Portfolio Loss Distribution

Risky assets in loan portfolio

- highly illiquid assets
- "hold-to-maturity" in the bank's balance sheet

Outstandings

The portion of the bank asset that has already been extended to borrowers.

Commitment

A commitment is an amount the bank has committed to lend. Should the borrower encounter financial difficulties, it would draw on this committed line of credit.

Adjusted exposure and expected loss

Let α be the amount of drawn down or usage given default.

Asset value at later time H, V_H

- Outstanding + $\alpha \times$ commitment, Risky
- $(1-\alpha)$ commitment, Riskless

Adjusted exposure is the risky part of V_H.

Expected loss = adjusted exposure $\times$ loss given default $\times$ probability of default

* Normally, practitioners treat the uncertain draw-down rate as a known function of the obligor's end-of-horizon credit class rating.

Example
calculation of expected loss

Commitment	\$10,000,000
Outstanding	\$3,000,000
Internal risk rating	3
Maturity	1 year
Type	Non-secured
Unused drawn-down on default (for internal rating = 3)	65%
Adjusted exposure on default	\$8,250,000
EDF for internal rating = 3	0.15%
Loss given default for non-secured asset	50%
Expected loss	\$6,188

Unexpected loss

Unexpected loss is the estimated volatility of the potential loss in value of the asset around its expected loss.

$$UL = AE \times \sqrt{\mathrm{EDF} \times \sigma^2_{\mathrm{LGD}} + \mathrm{LGD}^2 \times \sigma^2_{\mathrm{EDF}}}$$

where

$$\sigma^2_{\mathrm{EDF}} = \mathrm{EDF} \times (1 - \mathrm{EDF}).$$

Assumptions

* The random risk factors contributing to an obligor's default (resulting in EDF) are statistically independent of the severity of loss (as given by LGD).

* The default process is two-state event.

