Nummerical Simulation of the 2-Body Problem

May 4, 2023

1 Numerical Simulation of the 2-Body Problem

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Q inich Pakal Figueroa Coc: Gruppe 5

[ ]:
```
import numpy as np
import matplotlib.pyplot as plt
```

1.1 *i)*

[ ]:
```
def euler(w0,s0,h,e):
      #the function will take the initial velocity
    # initial position, the whished step size
    # and the desired upper bondary for the integral

    G = 1 #grativational constant
    m = 1 #mass of both bodies
    N = int(e/h) #time steps

    s = np.zeros((N,2))
    w = np.zeros((N,2))
    LRL = np.zeros((N,2))
    s[0] = s0
    w[0] = w0
    #leap-frog algorith
    for i in range(N-1):
        s[i+1] = s[i] + h*w[i]
        w[i+1] = w[i]-h*s[i]/np.linalg.norm(s[i])**3
         #calculatin LRl for every time step
        LRL[i] = s[i]*(np.dot(w[i],w[i])) - w[i]*(np.dot(s[i],w[i])) - s[i]/np.
 ↪linalg.norm(s[i])

    #calculating eccentrisity with LRL
    epsilon = np.mean([np.linalg.norm(x) for x in LRL])
    print(np.linalg.norm(LRL[1]))
    #calculatin eccentrisity geometrically
    a = max(s[:,0])- min(s[:,0])
```

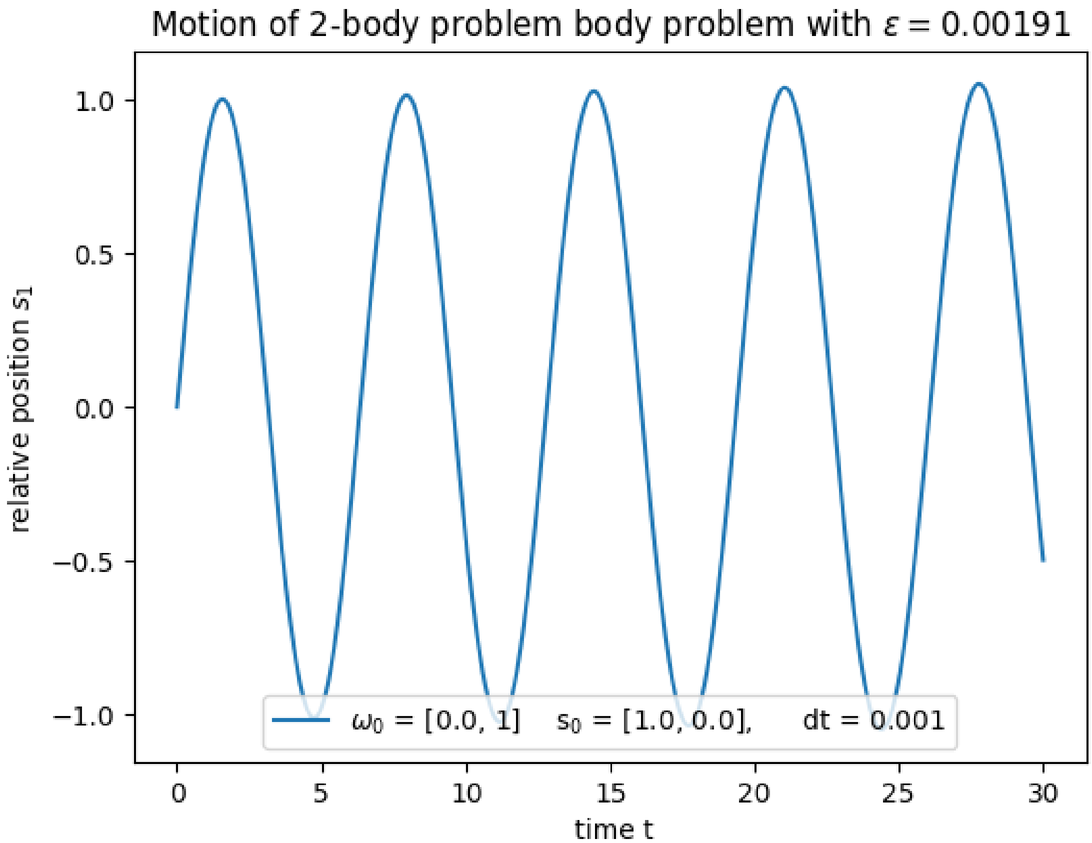
```
    b = max(s[:,1])-min(s[:,1])
    print(a)
    print(b)
    epsilon2 = np.sqrt(abs(1-a**2/b**2))
    print('epislon calculated geometrically :{:.3}'.format(epsilon2))
    #plot
    fig1, ax1 = plt.subplots()
    t = np.linspace(0,e,N)
    ax1.plot(t,s[:,1], label= r'$\omega$$_0$ = {} $\quad$s$_0$ = {},$\quad$ $\;
 ↪$ dt = {}'.format(w0,s0,h))
    plt.title('Motion of 2-body problem body problem with $\epsilon = {:.3}$'.
 ↪format(epsilon))
    plt.xlabel(' time t')
    plt.ylabel('relative position $s_1$')
    plt.legend(loc = 8)
    fig2, ax2 = plt.subplots()
    ax2.plot(s[:,0], s[:,1],  label= r'$\omega$$_0$ = {} $\quad$s$_0$ =␣
 ↪{},$\quad$ $\;$ dt = {}'.format(w0,s0,h))
    plt.title('Orbit of the 2-body problem with $\epsilon = {:.3}$'.
 ↪format(epsilon))
    plt.xlabel('relative position $s_1$')
    plt.ylabel('relative position $s_2$')
    plt.legend(loc = 8)
    return s,w
```

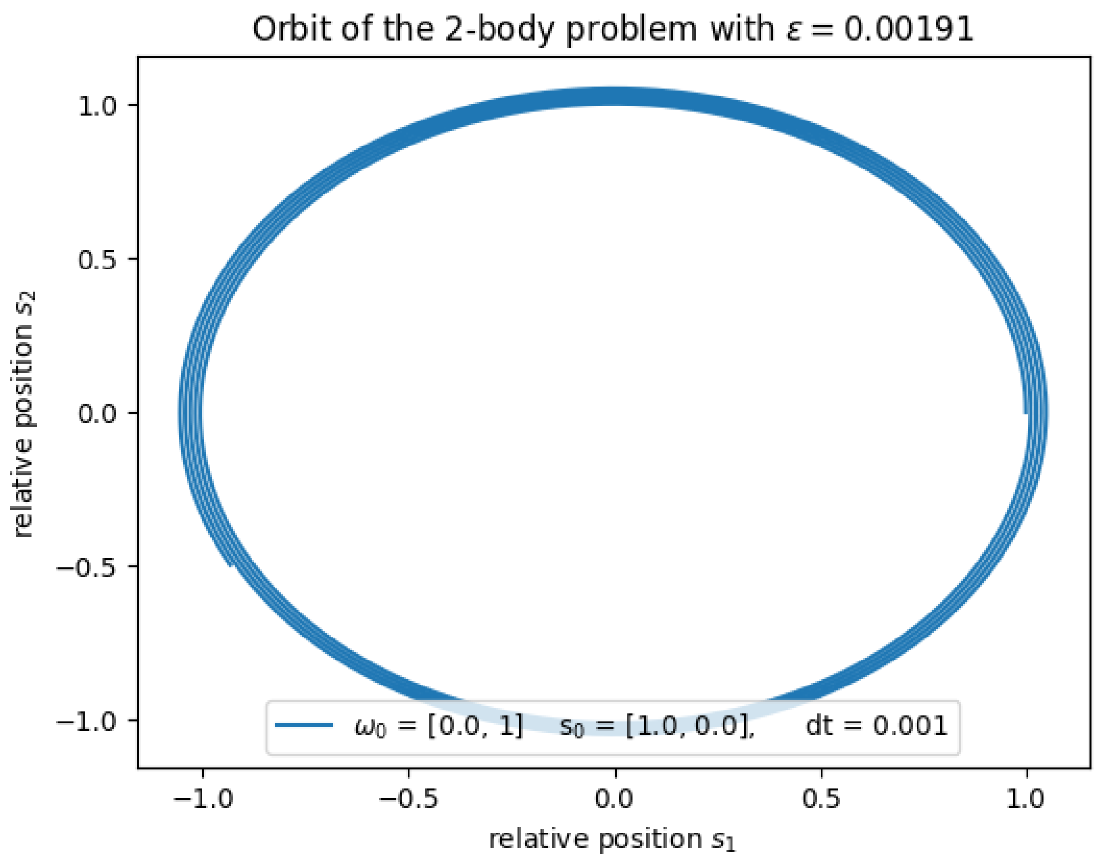
[ ]:
```
w1 = [0.0,1]
s1=[1.0,0.0]
euler(w1,s1,0.001,30)
```

```
1.5000003749539096e-06
2.1054507526942885
2.0993082373604404
epislon calculated geometrically :0.0766
```

[ ]:
```
(array([[ 1.        ,  0.        ],
        [ 1.        ,  0.001     ],
        [ 0.999999  ,  0.002     ],
        …,
        [-0.93439238, -0.49522218],
        [-0.93393954, -0.49608233],
        [-0.93348591, -0.49694207]]),
 array([[ 0.        ,  1.        ],
        [-0.001     ,  1.        ],
        [-0.002     ,  0.999999  ],
        …,
        [ 0.4528374 , -0.86015207],
        [ 0.45362748, -0.85973333],
```

```
[ 0.45441717, -0.85931387]]))
```



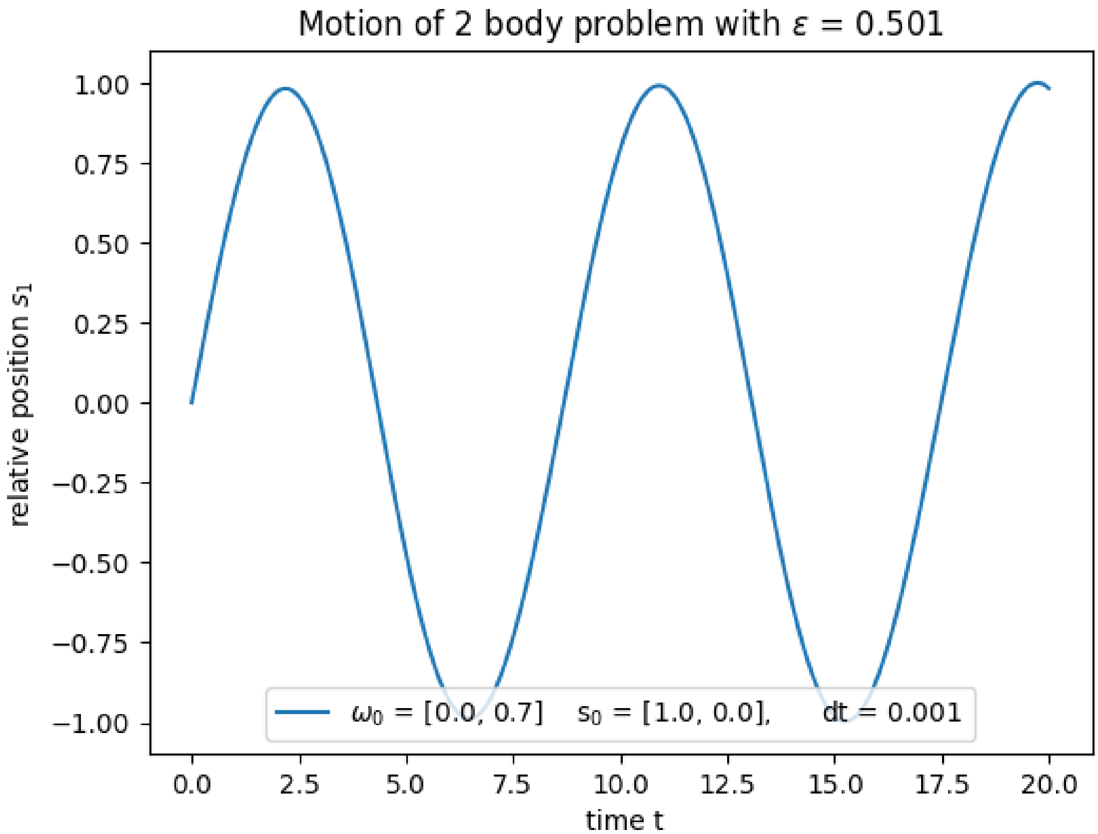


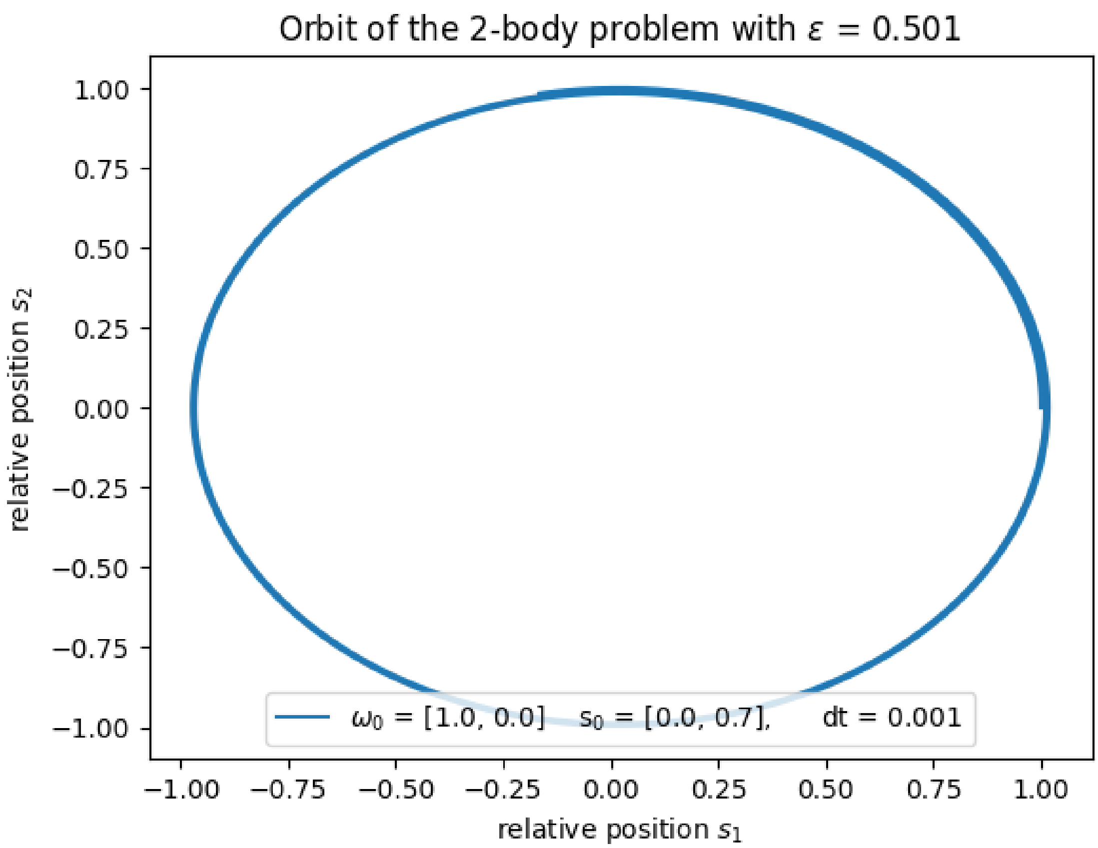
[ ]:
```
def leap_frog(w0,s0,h,e):
     #the function will take the initial velocity
    # initial position, the whished step size
    # and the desired upper bondary for the integral
    G = 1 #grativational constant
    m = 1 #mass of both bodies
    N = int(e/h) #time steps

    s = np.zeros((N,2))
    w = np.zeros((N,2))
    LRL = np.zeros((N,2))

    w[0] = w0
    s[0] = s0

    #First the programm will compute the s_1/2 step using the
    #euler forward methode
    a = 0.5*h/np.linalg.norm(s0)**3
    wn = w[0] - h/2*s[0]/np.linalg.norm(s[0])
```

```
    s[0] = s0 + h*wn

    #leap-frog algorith
    for i in range(N-1):
        w[i + 1] = w[i]-h/2*s[i]/np.linalg.norm(s[i])**3
        s[i + 1] = s[i] + w[i]*h
         #calculatin LRl for every time step
        LRL[i] = s[i]*(np.dot(w[i],w[i])) - w[i]*(np.dot(s[i],w[i])) - s[i]/np.
 ↪linalg.norm(s[i])

    #calculating eccentrisity with LRL
    epsilon = np.mean([np.linalg.norm(x) for x in LRL])
    #calculatin eccentrisity geometrically
    a = max(s[:,0])- min(s[:,0])
    b = max(s[:,1])-min(s[:,1])
    print(a)
    print(b)
    epsilon2 = np.sqrt(abs(1-b**2/a**2))
    print('epislon calculated geometrically :{:.3}'.format(epsilon2))
    #plot
    fig1, ax1 = plt.subplots()
    t = np.linspace(0,e,N)
    ax1.plot(t,s[:,1], label= r'$\omega$$_0$ = {} $\quad$s$_0$ = {},$\quad$ $\;
 ↪$ dt = {}'.format(w0,s0,h))
    plt.title('Motion of 2 body problem with $\epsilon$ = {:.3}'.
 ↪format(epsilon))
    plt.xlabel(' time t')
    plt.ylabel('relative position $s_1$')
    plt.legend(loc = 8)
    fig2, ax2 = plt.subplots()
    ax2.plot(s[:,0], s[:,1],  label= r'$\omega$$_0$ = {} $\quad$s$_0$ =␣
 ↪{},$\quad$ $\;$ dt = {}'.format(s0,w0,h))
    plt.title('Orbit of the 2-body problem with $\epsilon$ = {:.3}'.
 ↪format(epsilon))
    plt.xlabel('relative position $s_1$')
    plt.ylabel('relative position $s_2$')
    plt.legend(loc = 8)
```

