Electrons with energies of 1.0 eV and 2.0 eV are incident on a barrier 10.0 eV high and 0.50 nm wide. (a) find their respective transmission probabilities (b) How are these affected if the barrier is doubled in width.

8 Solution

Transmission probability is given by

$$T = e^{-2kL}$$

where k is wave number inside barrier and is given by $k = \frac{\sqrt{2m(U-E)}}{\hbar}$

$\hbar = h/2\pi$ reduced planck's const.

for e^- with 1.0 eV energy

$$k_1 = \frac{\sqrt{2m(10-1)\times 1.6\times10^{-19}}}{1.054\times10^{-34}\ \text{Js}} = \frac{\sqrt{2\times(9.1\times10^{-31}\ \text{kg})\times 9\times1.6\times10^{-19}\ \text{J/eV}}}{1.054\times10^{-34}}$$

$$k_1 = 1.542\times10^{10}\ m^{-1}$$

$$\Rightarrow T_1 = e^{-2k_1 L}$$

$$= e^{-2\times1.542\times10^{10}\times0.50\times10^{-9}}$$

$$\boxed{T_1 = e^{-15.42} = 2.05\times10^{-7}}$$

II For 2.0 eV

$$k_2 = \frac{\sqrt{2\times9.1\times10^{-31}\times8\times1.6\times10^{-19}}}{1.054\times10^{-34}} = 1.448\times10^{10}\ m^{-1}$$

$$T_2 = e^{-2k_2L} = e^{-2\times1.448\times10^{10}\times0.50\times10^{-9}}$$

$$= e^{-14.48} = 5.14\times10^{-7}$$

2nd Part: if barrier is doubled in width to 1.0 nm find T_1 & T_2

$$T_1 = e^{-30.84} = 4.039\times10^{-14}$$

$$T_2 = e^{-28.96} = 2.647\times10^{-13}$$

$$\left[\frac{4x^2}{L^4} - \frac{2}{L^2}\right]\psi + \frac{2m}{\hbar^2}\left[E - U\right]\psi = 0$$

$$\frac{4x^2}{L^4} - \frac{2}{L^2} + \frac{2m}{\hbar^2}(-U) = 0$$

$$\Rightarrow \frac{4x^2}{L^4} - \frac{2}{L^2} = \frac{2m}{\hbar^2}U$$

$$\Rightarrow \boxed{U = \frac{\hbar^2}{2m}\left(\frac{4x^2}{L^4} - \frac{2}{L^2}\right)}$$

(7) Solution

The Energy of Particle in an infinite square well is given by

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

The energy E and wavelength λ of a Photon emitted as the Particle makes a transition from the $n=2$ State to the $n=1$ State are

$$E = E_2 - E_1 = \frac{2^2\pi^2\hbar^2}{2mL^2} - \frac{1^2\pi^2\hbar^2}{2mL^2}$$

$$E = (2^2 - 1^2)\frac{\pi^2\hbar^2}{2mL^2} = 3\frac{\pi^2\hbar^2}{2mL^2} = 6.15\ \text{MeV}$$

and λ of Photon $= \frac{hc}{E} = 2.02 \times 10^{-7-6}\ m \times \frac{10^{-15}}{\cancel{10^{-15}}}$

$= \cancel{202\text{Å}}\ 202\ fm$

$$\frac{6.62\times10^{-34}\times3\times10^{8}}{6.15\times10^{6}\times1.6\times10^{-19}} = 2.018\times10^{-13} = 202\ fm$$

Q.7 A proton is confined in an infinite square well of width 10 fm. Calculate the energy and wavelength of the photon emitted when the proton undergoes a transition from the first excited state ($n=2$) to ground state ($n=1$).

(b) The expectation value $\langle x \rangle$ of particle's position is

$$\langle x\rangle = \int_0^1 \psi^* \hat{x} \psi \, dx$$

$$= \int_0^1 \psi^* x \psi \, dx$$

$$= \int_0^1 ax \cdot x \cdot ax \, dx = \int_0^1 a^2 x^3 \, dx = a^2 \int_0^1 x^3 dx$$

$$= \frac{a^2}{4} \Big| x^4 \Big|_0^1$$

$$= \frac{a^2}{4} \text{ Ans}$$

Q6 In a region of space, a particle with zero energy has a wave function $\psi = Ae^{-\left(\frac{x^2}{L^2}\right)}$. Determine the steady state potential energy as a function of x.

