1, 2 and 3 Qubit Gate Basis States

Number of qubits	Basis ket notation	Basis vector notation	Basis states
Single Qubit	$\{ \|0>, \|1> \}$	$\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$
Two Qubits	$\{ \|00>, \|01>, \|10>, \|11> \}$	$\|0>\otimes\|0> = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ $\|0>\otimes\|1> = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ $\|1>\otimes\|0> = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ $\|1>\otimes\|1> = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$	$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$
Three Qubits	$\{ \|000>, \|001>, \|010, \|011>,$ $\|100>, \|101>, \|110, \|111> \}$	$\{ \|0>\otimes\|0>\otimes\|0>,\ \|0>\otimes\|0>\otimes\|1>,$ $\|0>\otimes\|1>\otimes\|0>,\ \|0>\otimes\|1>\otimes\|1>,$ $\|1>\otimes\|0>\otimes\|0>,\ \|1>\otimes\|0>\otimes\|1>,$ $\|1>\otimes\|1>\otimes\|0>,\ \|1>\otimes\|1>\otimes\|1> \}$	$\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}$

Eigen values and Eigen Vectors of Pauli X, Y and Z matrices

Generally, to find the Eigen Vectors(X) and Eigen Values (λ) of a matrix A,

$\boldsymbol{AX = \lambda X}$
$(A - \lambda I) = 0 \; and \; \det(A - \lambda I) = 0 \; where, \det = determinant$
Note: If $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$, det(A) is given by det(A) = ad-bc

Pauli Gate	Matrix	Eigen Values	Eigen Vectors
X gate	$\sigma_x = \begin{bmatrix}0&1\\1&0\end{bmatrix}$	$\lambda = \pm 1$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$
Y gate	$\sigma_y = \begin{bmatrix}0&-i\\i&0\end{bmatrix}$	$\lambda = \pm 1$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix}-i\\-1\end{bmatrix}$
Z gate	$\sigma_z = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$	$\lambda = \pm 1$	$\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$

Rotational operator about an arbitrary direction and obtaining the Pauli matrices and IST gates

Unitary Operator U can be written as $U = e^{i\alpha} R_{\hat{n}}(\theta)$

Applying rotational Operator about an arbitrary direction, $U = e^{-i(\frac{\theta}{2})\vec{\sigma}\,\hat{n}}$

According to Euler's Formula, $e^{-i(\frac{\theta}{2})} = \cos(\frac{\theta}{2}) - i\sin(\frac{\theta}{2})$

Therefore, $U = e^{i\alpha}\,[I\ \cos(\frac{\theta}{2}) - i\,(\vec{\sigma}\,\hat{n})\sin(\frac{\theta}{2})]$

For $\alpha = \frac{\pi}{2}\ and\ \theta = \pi$,

$U = e^{i\frac{\pi}{2}}\,[I\ \cos(\frac{\pi}{2}) - i\,(\vec{\sigma}\,\hat{n})\sin(\frac{\pi}{2})]$

$U = i[-i\,(\vec{\sigma}\,\hat{n})] = \vec{\sigma}\,\hat{n}$

Therefore, for X, Y and Z axes, $U = \vec{\sigma}\,\hat{n} = \sigma_x n_x + \sigma_y n_y + \sigma_z n_z$

Pauli X	Pauli Y	Pauli Z
$\hat{n} = (1,0,0)$ $\sigma_x = \begin{bmatrix}0&1\\1&0\end{bmatrix}$	$\hat{n} = (0,1,0)$ $\sigma_y = \begin{bmatrix}0&-i\\i&0\end{bmatrix}$	$\hat{n} = (0,0,1)$ $\sigma_z = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$