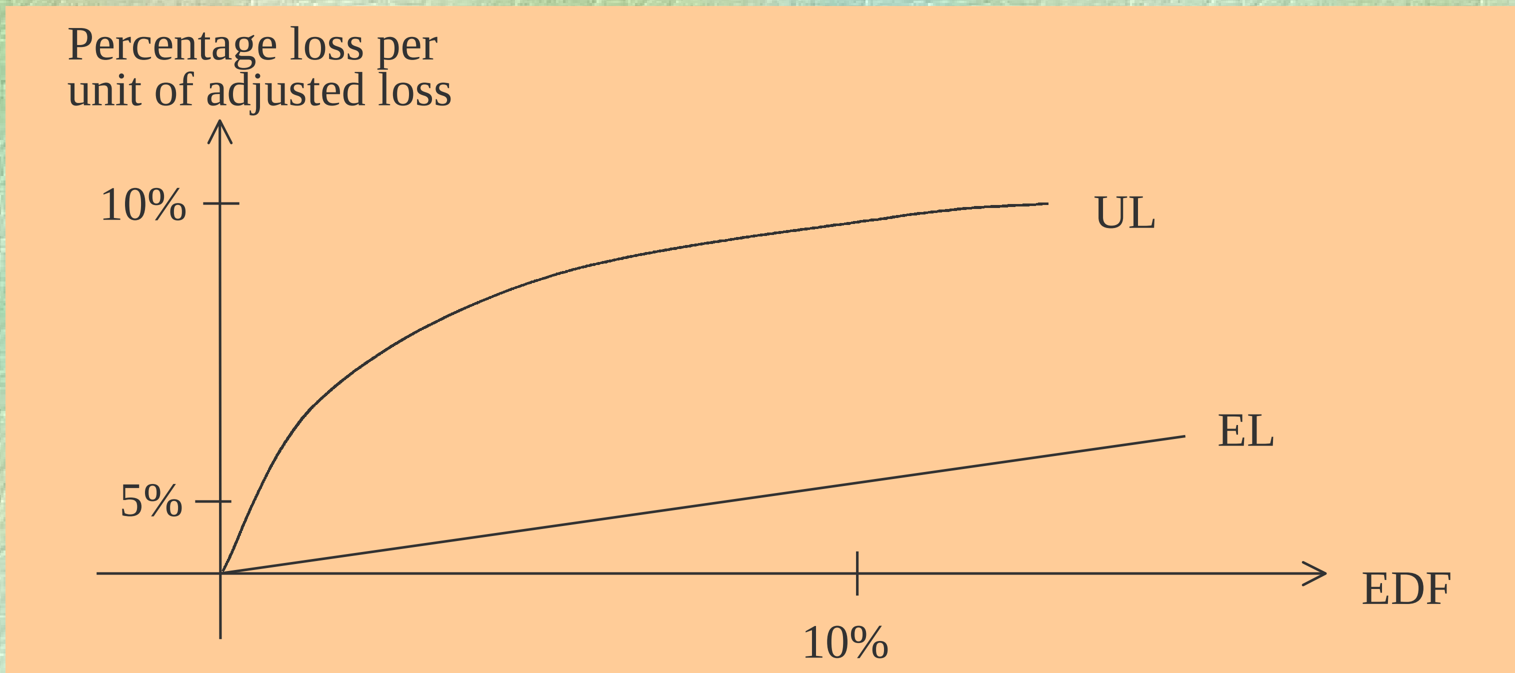
Example on unexpected loss calculation

Adjusted exposure	$8,250,000
EDF	0.15%
σ_{EDF}	3.87%
LGD	50%
σ_{LGD}	25%
Unexpected loss	$178,511

* The calculated unexpected loss is 2.16% of the adjusted exposure, while the expected loss is only 0.075%

Comparison between expected loss and unexpected loss

* The higher the recovery rate (lower LGD), the lower is the percentage loss for both EL and UL.
* EL increases linearly with decreasing credit quality (with increasing EDF)
* UL increases much faster than EL with increasing EDF.



Assets with varying terms of maturity

* The longer the term to maturity, the greater the variation in asset value due to changes in credit quality.

* The two-state default process paradigm inherently ignores the credit losses associated with defaults that occur beyond the analysis horizon.

* To mitigate some of the maturity effect, banks commonly adjust a risky asset's internal credit class rating in accordance with its terms to maturity.

Portfolio expected loss

$$\mathrm{EL}_p = \sum_i \mathrm{EL}_i = \sum_i \mathrm{AE}_i \times \mathrm{LGD}_i \times \mathrm{EDF}_i$$

where EL_p is the expected loss for the portfolio,
AE_i is the risky portion of the terminal value of the *i*th asset to which the bank is exposed in the event of default.

We may write

$$\frac{\mathrm{EL}_p}{\mathrm{AE}_p} = \sum_i w_i \frac{EL_i}{AE_i}$$

where the weights refer to

$$w_i = \frac{\mathrm{AE}_i}{\sum_i \mathrm{AE}_i} = \frac{\mathrm{AE}_i}{\mathrm{AE}_p}.$$

$$\frac{EL_p}{AE_p} = \frac{\sum EL_i}{AE_p} = \sum \frac{AE_i}{AE_p}\frac{EL_i}{AE_i} = w_i \frac{EL_i}{AE_i}$$

i	AE_i	w_i	EL_i	EL_i/AE_i
1	\$10 M	0.5	\$1	0.1
2	\$4 M	0.2	\$0.5	0.125
3	\$6 M	0.3	\$0.6	0.1
	$\sum AE_i = \$20M$	$\sum w_i = 1$		

$$\frac{EL_p}{AE_p} = 0.5 \times 0.1 + 0.2 \times 0.125 + 0.3 \times 0.1 = 0.105$$

Portfolio unexpected loss

$$\text{portfolio unexpected loss } = \mathrm{UL}_p = \sqrt{\sum_i \sum_j \rho_{ij} w_i w_j \mathrm{UL}_i \mathrm{UL}_j}$$

where

$$\mathrm{UL}_i = \mathrm{AE}_i \times \sqrt{\mathrm{EDF}_i \times \sigma^2_{\mathrm{LGD}_i} + \mathrm{L}GD_i^2 \times \sigma^2_{EDF_i}}$$

and ρ_{ij} is the correlation of default between asset i and asset j. Due to diversification effect, we expect

$$\mathrm{UL}_p << \sum_i \mathrm{UL}_i .$$

Risk contribution

The risk contribution of a risky asset *i* to the portfolio unexpected loss is defined to be the *incremental risk* that the exposure of a single asset contributes to the portfolio's total risk.

