\left[\frac{\sin^2 t}{t}\right] = \int_s^\infty e^{-st} \cdot \frac{\sin^2 t}{t} dt$

Putting $s = 1$

$$\int_0^\infty e^{-t} \cdot \frac{\sin^2 t}{t} dt \quad (\text{original question})$$

$$\therefore \text{putting } s = 1, \frac{1}{4} \log\left(\frac{1+4}{1}\right) = \frac{1}{4} \log(5) \quad //$$

Hence Proved.

Q. Evaluate $\int_0^\infty \frac{\cos(6t) - \cos(4t)}{t} dt$

(Ans) finding $L\left[\frac{\cos(6t) - \cos(4t)}{t}\right] = [(\frac{s}{s^2+36}) - (\frac{s}{s^2+16})]$

$$\text{Let } f(t) = \cos(6t) - \cos(4t)$$

$$\therefore L[f(t)] = L[\cos(6t)] - L[\cos(4t)] = \frac{s}{s^2+36} - \frac{s}{s^2+16} = \phi(s)$$

$$(\text{c}) \rightarrow (\text{d}) \rightarrow (\text{e}) \rightarrow (\text{f}) \rightarrow (\text{g}) \rightarrow (\text{h}) \rightarrow (\text{i}) \rightarrow (\text{j}) \rightarrow (\text{k}) \rightarrow (\text{l}) \rightarrow (\text{m})$$

$$\therefore L\left[\frac{f(t)}{t}\right] = \int_0^\infty \phi(s) \cdot ds$$

$$(\text{c}) \rightarrow (\text{d}) \rightarrow (\text{e}) \rightarrow (\text{f}) \rightarrow (\text{g}) \rightarrow (\text{h}) \rightarrow (\text{i}) \rightarrow (\text{j}) \rightarrow (\text{k}) \rightarrow (\text{l}) \rightarrow (\text{m})$$

$$= \int_s^\infty \frac{s}{s^2+36} \cdot ds - \int_s^\infty \frac{s}{s^2+16} \cdot ds$$

$$= \frac{1}{2} \left[\log(s^2+36) - \log(s^2+16) \right]_s^\infty$$

$$= \frac{1}{2} \left[\log\left(\frac{s^2+36}{s^2+16}\right) \right]_s^\infty$$

$$= \frac{1}{2} \left[\log\left(1 + \frac{20}{s^2+16}\right) \right]_s^\infty$$

$$= \frac{1}{2} \left[0 + \log\left(\frac{s^2+16}{s^2+36}\right) \right]$$

$$\therefore L\left[\frac{\cos(6t) - \cos(4t)}{t}\right] = \frac{1}{2} \log\left(\frac{s^2+16}{s^2+36}\right)$$

↓

$$\text{By first definition, } \rightarrow \int_0^\infty e^{-st} \cdot f(t) \cdot dt$$

$$\text{for } \int_0^\infty f(t) \cdot dt, \text{ put } s=0$$

$$\therefore \frac{1}{2} \log\left(\frac{0^2+16}{0^2+36}\right) = \log\left(\frac{4}{6}\right) = \log\left(\frac{2}{3}\right) //$$

• Laplace of derivative property \rightarrow

$$\mathcal{L}[f^{(n)}(t)] = s^n \cdot \mathcal{L}[f(t)] - s^{n-1} \cdot f(0) - s^{n-2} \cdot f'(0) - s^{n-3} \cdot f''(0) - \dots - f^{(n-1)}(0)$$

$$n=1, \mathcal{L}[f'(t)] = s \cdot \mathcal{L}[f(t)] - f(0)$$

$$n=2, \mathcal{L}[f''(t)] = s^2 \cdot \mathcal{L}[f(t)] - s \cdot f(0) - f'(0)$$

$$n=3, \mathcal{L}[f'''(t)] = s^3 \cdot \mathcal{L}[f(t)] - s^2 \cdot f(0) - s \cdot f'(0) - f''(0)$$

$$\mathcal{L}[(2+5s^2) \cos t] = (2s + 5s^3) \frac{1}{1-s^2}$$

$$2s + 5s^3 = \frac{1}{1-s^2}$$

$$\frac{1}{1-(2+5s^2)s^2} = \frac{1}{1-s^2}$$

$$\frac{1}{(2+5s^2)s^2} = \frac{1}{1-s^2}$$

$$\frac{(2+5s^2)s^2}{(2+5s^2)s^2 - s^2} = \frac{(2+5s^2)s^2}{(2+5s^2)s^2 - s^2}$$