[ ]:
```
w2 = [0.0,.70]
s2=[1.0,0.0]
leap_frog(w2,s2,0.001,20)
```

```
1.992265652834932
1.9961493469569298
epislon calculated geometrically :0.0625
```


Motion of 2 body problem with ε = 0.501
relative position s_1
time t
ω_0 = [0.0, 0.7] s_0 = [1.0, 0.0],   dt = 0.001
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0



2 *ii)*

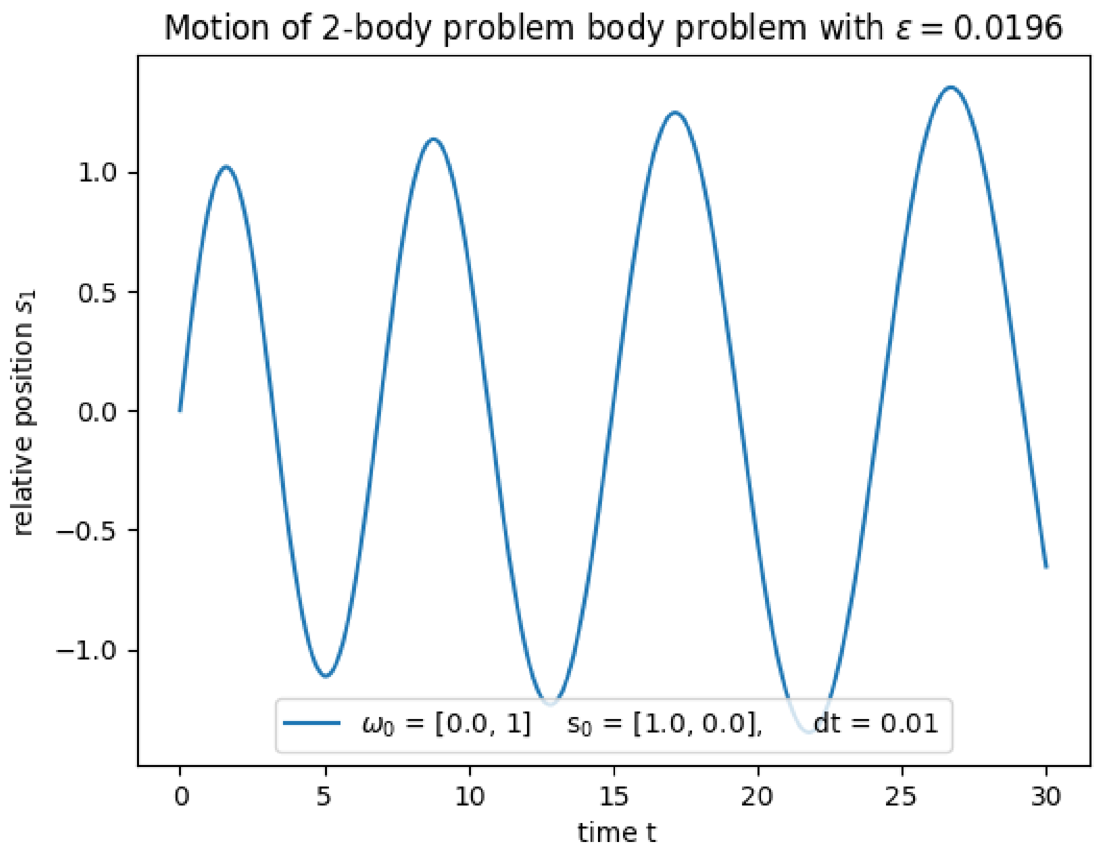
After changing the paramters of the Euler Forward Method it has been stablished, that due to the errors of this method it is not posible to get a perfect circle with radius 1. However with ω_0 =(0, 1.0) the programm delivers a good approximation. In the other hand, for the leap-frog algorith to deliver a good approximation the inital velocity has to be set to ω_0 = (0,0.7)

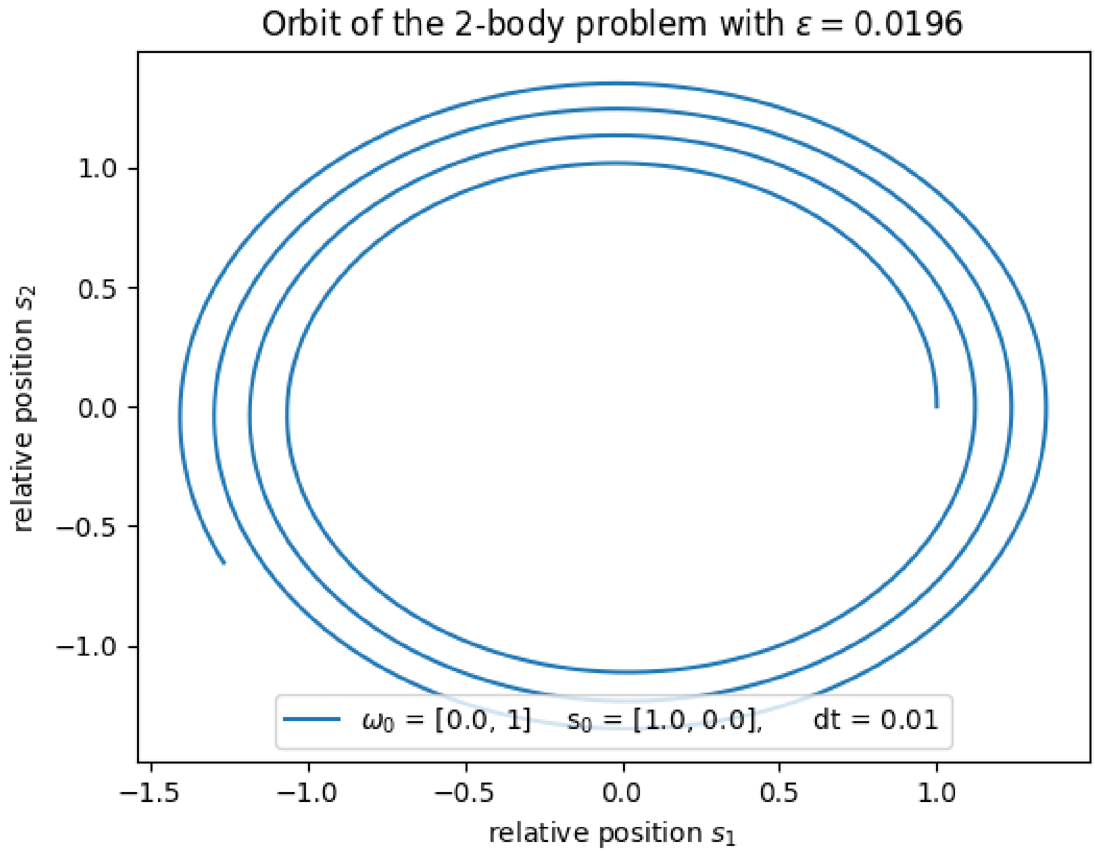
2.1 *iii)*

[ ]:
```
w1 = [0.0,1]
s1=[1.0,0.0]
euler(w1,s1,0.01,30)
```

```
0.00015000374993734167
2.7516943302449928
2.6964787836941415
episLon calculated geometrically :0.203
```

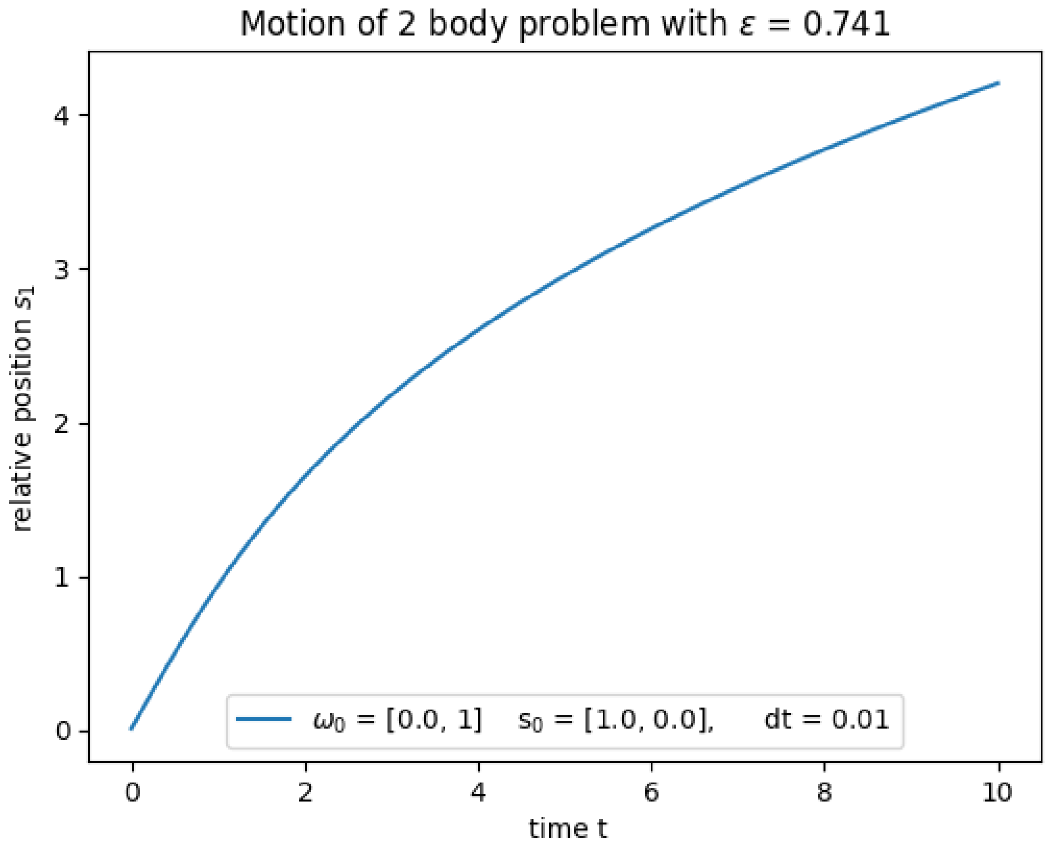
```
[ ]: (array([[ 1.        ,  0.        ],
            [ 1.        ,  0.01      ],
            [ 0.9999    ,  0.02      ],
            …,
            [-1.27251341, -0.63774383],
            [-1.26897081, -0.64530039],
            [-1.26538408, -0.65283482]]),
     array([[ 0.        ,  1.        ],
            [-0.01      ,  1.        ],
            [-0.0199985 ,  0.99990001],
            …,
            [ 0.35426036, -0.75565525],
            [ 0.35867302, -0.75344376],
            [ 0.36307115, -0.75120722]]))
```



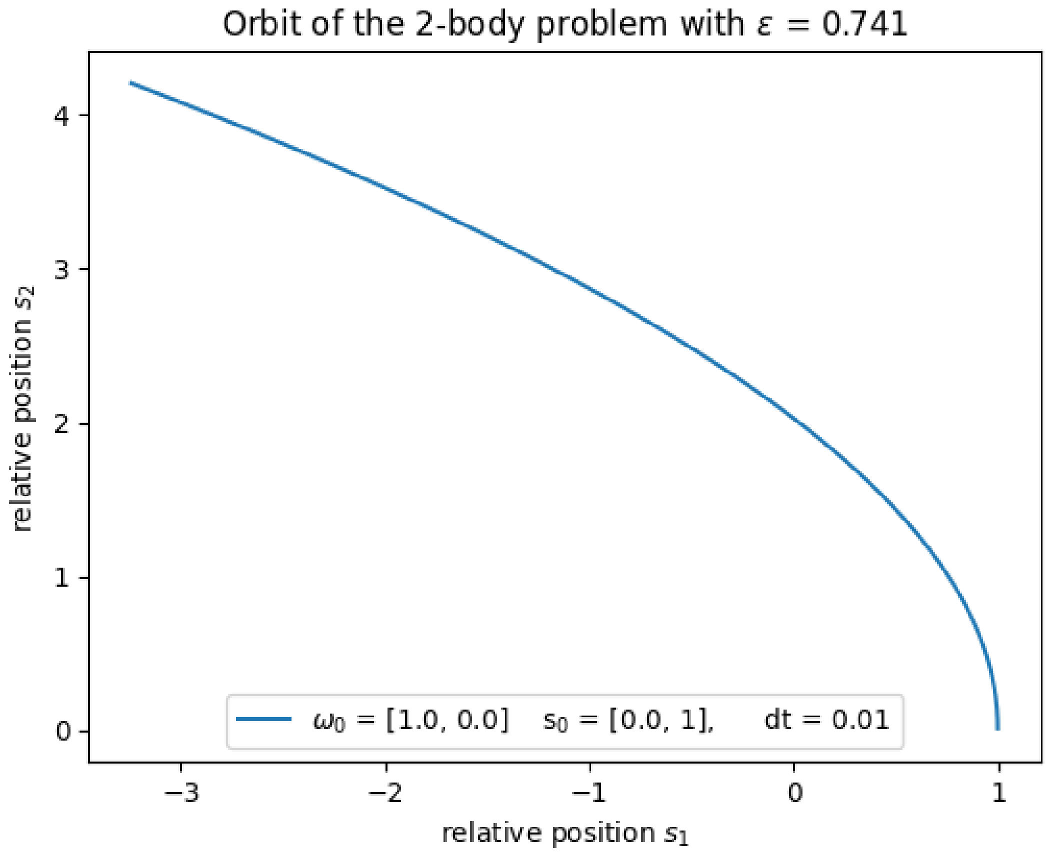



```
[ ]: leap_frog(w1,s1,0.01,10)
     #for the leap_frog algorith the increase in the distance is lower about 0.01%
```

```
4.241240714984339
4.1951887488190645
epislon calculated geometrically :0.147
```



Motion of 2 body problem with ε = 0.741
relative position s_1
time t
ω_0 = [0.0, 1] s_0 = [1.0, 0.0],   dt = 0.01
0
1
2
3
4
0
2
4
6
8
10



If the time step is increased the exactitude of both methods decrease greatlly. The Euler Method plots a piral trajectory, meanwhile the leap-frog algorithm shows a hyperbola.

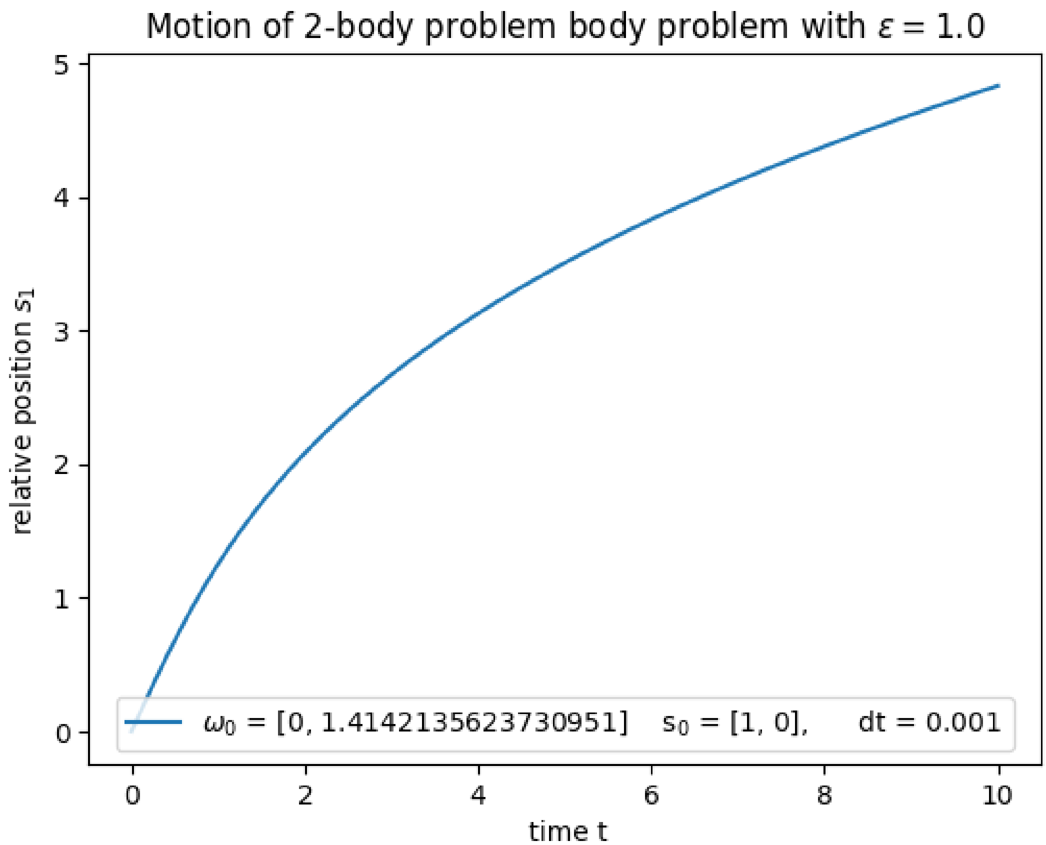
2.2 *v)*

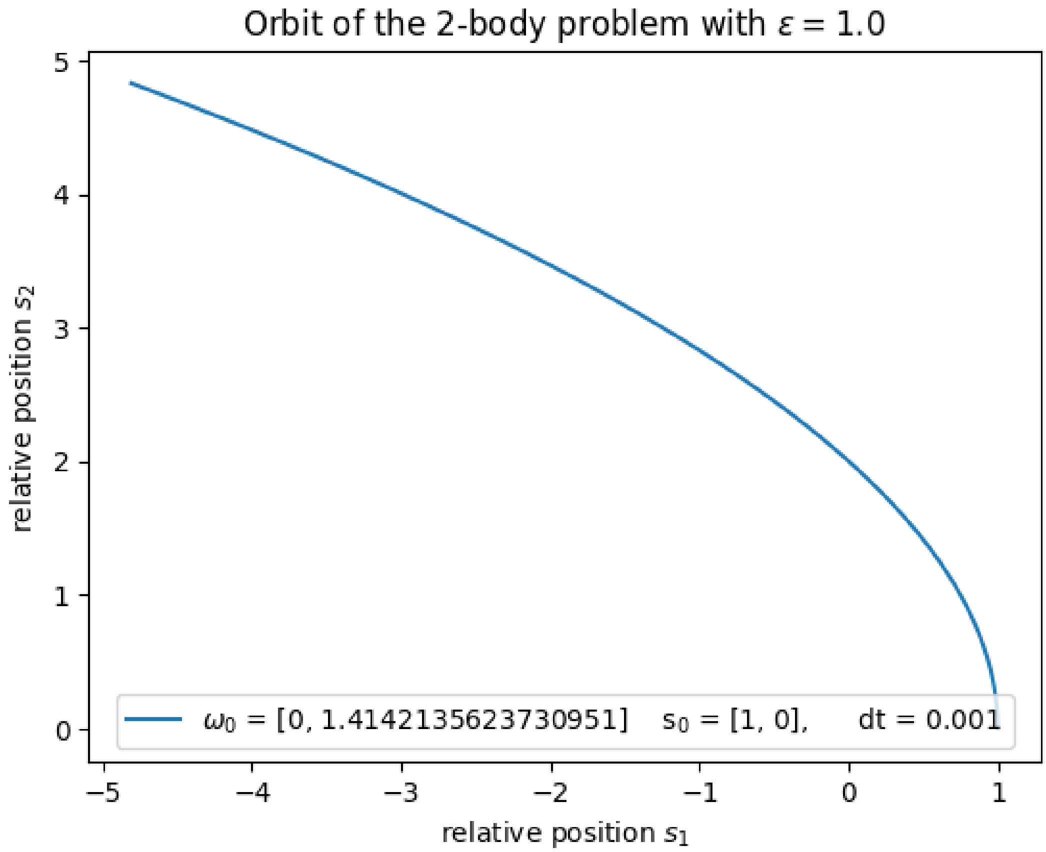
[ ]:
```
euler([0,np.sqrt(2)],[1,0], 0.001,10)
```

```
1.0000029999985007
5.808259981521099
4.827925218332602
epislon calculated geometrically :0.669
```

