→ **Conjugate** –

$$(x^*(n))_N \xrightarrow{DFT} X^*(N-k)$$

NOTE: Unitary transform ⇒

$$A \times A^H = I \quad \text{(any constant multiple of identity mat.)}$$

where $A^H = A^{*T}$, I = identity matrix

A^* is complex conjugate of A, if $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & +1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}$, $A^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & +1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix}$

Q. Find if DFT is a unitary transform.

Ans. kernel $= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}$ ∴ kernel$^H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix}$

∴ kernel × kernel$^H = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 4 \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ //

∴ DFT kernel satisfies Unitary transform condition. //

* Fast Fourier Transform (FFT)

[Algorithm to DFT]

In DFT, $W_N = e^{-j\frac{2\pi}{N}}$ is called toddle factor

Equations: (using kernel circle)

- → $W_N^N = 1 = W_N^0$
- → $W_N^{N/2} = -1$
- → $W_N^{N+k} = W_N^k$
- → $W_N^{N/2+k} = -W_N^k$
- → $W_{N/L}^{k/L} = W_N^k$

1) Decimation-in-time (DIT-FFT)
2) Decimation-in-frequency (DIF-FFT)

→ **DIT-FFT:**

$x(n)$ – 8 points $\quad x(0)\ x(1)\ x(2)\ x(3)\ x(4)\ x(5)\ x(6)\ x(7)$

$$\therefore \underbrace{G(k) = \sum_{n=0}^{N/2-1} x(2n)\, W_N^{2nk}}_{\text{Even}} \qquad \underbrace{H(k) = \sum_{n=0}^{N/2-1} x(2n+1)\, W_N^{(2n+1)k}}_{\text{Odd}}$$

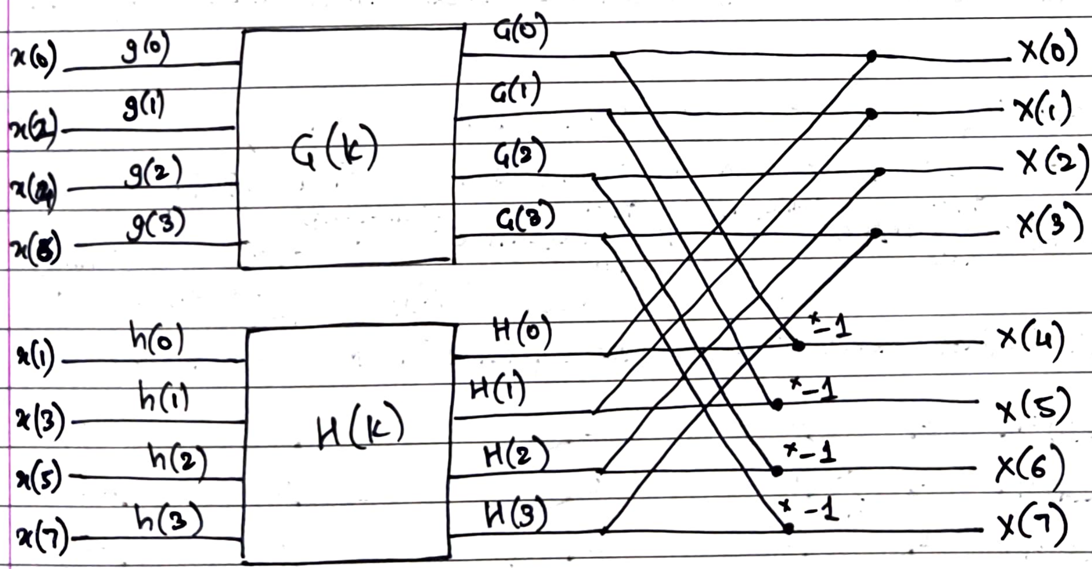
$$\therefore X(k) = \sum_{n=0}^{N/2-1} x(2n)\, W_N^{2nk} + \sum_{n=0}^{N/2-1} x(2n+1)\, W_N^{(2n+1)k}$$

$$= \sum_{n=0}^{N/2-1} x(2n)\, W_{N/2}^{nk} + \sum_{n=0}^{N/2-1} x(2n+1)\, W_{N/2}^{nk}\, W_N^{k}$$

$$\therefore X(k) = G(k) + W_N^{k} \cdot H(k) \longrightarrow [k = 0 \text{ to } N/2-1]$$

$$= G(N/2+k) + W_N^{(N/2+k)} \cdot H(N/2+k)$$

$$= G(k) - W_N^{k} \cdot H(k) \longrightarrow [k = N/2 \text{ to } N-1]$$



This is when we have divided 8 points into 4 points. We will continue ~~the~~ doing this 'Divide and Conquer' approach

