

Vector Algebra

$$[\vec{r} \vec{p} \vec{q}] = [(\vec{q} + \vec{r}) \times (\vec{r} + \vec{p})] \cdot (\vec{r} + \vec{q})$$

$$[(\vec{q} + \vec{r}) \times (\vec{r} + \vec{p})] \cdot \vec{r} + [(\vec{q} + \vec{r}) \times (\vec{r} + \vec{p})] \cdot \vec{q} = 2/11 \cdot 2/19$$

• **Scalar Triple Product** $\rightarrow [(\vec{q} + \vec{r}) \times \vec{r} + (\vec{q} + \vec{r}) \times \vec{p}] \cdot \vec{q} =$

$$[(\vec{q} \times \vec{r}) + (\vec{r} \times \vec{r}) + (\vec{q} \times \vec{p}) + (\vec{r} \times \vec{p})] \cdot \vec{q} =$$

Let \vec{a} , \vec{b} and \vec{c} be any 3 vectors. Then, $(\vec{a} \times \vec{b}) \cdot \vec{c}$ is called scalar triple product of \vec{a} , \vec{b} and \vec{c} . $[\vec{a} \vec{b} \vec{c}] + [\vec{b} \vec{c} \vec{a}] =$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = [\vec{a} \vec{b} \vec{c}]$$

\rightarrow **Properties:**

$$(1) (\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b} \quad (\text{cyclic})$$

$$(2) (\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$(3) [\vec{a} \vec{b} \vec{c}] = -[\vec{a} \vec{c} \vec{b}] = -[\vec{b} \vec{a} \vec{c}] = -[\vec{c} \vec{b} \vec{a}]$$

$$(4) [\vec{a} \vec{a} \vec{b}] = [\vec{a} \vec{b} \vec{a}] = [\vec{b} \vec{a} \vec{a}] = 0$$

$$(5) \text{ If } \vec{a}, \vec{b}, \vec{c} \text{ are coplanar, then } [\vec{a} \vec{b} \vec{c}] = 0.$$

$$(6) \text{ Volume of tetrahedron (if } \vec{a}, \vec{b}, \vec{c} \text{ are sides with same initial point),}$$

$$= \frac{1}{6} [\vec{a} \vec{b} \vec{c}]$$

$$(7) \text{ Volume of parallelepiped, whose coterminal edges are } \vec{a}, \vec{b}, \vec{c},$$

$$= [\vec{a}, \vec{b}, \vec{c}]$$

Q. Show that $(\bar{p} + \bar{q}) \cdot [(\bar{q} + \bar{r}) \times (\bar{r} + \bar{p})] = 2[\bar{p} \bar{q} \bar{r}]$

$$\begin{aligned} \text{Ans LHS} &= \bar{p} \cdot [(\bar{q} + \bar{r}) \times (\bar{r} + \bar{p})] + \bar{q} \cdot [(\bar{q} + \bar{r}) \times (\bar{r} + \bar{p})] \\ &= \bar{p} \cdot [\bar{q} \times (\bar{r} + \bar{p}) + \bar{r} \times (\bar{r} + \bar{p})] + \bar{q} \cdot [\bar{q} \times (\bar{r} + \bar{p}) + \bar{r} \times (\bar{r} + \bar{p})] \\ &= \bar{p} \cdot [\bar{q} \times \bar{r} + (\bar{q} \times \bar{p}) + (\bar{r} \times \bar{r}) + (\bar{r} \times \bar{p})] \\ &\quad + \bar{q} \cdot [(\bar{q} \times \bar{r}) + (\bar{q} \times \bar{p}) + (\bar{r} \times \bar{r}) + (\bar{r} \times \bar{p})] \quad \text{[}\bar{r} \times \bar{r} = 0\text{]} \\ &= [\bar{p} \bar{q} \bar{r}] + [\bar{p} \bar{q} \bar{p}] + [\bar{p} \bar{r} \bar{p}] + [\bar{q} \bar{q} \bar{r}] + [\bar{q} \bar{q} \bar{p}] + [\bar{q} \bar{r} \bar{p}] \end{aligned}$$

But $[\bar{a} \bar{a} \bar{b}] = 0$

$\therefore \text{LHS} = [\bar{p} \bar{q} \bar{r}] + [\bar{q} \bar{r} \bar{p}]$

But $[\bar{q} \bar{r} \bar{p}] = [\bar{p} \bar{q} \bar{r}]$ (cyclic)

$\therefore \text{LHS} = 2[\bar{p} \bar{q} \bar{r}] = \text{RHS}$

Hence Proved.