[ ]:
```
(array([[ 1.00000000e+00,  0.00000000e+00],
        [ 1.00000000e+00,  1.41421356e-03],
        [ 9.99999000e-01,  2.82842712e-03],
        …,
        [-4.80725699e+00,  4.82750700e+00],
        [-4.80775849e+00,  4.82771612e+00],
        [-4.80825998e+00,  4.82792522e+00]]),
 array([[ 0.00000000e+00,  1.41421356e+00],
```

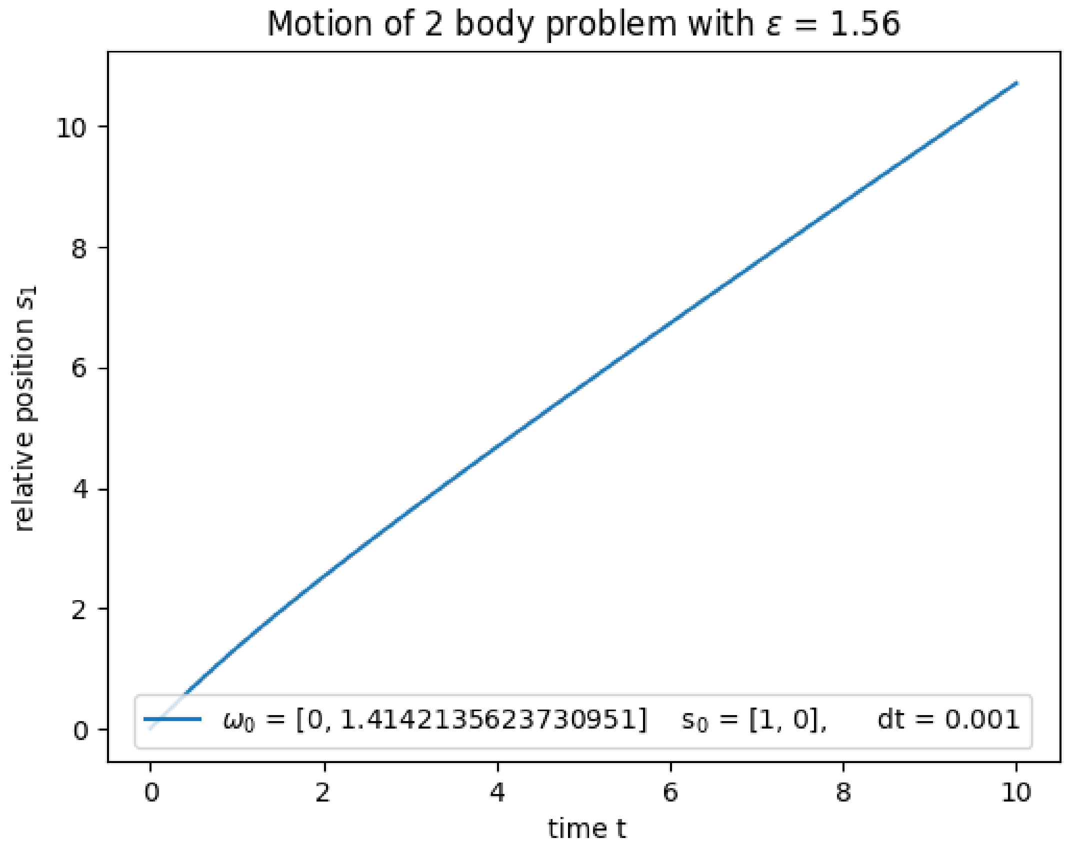
```
[-1.00000000e-03,  1.41421356e+00],
[-1.99999700e-03,  1.41421215e+00],
…,
[-5.01504064e-01,  2.09114958e-01],
[-5.01488862e-01,  2.09099691e-01],
[-5.01473661e-01,  2.09084427e-01]]))
```



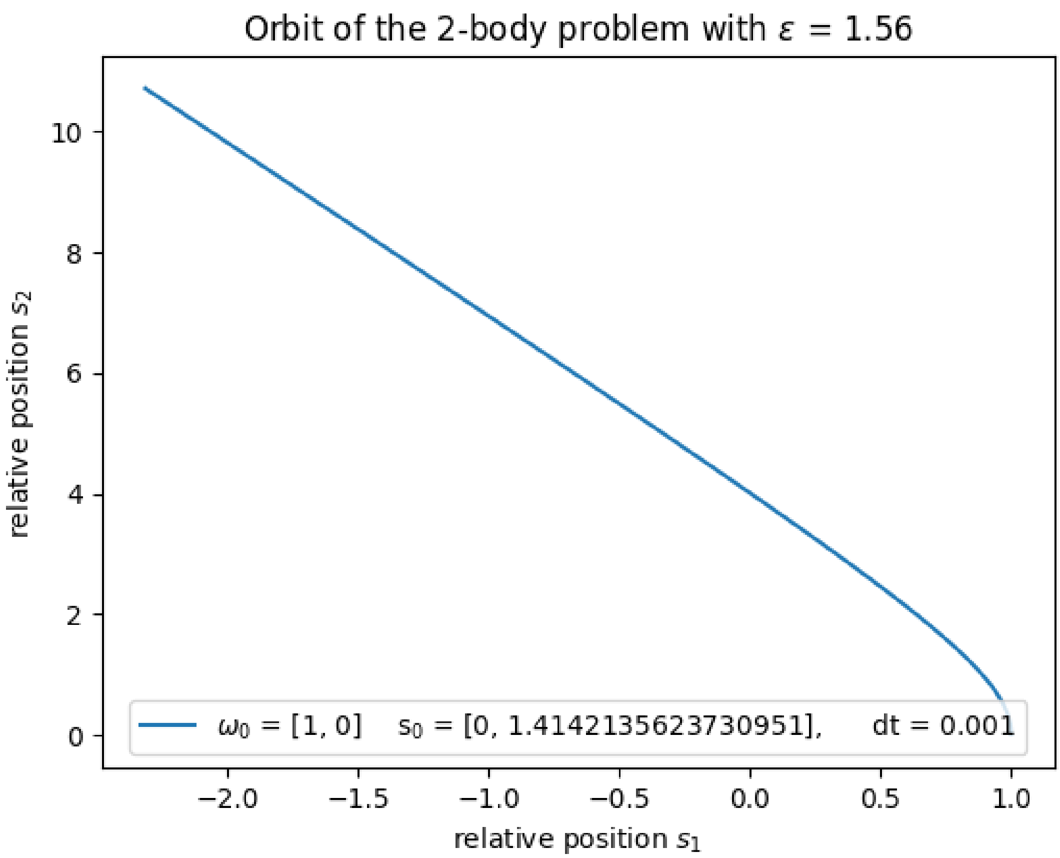


[ ]:
```
leap_frog([0,np.sqrt(2)],[1,0], 0.001,10)
```

```
3.3165115257055158
10.712525087721502
epislon calculated geometrically :3.07
```



Motion of 2 body problem with ε = 1.56
relative position s_1
time t
ω_0 = [0, 1.4142135623730951] s_0 = [1, 0],   dt = 0.001
0
2
4
6
8
10

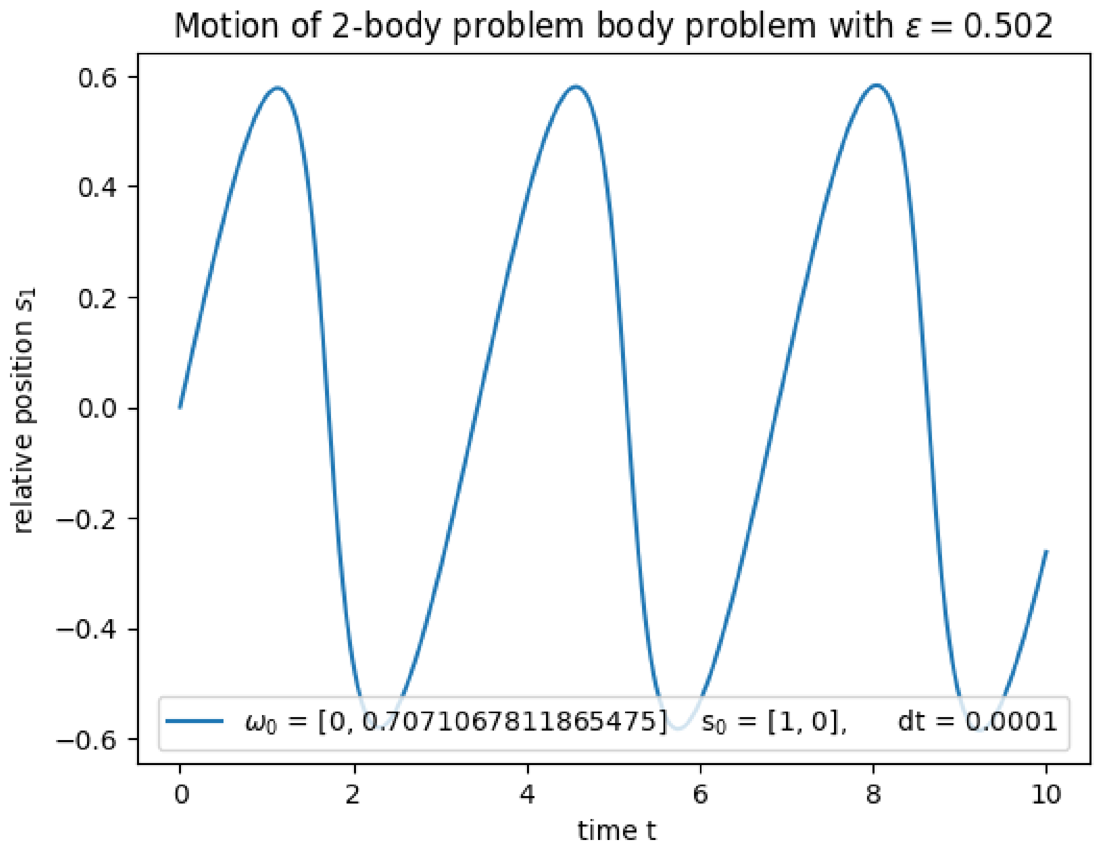


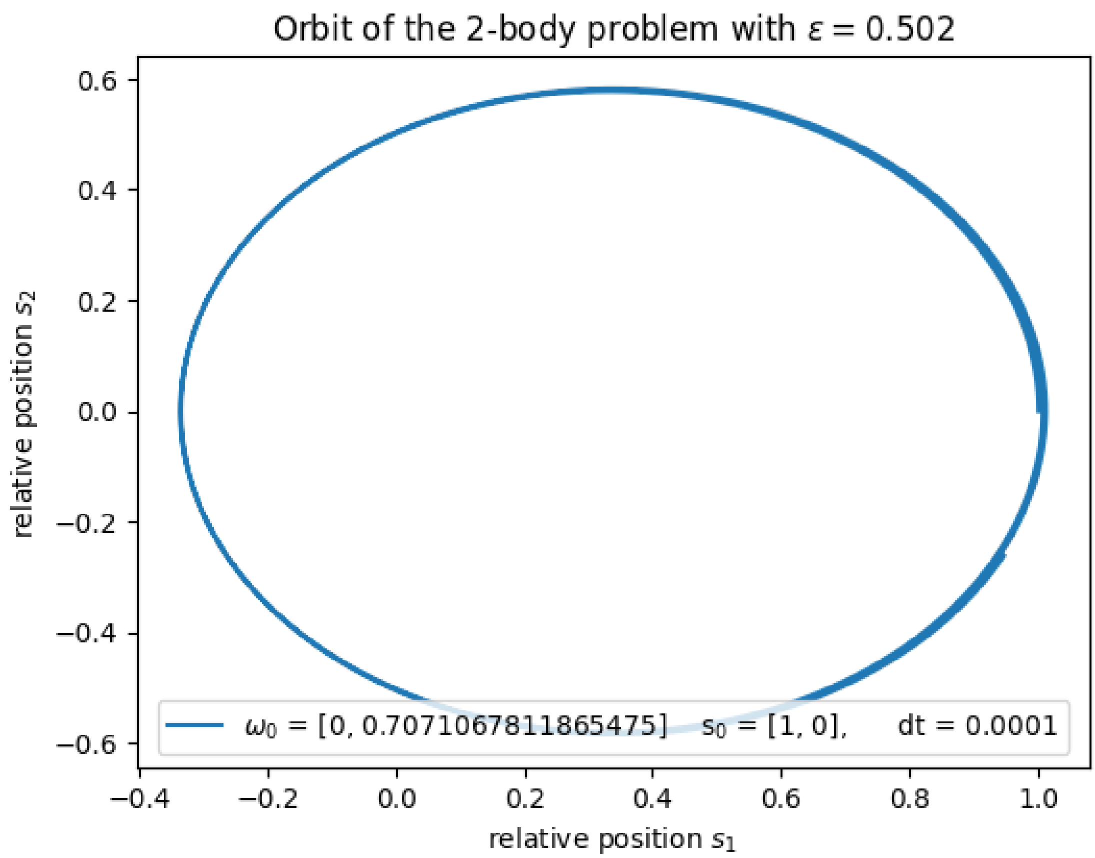

```
[ ]: euler([0,1/np.sqrt(2)],[1,0], 0.0001,10)
```

```
0.49999999250000005
1.3480258576967694
1.1675373818253267
epislon calculated geometrically :0.577
```

```
[ ]: (array([[ 1.00000000e+00,  0.00000000e+00],
            [ 1.00000000e+00,  7.07106781e-05],
            [ 9.99999990e-01,  1.41421356e-04],
            …,
            [ 9.47099038e-01, -2.61851117e-01],
            [ 9.47136525e-01, -2.61786428e-01],
            [ 9.47174002e-01, -2.61721736e-01]]),
      array([[ 0.00000000e+00,  7.07106781e-01],
            [-1.00000000e-04,  7.07106781e-01],
            [-1.99999999e-04,  7.07106774e-01],
            …,
            [ 3.74868772e-01,  6.46891334e-01],
            [ 3.74768950e-01,  6.46918932e-01],
```

```
[ 3.74669130e-01,  6.46946522e-01]]))
```