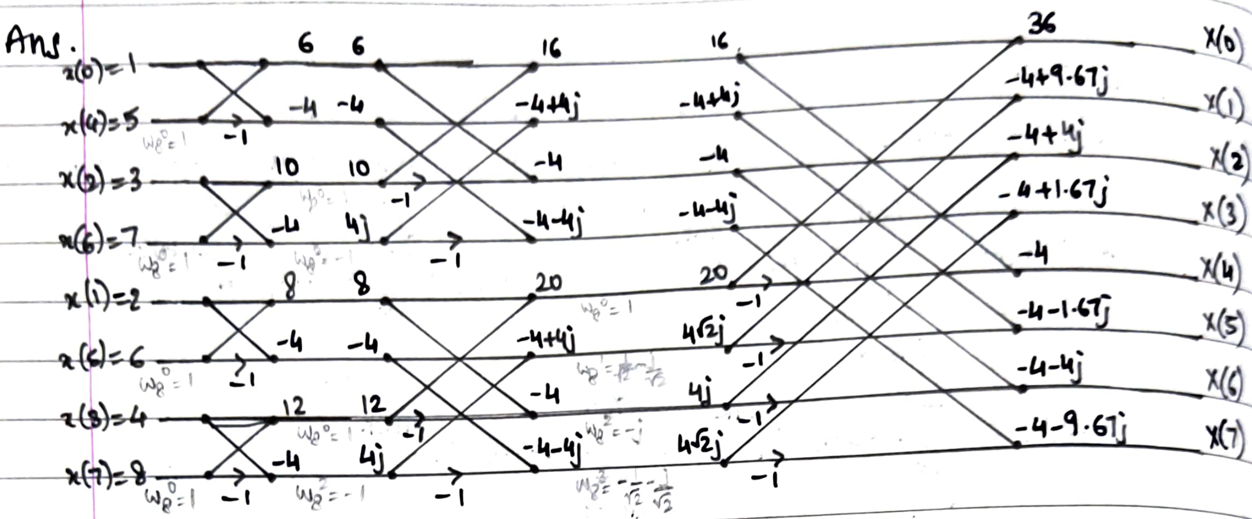
$(-4+4j)\left(\frac{1}{\sqrt2}-\frac{j}{\sqrt2}\right) = -\frac{4}{\sqrt2}+\frac{4j}{\sqrt2}+\frac{4j}{\sqrt2}+\frac{4}{\sqrt2} = 4\sqrt2 j$

$(-4-4j)\left(-\frac{1}{\sqrt2}-\frac{j}{\sqrt2}\right) = \frac{4}{\sqrt2}+\frac{4j}{\sqrt2}+\frac{4j}{\sqrt2}-\frac{4}{\sqrt2} = 4\sqrt2 j$

Q. Find DFT using DIT-FFT :

$x(n) = \{1,2,3,4,5,6,7,8\}$

Ans.



$\therefore X[k] = \{36, -4+9.67j, -4+4j, -4+1.67j, -4, -4-1.67j, -4-4j, -4-9.67j\}$ //

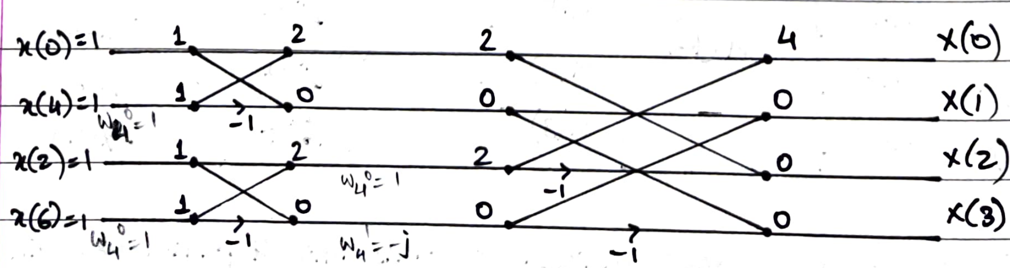
[To check, we know $|X[k]| = |X[N-k]|$ (magnitudes are equal)]

→ To calculate IDFT, flowchart will remain same but power of W_8 will become -ve everywhere, and final output divided by N.

Q. $x(n) = \begin{cases} 1, & n = \text{even in } 0 \le n \le N-1 \\ 0, & n = \text{odd in } 0 \le n \le N-1 \end{cases}$ $N=8$, Find DFT. //

Ans. $X[k] = \underbrace{\sum_{n=0}^{N/2-1} x(2n) e^{-j\frac{2\pi k(2n)}{N}}}_{\text{even}} + \underbrace{\sum_{n=0}^{N/2-1} x(2n+1) e^{-j\frac{2\pi k(2n+1)}{N}}}_{\text{odd}}$ $\therefore X[k] = \sum_{n=0}^{N/2-1} x(2n) e^{-j\frac{4\pi kn}{N}}$

∴ We just need to solve ~~for~~ 4-point DFT for even terms.



Now, we know same values will be repeated for ~~the~~ one more sequence.

$\therefore X[k] = \{4,0,0,0,4,0,0,0\}$ //

* Discrete Cosine Transform (DCT):

$$X[k] = \alpha(k) \sum_{n=0}^{N-1} x(n) \cos\left(\frac{(2n+1)\pi k}{2N}\right) \Rightarrow \left[\alpha(k) = \begin{cases} \sqrt{\frac{1}{N}}, & k=0 \\ \sqrt{\frac{2}{N}}, & k\neq 0 \end{cases}\right]$$

