Vector-Valued Image Regularization with PDEs: A Common Framework for Different Applications

Siddharth Khandelwal(37.5%)	190070062
Parin Senta (37.5%)	190070042
Aniket Gupta(25%)	190070007

Theory

The paper proposes a way of vector-valued image regularization, based on variational methods and PDEs.

Regularizing an image can be seen as minimizing a measure of global image variation. This can be written as

$$\min_{\mathbf{I}:\Omega\rightarrow\mathbb{R}^n} E(\mathbf{I}) = \int_\Omega \phi(\mathcal{N}(\mathbf{I}))\, d\Omega,$$

Where N(I) is a norm related to local image variation related to eigenvalues of the structure tensor of the image and Φ is an increasing function.

Theory Contd.

The previous equation can be converted to other forms shown here which is proved in the paper

Divergence Form:
(here n is number of channels and D is 2x2 diffusion tensor related to eigenvectors and values of the structure tensor)

$$\frac{\partial I_i}{\partial t} = \operatorname{div}\left(\mathbf{D}\nabla I_i\right) \qquad (i = 1..n)$$

$$\mathbf{D} = \frac{\partial \psi}{\partial \lambda_+}\,\theta_+\theta_+^T + \frac{\partial \psi}{\partial \lambda_-}\,\theta_-\theta_-^T.$$

Oriented Laplacian Form:
(where Hi is Hessian matrix of channel i of image)

$$\frac{\partial I_i}{\partial t} = c_1\, I_{i_{\xi\xi}} + c_2\, I_{i_{\eta\eta}} = \mathbf{trace}\left(\mathbf{T}\mathbf{H}_i\right) \qquad (i = 1..n)$$

$$I_{i_{(t)}} = I_{i_{(t=0)}} * G^{(\mathbf{T},t)} \qquad (i = 1..n)$$

$$G^{(\mathbf{T},t)}(\mathbf{x}) = \frac{1}{4\pi t}\exp\left(-\frac{\mathbf{x}^T\mathbf{T}^{-1}\mathbf{x}}{4t}\right) \quad \text{with} \quad \mathbf{x} = (x \;\; y)^T$$

Theory Contd.

The matrix T is chosen as :

$$\mathbf{T} = f_-\left(\sqrt{\lambda_+^* + \lambda_-^*}\right)\theta_-^*\theta_-^{*\,T} + f_+\left(\sqrt{\lambda_+^* + \lambda_-^*}\right)\theta_+^*\theta_+^{*\,T}$$

Where

$$f_+(s) = \frac{1}{1+s^2} \quad \text{and} \quad f_-(s) = \frac{1}{\sqrt{1+s^2}}.$$

and θ+/- is λ+/- are spectral

Elements of $G_\sigma = G * G_{\sigma'}$ a gaussian smoothed version of the structure tensor G.

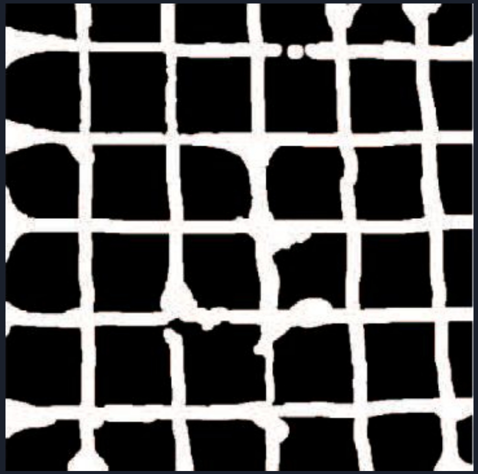
These equations can be performed locally by applying a spatially varying mask over the image which is equivalent to the trace. An illustration is shown below.

$$[\mathrm{Trace(TH)}](x,y) = \left(\begin{array}{|c|c|c|} \hline i(x-1,y-1) & i(x,y-1) & i(x+1,y) \\ \hline i(x-1,y) & i(x,y) & i(x+1,y) \\ \hline i(x-1,y+1) & i(x,y+1) & i(x+1,y+1) \\ \hline \end{array} * \begin{array}{|c|c|c|} \hline G(-1,-1) & G(0,-1) & G(1,-1) \\ \hline G(-1,0) & G(0,0) & G(1,0) \\ \hline G(-1,1) & G(0,1) & G(1,1) \\ \hline \end{array} \right)(0,0)$$