6 solution Steady State Potential ⇒ Steady State S.W.E Schrodinger Equation

given $\psi = Ae^{-x^2/L^2}$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{2m}{\hbar^2}(E-U)\psi = 0 \longrightarrow (1)$$ time independent

$E = 0$; $\psi = Ae^{-x^2/L^2}$

$$\Rightarrow \frac{\partial\psi}{\partial x} = \frac{\partial}{\partial x}\left(Ae^{-x^2/L^2}\right) = Ae^{-x^2/L^2}\cdot\left(\frac{-2x}{L^2}\right)$$

$$= \frac{-2A}{L^2} x \cdot e^{-x^2/L^2}$$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{-2}{L^2} A x \cdot e^{-x^2/L^2}\right)$$

$$= \frac{-2A}{L^2}\left[x\cdot e^{-x^2/L^2}\cdot\left(\frac{-2x}{L^2}\right) + e^{-x^2/L^2}\right]$$

$$= \frac{-2A}{L^2}\left[-\frac{2x^2}{L^2} e^{-x^2/L^2} + e^{-x^2/L^2}\right]$$

$$= -\frac{2}{L^2}\left[-\frac{2x^2}{L^2} + 1\right]\psi = \left[+\frac{4x^2}{L^4} - \frac{2}{L^2}\right]\psi$$

$$= \left[\frac{4x^2}{L^4} - \frac{2}{L^2}\right]\psi \longrightarrow (2)$$

Put this in eq. 1

(4) Which of the following are eigenfunctions of the operator $\frac{\partial^2}{\partial x^2}$? Find out the appropriate eigenvalue for them

(i) $\sin x$

(ii) $\sin^2 x$

Solution given that $f(x) = \sin x$

(i) operating $\frac{\partial^2}{\partial x^2}$ on $f(x)$, we get

$$\frac{\partial^2}{\partial x^2}(\sin x) = -\sin x = -f(x)$$

Hence, $\sin x$ is an eigenfunction having eigenvalue -1

(ii) $f(x) = \sin^2 x$

operating $\frac{\partial^2}{\partial x^2}$ on $f(x)$, we get

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}(2\sin^2 x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}(1 - \cos 2x)$$

$$= \frac{1}{2}\frac{\partial}{\partial x}(0 + 2\sin 2x)$$

$$= \frac{1}{2} \cdot 4\cos 2x$$

$$= 2\cos 2x$$

$\cos 2x = \frac{1 - \sin^2 x}{2}$

$1 - 2\cos 2x = 2\sin^2 x$

$2(1 - 2\sin^2 x)$

$= 2 - 4\sin 2x$

Hence it is not an eigenfunction ~~of~~ for $f(x) = \sin^2 x$ Not same function

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(5) **Solution** A particle limited to the x axis has the wave function $\psi = ax$ b/w $x = 0$ and $x = 1$; $\psi = 0$ elsewhere. (a) Find the probability that the particle can be found b/w $x = 0.45$ and $x = 0.55$ (b) find the expectation value $\langle x \rangle$ of the particle's position.

given $\psi = ax \quad 0 \le x \le 1$

$\psi = 0$ elsewhere

To find:- Probability of particle b/w 0.45 and 0.55

$$\text{Probability} = \int_{x_1}^{x_2} \psi^* \psi \, dx = \int_{0.45}^{0.55} a^2 x^2 \, dx = a^2 \int_{0.45}^{0.55} x^2 dx = \frac{a^2}{3} \left| x^3 \right|_{0.45}^{0.55}$$

$$= 0.0251 a^2$$

③ The wave function of a free particle in normalized state is represented by $\psi = Ne^{-(x^2/2a^2)+ikx}$

Calculate the normalization factor N and the maximum probability of finding the particle.

<u>Solution</u> The normalization condition is

$$\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1$$

Putting the values of ψ & ψ^* in the above equation, we get

$$\Rightarrow \int_{-\infty}^{\infty} Ne^{-(x^2/2a^2)\cancel{+ikx}} \cdot Ne^{-(x^2/2a^2)+ikx} \, dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} N^2 e^{-2x^2/2a^2} \, dx = 1$$

$$\Rightarrow N^2 \int_{-\infty}^{\infty} e^{-x^2/a^2} \, dx = 1 \qquad \left[\because \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \right.$$

$$\sqrt{\frac{\pi}{1/a^2}}$$

$$a\sqrt{\pi}$$

$$\Rightarrow N^2 \cdot a\sqrt{\pi} = 1$$

$$\Rightarrow \boxed{N^2 = \frac{1}{a\sqrt{\pi}} \Rightarrow N = \frac{1}{a^{1/2}\pi^{1/4}}}$$

→ The maximum probability $P(x)$ can be given as

$$P(x) = |\psi^*(x)\psi(x)|$$

$$= N^2 e^{-x^2/a^2}$$

$$= \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2}$$

Q. Write down the conditions for acceptable wave function and Prove that $\psi = Ae^{-x^2}$ is an acceptable wave function.