Finding kernel, for $N=4$:

for $k=0$,

$$\frac{1}{\sqrt{4}} \cdot \cos(0) = \frac{1}{2} \rightarrow \text{for all } n \text{ values}$$

for $k=1$,

$n=0$	$n=1$	$n=2$	$n=3$
$\frac{\sqrt{2}}{\sqrt{4}}\cos\left(\frac{\pi}{8}\right)$	$\frac{\sqrt{2}}{\sqrt{4}}\cos\left(\frac{3\pi}{8}\right)$	$\frac{1}{\sqrt{2}}\cos\left(\frac{5\pi}{8}\right)$	$\frac{1}{\sqrt{2}}\cos\left(\frac{7\pi}{8}\right)$

for $k=2$,

$n=0$	$n=1$	$n=2$	$n=3$
$\frac{1}{\sqrt{2}}\cos\left(\frac{2\pi}{8}\right)$	$\frac{1}{\sqrt{2}}\cos\left(\frac{6\pi}{8}\right)$	$\frac{1}{\sqrt{2}}\cos\left(\frac{10\pi}{8}\right)$	$\frac{1}{\sqrt{2}}\left(\frac{14\pi}{8}\right)$

for $k=3$,

$n=0$	$n=1$	$n=2$	$n=3$
$\frac{1}{\sqrt{2}}\cos\left(\frac{3\pi}{8}\right)$	$\frac{1}{\sqrt{2}}\cos\left(\frac{9\pi}{8}\right)$	$\frac{1}{\sqrt{2}}\cos\left(\frac{15\pi}{8}\right)$	$\frac{1}{\sqrt{2}}\cos\left(\frac{21\pi}{8}\right)$

Q. Find DCT of $f(n) = [1, 2, 4, 7]$

Ans. ~~DCT kernel = [1/2 1/2 1/2 1/2 DCT kernel =~~ ~~$\frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{8}\right)$~~ ~~$\frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{8}\right)$ $\frac{1}{\sqrt{2}}\cos\left(\frac{3\pi}{8}\right)$ $\frac{1}{\sqrt{2}}\cos\left(\frac{5\pi}{8}\right)$ $\frac{1}{\sqrt{2}}$~~

$$\text{DCT kernel} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{3\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{5\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{7\pi}{8}\right) \\ \frac{1}{\sqrt{2}}\cos\left(\frac{2\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{6\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{10\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{14\pi}{8}\right) \\ \frac{1}{\sqrt{2}}\cos\left(\frac{3\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{9\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{15\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{21\pi}{8}\right) \end{bmatrix}$$

$$\therefore X[k] = \cancel{[1\ 2\ 4\ 7]} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{3\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{5\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{7\pi}{8}\right) \\ \frac{1}{\sqrt{2}}\cos\left(\frac{2\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{6\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{10\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{14\pi}{8}\right) \\ \frac{1}{\sqrt{2}}\cos\left(\frac{3\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{9\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{15\pi}{8}\right) & \frac{1}{\sqrt{2}}\cos\left(\frac{21\pi}{8}\right) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \\ 7 \end{bmatrix}$$

$$\therefore X[k] = [7, -4.46, 1, -0.37]$$

(remember calculator radians setting)

Q. Calculate DCT of $f(m,n) = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 \end{bmatrix}$

Ans. $\therefore X[k] = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.6532 & 0.2706 & -0.2706 & -0.6532 \\ 0.5 & -0.5 & -0.5 & 0.5 \\ 0.2706 & -0.6532 & 0.6532 & -0.2706 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.6532 & 0.5 & 0.2706 \\ 0.5 & 0.2706 & -0.5 & -0.6532 \\ 0.5 & -0.2706 & -0.5 & 0.652 \\ 0.5 & -0.6532 & 0.5 & -0.2706 \end{bmatrix}$

$\therefore X[k] = \begin{bmatrix} 6 & 0.3826 & -1 & 0.9238 \\ 0 & -0.146 & -0.382 & -0.353 \\ 0 & 0 & 0 & 0 \\ 0 & -0.353 & -0.923 & -0.853 \end{bmatrix}$ //

* Hadamard Transform :

→ $H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ for H_4, $H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix}$

$\therefore H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ and so on for larger kernels.

Q. Find Hadamard Transform of $[1, 2, 0, 3]$

Ans. $X[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 0 \\ 2 \end{bmatrix}$ //

Q. Find Hadamard Transform of $f(k,l) = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 2 \end{bmatrix}$

Ans. $F[k,l] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

$\therefore F[k,l] = \begin{bmatrix} 34 & 2 & -6 & -6 \\ 2 & 2 & 2 & 2 \\ -6 & 2 & 2 & 2 \\ -6 & 2 & 2 & 2 \end{bmatrix}$ //

* **Walsh Transform:**

Kernel = $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$ ← 1 sign change, ← 2 sign changes, ← 3 sign changes — from Hadamard Transform itself

Q. Find Walsh Transform of [1, 2, 0, 3]

Ans. $X[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \\ -4 \end{bmatrix}$ //

Q. Find Walsh Transform of ...

Ans. $X[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 34 & -6 & -6 & 2 \\ -6 & 2 & 2 & 2 \\ -6 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}$ //

* Harr Transform:

Kernel for 2x2: $\frac{1}{\sqrt{2}} \times H_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Kernel for 4x4:

$$\frac{1}{2} \times \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

* Karshunan - Loeve Transform (KL):

→ Hotelling

→ Principal Component Analysis (Eigen vector component analysis)

① Find X vector (object position)

② μ_x (mean) $= \frac{1}{N}\sum_{i=0}^{N-1} x_i$

③ covariance matrix $C_x = \frac{1}{N}\sum_{i=0}^{N-1}(x_i-\mu_x)(x_i-\mu_x)^T$

④ Eigen values → $|C_x - \lambda I| = 0$ (characteristic equation)

