Julian Salazar Pantoja

Taller 3

Capitulo 6

6.1.10

$$x' = y + y^2 \quad (1)$$

$$y' = -\frac{x}{2} + \frac{y}{5} - xy + \frac{6}{5}y^2 \quad (2)$$

igualando (1) a 0

$$y + y^2 = 0 \qquad y(y+1) = 0 \qquad y = 0 \qquad y = -1$$

(2)

$$-\frac{x}{2} + \frac{y}{5} - xy + \frac{6}{5}y^2 = 0 \qquad -x\left(\frac{1}{2} + y\right) = -\frac{y}{5}(6y+1)$$

$$x = \frac{y}{5}\frac{(6y+1)}{(y+1/2)} = \frac{y(6y+1)2}{5(2y+1)} \qquad x = \frac{12y^2 + 2y}{10y+5}$$

6.3.12

$$\dot{\theta} = \frac{d\theta}{dr} = \frac{d}{dr}\tan^{-1}(y/x)$$

$$= \frac{d}{d(y/x)}\tan^{-1}(y/x) \cdot \frac{d(y/x)}{dr} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{d\,y/x}{dr}$$

$$= \frac{1}{\frac{x^2+y^2}{x^2}} \; \frac{\left(x\frac{dy}{dr} - y\frac{dx}{dr}\right)}{x^2}$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{x\dot{y} - y\dot{x}}{x^2} = \frac{x\dot{y} - y\dot{x}}{x^2+y^2} = \frac{x\dot{y} - y\dot{x}}{r^2} \quad (r^2 = x^2 + y^2)$$

$$\Rightarrow \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$$

6.3.14

$$x' = -y + ax^3 \quad (1)$$

$$y' = x + ay^3 \quad (2)$$

$$\text{Jacobiano} = \begin{vmatrix} 3ax^2 & -1 \\ 1 & 3ay^2 \end{vmatrix} \Rightarrow \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$

$x = \pm i$

$Ec^n(1)\,x + Ec^n(2)\,y \Rightarrow xx' + yy' = ax^4 + ay^4$

$\Rightarrow rr' = a(x^4 + y^4)$

Por $r>0$, $r'>0$ para $a>0$ y $r'<0$ para $a<0$

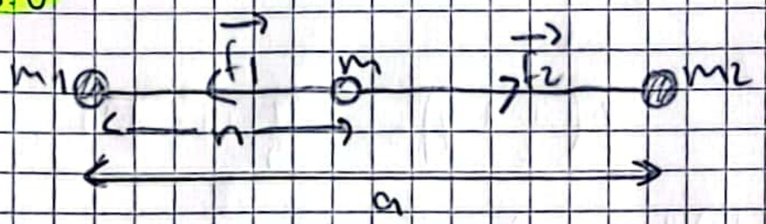
r crece para $a>0$ y decrece para $a<0$

$\Rightarrow \quad \theta' = 1 - a r^2 \cos\theta \cdot \sin\theta\,(\cos^2\theta - \sin^2\theta)$

θ es una función de incremento.

El origen es una espiral inestable para $a>0$ y estable para $a<0$.

6.3.8



$\Rightarrow \vec{F_1} = \dfrac{Gm_1m}{r^2}(-\hat{i}) \qquad \vec{F_2} = \dfrac{Gm_2m}{(a-r)^2}\,\hat{i}$

Aplicando la segunda ley de Newton

$F = ma$

$\Rightarrow \dfrac{Gm_2m}{(a-r)^2} - \dfrac{Gm_1m}{r^2} = m\vec{r}'' \quad \Rightarrow \vec{r}'' = \dfrac{Gm_2}{(a-r)^2} - \dfrac{Gm_1}{r^2}$

$\Rightarrow \vec{r}'' = \dfrac{Gm_2}{(r-a)^2} - \dfrac{Gm_1}{r^2}$

b) En el punto de equilibrio.

$\Rightarrow \dfrac{Gm_2}{(r-a)^2} = \dfrac{Gm_1}{r^2} \Rightarrow \dfrac{r^2}{(r-a)^2} = \dfrac{m_1}{m_2} \Rightarrow \dfrac{r}{r-a} = \sqrt{\dfrac{m_1}{m_2}}$

$\Rightarrow \dfrac{r-a}{r} = \sqrt{\dfrac{m_2}{m_1}} \Rightarrow 1 - \dfrac{a}{r} = \sqrt{\dfrac{m_2}{m_1}} \Rightarrow 1 - \sqrt{\dfrac{m_2}{m_1}} = \dfrac{a}{x}$

$\Rightarrow x = \dfrac{a}{1 - \sqrt{\dfrac{m_2}{m_1}}} \Rightarrow x = \sqrt{\dfrac{m_1}{m_1 - m_2}} \cdot a$ Inestable.