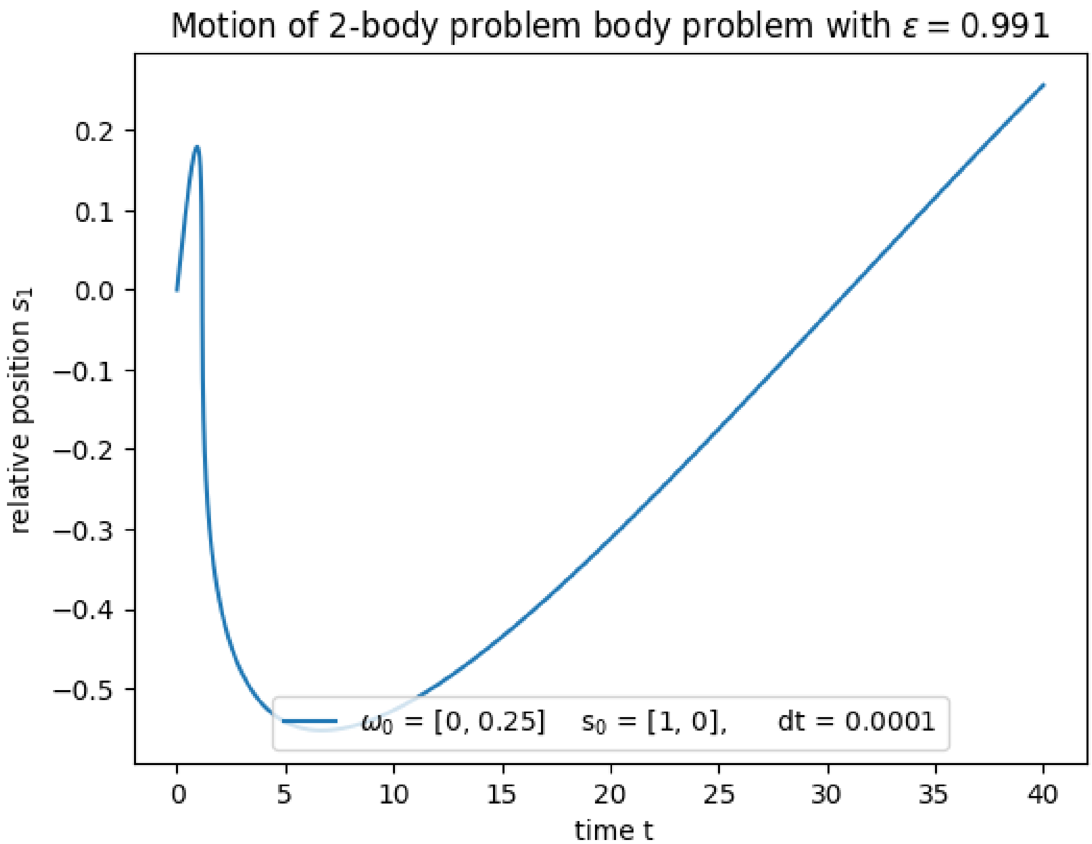


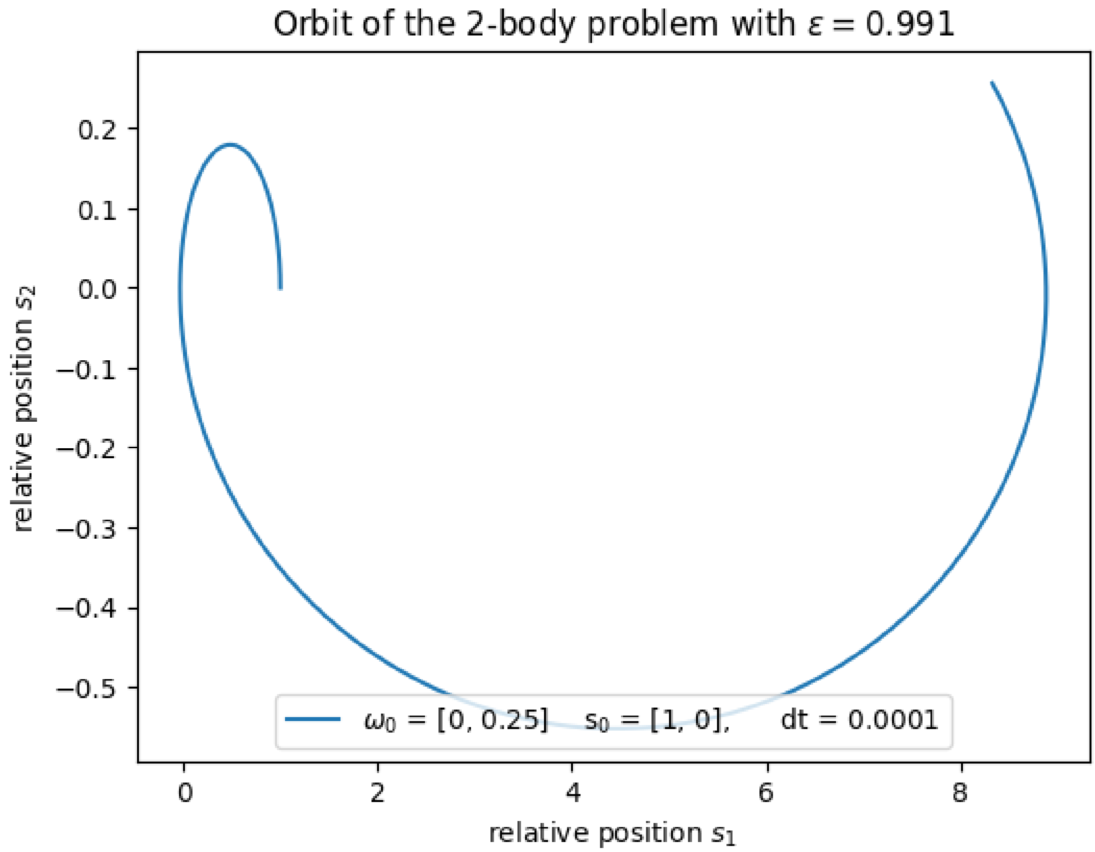
[ ]:
```
euler([0,1/4],[1,0], 0.0001,40)
```

```
0.9374999990624999
8.916320592151196
0.8093261724427312
epislon calculated geometrically :11.0
```

[ ]:
```
(array([[1.00000000e+00, 0.00000000e+00],
        [1.00000000e+00, 2.50000000e-05],
        [9.99999990e-01, 5.00000000e-05],
        …,
        [8.32886475e+00, 2.56626550e-01],
        [8.32885251e+00, 2.56629291e-01],
        [8.32884027e+00, 2.56632031e-01]]),
 array([[ 0.00000000e+00,  2.50000000e-01],
        [-1.00000000e-04,  2.50000000e-01],
        [-2.00000000e-04,  2.49999998e-01],
        …,
        [-1.22373708e-01,  2.74071714e-02],
        [-1.22375148e-01,  2.74071271e-02],
```

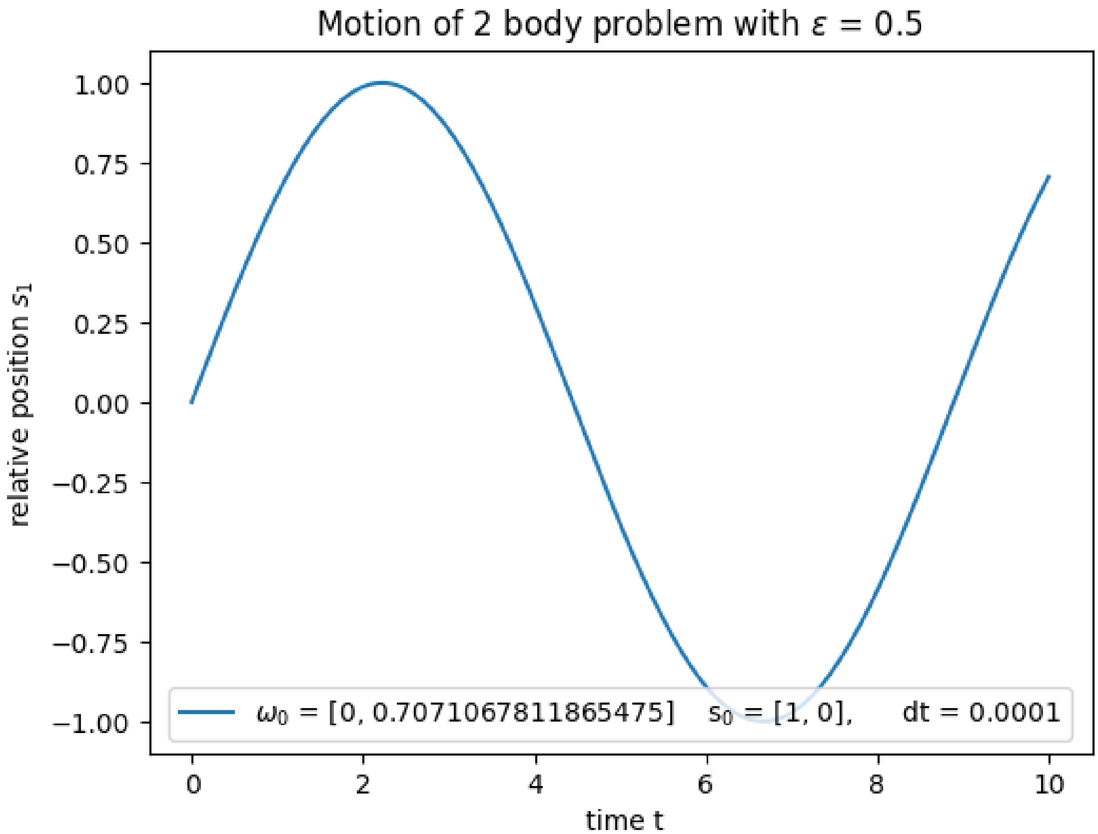
```
[-1.22376587e-01,  2.74070827e-02]]))
```

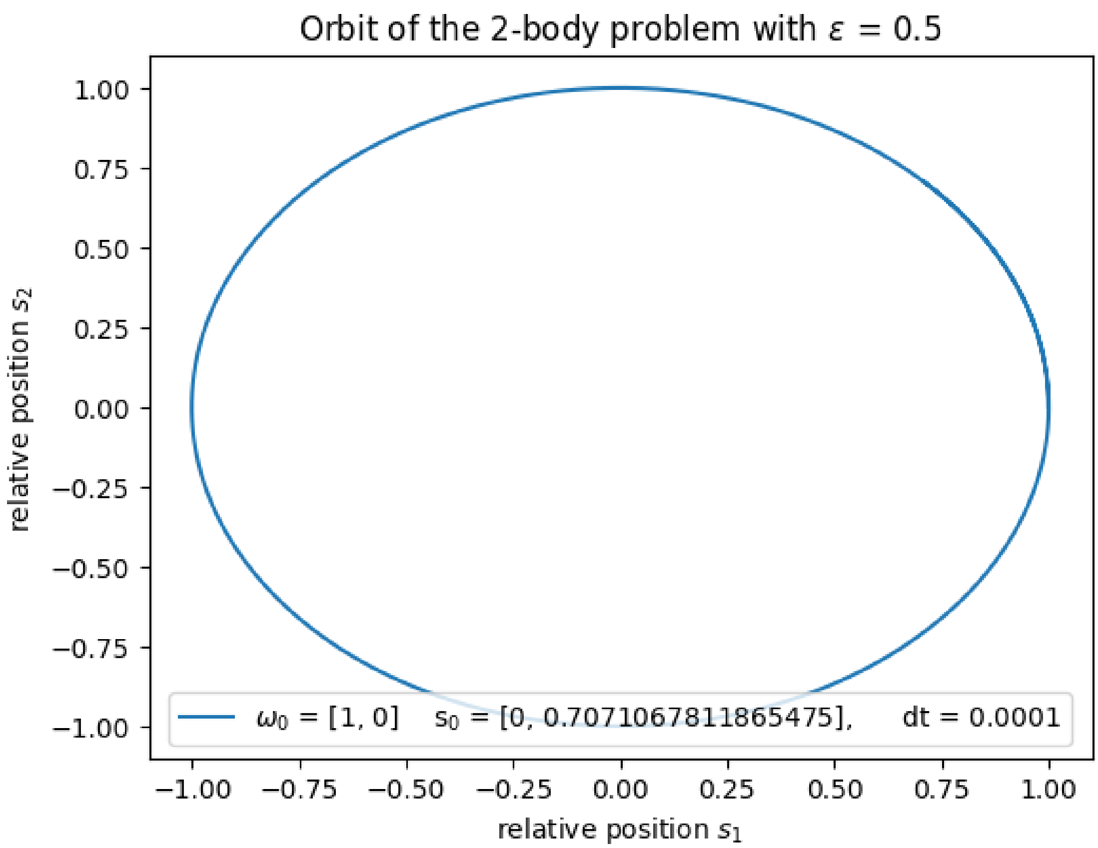





```
[ ]: leap_frog([0,1/np.sqrt(2)],[1,0], 0.0001,10)
```

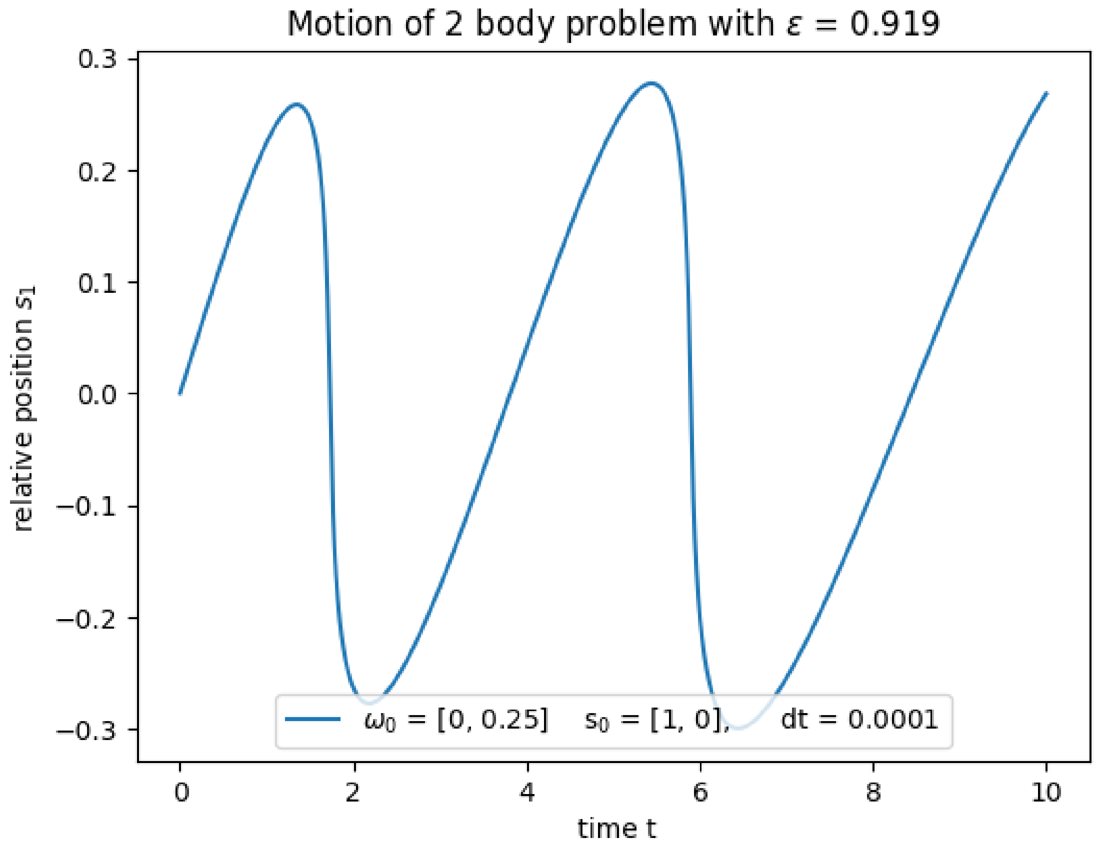
```
2.001332646540053
2.000888510189057
epislon calculated geometrically :0.0211
```

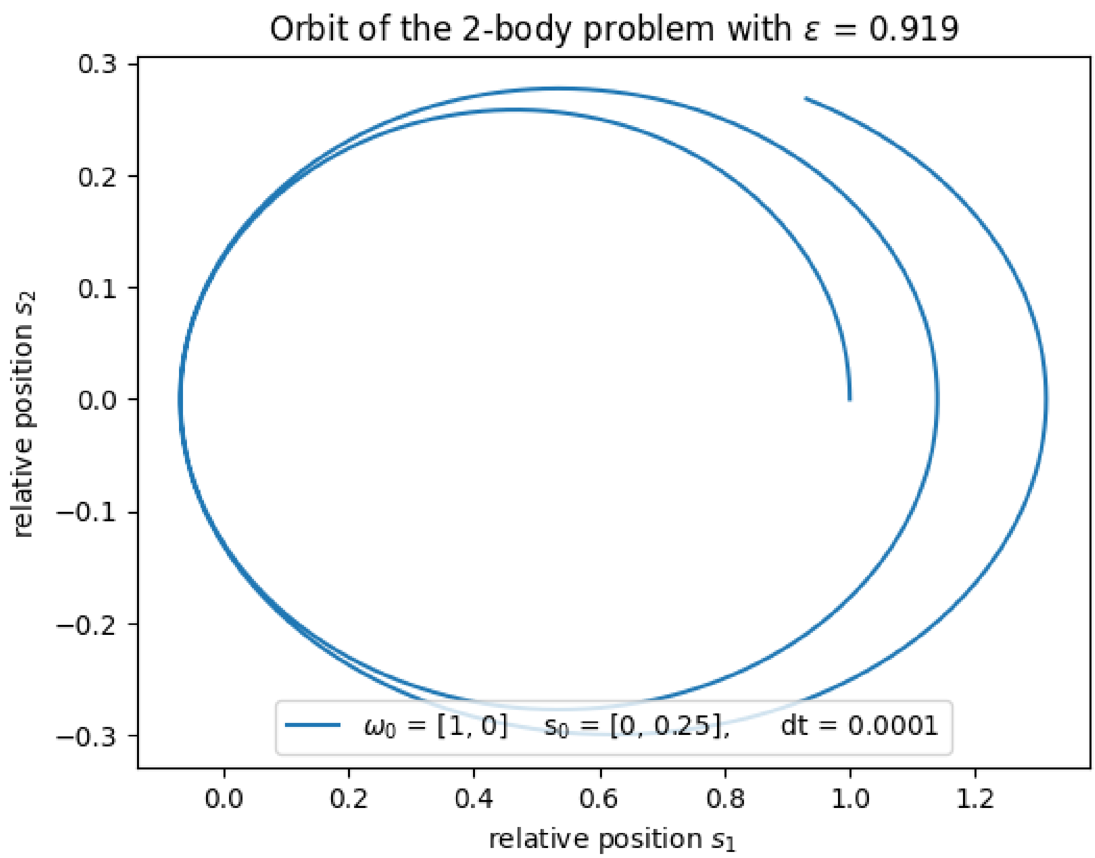

Motion of 2 body problem with ε = 0.5
relative position s_1
time t
ω_0 = [0, 0.7071067811865475] s_0 = [1, 0],   dt = 0.0001
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0
2
4
6
8
10