$$\frac{1}{(2+5s^2)s^2} = \frac{1}{(2+5s^2)s^2 - s^2}$$

$$\frac{1}{(2+5s^2)s^2} = \frac{1}{(2+5s^2)s^2 - s^2}$$

Q. Using Laplace transform of $\cos(at)$, find Laplace transform of $\sin(at)$. [Derivative Property]

$$(2) \Phi = [(+)f] \cup \text{LT}$$

Ans. We know that $L[\cos(at)] = \frac{us \cdot (u)f^+}{s^2 + a^2} \cup \text{mult}$

$$\text{but } -a \sin(at) = \frac{d}{dt} (\cos(at))$$

(natural diff $\frac{dt}{dt}$ parallel to natural angle shift)

$$\therefore -\frac{1}{a} L\left[\frac{d}{dt} (\cos(at))\right] = L[\sin(at)] \quad (3) \text{ frs } (i)$$

$$u \cdot u \cdot (u)^{200} \int_0^\infty \frac{du}{u} \quad (ii)$$

$$u \cdot (u)^{200} \int_0^\infty s^+ \cdot (+) \cdot 200 \int_0^\infty \frac{du}{u} \quad (iii)$$

$$\therefore L[\sin(at)] = -\frac{1}{a} \left(s \cdot L[f(t)] - f(0) \right) \quad (us)^{200} \cdot u \cdot u \int_0^\infty \frac{du}{u} \quad (iv)$$

\therefore When $f(t) = \cos(at)$

$$[(+)f] \cup \text{LT} \quad (3) \text{ frs } -1 = (3) \text{ frs } \quad (v)$$

$$= -\frac{1}{a} \left(s \left(\frac{s}{s^2 + a^2} \right) \int_0^\infty \frac{du}{u} \right) \cup -(-1) \cup = [(3) \text{ frs}] \cup \therefore$$

which

$$[(\text{natural rcs}) \text{ frs} \equiv \frac{1}{a} \int_0^\infty \frac{\cos(at)}{s^2 + a^2} \frac{du}{u}] \cup \frac{s}{s^2 + a^2} \int_0^\infty \frac{du}{u} \cup s \equiv (3) \text{ frs}$$

$$\frac{1}{a} \int_0^\infty \frac{du}{u} \cdot \frac{s^2 - a^2}{s^2 + a^2} \cup \frac{s^2 - a^2}{s^2 + a^2} \cdot \frac{1}{a} \int_0^\infty \frac{du}{u} \cup$$

$$\therefore L[\sin(at)] = \frac{a}{s^2 + a^2} \int_0^\infty \frac{du}{u} \cdot \frac{s}{s^2 + a^2} \cup = (3) \text{ frs} \therefore$$

$$\left[\int_0^\infty \frac{du}{u} \cdot \frac{s}{s^2 + a^2} \right] \cup \frac{1}{a} \int_0^\infty \frac{du}{u} \cup = [(3) \text{ frs}] \cup \therefore$$

$$[(sV - V_0) \cup = (3) \Phi \text{ (mult)}] \quad \frac{(3)V}{2} \cdot \frac{1}{a} \int_0^\infty \frac{du}{u} \cup$$

$$\frac{1}{a} \int_0^\infty \frac{du}{u} \cup = [sV - V_0] \cup \therefore$$

$$\frac{1}{\sqrt{1+2V}} \approx \frac{1}{2} \cdot \frac{1}{\sqrt{1+2V}} \cdot \frac{1}{a} \cup = [(3) \text{ frs}] \cup \therefore$$

$$\frac{1}{\sqrt{1+2V}} = \frac{1}{2} \cdot \frac{1}{\sqrt{1+2V}} \cup = [(3) \text{ frs}] \cup \therefore$$

• Laplace of integral property \Rightarrow constant multiple which is integer value