• Vector Triple Product

Let $\bar{a}, \bar{b}, \bar{c}$ be any 3 vectors. Then vector triple product of $\bar{a}, \bar{b}, \bar{c}$ is $(\bar{a} \times \bar{b}) \times \bar{c} = [\bar{a} \bar{b} \bar{c}] = [\bar{c} \bar{a} \bar{b}]$

$$(\bar{a} \times \bar{b}) \times \bar{c} = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{b} \cdot \bar{c})\bar{a}$$

$$[\bar{a} \bar{b} \bar{c}] = [\bar{c} \bar{a} \bar{b}]$$

(Take $\bar{a} = \hat{i}, \bar{b} = \hat{j}, \bar{c} = \hat{k}$ and verify the result)

$$[\hat{i} \hat{j} \hat{k}] = 1$$

$\bar{a} \cdot \bar{a} = 1, \bar{b} \cdot \bar{b} = 1, \bar{c} \cdot \bar{c} = 1$ and $\bar{a} \cdot \bar{b} = 0, \bar{b} \cdot \bar{c} = 0, \bar{c} \cdot \bar{a} = 0$

$$[\hat{i} \hat{j} \hat{k}] = 1$$

Q. If $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors, then prove that $(\vec{a} \times \vec{b}), (\vec{b} \times \vec{c}), (\vec{c} \times \vec{a})$ are also coplanar vectors.

Ans. Given $[\vec{a} \ \vec{b} \ \vec{c}] = 0$, we need to prove $[\vec{a} \times \vec{b} \ \vec{b} \times \vec{c} \ \vec{c} \times \vec{a}] = 0$

$$\therefore (\vec{a} \times \vec{b}) \cdot [(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})] = 0$$

$$\text{Let } \vec{c} \times \vec{a} = \vec{m} \cdot [\vec{a} \cdot \vec{b}] - (\vec{a} \times \vec{b}) \cdot [\vec{a} \cdot \vec{b}] = 0$$

$$\begin{aligned} \therefore (\vec{a} \times \vec{b}) \cdot [(\vec{b} \times \vec{c}) \times \vec{m}] \\ = (\vec{a} \times \vec{b}) \cdot [(\vec{b} \cdot \vec{m})\vec{c} - (\vec{c} \cdot \vec{m})\vec{b}] \\ = (\vec{a} \times \vec{b}) \cdot [(\vec{b} \cdot (\vec{c} \times \vec{a}))\vec{c} - (\vec{c} \cdot (\vec{c} \times \vec{a}))\vec{b}] \\ = (\vec{a} \times \vec{b}) \cdot [\vec{a} \ \vec{b} \ \vec{c}] \cdot \vec{c} \end{aligned}$$

$$\begin{aligned} &= [\vec{a} \ \vec{b} \ \vec{c}] (\underbrace{(\vec{a} \times \vec{b}) \cdot \vec{c}}_{\text{Scalar}}) = [\vec{a} \ \vec{b} \ \vec{c}] \cdot [\vec{a} \ \vec{b} \ \vec{c}] \\ &= [\vec{a} \ \vec{b} \ \vec{c}]^2 \end{aligned}$$

$$\text{But } [\vec{a} \ \vec{b} \ \vec{c}] = 0$$

$$\therefore [\vec{a} \times \vec{b} \ \vec{b} \times \vec{c} \ \vec{c} \times \vec{a}] = 0 //$$

Q. Show that $[\bar{a} \times (\bar{a} \times \bar{b})] \cdot (\bar{a} \times \bar{c}) = [\bar{a} \bar{b} \bar{c}] (\bar{a} \cdot \bar{a})$

Ans. LHS = $[(\bar{a} \cdot \bar{b})\bar{a} - (\bar{a} \cdot \bar{a})\bar{b}] \cdot (\bar{a} \times \bar{c})$

$$= \underbrace{[(\bar{a} \cdot \bar{b})\bar{a}]}_{\text{scalar}} \cdot (\bar{a} \times \bar{c}) - \underbrace{[(\bar{a} \cdot \bar{a})\bar{b}]}_{\text{scalar}} \cdot (\bar{a} \times \bar{c})$$

$$= [\bar{a} \cdot \bar{b}] (\bar{a} \cdot (\bar{a} \times \bar{c})) - [\bar{a} \cdot \bar{a}] (\bar{b} \cdot (\bar{a} \times \bar{c}))$$

$$\begin{aligned} \text{LHS} &= -[\bar{b} \bar{a} \bar{c}] (\bar{a} \cdot \bar{a}) - \bar{a} (\bar{a} \cdot \bar{a}) \cdot (\bar{b} \times \bar{c}) = \\ &= -[\bar{a} \bar{c} \bar{b}] (\bar{a} \cdot \bar{a}) - \bar{a} (\bar{a} \cdot \bar{a}) \cdot (\bar{b} \times \bar{c}) = \\ &= [\bar{a} \bar{b} \bar{c}] \cdot (\bar{a} \cdot \bar{a}) = \text{RHS} \end{aligned}$$