$$\mathrm{RC}_i = \mathrm{UL}_i \frac{\partial \mathrm{UL}_p}{\partial \mathrm{UL}_i}$$

and it can be shown that

$$\mathrm{RC}_i = \frac{\mathrm{UL}_i \sum_j \mathrm{UL}_j \rho_{ij}}{\mathrm{UL}_p}.$$

Undiversifiable risk

The risk contribution is a measure of the *undiversifiable risk* of an asset in the portfolio – the amount of credit risk which cannot be diversified away by placing the asset in the portfolio.

$$\mathrm{UL}_p = \sum_i \mathrm{RC}_i$$

To incorporate industry correlation, using $i \rightarrow$ industry α and $j \rightarrow$ industry β

$$\mathrm{RC}_i = \frac{\mathrm{U}L_i}{\mathrm{U}L_p}\left[\mathrm{U}L_{i\in\alpha}(1-\rho_{\alpha\alpha}) + \sum_{\beta\neq\alpha}\left(\sum_{k\in\beta}\mathrm{U}L_k\right)\rho_{\alpha\beta}\right].$$

Calculation of EL, UL and RC for a two-asset portfolio

ρ default correlation between the two exposures

EL_p portfolio expected loss

$$EL_p = EL_1 + EL_2$$

UL_p portfolio unexpected loss

$$UL_p = \sqrt{UL_1^2 + UL_2^2 + 2\rho UL_1 UL_2}$$

RC_1 risk contribution from Exposure 1

$$RC_1 = UL_1(UL_1 + \rho UL_2) / UL_p$$

RC_2 risk contribution from Exposure 2

$$RC_2 = UL_2(UL_2 + \rho UL_1) / UL_p$$

$$UL_p = RC_1 + RC_2$$

$$UL_p << UL_1 + UL_2$$

Fitting of loss distribution

The two statistical measures about the credit portfolio are

- portfolio expected loss;
- portfolio unexpected loss.

At the simplest level, the *beta distribution* may be chosen to fit the portfolio loss distribution.

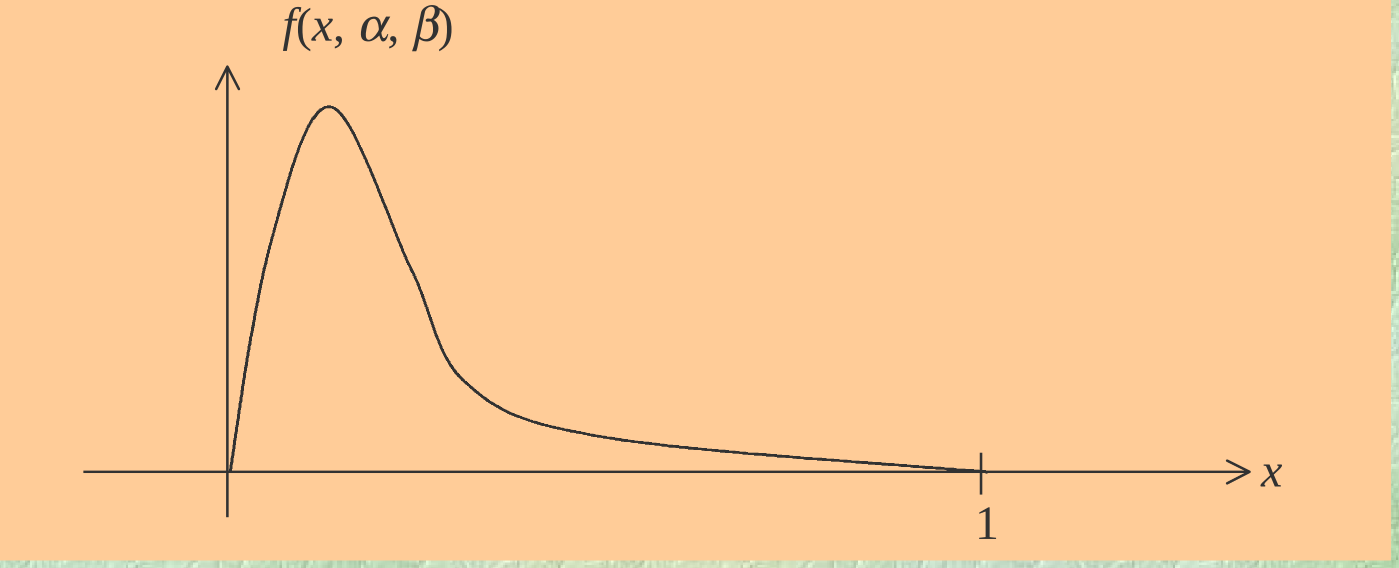
Reservation A beta distribution with only two degrees of freedom is perhaps insufficient to give an adequate description of the tail events in the loss distribution.