⑤ Eigen vector → $(C_x - \lambda I)(x) = 0 \rightarrow e_0, e_1 \dots e_w$

⑥ Transformation Matrix → $A = [e_0, e_1, \dots]$

⑦ Result: $Y = A[X - \mu_x]$ (Inverse: $X = A^T[Y + \mu_x]$)

Q. Perform KL Transform:

y ↑ \ x →	0	1	2	3
3	0	0	0	0
2	1	0	0	0
1	1	1	1	0
0	0	0	0	0

Ans. ① $X = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right]$

② $\mu_x = \begin{bmatrix} \left(\frac{0+0+1+2}{4}\right) \\ \left(\frac{1+2+1+1}{4}\right) \end{bmatrix} = \begin{bmatrix} 0.75 \\ 1.25 \end{bmatrix}$

(continued →)

③ $C_x = \frac{1}{N}\sum_{i=0}^{N-1}(x_i - \mu_x)(x_i - \mu_x)^T$

$x_0 = \begin{pmatrix}0\\1\end{pmatrix}$ $\quad\therefore x_0 - \mu_x = \begin{bmatrix}-0.75\\-0.25\end{bmatrix}$

$C_0 = \begin{bmatrix}-0.75\\-0.25\end{bmatrix}\begin{bmatrix}-0.75 & -0.25\end{bmatrix} = \begin{bmatrix}0.5625 & 0.1875\\0.1875 & 0.0625\end{bmatrix}$

Similarly,

$x_1 = \begin{pmatrix}0\\2\end{pmatrix}$ $\quad\therefore x_0 - \mu_x = \begin{bmatrix}-0.75\\0.75\end{bmatrix}$

$C_1 = \begin{bmatrix}-0.75\\0.75\end{bmatrix}\begin{bmatrix}-0.75 & 0.75\end{bmatrix} = \begin{bmatrix}0.5625 & -0.5625\\-0.5625 & 0.5625\end{bmatrix}$

$x_2 = \begin{pmatrix}1\\1\end{pmatrix}$ $\quad\therefore x_0 - \mu_x = \begin{bmatrix}0.25\\-0.25\end{bmatrix}$

$\therefore C_2 = \begin{bmatrix}0.25\\-0.25\end{bmatrix}\begin{bmatrix}0.25 & -0.25\end{bmatrix} = \begin{bmatrix}0.0625 & -0.0625\\-0.0625 & 0.0625\end{bmatrix}$

$x_3 = \begin{pmatrix}2\\1\end{pmatrix}$ $\quad\therefore x_0 - \mu_x = \begin{bmatrix}1.25\\-0.25\end{bmatrix}$

$\therefore C_3 = \begin{bmatrix}1.25\\-0.25\end{bmatrix}\begin{bmatrix}1.25 & -0.25\end{bmatrix} = \begin{bmatrix}1.5625 & -0.3125\\-0.3125 & 0.0625\end{bmatrix}$

$$C_x = \begin{bmatrix}\frac{0.5625+0.5625+0.0625+1.5625}{4} & \frac{0.1875-0.5625-0.0625-0.3125}{4}\\[2ex] \frac{0.1875-0.5625-0.0625-0.3125}{4} & \frac{0.0625+0.5625+0.0625+0.0625}{4}\end{bmatrix}$$

$C_x = \begin{bmatrix}0.6875 & -0.1875\\-0.1875 & 0.1875\end{bmatrix}$ //

④ Eigen values: $|C_x - \lambda I| = 0$

$\therefore \begin{bmatrix}0.6875-\lambda & -0.1875\\-0.1875 & 0.1875-\lambda\end{bmatrix} = 0$ $\quad\therefore \lambda_1 = 0.75$ //, $\lambda_2 = 0.125$ //

(continued →)

⑤ Eigen vectors:

$$\underset{\uparrow\,(\lambda_1)}{(C_x - \lambda I)} \begin{pmatrix} \phi_{11} \\ \phi_{12} \end{pmatrix} = 0$$

(Take first equation from this) → $\phi_{11} = 3\phi_{12}$

Let $\phi_{12} = 1$, $\therefore \phi_{11} = 3$

$$\begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} = \underbrace{\frac{1}{\sqrt{3^2+1^2}}}_{\uparrow\text{(normalisation)}} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.94 \\ 0.31 \end{bmatrix} //$$

λ_1

$$\underset{\uparrow\,(\lambda_2)}{(C_x - \lambda I)} \begin{pmatrix} \phi_{11} \\ \phi_{12} \end{pmatrix} = 0$$

(Take first equation from this) → $3\phi_{11} = \phi_{12}$

Let $\phi_{12} = 1$, $\therefore \phi_{11} = 0.33$

$$\begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix} = \frac{1}{\sqrt{0.33^2+1^2}} \begin{bmatrix} 0.33 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.9 \end{bmatrix} //$$