Experiments and Results

1. Inpainting

Input Image

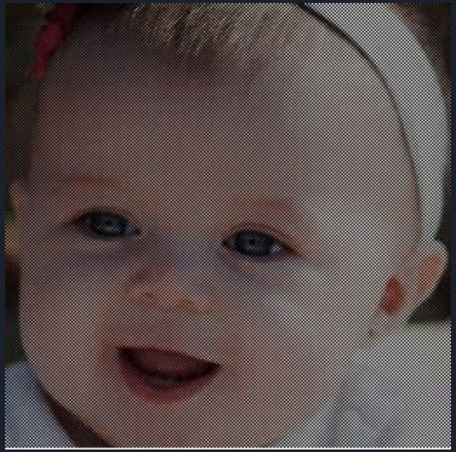


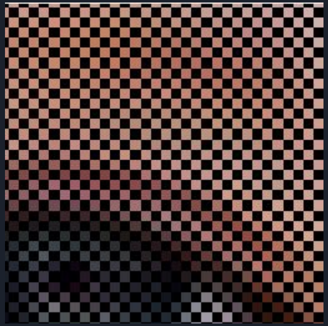
Input Mask

Output

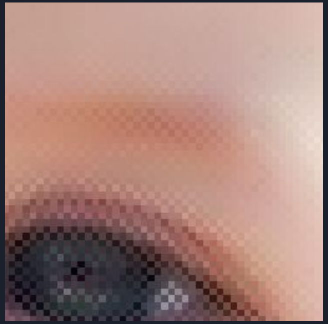
2. Image Reconstruction

Input image





50% pixels removed



Output image

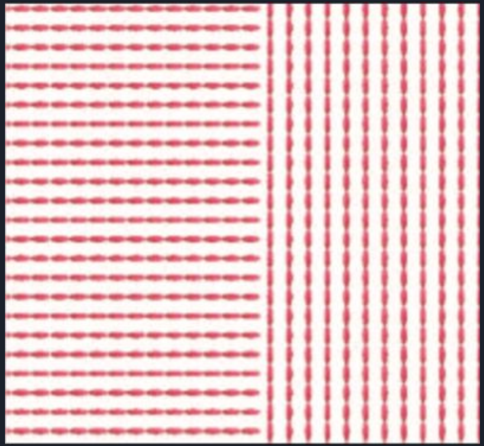
3. Flow Visualization

This is implemented through the equation :

$$\frac{\partial I_i}{\partial t} = \operatorname{trace}\left(\left[\frac{1}{\|\mathcal{F}\|}\mathcal{F}\mathcal{F}^T\right]\mathbf{H}_i\right) \quad (i = 1..n)$$

where the vector F gives the direction of the flow

Input Image



Flow Vectors

Output Image

4.1 Denoising

Image with Gaussian Noise

Gaussian Filtered Image

PDE Regularised Image

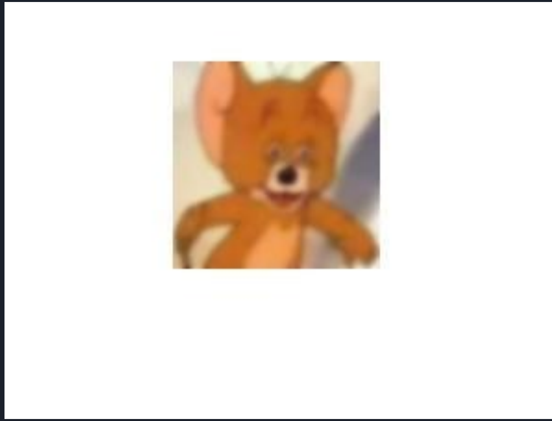
4.2 Denoising

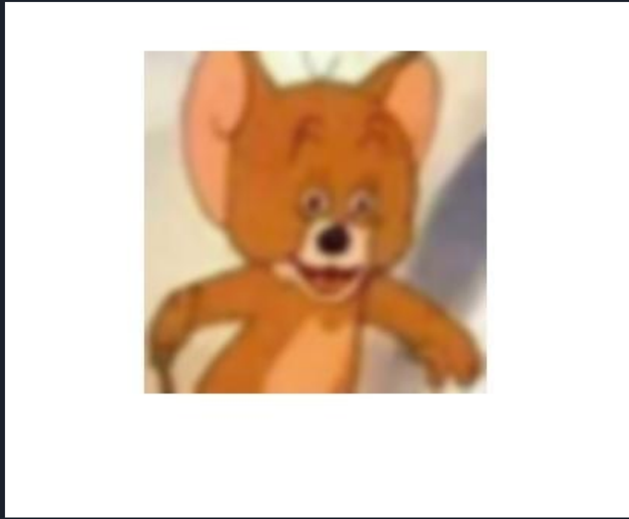
Image with Salt and Pepper Noise

Gaussian Filtered Image

PDE Regularised Image

5. Magnification





Original Image

2x Magnified Image

4x Magnified Image

Conclusion

We observe that the proposed new formulation for vector-valued image regularization based on PDEs can be used in various applications as shown and produces very good results.

References

https://tschumperle.users.greyc.fr/publications/tschumperle_pami05.pdf