Hence proved.

$$[\bar{a} \bar{b} \bar{c}] \cdot [\bar{a} \bar{b} \bar{c}] = (\bar{a} \cdot (\bar{b} \times \bar{c})) [\bar{a} \bar{b} \bar{c}] =$$

$$[\bar{a} \bar{b} \bar{c}]^2$$

$$0 = [\bar{a} \bar{b} \bar{c}]$$

$$0 = [\bar{a} \times \bar{b} \bar{c} \bar{c} \times \bar{a}]$$

Q. $(\bar{a} \times \bar{b}) \cdot (\bar{b} \times \bar{c}) \times (\bar{c} \times \bar{a}) = [\bar{a} \cdot (\bar{b} \times \bar{c})]^2$

Ans.

Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$, $\bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$

$$(\bar{b} \times \bar{c}) = (b_2c_3 - b_3c_2)\bar{i} + (b_3c_1 - b_1c_3)\bar{j} + (b_1c_2 - b_2c_1)\bar{k}$$

$$(\bar{a} \times (\bar{b} \times \bar{c})) = (a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1))\bar{i} + \dots$$

$$[\bar{a} \cdot (\bar{b} \times \bar{c})] = (a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1))$$

$$(\bar{b} \times \bar{c}) \cdot (\bar{a} \times (\bar{b} \times \bar{c})) = (\bar{b} \times \bar{c}) \cdot \bar{a} \cdot (\bar{b} \times \bar{c}) = [\bar{a} \cdot (\bar{b} \times \bar{c})]^2$$

$$\bar{a} \cdot (\bar{b} \times \bar{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$\bar{a} \cdot \bar{a} = 1$$

$$[\bar{a} \cdot (\bar{b} \times \bar{c})]^2$$

$$\bar{a} \cdot \bar{a} = 1$$

$$\bar{b} \cdot \bar{b} = 1$$

$$[\bar{a} \cdot (\bar{b} \times \bar{c})]$$

$$[\bar{a} \cdot (\bar{b} \times \bar{c})]$$

Q. Vectors $\vec{u}, \vec{v}, \vec{w}$ are non-coplanar. then show that $(\vec{u} \times \vec{v}), (\vec{v} \times \vec{w}), (\vec{w} \times \vec{u})$ are also non-coplanar. Hence obtain scalars $\exists L, m, n$ such that $\vec{u} = L(\vec{v} \times \vec{w}) + m(\vec{w} \times \vec{u}) + n(\vec{u} \times \vec{v})$

Ans. [first part same as question on page 93. !]

$$\therefore (\vec{u} \times \vec{v}) [\vec{u} \times \vec{v} \quad \vec{v} \times \vec{w} \quad \vec{w} \times \vec{u}] = [\vec{u} \vec{v} \vec{w}]^2 \neq 0 //$$

Taking 2nd parts

$$\vec{u} = L(\vec{v} \times \vec{w}) + m(\vec{w} \times \vec{u}) + n(\vec{u} \times \vec{v})$$

$$\therefore \vec{u} \cdot \vec{u} = L(\vec{v} \times \vec{w}) \cdot \vec{u} + m \underbrace{(\vec{w} \times \vec{u}) \cdot \vec{u}}_{=0} + n \underbrace{(\vec{u} \times \vec{v}) \cdot \vec{u}}_{=0}$$

$$\therefore L = \frac{\vec{u} \cdot \vec{u}}{[\vec{u} \vec{v} \vec{w}]} //$$

$$\text{Similarly, } m = \frac{\vec{u} \cdot \vec{v}}{[\vec{u} \vec{v} \vec{w}]} //, \quad n = \frac{\vec{u} \cdot \vec{w}}{[\vec{u} \vec{v} \vec{w}]} //$$

Q. Scalar Product of 4 vectors \rightarrow is to find out result.

$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$ is called Scalar product of 4 vectors, and is given by:

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{b} \cdot \vec{c} \\ \vec{a} \cdot \vec{d} & \vec{b} \cdot \vec{d} \end{vmatrix} \quad \leftarrow \text{Lagrange's Identity}$$

Q. Prove that $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) + (\vec{b} \times \vec{c}) \cdot (\vec{a} \times \vec{d})$

$$+ (\vec{c} \times \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$$

Ans. $\therefore \text{LHS} = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{b} \cdot \vec{c} \\ \vec{a} \cdot \vec{d} & \vec{b} \cdot \vec{d} \end{vmatrix} + \begin{vmatrix} \vec{b} \cdot \vec{a} & \vec{c} \cdot \vec{a} \\ \vec{b} \cdot \vec{d} & \vec{c} \cdot \vec{d} \end{vmatrix} + \begin{vmatrix} \vec{c} \cdot \vec{b} & \vec{a} \cdot \vec{b} \\ \vec{c} \cdot \vec{d} & \vec{a} \cdot \vec{d} \end{vmatrix}$

$$= [(\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})] + [(\vec{b} \cdot \vec{a})(\vec{c} \cdot \vec{d}) - (\vec{b} \cdot \vec{d})(\vec{c} \cdot \vec{a})] + [(\vec{c} \cdot \vec{b})(\vec{a} \cdot \vec{d}) - (\vec{c} \cdot \vec{d})(\vec{a} \cdot \vec{b})]$$

$$= 0 \quad (\text{all terms get cancelled})$$

- Vector Product of 4 vectors \rightarrow 16 different results

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ is vector product of 4 vectors.

$$(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) = [\bar{a} \bar{c} \bar{d}] \bar{b} - [\bar{b} \bar{c} \bar{d}] \bar{a}$$

$$= [\bar{a} b \bar{c}] - [\bar{a} b c]$$

Q. Prove that $[(\bar{a} \times \bar{b}) \times (\bar{a} \times \bar{c})] \cdot \bar{a} = (\bar{a} \cdot \bar{a}) [\bar{a} \bar{b} \bar{c}]$

Ans. LHS = $([\bar{a} \bar{a} \bar{c}] \bar{b} - [\bar{b} \bar{a} \bar{c}] \bar{a}) \cdot \bar{d} \bar{c} +$

$$\overline{a \cdot b} = \overline{a} \cdot \overline{b} \quad \text{or} \quad \overline{a \cdot b} = \overline{a} \cdot \overline{b} \quad \text{or} \quad \overline{a \cdot b} = \overline{a} \cdot \overline{b} \quad \text{or} \quad \overline{a \cdot b} = \overline{a} \cdot \overline{b}$$

$$[(5.7) \text{ Hence Proved } (5.5) = \\ [(5.3)(5.7) - (5.3)(5.9)] + \\ [(5.3) - (5.9)(5.5)] +$$

(bellemeo. top mist 1/2) 0 =

Q. $\vec{d} \cdot [\vec{a} \times (\vec{b} \times (\vec{c} \times \vec{d}))] = (\vec{b} \cdot \vec{d}) [\vec{a} \cdot \vec{c}]$

Ans. LHS = $\vec{d} \cdot [\vec{a} \times ((\vec{b} \cdot \vec{d})\vec{c} - (\vec{b} \cdot \vec{c})\vec{d})]$
 $= \vec{d} \cdot [(\vec{b} \cdot \vec{d})(\vec{a} \times \vec{c}) - (\vec{b} \cdot \vec{c})(\vec{a} \times \vec{d})]$
 $= (\vec{b} \cdot \vec{d}) [\vec{d} \cdot (\vec{a} \times \vec{c})] - (\vec{b} \cdot \vec{c}) [\vec{d} \cdot (\vec{a} \times \vec{d})]$
 $= (\vec{b} \cdot \vec{d}) [\vec{d} \cdot \vec{a} \cdot \vec{c}] - (\vec{b} \cdot \vec{c}) [\vec{d} \cdot \vec{a} \cdot \vec{d}]$
 $= (\vec{b} \cdot \vec{d}) [\vec{a} \cdot \vec{c}] = \text{RHS}$

Hence Proved

Directional derivative of function f in direction of unit vector \vec{u} at point (a, b) is

$$\nabla f(a, b) \cdot \vec{u} = \frac{df}{ds}$$

Directional Derivatives

$$\frac{df}{ds} = \nabla f \cdot \vec{u}$$

$$|\vec{u}| = 1$$

$$\frac{df}{ds} = |\nabla f| \cos \theta$$

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