λ_2

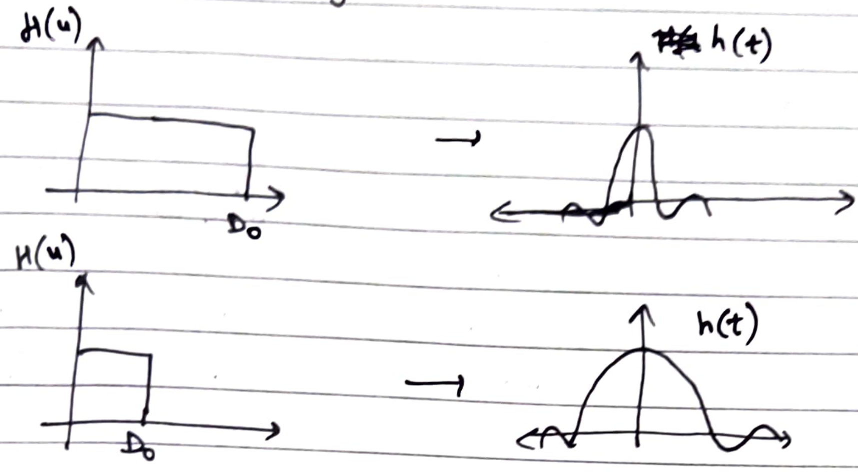
⑥ $A = \begin{bmatrix} 0.94 & 0.3 \\ 0.31 & 0.9 \end{bmatrix} //$

⑦ $Y = A[x - \mu_x]$ [Inverse is $X = A^T[Y + \mu_x]$]

(wavelet Transform also in syllabus!)
[Read up what is wavelet, why is it useful, etc.]

* Filtering in frequency Domains:

→ D_0 is cutoff - frequency.



Low-Pass Filters :-

→ Ideal Low-Pass filter (ILPF)

→ Butterworth Low-Pass filter (BLPF)

→ Gaussian Low-Pass filter (GLPF)

(Ideal LPF function given before)

⇒ BLPF : $H(u,v) = \dfrac{1}{1+\left(\frac{\sqrt{u^2+v^2}}{D_0}\right)^{2n}}$

⇒ GLPF : $H(u,v) = e^{-(u^2+v^2)/(2D_0^2)}$

High-Pass Filters :-

→ $H_{HP}(u,v) = 1 - H_{LP}(u,v)$

BHPF : $H(u,v) = 1/\left[1+\left(\frac{D_0}{\sqrt{u^2+v^2}}\right)^{2n}\right]$

Homomorphic Filtering :-

→ Enhance high frequencies, attenuate low frequencies but preserve fine detail.

$f(x,y) = i(x,y)\, r(x,y)$ → $i(x,y)$ = illumination, $r(x,y)$ = reflection

Steps: ① $\ln(f(x,y)) = \ln(i(x,y)) + \ln(r(x,y))$

② Apply Fourier Transform : $F(\ln(x,y)) = F(\ln(i(x,y))) + F(\ln(r(x,y)))$

③ Apply $H(u,v)$: $Z(u,v)H(u,v) = Illum(u,v)H(u,v) + Ref(u,v)H(u,v)$

④ Take Inverse FT : $F^{-1}(Z(u,v)H(u,v)) = F^{-1}(Illum(u,v)H(u,v)) + F^{-1}(Ref(u,v)H(u,v))$

⑤ Take exp() : $e^{s(x,y)} = e^{i'(x,y)}\, e^{r'(x,y)}$ or $g(x,y) = i_0(x,y)\, r_0(x,y)$

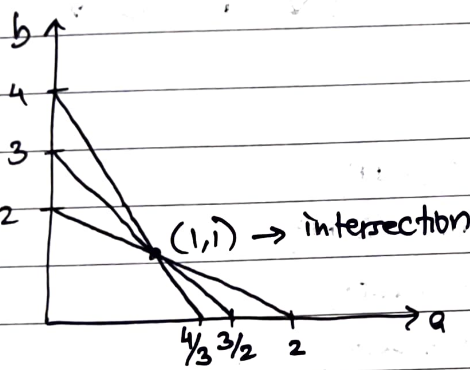
[Edge detection and Image Segmentation done through PPT]

Q. Using Hough Transform show that following points are ~~collinear~~ co-linear.
Points are (1,2), (2,3) and (3,4)

Ans. we know ~~y=mx+c~~ $y=ax+b$

$\therefore b = -ax + y$

x	y	b	(when $a=0$) b values	(when $b=0$) a values
1	2	$-a+2$	2	2
2	3	$-2a+3$	3	$3/2$
3	4	$-3a+4$	4	$4/3$

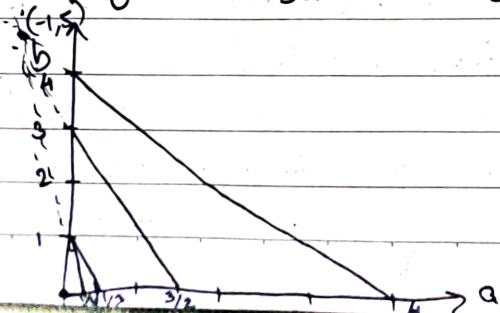


$\therefore$ Equation: $y = x+1$

Q. Given a set of points (1,4), (2,3), (3,1), (4,1), (5,0). Find collinear points.

Ans. we know $y=ax+b$ $\therefore b=-ax+y$

x	y	b	b values ($a=0$)	a values ($b=0$)
1	4	$-a+4$	4	4
2	3	$-2a+3$	3	$3/2$
3	1	$-3a+1$	1	$1/3$
4	1	$-4a+1$	1	$1/4$
5	0	$-5a$	0	0



Equation: $\therefore y=-x+5$ (passes through all except (3,1))