The wave ... represented ... Calculate ... Prob... Solut...

Solution Conditions

(i) The wave function must be finite everywhere
(ii) Single value
(iii) ψ must be continuous.
(iv) Derivative of a given function should also be continuous.

Double

Proof given function is $\psi = Ae^{-x^2}$

$A \frac{1}{e^{-x^2}}$ $\frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$

Finite (i) $\lim_{x \to \pm\infty} \psi(x) = Ae^{-x^2}$

$= 0 \Rightarrow$ this function is finite everywhere

Single Valued (ii) Test for Some values $x = 1, 2, 3 --$

$\psi(1) = Ae^{-1}$
$\psi(-1) = Ae^{-1}$
$\psi(2) = Ae^{-2}$

Hence it is Single Valued

$\lim_{x \to c} \frac{f(x)}{g(x)}$ $\lim_{x \to c} \frac{f'(x)}{g'(x)}$

Continuous :- (iii) $\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial x}(Ae^{-x^2}) = -2Axe^{-x^2}$

$\lim_{x \to \pm\infty} \frac{\partial\psi}{\partial x} = -2A \frac{x}{e^{x^2}} = \frac{\infty}{\infty}$

Apply L-Hospital Rule

$\Rightarrow \lim_{x \to \pm\infty} \frac{\partial\psi}{\partial x} = -\frac{2A \cdot 1}{2x \cdot e^{x^2}} = \frac{-A}{x \cdot e^{x^2}} = -\frac{A}{\infty} = 0$

(iv) $-2A\frac{\partial}{\partial x}(x \cdot e^{-x^2}) = -2A\left[x \cdot e^{-x^2} \cdot (-2x) + e^{-x^2}\right]$

$\lim_{x \to \pm\infty} \frac{\partial^2\psi}{\partial x^2} = -2A\left[-2x^2e^{-x^2} + e^{-x^2}\right]$

$= 4Ax^2e^{-x^2} - 2Ae^{-x^2}$

$= \frac{4Ax^2}{e^{x^2}} \times \cancel{2Ae^{x^2}}$

$\frac{4Ax^2}{e^{x^2}} - 2Ae^{-x^2} = \frac{4Ax^2}{e^{x^2}}$

Applying Hospital Rule

$\frac{8Ax}{2x \cdot e^{x^2}} = \frac{8A}{2[e^{x^2} + 2x \cdot 2x e^{x^2}]}$

$= 0$

Que: Kinetic energy of an electron and Photon is 4.55×10^{-25} J. Calculate the velocity, momentum and wavelength of electron and Photon.

Solution if m_0 is the rest mass and v is the velocity of electron then its kinetic energy (E_k) is given by

$$E_k = \frac{1}{2} m_0 v^2$$

given that $E_k = 4.55 \times 10^{-25}$ J

$$\Rightarrow 4.55 \times 10^{-25} = \frac{1}{2} \times 9.1 \times 10^{-31} kg \times v^2$$

$$\Rightarrow v^2 = \frac{4.55 \times 10^{-25}}{9.1 \times 10^{-31}} \times 2 = 1.00 \times 10^6$$

$$\Rightarrow v = 10^3 \ ms^{-1}$$

momentum of electron is given as $p = m_0 v$

$$= 9.1 \times 10^{-31} \times 10^3$$

$$= 9.1 \times 10^{-28} \ kg\,ms^{-1}$$

wavelength of electron :- $\lambda = h/p = \frac{6.62 \times 10^{-34}}{9.1 \times 10^{-28}} = 7.274 \times 10^{-7}$

$\lambda = \frac{h}{mv}$

wavelength of electron :- $\lambda = h/p = \frac{6.62 \times 10^{-34}}{9.1 \times 10^{-28}} = 7.274 \times 10^{-7}$ m

IInd Part For Photon $E = hc/\lambda = 4.55 \times 10^{-25} = \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{\lambda}$

$$\Rightarrow \lambda = 4.365 \times 10^{-1}$$

$v = 3 \times 10^8$

$$p = h/\lambda = \frac{6.62 \times 10^{-34}}{4.365 \times 10^{-1}} = 1.517 \times 10^{-33} \ kg\,ms^{-1}$$