If $L[f(t)] = \phi(s)$,

$$\text{then } L\left[\int_0^t f(u) \cdot du\right] = \frac{\phi(s)}{s}$$

Q. Find Laplace transform of following functions:

$$(i) \operatorname{erf}_c(rt)$$

$$(ii) \int_t^\infty \cos(u) \cdot du$$

$$(iii) \cosh(t) \cdot \int_0^t e^u \cosh(u) \cdot du$$

$$(iv) \int_0^t u \cdot e^{-3u} \cdot \cos^2(2u) \cdot du$$

$$(i) \operatorname{erf}_c(rt) = 1 - \operatorname{erf}(rt) \quad (\text{is a property})^*$$

$$\therefore L[\operatorname{erf}_c(rt)] = L(1) - L[\operatorname{erf}(rt)]$$

NOW,

$$\operatorname{erf}(rt) = \frac{2}{\sqrt{\pi}} \int_0^{rt} e^{-u^2} \cdot du \quad [\text{definition of erf (error function)}]$$

$$\text{Let } u^2 = v \therefore 2u \cdot du = dv \quad du = \frac{dv}{2\sqrt{v}}$$

$$\therefore \operatorname{erf}(rt) = \frac{2}{\sqrt{\pi}} \int_0^{r^2 t^2} e^{-v} \cdot \sqrt{v} \cdot \frac{1}{2} dv$$

$$\therefore L[\operatorname{erf}(rt)] = \frac{1}{\sqrt{\pi}} L\left[\int_0^{r^2 t^2} e^{-v} \cdot \sqrt{v} \cdot dv\right]$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{\phi(s)}{s} \quad [\text{where } \phi(s) = L[e^{-v} \cdot \sqrt{v}]]$$

$$\therefore L[e^{-v} \cdot \sqrt{v}] = \frac{\sqrt{\pi}}{(s+1)^{3/2}}$$

$$\therefore L[\operatorname{erf}(rt)] = \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{s+1}} \cdot \frac{1}{s} = \frac{1}{s\sqrt{s+1}}$$

$$\therefore L[\operatorname{erf}_c(rt)] = \frac{1}{s} - \frac{1}{s\sqrt{s+1}} = \frac{\sqrt{s+1} - 1}{s\sqrt{s+1}}$$

$$\textcircled{2} \quad t \int_0^\infty \frac{\cos(u)}{u} \cdot du$$

$$\text{nb. } (u) \text{ d}u \cdot u \Big|_0^\infty \cdot (t) \text{ d}u \quad \text{(1)}$$

$$\text{let } f(t) = t \int_0^\infty \frac{\cos u}{u} \cdot du$$

\therefore Put $u=vt$, $du=t \cdot dv$ $\quad \text{(2)}$

$$\therefore f(t) = \int_{v=1}^{\infty} \frac{\cos(vt)}{vt} \cdot t \cdot dv \quad [(t) + t_0 + (t) + t_0] \perp \frac{1}{s} =$$

$$\therefore L[f(t)] = \int_0^\infty e^{-st} \cdot f(t) \cdot dt \quad [(\text{+}) + (\text{-})] \perp \frac{1}{s} =$$

$$= \int_0^\infty e^{-st} \left[\int_{v=1}^{\infty} \frac{\cos(vt)}{v} dv \right] dt \quad [(1+2)\phi + (1-2)\phi] \perp \frac{1}{s} =$$

$$\quad \quad \quad \text{nb. } (u) \text{ d}u \cdot u \Big|_0^\infty \cdot u \Big|_0^\infty \Big] = [(\text{+})\phi] \perp = (2)\phi \quad \text{, similarly}$$

$$= \int_{v=1}^{\infty} \frac{dv}{v} \left[\int_{t=0}^{\infty} e^{-st} \cdot \cos(vt) \cdot dt \right] \quad = (2)\phi \rightarrow [(\omega) \text{ d}\omega \cdot \omega \Big|_0^\infty] \perp \text{, similarly}$$

$$= \int_1^\infty \frac{dv}{v} - L[\cos(vt)] \quad = \int_1^\infty \frac{dv}{v} \perp \frac{s(2)\phi}{\sqrt{t^2+s^2}}$$

$$= s \int_1^\infty \frac{1}{v\sqrt{v^2+s^2}} dv \quad \left[(v) \text{ d}v \Big|_1^\infty \right] \text{ (partial fraction)} \quad 2 = [(\omega) \text{ d}\omega] \perp$$

$$= \frac{1}{s} \int_1^\infty \left(\frac{1}{v} - \frac{v}{v^2+s^2} \right) dv \quad (\text{by partial fractions})$$

$$\left[\frac{1}{v} \cdot 2 + \frac{1}{2} \log(v^2+s^2) \right]_1^\infty = \text{cancel last term} \therefore$$