6.4.2

$$\dot{x} = x(3-2x-y) = 3x - 2x^2 - xy$$
$$\dot{y} = y(2-x-y) = 2y - xy - y^2$$

Puntos de equilibrio $(0,0)$; $(1,1)$; $(3/2,0)$; $(0,2)$

Jacobiano $fx = \begin{bmatrix} 3-4x-y & -x \\ -y & 2-x-2y \end{bmatrix}$

$fx\big|_{x=(0,0)} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \begin{matrix} \lambda_1 = 3 \ ; & v_1 = [1 \ 0] \\ \lambda_2 = 2 \ ; & v_2 = [0, 1] \end{matrix}$

$fx\big|_{x=(1,1)} = \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix} \rightarrow \begin{matrix} \lambda_1 = -2{,}618 \ ; & v_1 = [1{,}6 \ \ 1] \\ \lambda_2 = -0{,}38 \ ; & v_2 = [-0{,}6 \ \ 1] \end{matrix}$

$fx\big|_{x=(3/2,0)} = \begin{bmatrix} -3 & -1{,}5 \\ 0 & -0{,}5 \end{bmatrix} \rightarrow \begin{matrix} \lambda_1 = -3 \ ; & v_1 = [1 \ \ 0] \\ \lambda_2 = 0{,}5 \ ; & v_2 = [-0{,}42 \ \ 1] \end{matrix}$

$fx\big|_{x=(0,2)} = \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix} \rightarrow \begin{matrix} \lambda_1 = -2 \ ; & v_1 = [0 \ \ 1] \\ \lambda_2 = 1 \ ; & v_2 = [-3 \ \ 2] \end{matrix}$

6.5.4

$\ddot{x} = ax - x^2$ para $a<0$, $a>0$ y $a=0$

Puntos de equilibrio

$f(x) = ax - x^2 = 0$ $\quad x(a-x) = 0$ $\quad f'(x) = a - 2x$

$x=0$, $x=a$

Caso ①
Cuando $a=0$ el unico punto es $x=0$

$f'(x) = a - 2x$ $\quad f'(0) = a - 0 = a$

Caso ② si $a>0$, $x=0$, $x=a$

$f'(x) = a - 2x$ $\quad a>0$, $f'(x)>0$

Caso ③ si $a<0$, $x=0$, $x=a$

$f'(x) = a - 2x$ si $a<0$ $\quad f'(x)<0$

6.5.8

$$H = \frac{p^2}{2m} + \frac{kx^2}{2}$$

Ecuaciones de hamilton ① $\frac{\partial p}{\partial t} = \frac{\partial H}{\partial x}$

② $\frac{\partial x}{\partial t} = \frac{\partial H}{\partial p}$

$$-\frac{\partial H}{\partial x} = \frac{-2kx}{2} = -kx$$

$$\frac{\partial p}{\partial t} = F = -kx \Rightarrow \frac{\partial p}{\partial t} = -kx = -\frac{\partial H}{\partial x}$$

$$\frac{\partial H}{\partial p} = \frac{2p}{2m} = \frac{p}{m} \qquad p = mv \Rightarrow \frac{p}{m} = v \qquad \frac{\partial x}{\partial t} = v$$

$$\Rightarrow \frac{\partial x}{\partial t} = \frac{\partial H}{\partial p} = v$$

Simple harmonico $PE = \frac{kx^2}{2}$; $KE = \frac{p^2}{2m} = \frac{1}{2}mv^2 \Rightarrow p = mv$

Donde $PE + KE =$ Energía total $= \frac{p^2}{2m} + \frac{1}{2}kx^2 = H$

6.5.12

$$\dot{x} = xy, \quad \dot{y} = -x^2$$

a) $E = x^2 + y^2$

$\Rightarrow \dot{E} = 2x\dot{x} + 2y\dot{y} = 2x(xy) + 2y(-x^2)$

$\dot{E} = 0 \therefore E$ es conservada

b) $\dot{x} = 0$ y $\dot{y} = 0$ $xy = 0$ y $x^2 = 0$

$\Rightarrow (x = 0$ ó $y = 0)$ y $(x^2 = 0 \Rightarrow x = 0)$, $y \in \mathbb{R}$

$\therefore (x, y) = (0,0)$ es un punto fijo.