[ ]:
```
leap_frog([0,1/4],[1,0], 0.0001,10)
```

```
1.3808830743106262
0.5770487824456738
episilon calculated geometrically :0.909
```


Motion of 2 body problem with ε = 0.919
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
relative position s_1
0
2
4
6
8
10
time t
ω_0 = [0, 0.25] s_0 = [1, 0],   dt = 0.0001



3 Error Analysis

3.1 a) Euler methode

[ ]:
```
euler([0.0,1.0],[1,0], 0.1,30)
```

```
0.015037437733209797
4.752794399746748
5.07943758093948
epislon calculated geometrically :0.353
```

[ ]:
```
(array([[ 1.00000000e+00,  0.00000000e+00],
        [ 1.00000000e+00,  1.00000000e-01],
        [ 9.90000000e-01,  2.00000000e-01],
        [ 9.70148147e-01,  2.99014815e-01],
        [ 9.40687451e-01,  3.96088449e-01],
        [ 9.01954090e-01,  4.90304103e-01],
        [ 8.54374015e-01,  5.80794736e-01],
        [ 7.98457639e-01,  6.66753737e-01],
```

```
[ 7.34792507e-01,  7.47445214e-01],
[ 6.64033986e-01,  8.22213342e-01],
[ 5.86894173e-01,  8.90490295e-01],
[ 5.04129334e-01,  9.51802285e-01],
[ 4.16526338e-01,  1.00577338e+00],
[ 3.24888623e-01,  1.05212687e+00],
[ 2.30022234e-01,  1.09068418e+00],
[ 1.32722525e-01,  1.12136137e+00],
[ 3.37619908e-02,  1.14416352e+00],
[-6.61203620e-02,  1.15917729e+00],
[-1.66227826e-01,  1.16656226e+00],
[-2.65912845e-01,  1.16654122e+00],
[-3.64581885e-01,  1.15939018e+00],
[-4.61698396e-01,  1.14542831e+00],
[-5.56784057e-01,  1.12500823e+00],
[-6.49418509e-01,  1.09850693e+00],
[-7.39237845e-01,  1.06631757e+00],
[-8.25932122e-01,  1.02884209e+00],
[-9.09242149e-01,  9.86484978e-01],
[-9.88955794e-01,  9.39647949e-01],
[-1.06490399e+00,  8.88725588e-01],
[-1.13695665e+00,  8.34101917e-01],
[-1.20501859e+00,  7.76147744e-01],
[-1.26902553e+00,  7.15218725e-01],
[-1.32894041e+00,  6.51654023e-01],
[-1.38474982e+00,  5.85775488e-01],
[-1.43646077e+00,  5.17887249e-01],
[-1.48409775e+00,  4.48275646e-01],
[-1.52770009e+00,  3.77209434e-01],
[-1.56731955e+00,  3.04940182e-01],
[-1.60301824e+00,  2.31702839e-01],
[-1.63486676e+00,  1.57716401e-01],
[-1.66294257e+00,  8.31846496e-02],
[-1.68732860e+00,  8.29694296e-03],
[-1.70811201e+00, -6.67709755e-02],
[-1.72538318e+00, -1.41856164e-01],
[-1.73923477e+00, -2.16807680e-01],
[-1.74976099e+00, -2.91485793e-01],
[-1.75705695e+00, -3.65761231e-01],
[-1.76121809e+00, -4.39514454e-01],
[-1.76233980e+00, -5.12634966e-01],
[-1.76051696e+00, -5.85020663e-01],
[-1.75584371e+00, -6.56577222e-01],
[-1.74841314e+00, -7.27217523e-01],
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[ 3.79846375e-01, -2.45822947e-01]]))
```
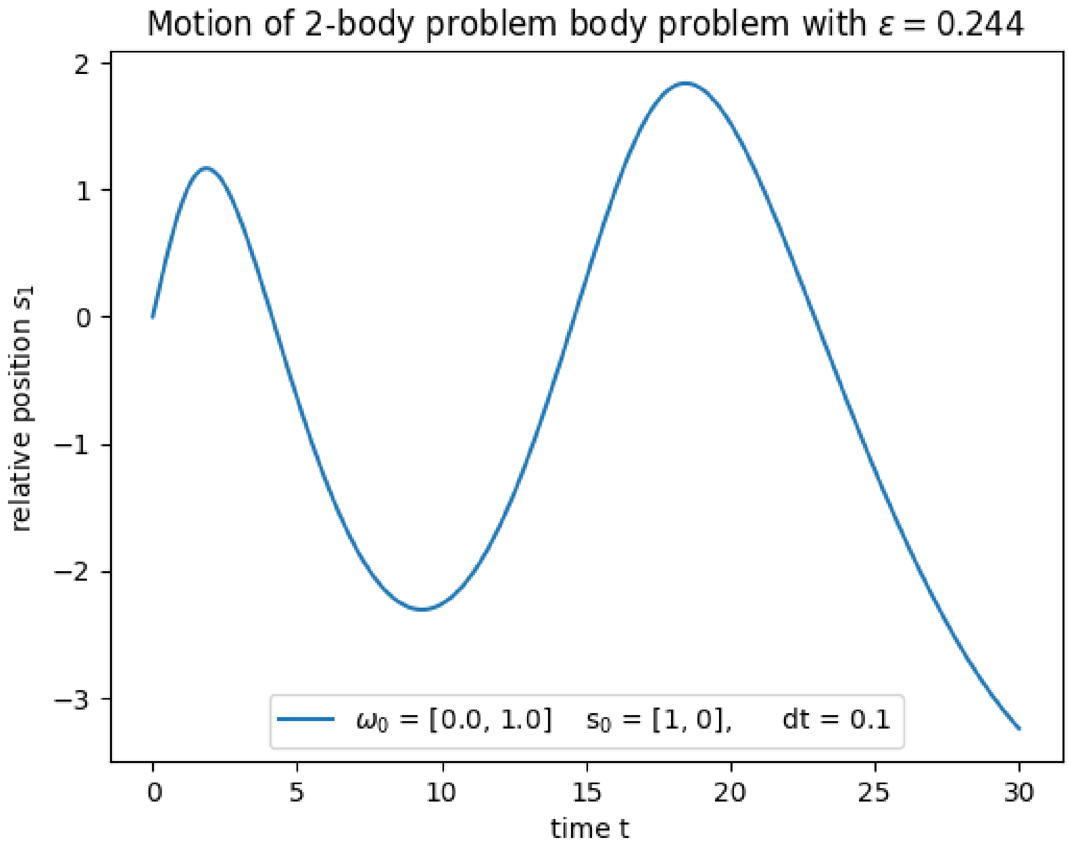
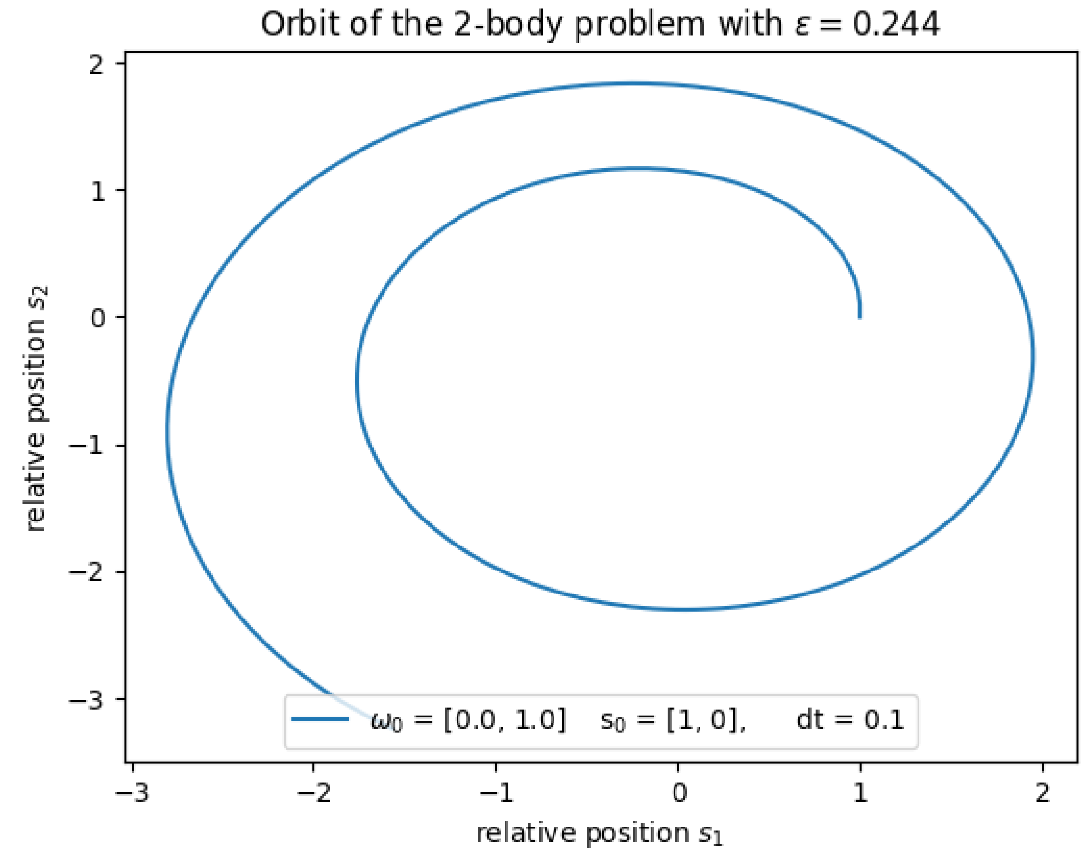



Motion of 2-body problem body problem with ε = 0.244
relative position s_1
time t
ω_0 = [0.0, 1.0] s_0 = [1, 0], dt = 0.1
2
1
0
−1
−2
−3
0
5
10
15
20
25
30



[ ]:
```
euler([0,1],[1,0], 0.01,30)
```

```
0.00015000374993734167
2.7516943302449928
2.6964787836941415
epislon calculated geometrically :0.203
```

[ ]:
```
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        [ 1.        ,  0.01      ],
        [ 0.9999    ,  0.02      ],
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        [-1.26897081, -0.64530039],
        [-1.26538408, -0.65283482]]),
 array([[ 0.        ,  1.        ],
        [-0.01      ,  1.        ],
        [-0.0199985 ,  0.99990001],
        …,
        [ 0.35426036, -0.75565525],
        [ 0.35867302, -0.75344376],
```
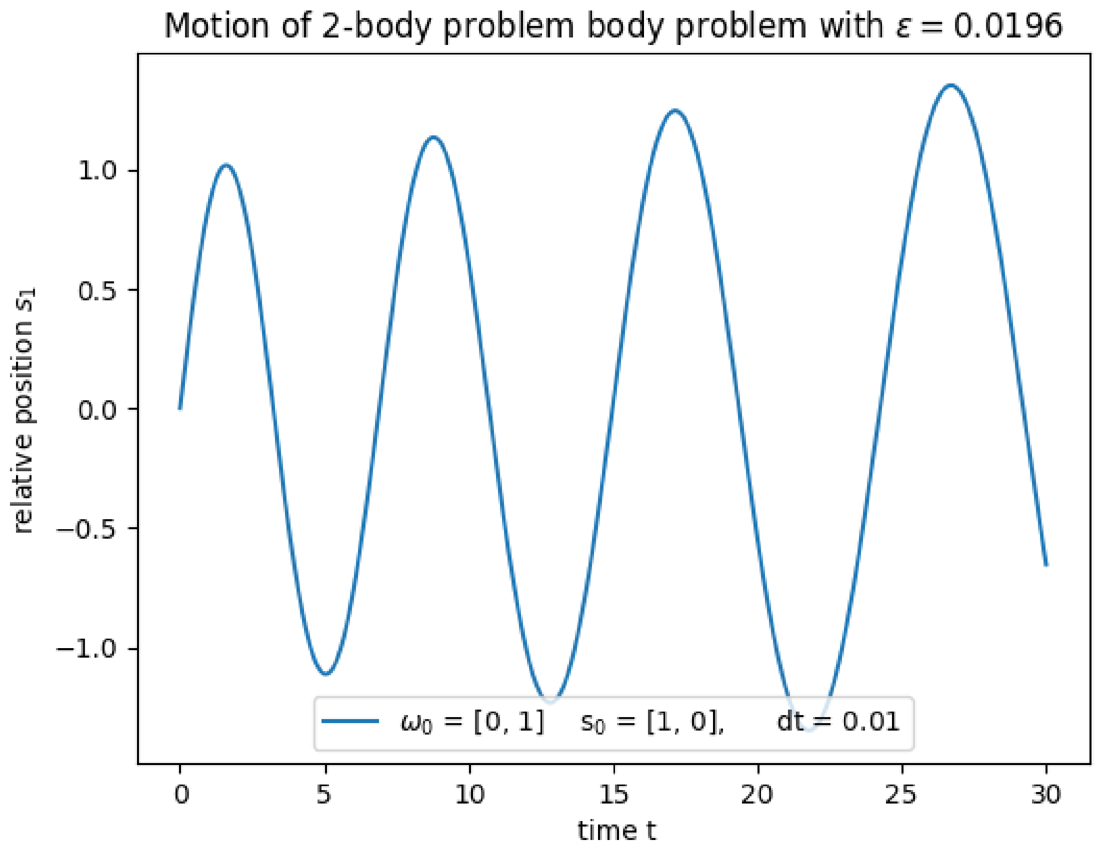
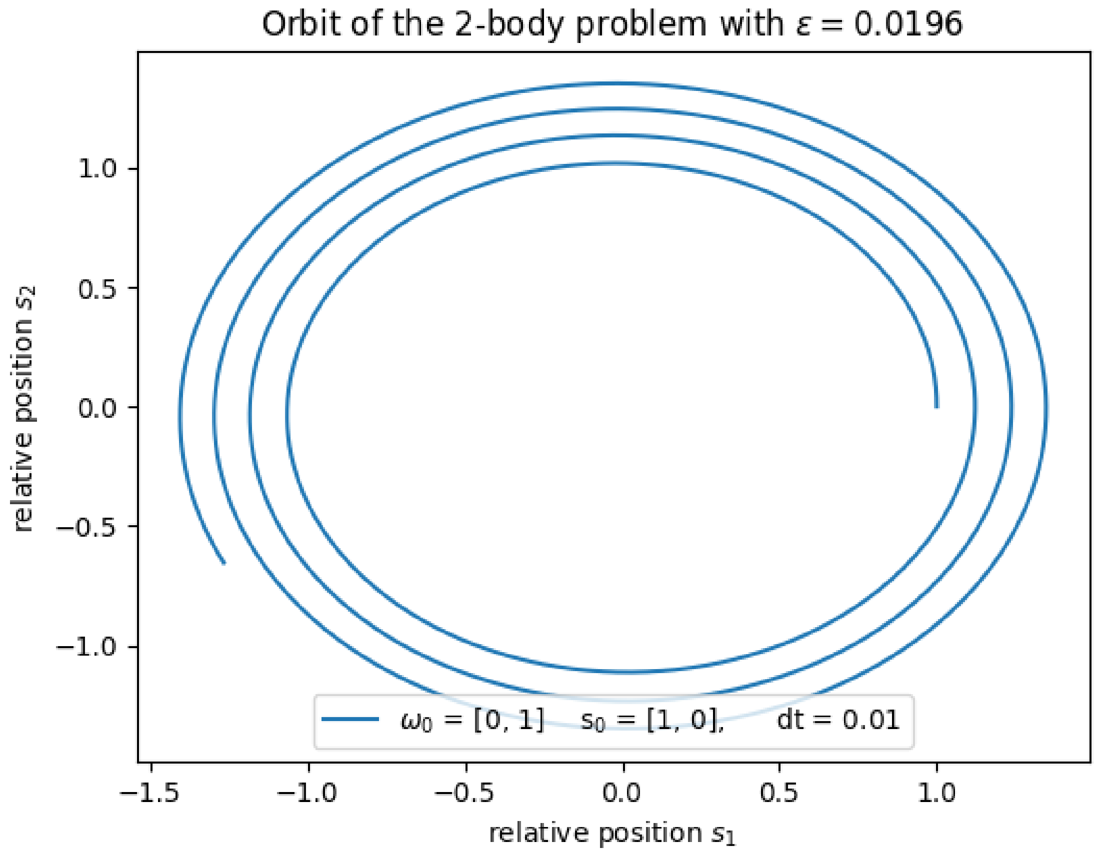

```
[ 0.36307115, -0.75120722]]))
```






```
[ ]: euler([0,1],[1,0], 0.001,30)
```

```
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2.1054507526942885
2.0993082373604404
episIon calculated geometrically :0.0766
```

```
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             [ 0.999999  ,  0.002     ],
             …,
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             [-0.93393954, -0.49608233],
             [-0.93348591, -0.49694207]]),
      array([[ 0.        ,  1.        ],
             [-0.001     ,  1.        ],
             [-0.002     ,  0.999999  ],
             …,
             [ 0.4528374 , -0.86015207],
             [ 0.45362748, -0.85973333],
```
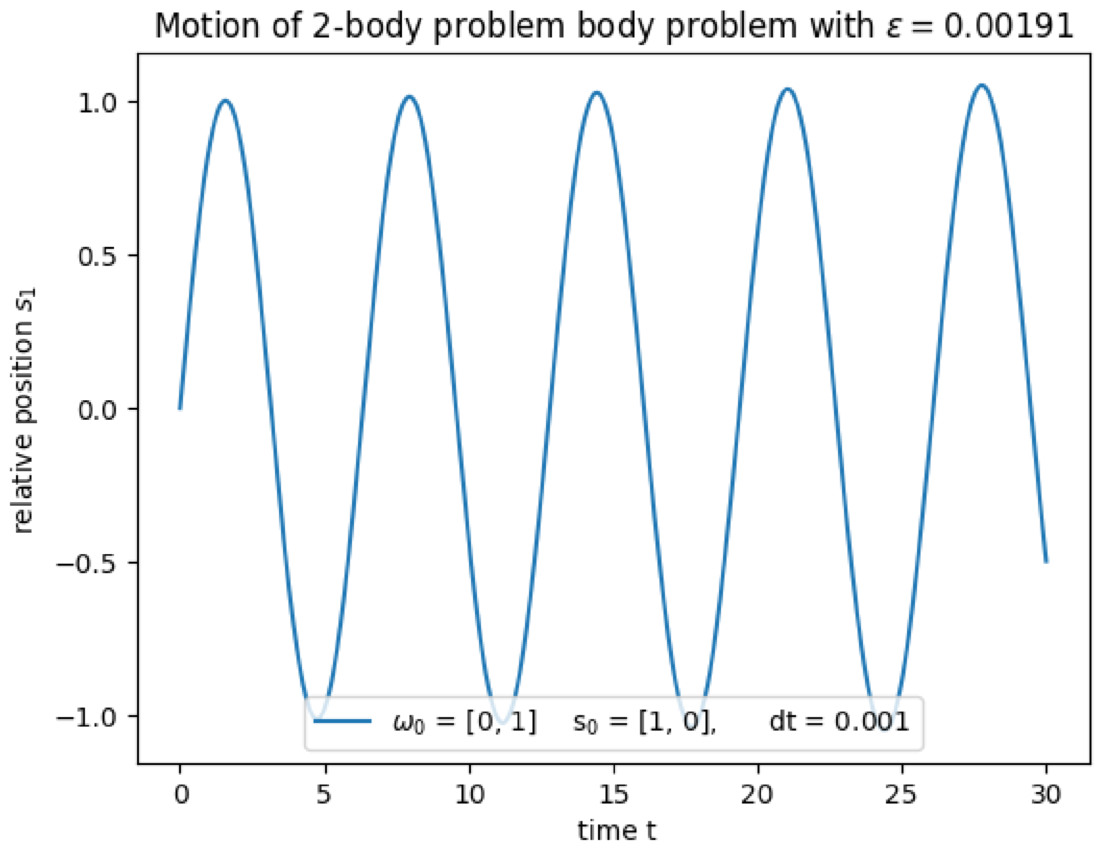
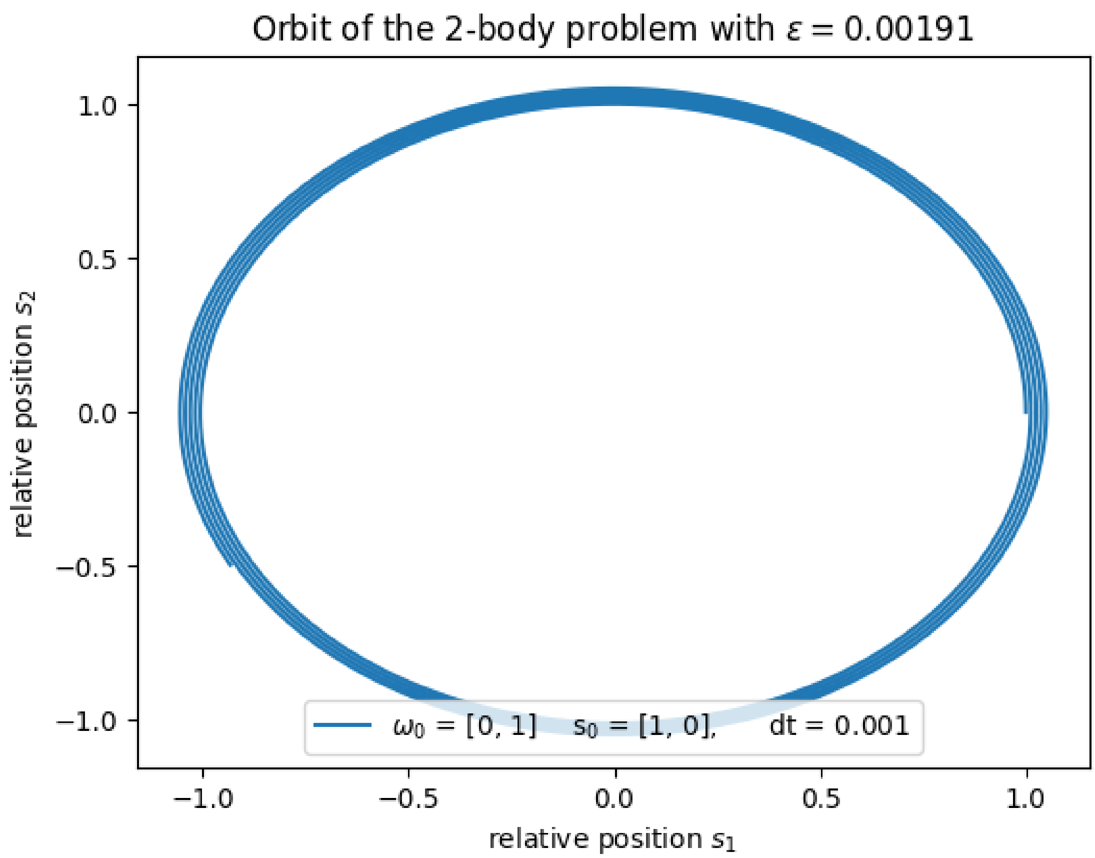

```
[ 0.45441717, -0.85931387]]))
```