$$= \frac{1}{s} \left[\log(v) - \frac{1}{2} \log(v^2+s^2) \right]_1^\infty$$

$$= \frac{1}{2s} \left[\log\left(\frac{v^2}{v^2+s^2}\right) \right]_1^\infty = \frac{1}{2s} \left[\log\left(1 + \frac{s^2}{v^2}\right) \right]_1^\infty$$

$$= \frac{1}{2s} \left[\log(1+s^2) - \log(1) \right]$$

$$= \frac{1}{2s} \cdot \log(1+s^2)$$

$$(3) \cosh(t) \cdot \int_0^t e^u \cosh(u) du$$

$$\rightarrow L \left[\frac{(e^t + e^{-t})}{2} \cdot \int_0^t e^u \cosh(u) du \right]$$

f(t)

$$= \frac{1}{2} L [e^t f(t) + e^{-t} f(t)]$$

$$= \frac{1}{2} L [e^t \cdot f(t)] + \frac{1}{2} L [e^{-t} \cdot f(t)]$$

$$= \frac{1}{2} [\phi(s-1) + \phi(s+1)]$$

$$\text{where, } \phi(s) = L[f(t)] = L \left[\int_0^t e^u \cosh(u) du \right]$$

$$\text{Let } L[e^u \cosh(u)] = \phi(s) = \frac{1}{2} \left[s - \frac{1}{s-1} \right]$$

$$\therefore \phi(s) = \frac{s}{s-1}$$

To find $L[e^u \cosh(u)]$

$$L[\cosh(u)] = s \quad \therefore L[e^u \cosh(u)] = \frac{s-1}{(s-1)^2 - 1}$$

(condition $s > 1$)

$$\therefore \text{original Laplace} = \frac{1}{2} \left[\frac{s-2}{(s-2)^2 - 1} \left(\frac{1}{s-1} \right) + \frac{s}{s^2 - 1} \left(\frac{1}{s+1} \right) \right]$$

$$\left[\frac{(s_2 - 2)(s_1)}{(s_2 - 2)^2 - 1} \right] \cdot \frac{1}{s-1} = \left[\frac{(s_1 - 1)}{(s_1 - 1)^2 - 1} \right] \cdot \frac{1}{s-1} =$$

$$\left[(s_2 - 2)(s_1) - (s_2 + 1)(s_1) \right] \cdot \frac{1}{s-1} =$$

$$(s_2 + 1)(s_1) \cdot \frac{1}{s-1} =$$

$$(4) \int_0^t u \cdot e^{-3u} \cdot \cos^2(2u) du$$

$$(2)\phi = [(0)] \downarrow \text{FT} *$$

$$f(t) \cdot t \cdot t^2 \cdot s^2 =$$

(2) ϕ to mroth2N0IT soal gelar struktur ti (3)t next

$$[(2)\phi]^{1-1} = (t) \downarrow$$

soal gelar rot gelarrot

$$\downarrow \quad [1] = \begin{bmatrix} 1 \\ 2 \end{bmatrix}^{1-1} \downarrow \quad \frac{1}{2} = [1] \downarrow \leftarrow$$

$$\downarrow \quad f_{09} = \left[\frac{1}{p+2} \right]^{1-1} \downarrow \quad \frac{1}{p+2} = [f_9] \downarrow \leftarrow$$

$$\downarrow \quad f_{09} = \begin{bmatrix} 1 \\ p+2 \end{bmatrix}^{1-1} \downarrow \text{vba/12}$$

$$\downarrow \quad (f_0)_{200} = \begin{bmatrix} 2 \\ s_0 + s_2 \end{bmatrix}^{1-1} \downarrow \quad 2 = [(f_0)_{200}] \downarrow \leftarrow$$

$$\downarrow \quad (f_0)_{1128} = \begin{bmatrix} 1 \\ s_0 + s_3 \end{bmatrix}^{1-1} \downarrow \quad 0 = [(f_0)_{1128}] \downarrow$$

$$\downarrow \quad (f_0)_{11200} = \begin{bmatrix} 2 \\ s_0 + s_2 \end{bmatrix}^{1-1} \downarrow \quad 2 = [(f_0)_{11200}] \downarrow \leftarrow$$

$$\downarrow \quad (f_0)_{11012} = \begin{bmatrix} 1 \\ s_0 + s_2 \end{bmatrix}^{1-1} \downarrow \quad 0 = [(f_0)_{11012}] \downarrow$$