Rotation about X axis	Rotation about Y axis	Rotation about Z axis
$R_x(\emptyset) = e^{-i(\frac{\emptyset}{2})}\sigma_x$ $R_x(\emptyset) = [I \cos(\frac{\emptyset}{2}) - i\sin(\frac{\emptyset}{2})\sigma_x]$ $R_x(\emptyset) = \begin{bmatrix} \cos(\frac{\emptyset}{2}) & 0 \\ 0 & \cos(\frac{\emptyset}{2}) \end{bmatrix} - \begin{bmatrix} 0 & i\sin(\frac{\emptyset}{2}) \\ i\sin(\frac{\emptyset}{2}) & 0 \end{bmatrix}$ $R_x(\emptyset) = \begin{bmatrix} \cos(\frac{\emptyset}{2}) & -i\sin(\frac{\emptyset}{2}) \\ -i\sin(\frac{\emptyset}{2}) & \cos(\frac{\emptyset}{2}) \end{bmatrix}$	$R_y(\emptyset) = e^{-i(\frac{\emptyset}{2})}\sigma_y$ $R_y(\emptyset) = [I \cos(\frac{\emptyset}{2}) - i\sin(\frac{\emptyset}{2})\sigma_y]$ $R_y(\emptyset) = \begin{bmatrix} \cos(\frac{\emptyset}{2}) & 0 \\ 0 & \cos(\frac{\emptyset}{2}) \end{bmatrix} - \begin{bmatrix} 0 & \sin(\frac{\emptyset}{2}) \\ \sin(\frac{\emptyset}{2}) & 0 \end{bmatrix}$ $R_y(\emptyset) = \begin{bmatrix} \cos(\frac{\emptyset}{2}) & -\sin(\frac{\emptyset}{2}) \\ -\sin(\frac{\emptyset}{2}) & \cos(\frac{\emptyset}{2}) \end{bmatrix}$	$R_z(\emptyset) = e^{-i(\frac{\emptyset}{2})}\sigma_z$ $R_z(\emptyset) = [I \cos(\frac{\emptyset}{2}) - i\sin(\frac{\emptyset}{2})\sigma_z]$ $R_z(\emptyset) = \begin{bmatrix} \cos(\frac{\emptyset}{2}) & 0 \\ 0 & \cos(\frac{\emptyset}{2}) \end{bmatrix} - \begin{bmatrix} i\sin(\frac{\emptyset}{2}) & 0 \\ 0 & -i\sin(\frac{\emptyset}{2}) \end{bmatrix}$ $R_z(\emptyset) = \begin{bmatrix} \cos(\frac{\emptyset}{2}) - i\sin(\frac{\emptyset}{2}) & 0 \\ 0 & \cos(\frac{\emptyset}{2}) + i\sin(\frac{\emptyset}{2}) \end{bmatrix}$ $R_z(\emptyset) = \begin{bmatrix} e^{-i(\frac{\emptyset}{2})} & 0 \\ 0 & e^{i(\frac{\emptyset}{2})} \end{bmatrix}$ Adding a global phase $e^{i(\frac{\emptyset}{2})}$ results in $R_z(\emptyset) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\emptyset} \end{bmatrix}$
Identity Gate	**S and SDG Gates**	**T and TDG Gates**
Consider $\emptyset = 2\pi$ in $R_z(\emptyset)$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	Consider $\emptyset = \pi/2$ in $R_z(\emptyset)$ $S = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{bmatrix}$ $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ $S^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi/2} \end{bmatrix}$	Consider $\emptyset = \pi/4$ in $R_z(\emptyset)$ $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$ $T^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

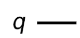
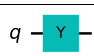
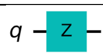
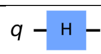
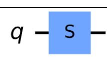
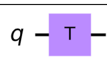
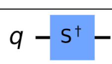
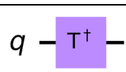
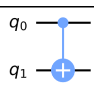
Hermitian And Unitary Matrices

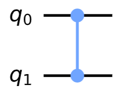
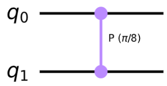
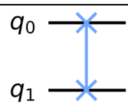
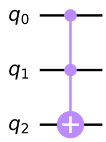
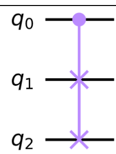
Hermitian Matrix	Unitary Matrix
If A is a (n x n) matrix, $\boldsymbol{A = A^\dagger}$ Where, $A^\dagger = (A^T)^*$ (read as Conjugate of A transpose) E.g. Let, A = Pauli Y $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\sigma_y{}^T = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ $(\sigma_y{}^T)^* = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\sigma_y = (\sigma_y{}^T)^* = \sigma_y{}^\dagger$ Therefore, Pauli Y matrix is Hermitian	If A is a (n x n) matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, **A = $\boldsymbol{A^{-1} = A^\dagger}$** Where, $A^\dagger = (A^T)^*$ and $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ det(A) is given by det(A) = ad-bc E.g. Let, A = Pauli Y $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\sigma_y{}^\dagger = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ We know that $AA^{-1} = I$ $\sigma_y \sigma_y{}^\dagger = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $\sigma_y \sigma_y{}^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ We can say that $\sigma_y = \sigma_y{}^\dagger = \sigma_y{}^{-1}$ Therefore, Pauli Y matrix is Unitary

Bracket Notations

<a|b> → Inner product → $\|a>^{\dagger}.\|b>$
|a><b| → Outer product → $\|a>.\|b>^{\dagger}$
|ab> → Tensor product → $\|a>\otimes\|b>$