[ ]:
```
euler([0,1],[1,0], 0.0001,30)
```

```
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2.010667350537004
2.011294139221177
epislon calculated geometrically :0.025
```

[ ]:
```
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        …,
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        [ 2.12869464e-02, -1.00589078e+00],
        [ 2.13866125e-02, -1.00588869e+00]]),
 array([[ 0.00000000e+00,  1.00000000e+00],
        [-1.00000000e-04,  1.00000000e+00],
        [-1.99999999e-04,  9.99999990e-01],
        …,
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        [ 9.96660768e-01,  2.09465975e-02],
```
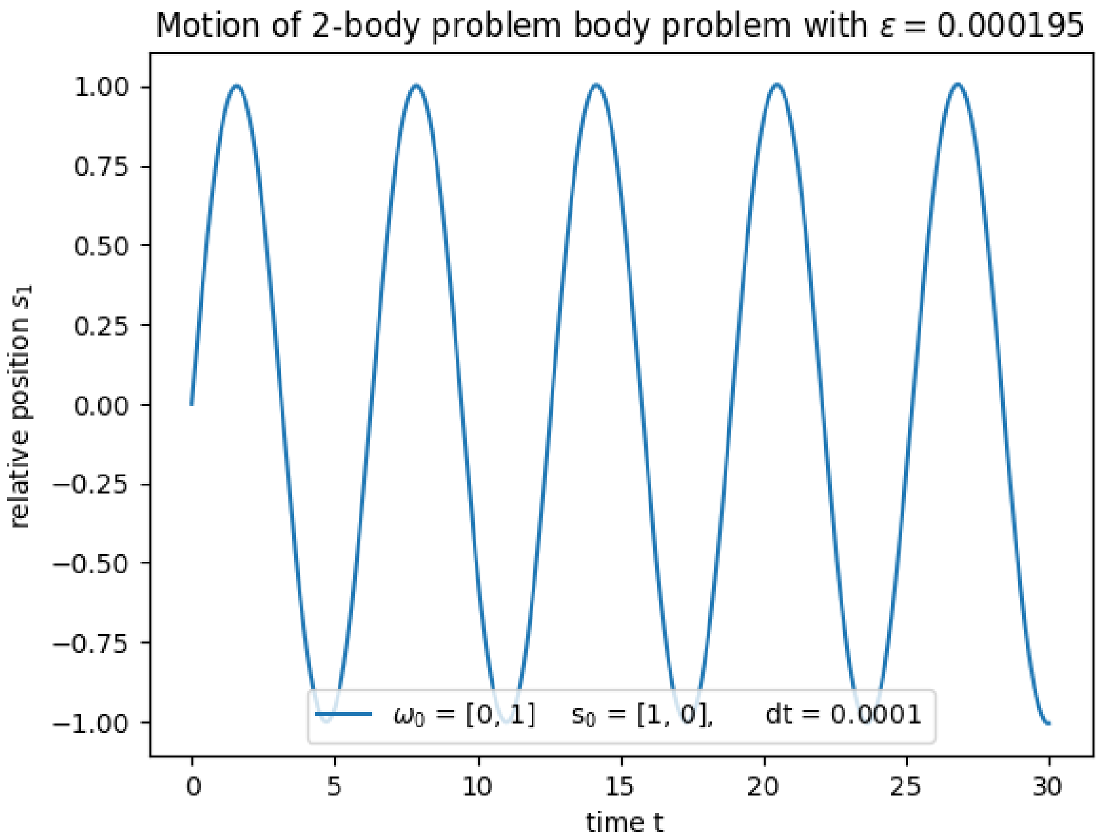
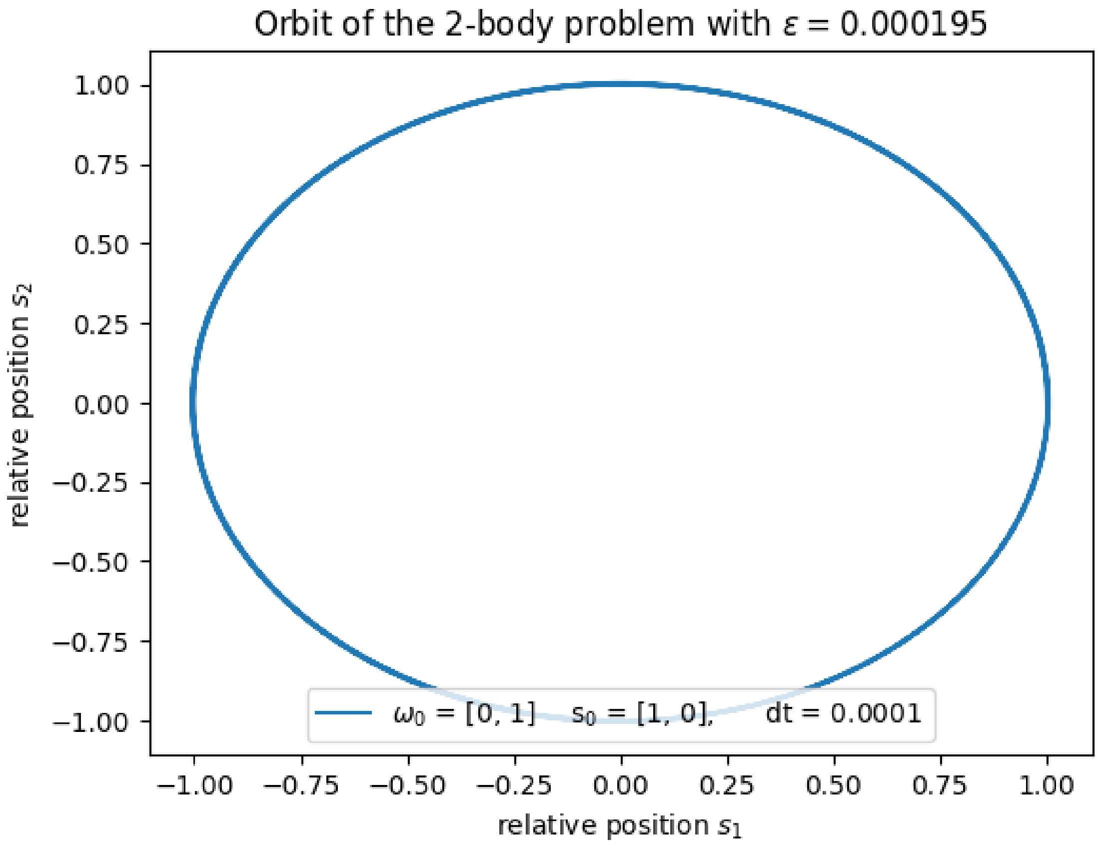

```
[ 9.96658677e-01,  2.10453633e-02]]))
```





[ ]:
```
euler([0,1/np.sqrt(2)],[1,0], 0.1,10)
```

```
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6.78091153839325
episloncalculated geometrically :0.898
```

[ ]:
```
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```

```
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       [ 1.87839276, -5.73556147],
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       [ 1.95863857, -5.84228215],
       [ 1.99863417, -5.89525684],
       [ 2.03854605, -5.9479818 ],
       [ 2.07837507, -6.00046236],
       [ 2.11812209, -6.05270363],
       [ 2.15778794, -6.10471058]]),
array([[ 0.        ,  0.70710678],
       [-0.1       ,  0.70710678],
       [-0.19925466,  0.70008842],
       [-0.29823981,  0.6859484 ],
       [-0.39735876,  0.66434518],
       [-0.49693428,  0.63468964],
       [-0.59718641,  0.59611913],
```
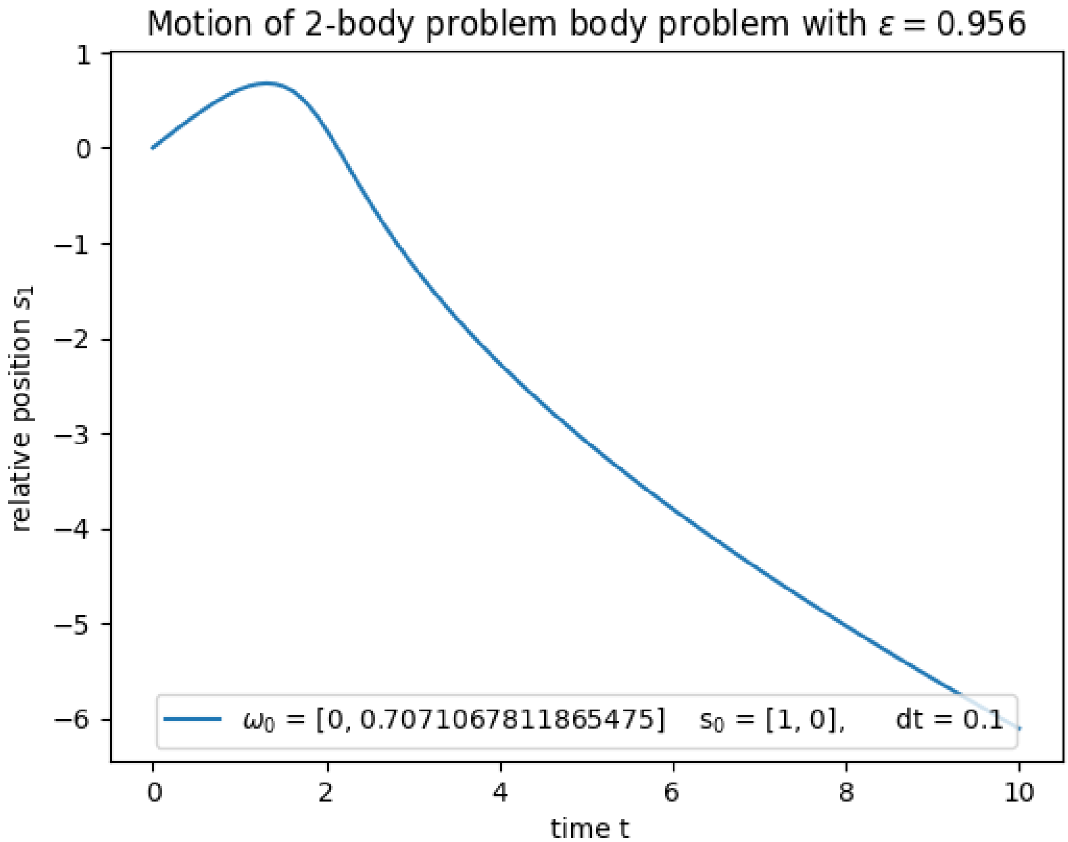
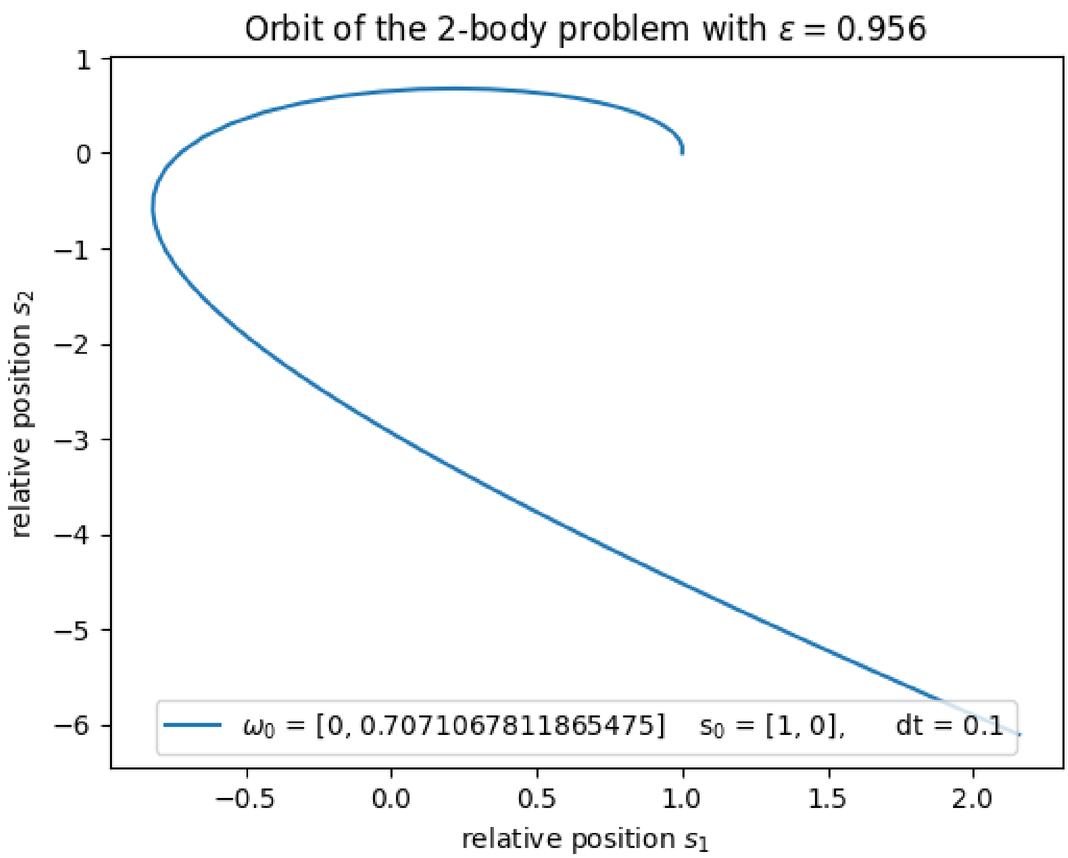