Beta distribution

The density function of a beta distribution is

$$F(x,\alpha,\beta)=\begin{cases}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}, & 0<x<1 \\ & & \alpha>0,\beta>0 \\ 0 & \text{otherwise}\end{cases}$$

Mean $\mu=\dfrac{\alpha}{\alpha+\beta}$ and variance $\sigma^2=\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

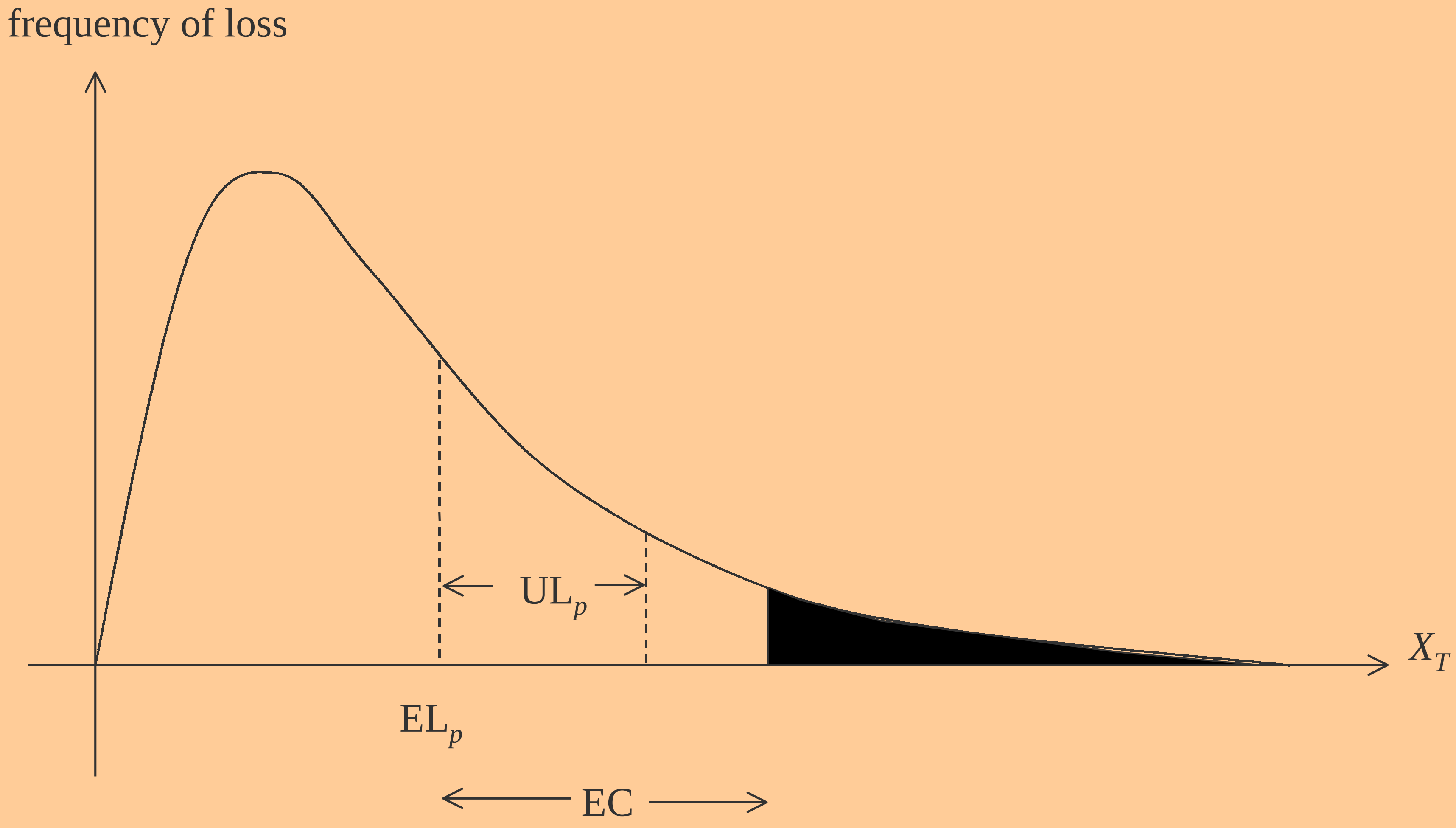


Economic Capital

If X_T is the random variable for loss and z is the percentage probability (confidence level), what is the quantity v of minimum economic capital EC needed to protect the bank from insolvency at the time horizon T such that

$$\Pr[X_T \leq v] = z.$$

Here, z is the desired debt rating of the bank, say, 99.97% for an AA rating.



frequency of loss
UL_p
X_T
EL_p
EC

Capital multiplier

Given a desired level of z, what is EC such that

$$\Pr[X_T - \mathrm{EL}_p \leq \mathrm{EC}] = z.$$

Let CM (capital multiplier) be defined by

$$\mathrm{EC} = \mathrm{CM} \times \mathrm{UL}_p$$

then

$$\Pr\left[\frac{X_T - \mathrm{EL}_p}{\mathrm{UL}_p} \leq \mathrm{CM}\right] = z.$$

Monte Carol simulation of loss distribution of a portfolio

1. *Estimate default and losses*

Assign risk ratings to loss facilities and determine their default probability

+ Assign LGD and σ_{LGD}

→

2. *Estimate asset correlation between obligors*

Determine pairwise asset correlation whenever possible

OR

Assign obligors to industry groupings, then determine industry pair correlation

3. Generate random loss given default

Determine stochastic loss given default

+

4. Generate correlated default events

Correlated default events + Decomposition of covariance matrix

+

Simulate default point

5. *Loss calculation*

Calculate facility loss for each scenario and obtain portfolio loss

6. *Loss distribution*

Construct simulated portfolio loss distribution

Generation of correlated default events

- Generate a set of random numbers drawn from a standard normal distribution.