1, 2 and 3 Qubit Gates

Gate	Gate Notation	Matrix Representation	Action -> Result	Resultant State Vector
Identity (Do nothing/no operation gate)		$I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$I\|0> \rightarrow \|0>$	[1.+0.j, 0.+0.j]
			$I\|1> \rightarrow \|1>$	[0.+0.j, 1.+0.j]
Pauli X (Classical NOT also called bit flip)		$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$X\|0> \rightarrow \|1>$	[0.+0.j, 1.+0.j]
			$X\|1> \rightarrow \|0>$	[1.+0.j, 0.+0.j]
Pauli Y (Bit and phase flip)		$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$	$Y\|0> \rightarrow i\|1$	[0.+0.j, 0.+1.j]
			$Y\|1> \rightarrow -i\|0>$	[0.-1.j, 0.+0.j]
Pauli Z (Phase flip) ($\emptyset = \pi$)		$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$Z\|0> \rightarrow \|0>$	[1.+0.j, 0.+0.j]
			$Z\|1> \rightarrow -\|1>$	[0.+0.j, -1.+0.j]
Hadamard (H) (Gate to put a qubit in a superposition state)		$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$	$H\|0> \rightarrow \frac{1}{\sqrt{2}}\{\|0> + \|1>\}$	[0.70710678+0.j, 0.70710678+0.j]
			$H\|1> \rightarrow \frac{1}{\sqrt{2}}\{\|0> - \|1>\}$	[0.70710678+0.j, -0.70710678+0.j]
S (Also called $\sqrt{Z}$ gate) ($\emptyset = \frac{\pi}{2}$)		$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$	S $\|0> \rightarrow \|0>$	[1.+0.j, 0.+0.j]
			$S\|1> \rightarrow i\|1>$	[0.+0.j, 0.+1.j]
T (Also called $\sqrt{S}$ gate) ($\emptyset = \frac{\pi}{4}$)		$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$	T $\|0> \rightarrow \|0>$	[1.+0.j, 0.+0.j]
			$T\|1> \rightarrow e^{i\frac{\pi}{4}}\|1>$	[0. +0.j, 0.70710678+0.70710678j]
$S^{\dagger}$ (SDG - SDagger)		$S^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$	$S^{\dagger}\|0> \rightarrow \|0>$	[1.+0.j, 0.+0.j]
			$S^{\dagger}\|1> \rightarrow -i\|1>$	[0.+0.j, 0.-1.j]
$T^{\dagger}$ (TDG - TDagger)		$T^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\frac{\pi}{4}} \end{bmatrix}$	$T^{\dagger}\|0> \rightarrow \|0>$	[1.+0.j, 0.+0.j]
			$T^{\dagger}\|1> \rightarrow e^{-i\frac{\pi}{4}}\|1>$	[0.+0.j , 0.70710678-0.70710678j]
Controlled Not (CX)		$CX = \begin{bmatrix} I_2 & 0_2 \\ 0_2 & X \end{bmatrix}$ where, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $0_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	CX $\|10> \rightarrow \|11>$ $2 \rightleftharpoons 3$	[0.+0.j, 0.+0.j, 0.+0.j, 1.+0.j]
			CX $\|11> \rightarrow \|10>$	[0.+0.j, 0.+0.j, 1.+0.j, 0.+0.j]

Controlled Z (CZ)		$CZ = \begin{bmatrix} I_2 & 0_2 \\ 0_2 & Z \end{bmatrix}$	CZ $\|11> \rightarrow -\|11>$ 3 -> -3	[0.+0.j, 0.+0.j, 0.+0.j, -1.+0.j]
Controlled Phase (CP)		$CP = \begin{bmatrix} I_2 & 0_2 \\ 0_2 & S \end{bmatrix}$	CP $\|11> \rightarrow i\|11>$ 3 -> -i3	[0. +0.j, 0.+0.j ,0. +0.j, 0.92387953+0.38268343j]
Swap		$Swap = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	Swap $\|01> \rightarrow \|10>$	[0.+0.j, 0.+0.j, 1.+0.j, 0.+0.j]
			Swap $\|10> \rightarrow \|01>$	[0.+0.j, 1.+0.j, 0.+0.j, 0.+0.j]
Toffoli (CCX – Controlled-Controlled NOT)		$CCX = \begin{bmatrix} I_4 & 0_4 \\ 0_4 & CX \end{bmatrix}$ where, $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $0_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	CCX $\|110> \rightarrow \|111>$ 6 -> 7	[0.+0.j, 0.+0.j, 0.+0.j, 0.+0.j, 0.+0.j, 0.+0.j, 0.+0.j, 1.+0.j]
Fredkin (CSWAP- Controlled Swap)		$CSWAP = \begin{bmatrix} I_4 & 0_4 \\ 0_4 & SWAP \end{bmatrix}$	CSWAP $\|101> \rightarrow \|110>$ $5 \rightleftharpoons 6$	[0.+0.j, 0.+0.j, 0.+0.j, 1.+0.j, 0.+0.j, 0.+0.j, 0.+0.j, 0.+0.j]