```
[-0.69818986,  0.54745533],
[-0.79980008,  0.48714696],
[-0.90153084,  0.413201  ],
[-1.00235584,  0.32311451],
[-1.10039498,  0.21383859],
[-1.19243516,  0.08184885],
[-1.27324787, -0.07652039],
[-1.33475598, -0.26418216],
[-1.36539744, -0.48140132],
[-1.35068629, -0.72257716],
[-1.27675855, -0.97249368],
[-1.13799209, -1.20545513],
[-0.9450793 , -1.39222167],
[-0.72460383, -1.5135628 ],
[-0.50704613, -1.56897007],
[-0.31382638, -1.57255913],
[-0.1535586 , -1.54268032],
[-0.02576025, -1.4947076 ],
[ 0.07418475, -1.43901621],
[ 0.15180701, -1.38173728],
[ 0.21211078, -1.32614204],
[ 0.25915483, -1.27378513],
[ 0.29607259, -1.22526054],
[ 0.32523139, -1.18065363],
[ 0.34840687, -1.13979746],
[ 0.36693161, -1.10241312],
[ 0.38181049, -1.06818432],
[ 0.39380665, -1.03679553],
[ 0.40350446, -1.00795014],
[ 0.41135525, -0.981378  ],
[ 0.41771072, -0.95683738],
[ 0.4228472 , -0.93411411],
[ 0.42698366, -0.91301943],
[ 0.43029496, -0.89338739],
[ 0.43292176, -0.87507218],
[ 0.43497801, -0.85794561],
[ 0.43655664, -0.84189485],
[ 0.43773389, -0.82682034],
[ 0.43857268, -0.81263409],
[ 0.4391252 , -0.79925814],
[ 0.43943497, -0.78662321],
[ 0.43953845, -0.77466759],
[ 0.4394663 , -0.76333618],
[ 0.43924444, -0.75257963],
[ 0.43889484, -0.74235366],
[ 0.4384362 , -0.73261841],
[ 0.43788449, -0.7233379 ],
```

```
[ 0.43725337, -0.71447964],
[ 0.43655459, -0.70601413],
[ 0.43579825, -0.69791462],
[ 0.43499308, -0.69015672],
[ 0.43414663, -0.68271821],
[ 0.43326543, -0.67557875],
[ 0.43235517, -0.66871974],
[ 0.43142078, -0.66212409],
[ 0.43046657, -0.65577611],
[ 0.42949628, -0.64966134],
[ 0.4285132 , -0.64376644],
[ 0.42752017, -0.63807909],
[ 0.42651971, -0.63258789],
[ 0.42551401, -0.62728228],
[ 0.42450497, -0.62215246],
[ 0.42349427, -0.61718931],
[ 0.42248339, -0.61238434],
[ 0.4214736 , -0.60772967],
[ 0.42046603, -0.60321791],
[ 0.41946165, -0.59884219],
[ 0.41846132, -0.59459604],
[ 0.41746579, -0.59047345],
[ 0.41647569, -0.58646876],
[ 0.4154916 , -0.58257666],
[ 0.41451398, -0.57879217],
[ 0.41354327, -0.5751106 ],
[ 0.41257981, -0.57152753],
[ 0.4116239 , -0.56803881],
[ 0.41067581, -0.56464051],
[ 0.40973575, -0.56132895],
[ 0.40880389, -0.55810061],
[ 0.40788038, -0.5549522 ],
[ 0.40696534, -0.55188058],
[ 0.40605884, -0.5488828 ],
[ 0.40516096, -0.54595604],
[ 0.40427175, -0.54309764],
[ 0.40339122, -0.54030507],
[ 0.40251939, -0.53757593],
[ 0.40165626, -0.53490794],
[ 0.40080181, -0.53229893],
[ 0.39995602, -0.52974683],
[ 0.39911884, -0.52724967],
[ 0.39829023, -0.52480557],
[ 0.39747014, -0.52241274],
[ 0.39665851, -0.52006948],
[ 0.39585527, -0.51777416]]))
```


Motion of 2-body problem body problem with $\varepsilon = 0.956$
relative position s_1
time t
ω_0 = [0, 0.7071067811865475] s_0 = [1, 0],   dt = 0.1
1
0
−1
−2
−3
−4
−5
−6
0
2
4
6
8
10

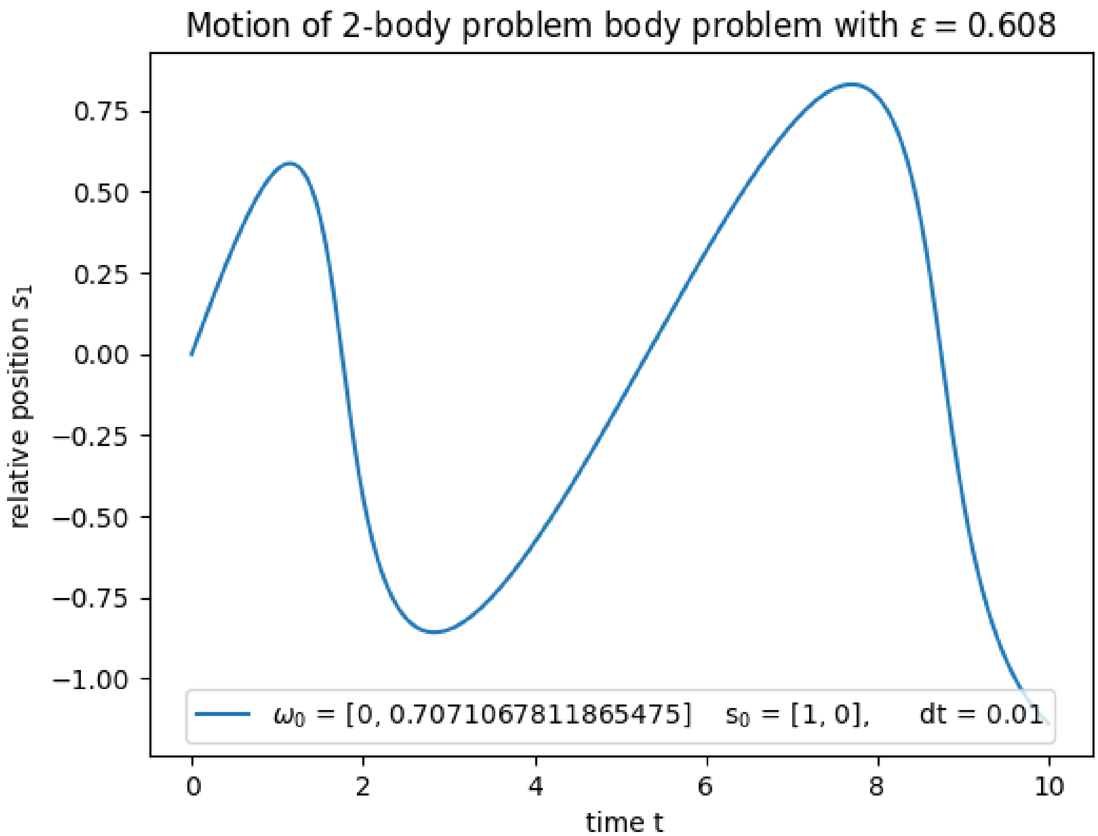


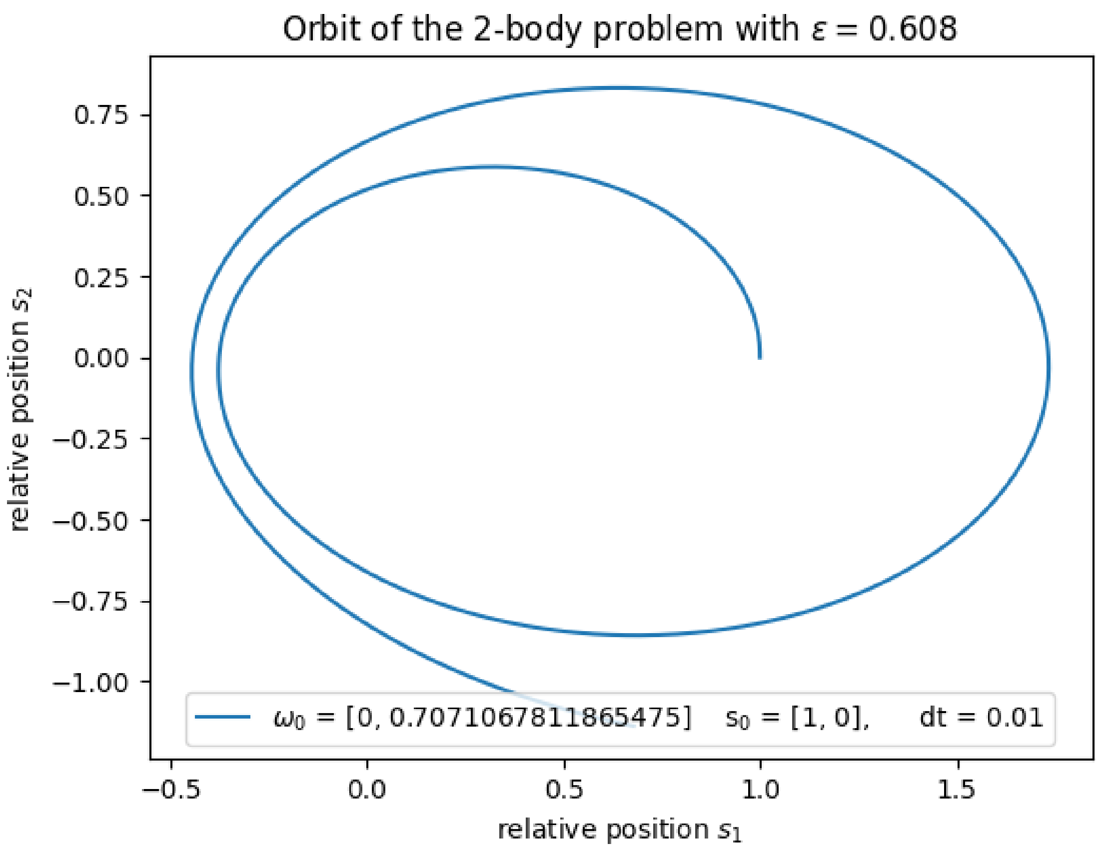
[ ]:
```
euler([0,1/np.sqrt(2)],[1,0], 0.01,10)
```

```
0.4999250009382423
2.1810763825379444
1.9697271277033543
epislon calculated geometrically :0.476
```

[ ]:
```
(array([[ 1.        ,  0.        ],
        [ 1.        ,  0.00707107],
        [ 0.9999    ,  0.01414214],
        …,
        [ 0.66002372, -1.13378841],
        [ 0.66931768, -1.13618733],
        [ 0.67858241, -1.13853604]]),
 array([[ 0.        ,  0.70710678],
        [-0.01      ,  0.70710678],
        [-0.01999925,  0.70703608],
        …,
        [ 0.92939642, -0.23989199],
        [ 0.92647329, -0.23487064],
```

```
[ 0.9235544 , -0.22991574]]))
```

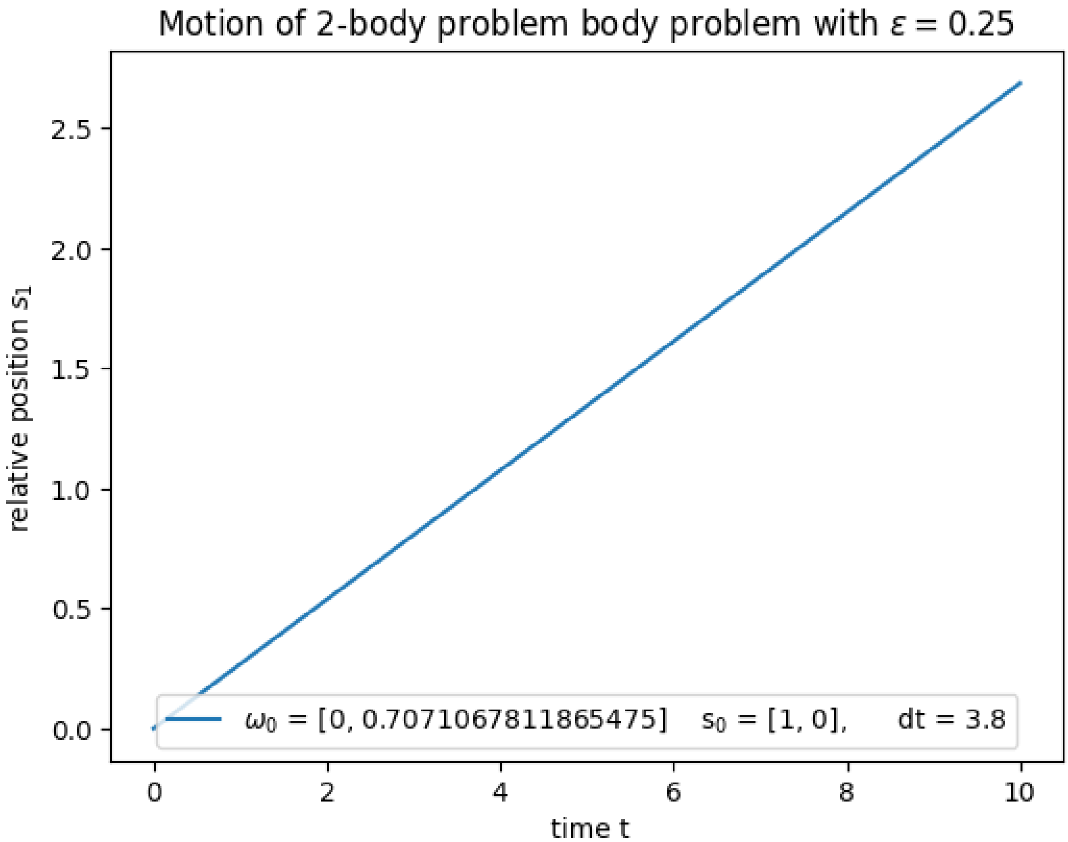


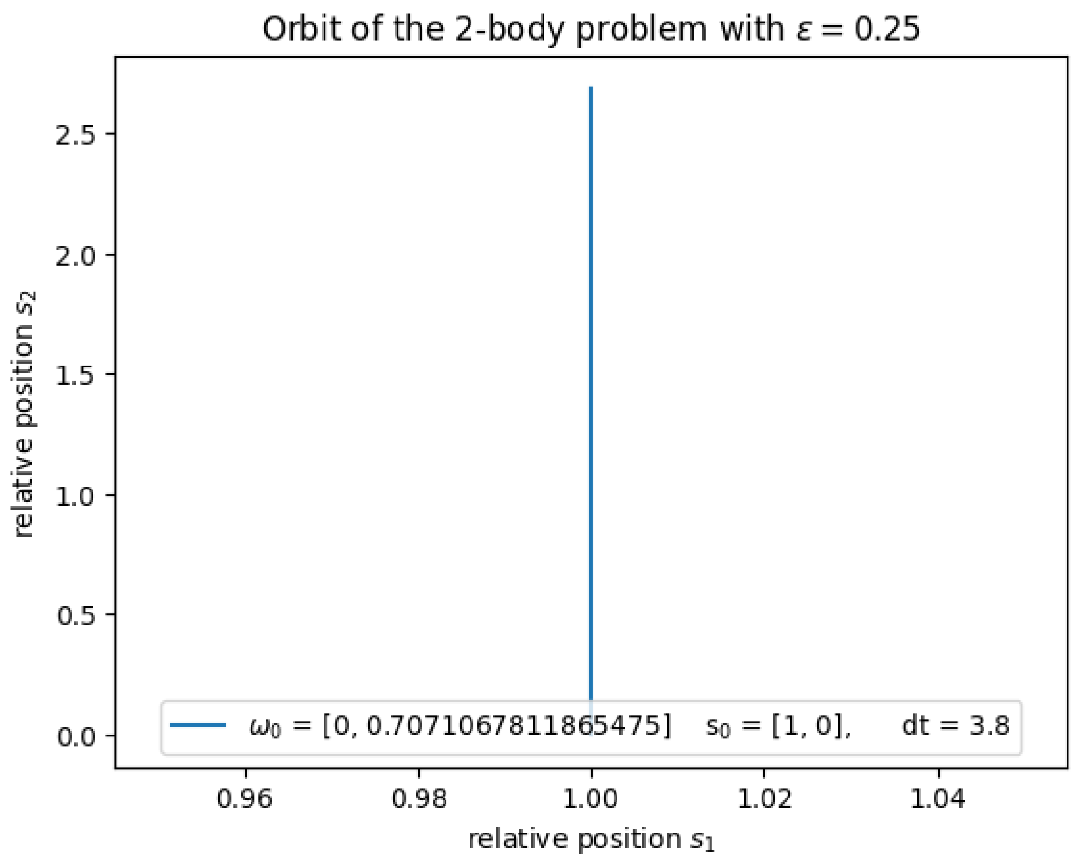


[ ]:
```
euler([0,1/np.sqrt(2)],[1,0], 3.8,10)
```