Q. Pixel coordinates of interest are (0,5), (1,4), (3,2), (3,3), (5,0).

Ans. ~~Eq~~ We know that $P = a\cos\theta + b\sin\theta$

∴ All equations: $(0,5) \rightarrow 5\sin\theta$

$(1,4) \rightarrow \cos\theta + 4\sin\theta$

$(3,2) \rightarrow 3\cos\theta + 2\sin\theta$

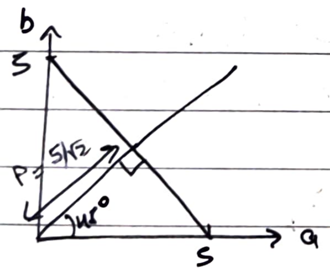
$(3,3) \rightarrow 3\cos\theta + 3\sin\theta$

$(5,0) \rightarrow 5\cos\theta$

x	y	$P = a\cos\theta + b\sin\theta$	$P(\theta = -\pi/2)$	$P(\theta = -\pi/4)$	$P(\theta = 0)$	$P(\theta = \pi/4)$	$P(\theta = \pi/2)$
0	5	$5\sin\theta$	-5	$-5/\sqrt{2}$	0	$5/\sqrt{2}$	5
1	4	$\cos\theta + 4\sin\theta$	-4	$-3/\sqrt{2}$	1	$5/\sqrt{2}$	4
3	2	$3\cos\theta + 2\sin\theta$	-2	$1/\sqrt{2}$	3	$5/\sqrt{2}$	2
3	3	$3\cos\theta + 3\sin\theta$	-3	0	3	$6/\sqrt{2}$	3
5	0	$5\cos\theta$	0	$5/\sqrt{2}$	5	$5/\sqrt{2}$	0

Now, plotting these points, we can see intersection is at $\theta = \pi/4$ and $\rho = 5/\sqrt{2}$.

Then, on another ~~gh~~ graph of a vs. b, draw a line with $\theta = 45°$, and at length of $5/\sqrt{2}$ from origin, draw a perpendicular line and its intercepts will give (a,b).



$\therefore P = 5\cos\theta + 5\sin\theta$

⇒ Thresholding (based on histogram)

→ **Global Processing:**

(1) Assume a threshold (min, max) → T
(2) Group G1 ≤ T, Group G2 > T
(3) Mean of G1 = m_1, mean of G2 = m_2
(4) Threshold $\frac{m_1+m_2}{2} = T_2$.
(5) Repeat until threshold stabilises.

Q.

5	3	9
2	1	7
8	4	2

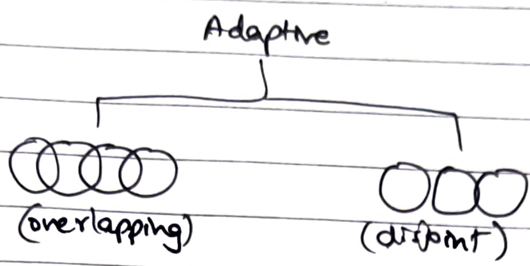
Ans. (1) Assume T=4
(2) ∴ G1 = [3, 2, 1, 4, 2]
G2 = [5, 9, 7, 8]
(3) ∴ $m_1 = \frac{3+2+1+4+2}{5} \approx 2$, $m_2 = \frac{5+9+7+8}{4} \approx 7$
(4) ∴ Threshold $T_2 = \frac{2+7}{2} \approx 5$

Repeating,
(1) T = 5
(2) $G_1 = [~~5~~, 3, 2, 1, 4, 2]$, $G_2 = [9, 7, 8]$
(3) $m_1 = \frac{5+3+2+1+4+2}{6} \approx 3$, $m_2 = \frac{9+7+8}{3} \approx 8$
(4) ∴ Threshold $T_2 = \frac{3+8}{2} \approx 6$

Repeating,
(1) T = 6
(2) $G_1 = [5, 3, 2, 1, 4, 2]$, $G_2 = [9, 7, 8]$
(3) $m_1 = \frac{5+3+2+1+4+2}{6} \approx 3$, $m_2 = \frac{9+7+8}{3} \approx 8$
(4) ∴ Threshold $T_2 = \frac{3+8}{2} \approx 6$

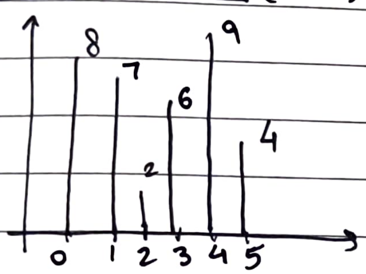
∴ Threshold = 6

→ **Local Processing:**



Use features like mean, median, etc for threshold

→ **Optimal Processing (otsu):**



(sum of no. of pixels $= 8+7+2+6+9+4 = 36$)

Let $T=3$

$\therefore$ weight $w_1 = \dfrac{8+7+2+6}{\text{sum of no. of pixels}} = \dfrac{23}{36} \approx 0.64$

Mean $\mu_1 = \dfrac{(0\times8)+(1\times7)+(2\times2)+(3\times6)}{\cancel{23}\ 8+7+2+6} = \dfrac{29}{23} \approx 1.26$

Variance $\sigma_1 = \dfrac{(0-1.26)^2\times8+(1-1.26)^2\times7+(2-1.26)^2\times2+(3-1.26)^2\times6}{8+7+2+6}$

$= \dfrac{32.4348}{23} \approx 1.41$

Similarly,

weight $w_2 = \dfrac{9+4}{36} = \dfrac{13}{36} \approx 0.36$

Mean $\mu_2 = \dfrac{(4\times9)+(5\times4)}{9+4} = \dfrac{56}{13} \approx 4.31$

Variance $\sigma_2 = \dfrac{(4-4.31)^2\times9+(5-4.31)^2\times4}{9+4} = \dfrac{2.7693}{13} \approx 0.213$