$\therefore \forall y \in \mathbb{R}$, (x,y) es un punto fijo para cada vecindario de $(0,0)$

Contiene infinitos puntos $(0,y)$, $y \neq 0 \Rightarrow (0,0)$ es no aislado.

6.7.4

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2\right] = \frac{d\theta^2}{dt^2}\cdot\frac{d\theta}{dt}$$

y, $\int \frac{d\theta^2}{dt^2}\cdot\frac{d\theta}{dt}\,dt = \frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 + C$

multiplicando la ecuación diferencial $\frac{d\theta^2}{dt^2}\cdot\frac{d\theta}{dt} = -\operatorname{sen}\theta\cdot\frac{d\theta}{dt}$

$\therefore \int \frac{d\theta^2}{dt^2}\cdot\frac{d\theta}{dt}\cdot dt = \int -\operatorname{sen}\theta\frac{d\theta}{dt}\cdot dt \Rightarrow \frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = \cos\theta + C$

pero, $\frac{d\theta}{dt} = 0$ cuando $\theta = \alpha$ $\quad C = -\cos\alpha$

$\therefore \frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = \cos\theta - \cos\alpha \Rightarrow \dot{\theta}^2 = 2(\cos\theta - \cos\alpha)$

Sabemos que $\cos\theta = 1 - 2\operatorname{sen}^2\frac{\theta}{2}$, y $\cos\alpha = 1 - 2\operatorname{sen}^2\frac{\alpha}{2}$

$\therefore \frac{d\theta}{dt} = \dot{\theta} = 2\sqrt{\operatorname{sen}^2\frac{\alpha}{2} - \operatorname{sen}^2\frac{\theta}{2}}$

Dejando $k = \operatorname{sen}\frac{\alpha}{2}$ $\quad\therefore \frac{d\theta}{dt} = 2\sqrt{k^2 - \operatorname{sen}^2\frac{\theta}{2}}$

$\Rightarrow \frac{dt}{d\theta} = \frac{1}{2\sqrt{k^2 - \operatorname{sen}^2\frac{\theta}{2}}}$

Integrando $\Rightarrow \int_0^{T/4}\frac{dt}{d\theta}\cdot d\theta = \int_0^{\alpha}\frac{d\theta}{2\sqrt{k^2 - \operatorname{sen}^2\frac{\theta}{2}}}$

$\therefore T = 2\int_0^{\alpha}\frac{d\theta}{\sqrt{k^2 - \operatorname{sen}^2\frac{\theta}{2}}}$

$\therefore \operatorname{sen}\left(\frac{1}{2}\alpha\right)\cdot\cos\theta\, d\theta = \frac{1}{2}\cos\left(\frac{1}{2}\theta\right)d\theta$

$\Rightarrow T = 2\int_0^{\alpha}\frac{d\theta}{\sqrt{\frac{1}{4}\operatorname{sen}^2\alpha - \operatorname{sen}^2\frac{\theta}{2}}} = 4\int_0^{\alpha}\frac{d\theta}{\sqrt{4\left(\operatorname{sen}^2\frac{\alpha}{2} - \operatorname{sen}^2\frac{\theta}{2}\right)}}$

$$= 4\int_0^{\pi/2} \frac{d\theta}{\cos^{1/2}\theta} \quad \therefore \quad T = 4\int_0^{\pi/2} \frac{d\theta}{\cos\frac{\theta}{2}} \quad (1)$$

$$\Rightarrow K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m\,\mathrm{sen}^2\theta}} \,;\; K\left(\mathrm{sen}^2\frac{\alpha}{2}\right) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \mathrm{sen}^2\frac{\alpha}{2}\,\mathrm{sen}^2\theta}}$$

$$= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \mathrm{sen}^2\left(\frac{\theta}{2}\right)}} \qquad \left(\mathrm{sen}^2\frac{\alpha}{2}\,\mathrm{sen}^2\theta = \mathrm{sen}^2\frac{\theta}{2}\right)$$

$$\Rightarrow \int_0^{\pi/2} \frac{d\theta}{\cos\theta/2}$$ Sustituyendo en (1) $\Rightarrow T = 4K\left(\mathrm{sen}^2\frac{1}{2}\alpha\right)$