- Perform a decomposition (Cholesky, SVD or eigenvalue) on the asset correlation matrix to transform the independent set of random numbers (stored in the vector e) into a set of correlated asset values (stored in the vector e'). Here, the transformation matrix is M, where

$$e' = M e \,.$$

The covariance matrix Σ and M are related by

$$M^T M = \Sigma.$$

Calculation of the default point

The default point threshold, DP, of the i^{th} obligor can be defined as $DP = N^{-1}(\mathrm{EDF}_i, 0, 1)$. The criterion of default for the i^{th} obligor is

$$\begin{array}{lll} \text{default} & \text{if} & e'_i < DP_i \\ \text{no default} & \text{if} & e'_i \geq DP_i. \end{array}$$

Generate loss given default

The LGD is a stochastic variable with an unknown distribution.

A typical example may be

	Recovery rate (%)	LGD (%)	σ_{LGD} (%)
secured	65	35	21
unsecured	50	50	28

$$\text{LGD}_i = \text{LGD}_s + f_i \times \sigma^{s}_{\text{LGD}}$$

where f_i is drawn from a uniform distribution whose range is selected so that the resulting LGD has a standard deviation that is consistent with historical observation.

Calculation of loss

Summing all the simulated losses from one single scenario

$$\text{Loss} = \sum_{\substack{\text{Obligors} \\ \text{in default}}} \text{Adjusted exposure}_i \times \text{LGD}$$

Simulated loss distribution

The simulated loss distribution is obtained by repeating the above process sufficiently number of times.

Features of portfolio risk

- The variability of default risk within a portfolio is substantial.
- The correlation between default risks is generally low.
- The default risk itself is dynamic and subject to large fluctuations.
- Default risks can be effectively managed through diversification.
- Within a well-diversified portfolio, the loss behavior is characterized by lower than expected default credit losses for much of the time, but very large losses which are incurred infrequently.