$\therefore$ Intraclass variance $= w_1\sigma_1 + w_2\sigma_2 = (0.64\times1.41)+(0.36\times0.213)$

≈ 0.98

⇒ **Region-based Segmentation** ↴

1) Region growing
2) Region splitting
3) Merging
4) S and merge

[Union of all regions should give R]
[All subdivisions have to be connected]
[Intersections of regions should be null (disjoint)]

(1) **Region growing** ↴

→ seed pixel
→ connectivity
→ max. value − min value $\leq T$

Example: seed = 7, 4-connectivity, $T \leq 2$

3	4	5	1
3	7	6	0
4	8	4	2
2	7	6	5

Q. Apply region-growing for the image:

5	6	6	7	6	7	6	6
6	7	6	7	5	5	4	7
6	6	4	4	3	2	5	6
5	4	5	4	2	3	4	6
0	3	2	3	3	2	4	7
0	0	0	0	2	2	5	6
1	1	0	1	0	3	4	4
1	0	1	0	2	3	5	4

$T \leq 4$, seed = 6,
if not mentioned,
take 8-connectivity

(2) Region splitting ↴
Keep splitting into quadrants to see which regions satisfy as being a single "connected" region.

~~(3)~~

* Image Compression ↴

- Image Compression
  - Lossless
    - Runlength Coding
    - Huffman coding
    - Arithmetic Coding
    - DC
  - Lossy

→ Runlength Coding ↴

Example:

					Horizontal
0	0	0	0	0	(0,5)
0	0	0	1	1	(0,3) (1,2)
1	1	1	1	1	(1,5)
1	1	1	1	1	(1,5)
1	1	1	1	1	(1,5)

Max. value = 5 (3 bits), Bit depth = 1
vectors = 6

∴ Total bits required = $6 \times (3+1)$ = 24 bits
Original bits required = $5 \times 5 \times 1$ = 25 bits
Compression = $(25-24)/25$ = $\frac{25}{24} : 1$

vertically → Max. value = 4 (3 bits), Bit depth = 1, vectors = 10
Total bits required = $10 \times (3+1)$ = 40 bits
Original bits required = $5 \times 5 \times 1$ = 25 bits

→ Huffman Coding ↓

Example: $10 \times 10 \rightarrow$ 6 symbols (5 bits)

a_1	a_2	a_3	a_4	a_5	a_6
10	40	6	10	4	30

Probability: 0.1   0.4   0.06   0.1   0.04   0.3

Symbols	Probability				
(1) a_2	0.4	0.4	0.4	0.4	0.6 (0)
(00) a_6	0.3	0.3	0.3	0.3 (00)	0.4 (1)
(011) a_1	0.1	0.1	0.2 (010)	0.3 (01)	
(0100) a_4	0.1	0.1 (0100)	0.1 (011)		
(01010) a_3	0.06 (01010)	0.1 (0101)			
(01011) a_5	0.04 (01011)				

∴ Average length of code:

$$(0.4 \times 1) + (0.3 \times 2) + (0.1 \times 3) + (0.1 \times 4) + (0.06 \times 5) + (0.04 \times 5)$$

$$= 2.2 \text{ bits/code}$$

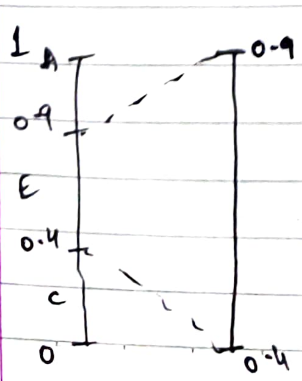
∴ Total number of bits transmitted $= 10 \times 10 \times 2.2 = 220$ bits

Original number of bits transmitted $= 10 \times 10 \times 5 = 500$ bits

→ Arithmetic Coding ↓

Example: Symbols → A   C   E   To transmit ECEA

A	C	E
0.1	0.4	0.5



Range $= 0.9 - 0.4 = 0.5$

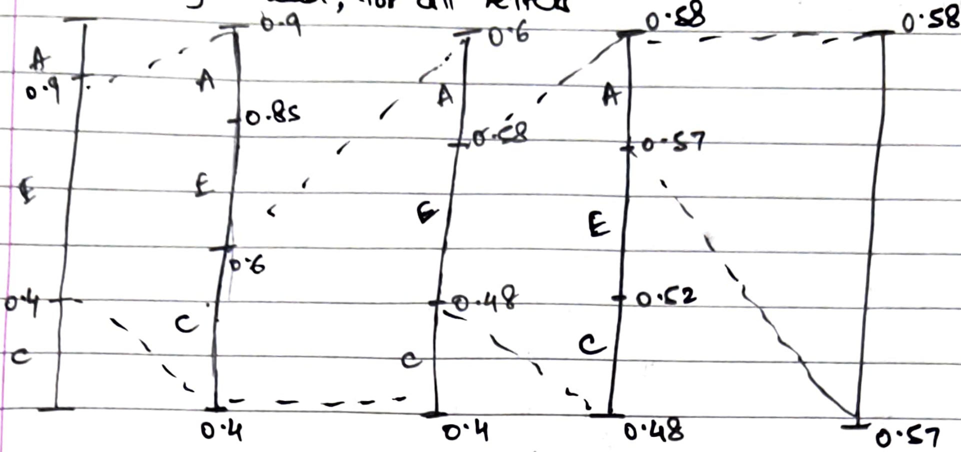
∴ Range of different symbols: (low level + (range × prob.))

$C = 0.4: 0.4 + (0.5 \times 0.4) = 0.6 \rightarrow 0.4$ to 0.6

$E = 0.6: 0.6 + (0.5 \times 0.5) = 0.85 \rightarrow 0.6$ to 0.85

$A = 0.85: 0.85 + (0.5 \times 0.1) = 0.9 \rightarrow 0.85$ to 0.9

continuing ahead, for all letters



$\therefore Tag_1 = \frac{0.58 + 0.57}{2} = \underline{0.575} \rightarrow \therefore E$ //

$Tag_2 = \frac{\cancel{tag_1} \; tag_1 - low}{0.5 \text{ (range)}} = 0.35 \rightarrow \therefore C$ //

$Tag_3 = \frac{tag_2 - low}{0.4 \text{ (range)}} = \cancel{0.85} \; 0.875 \rightarrow \therefore E$ //

$Tag_4 = \frac{tag_3 - low}{0.5 \text{ (range)}} = 0.95 \rightarrow \therefore A$ //

string = ECEA //

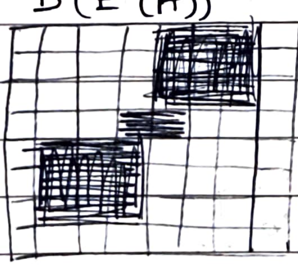
* Morphological Processing ↓

→ Dilation (OR) : thickens image

→ Erosion (AND) : thinner image (opposite of dilation)

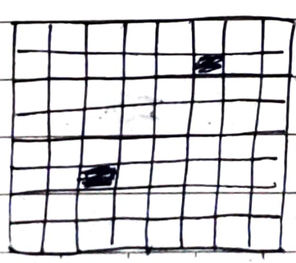
→ Opening : D(E(A))

Example :-

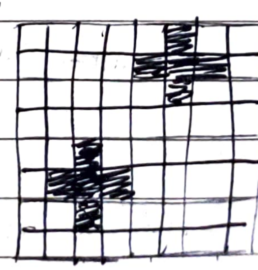


structural element :

After erosion:



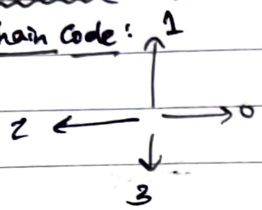
After dilation on erosion result:



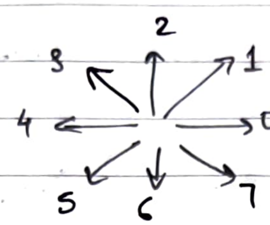
→ Closing: $E(D(A))$

* Boundary Extraction ↴

→ Chain Code:



4-connectivity



8-connectivity

→ Signature:

Plotting distance of ~~boundary~~ boundary from centroid as function of angle.

→ Image Moments:

Image moment M_{ij} of order (i,j) for grayscale image with pixel intensities $I(x,y)$ is:

$$\cancel{M_{ij} = \sum_x \sum_y x_i y_j I(x,y)}$$

$$M_{ij} = \sum_x \sum_y x^i y^j I(x,y)$$

→ For binary image, zeroth moment corresponds to area.

$$\therefore M_{00} = \sum_x \sum_y I(x,y)$$

→ Centroid $\{\bar{x}, \bar{y}\} = \left\{\frac{M_{10}}{M_{00}}, \frac{M_{01}}{M_{00}}\right\}$

→ Central Moments:

$$\mu_{pq} = \sum_x \sum_y (x-\bar{x})^p (y-\bar{y})^q I(x,y)$$ [$\bar{x}, \bar{y}$ are centroid components]

→ Normalized central moment: $(\mu_{ij})_n = \frac{\mu_{ij}}{(\mu_{00})^\lambda}$ $\left[\lambda = \left(\frac{i+j}{2}\right)+1\right]$

~~→ Rotated central moment~~

→ $\mu_{00} = M_{00}$

→ $\mu_{10} = \mu_{01} = 0$

→ $\mu_{20} = M(2,0) - \bar{x}M(1,0)$

→ $\mu_{02} = M(0,2) - \bar{y}M(0,1)$

* Fidelity Criteria ↓

→ Objective Fidelity ↓

– Mean Square Error (MSE) $= \frac{1}{M \times N} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} [f(m,n) - g(m,n)]^2$

– Signal-to-Noise Ratio (SNR) $= 10 \log_{10} \left[\frac{\frac{1}{M \times N} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f(m,n)^2}{\frac{1}{M \times N} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} [f(m,n) - g(m,n)]^2} \right]$

– Peak Signal-to-Noise Ratio (PSNR) $= 10 \log_{10} \left[\frac{2^b - 1}{\frac{1}{M \times N} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} [f(m,n) - g(m,n)]^2} \right]$

(b = no. of bits to represent image)

→ Subjective Fidelity ↓

– Checking by human eye

– Energy compaction transform coding

$$GTC = \frac{(1/N) \sum_{i=0}^{N-1} \sigma_i^2}{\left[\prod_{i=0}^{N-1} \sigma_i^2 \right]^{1/N}}$$

(← arithmetic mean)

(← geometric mean)