```
0.0
0.0
2.68700576850888
epislon calculated geometrically :1.0
```

[ ]:
```
(array([[1.        , 0.        ],
        [1.        , 2.68700577]]),
 array([[ 0.        ,  0.70710678],
        [-3.8       ,  0.70710678]]))
```


Motion of 2-body problem body problem with ε = 0.25
relative position s_1
time t
ω_0 = [0, 0.7071067811865475] s_0 = [1, 0],   dt = 3.8
0.0
0.5
1.0
1.5
2.0
2.5
0
2
4
6
8
10

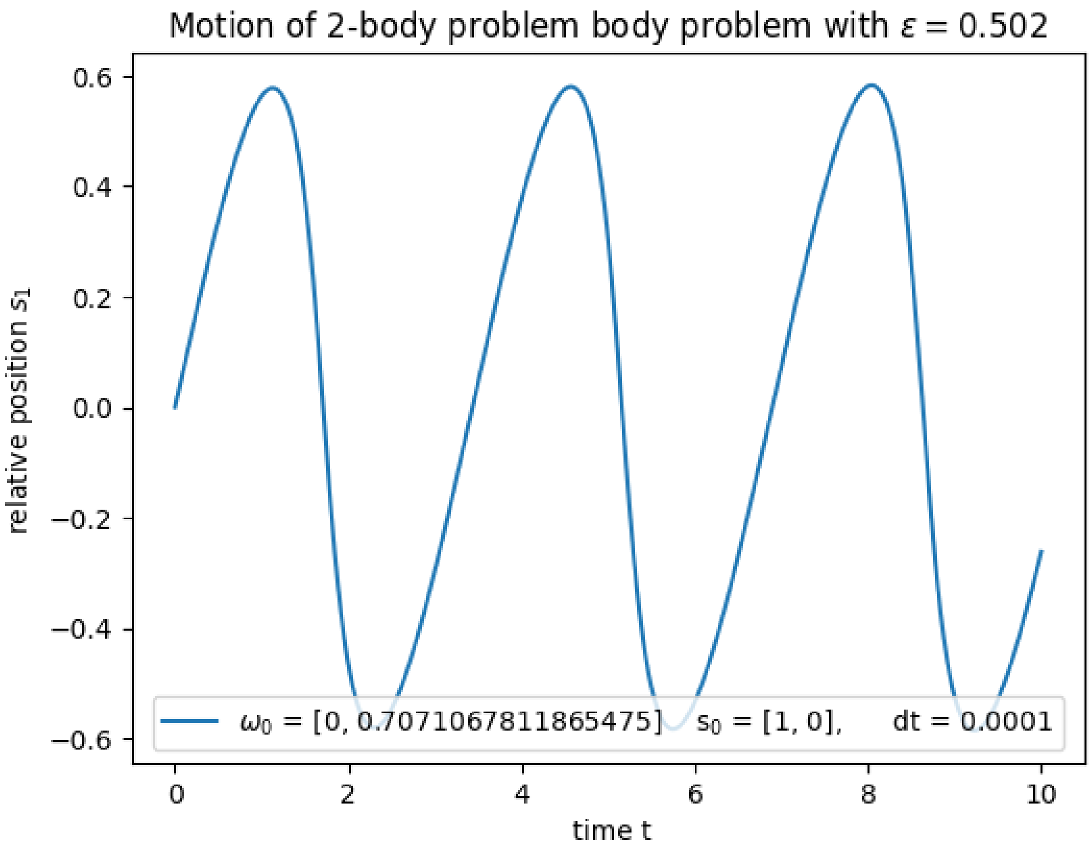


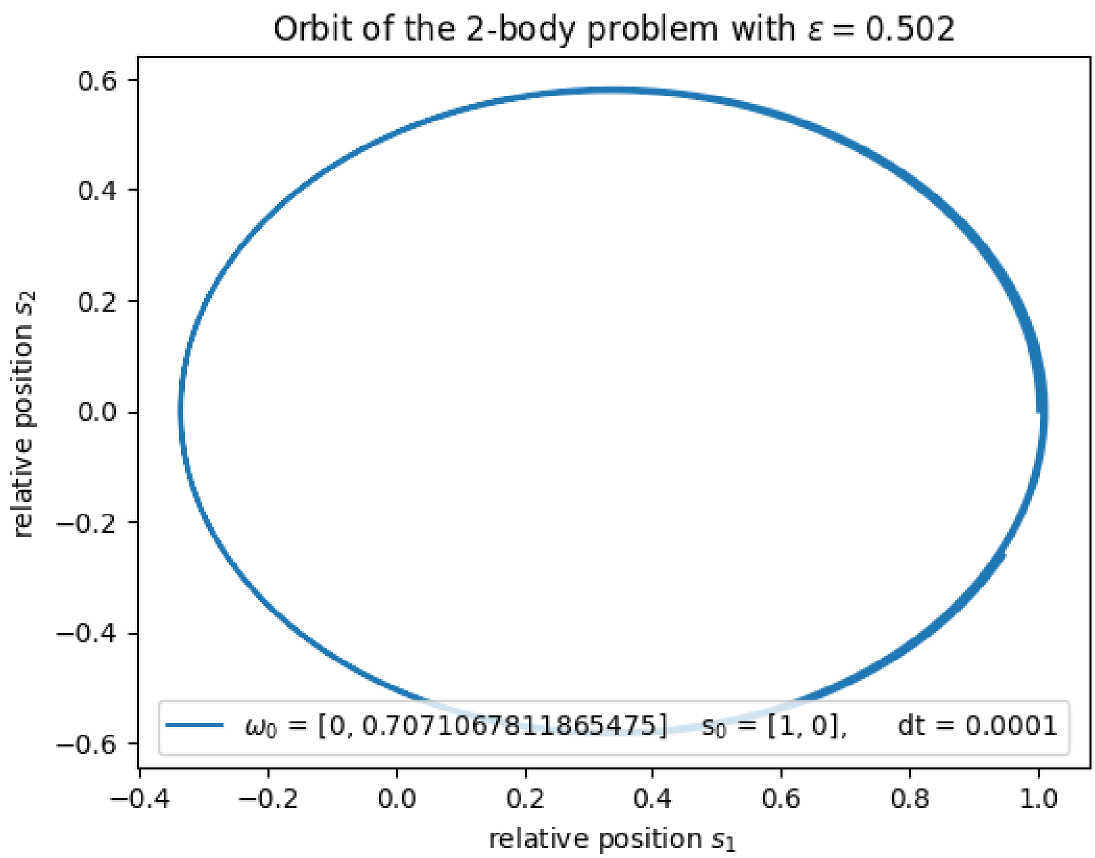
[ ]:
```
euler([0,1/np.sqrt(2)],[1,0], 0.0001,10)
```

```
0.49999999250000005
1.3480258576967694
1.1675373818253267
epislon calculated geometrically :0.577
```

[ ]:
```
(array([[ 1.00000000e+00,  0.00000000e+00],
        [ 1.00000000e+00,  7.07106781e-05],
        [ 9.99999990e-01,  1.41421356e-04],
        …,
        [ 9.47099038e-01, -2.61851117e-01],
        [ 9.47136525e-01, -2.61786428e-01],
        [ 9.47174002e-01, -2.61721736e-01]]),
 array([[ 0.00000000e+00,  7.07106781e-01],
        [-1.00000000e-04,  7.07106781e-01],
        [-1.99999999e-04,  7.07106774e-01],
        …,
        [ 3.74868772e-01,  6.46891334e-01],
        [ 3.74768950e-01,  6.46918932e-01],
```

```
[ 3.74669130e-01,  6.46946522e-01]]))
```





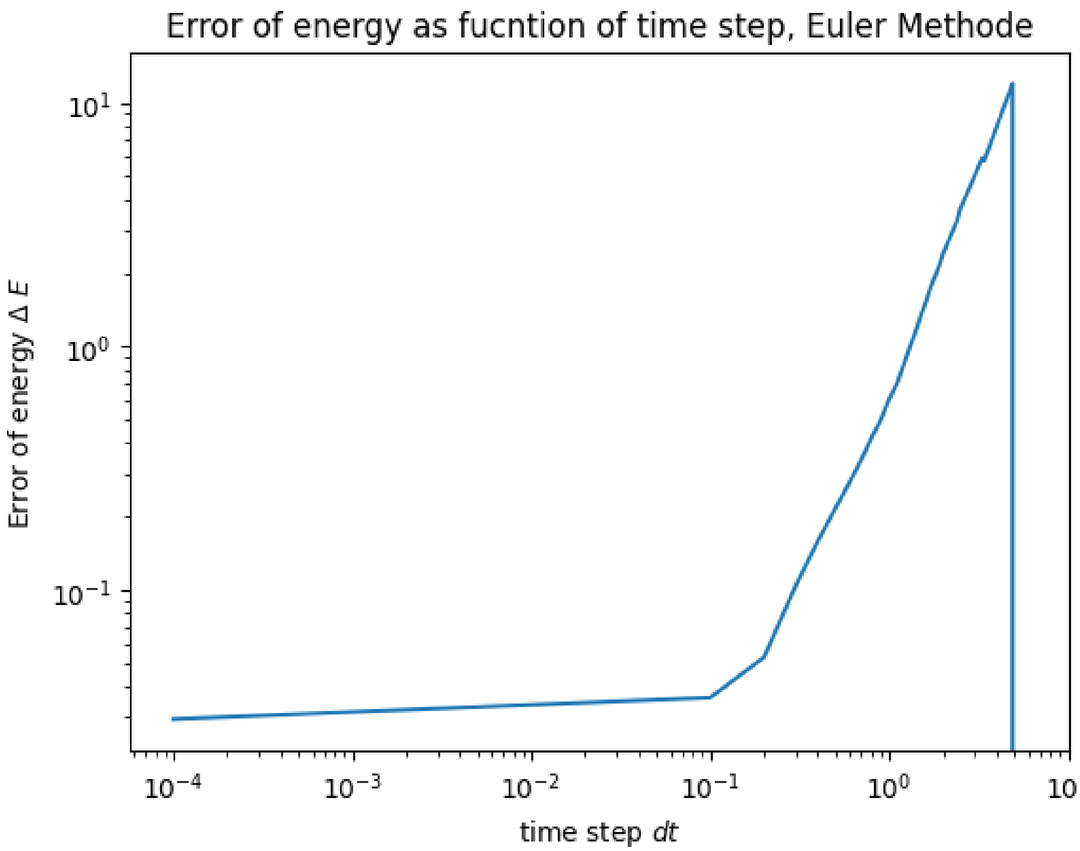
[ ]:
```
#computing the energie of the motion and plotting it

def euler_0(w0,s0,h,e):
    #the function will take the initial velocity
    # initial position, the whished step size
    # and the desired upper bondary for the integral

    G = 1 #grativational constant
    m = 1 #mass of both bodies
    N = int(e/h) #time steps

    s = np.zeros((N,2))
    w = np.zeros((N,2))
    LRL = np.zeros((N,2))
    s[0] = s0
    w[0] = w0
    #leap-frog algorith
    for i in range(N-1):
        s[i+1] = s[i] + h*w[i]
```

```
        w[i+1] = w[i]-h*s[i]/np.linalg.norm(s[i])**3
    return w

def energy_em(w0,s0,dt,e):
    # dt should be a list with initila and final ime step for which the␣
↪difference of energy has
    # to be computed
    # w0 is the initial velocity, s0, initila position
    # and e the time span in which the motion will take place

    dt = np.arange(dt[0], dt[-1], 0.1) #list of dt to be consider
    E = np.array([])

    #computiong the error in the energy
    for i in dt:
        w = euler_0(w0,s0,i,e)
        e_i = 1/2*np.dot(w[0],w[0]) #energy at beginning of motion
        e_f = 1/2*np.dot(w[-1],w[-1]) #energy at end of motion
        E = np.append(E, abs(e_f-e_i))

    #plotting error as funktion of dt
    plt.loglog(dt,E)
    plt.title(r'Error of energy as fucntion of time step, Euler Methode')
    plt.xlabel(r'time step $dt$')
    plt.ylabel(r'Error of energy $\Delta$ $E$')
    print(dt)
    print()
    print(E)
```

[ ]:
```
energy_em([0,1/np.sqrt(2)],[1,0], [0.0001,6],10)
```

```
[1.0000e-04 1.0010e-01 2.0010e-01 3.0010e-01 4.0010e-01 5.0010e-01
 6.0010e-01 7.0010e-01 8.0010e-01 9.0010e-01 1.0001e+00 1.1001e+00
 1.2001e+00 1.3001e+00 1.4001e+00 1.5001e+00 1.6001e+00 1.7001e+00
 1.8001e+00 1.9001e+00 2.0001e+00 2.1001e+00 2.2001e+00 2.3001e+00
 2.4001e+00 2.5001e+00 2.6001e+00 2.7001e+00 2.8001e+00 2.9001e+00
 3.0001e+00 3.1001e+00 3.2001e+00 3.3001e+00 3.4001e+00 3.5001e+00
 3.6001e+00 3.7001e+00 3.8001e+00 3.9001e+00 4.0001e+00 4.1001e+00
 4.2001e+00 4.3001e+00 4.4001e+00 4.5001e+00 4.6001e+00 4.7001e+00
 4.8001e+00 4.9001e+00 5.0001e+00 5.1001e+00 5.2001e+00 5.3001e+00
 5.4001e+00 5.5001e+00 5.6001e+00 5.7001e+00 5.8001e+00 5.9001e+00]

[ 0.02945838  0.03610472  0.0529069   0.10374135  0.15885144  0.2156045
  0.27591036  0.34284985  0.42363654  0.50068126  0.60675802  0.69698256
  0.82866947  0.98541209  1.13792351  1.33209797  1.50990621  1.73431401
  1.92695346  2.12763023  2.39696296  2.60742929  2.82609298  3.05307877
  3.28856534  3.68189761  3.92516049  4.17829417  4.4414105   4.71459728
  4.99792264  5.29143866  5.59518446  5.90918866  5.78034001  6.12535001
```

```
  6.48036001   6.84537001   7.22038001   7.60539001   8.0004       8.40541001
  8.82042      9.24543      9.68044001  10.12545     10.58046001  11.04547001
 11.52048001  12.00549001   0.           0.           0.           0.
  0.           0.           0.           0.           0.           0.        ]
```

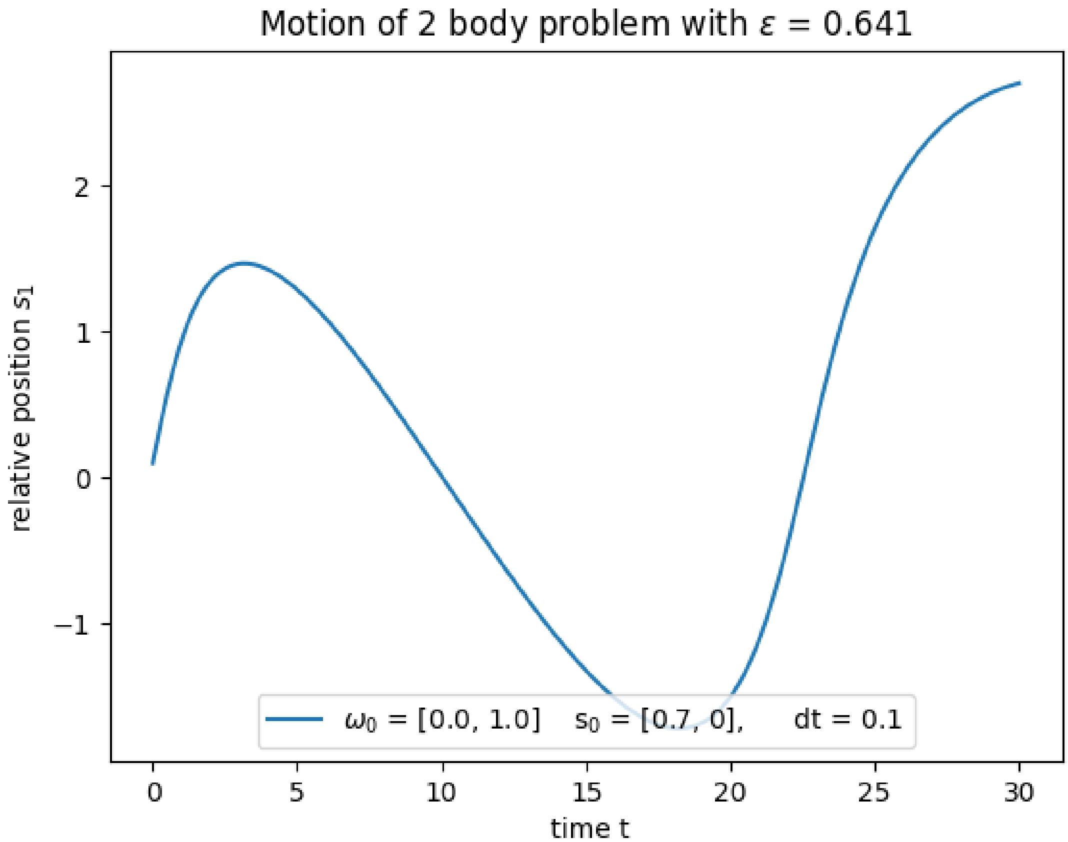


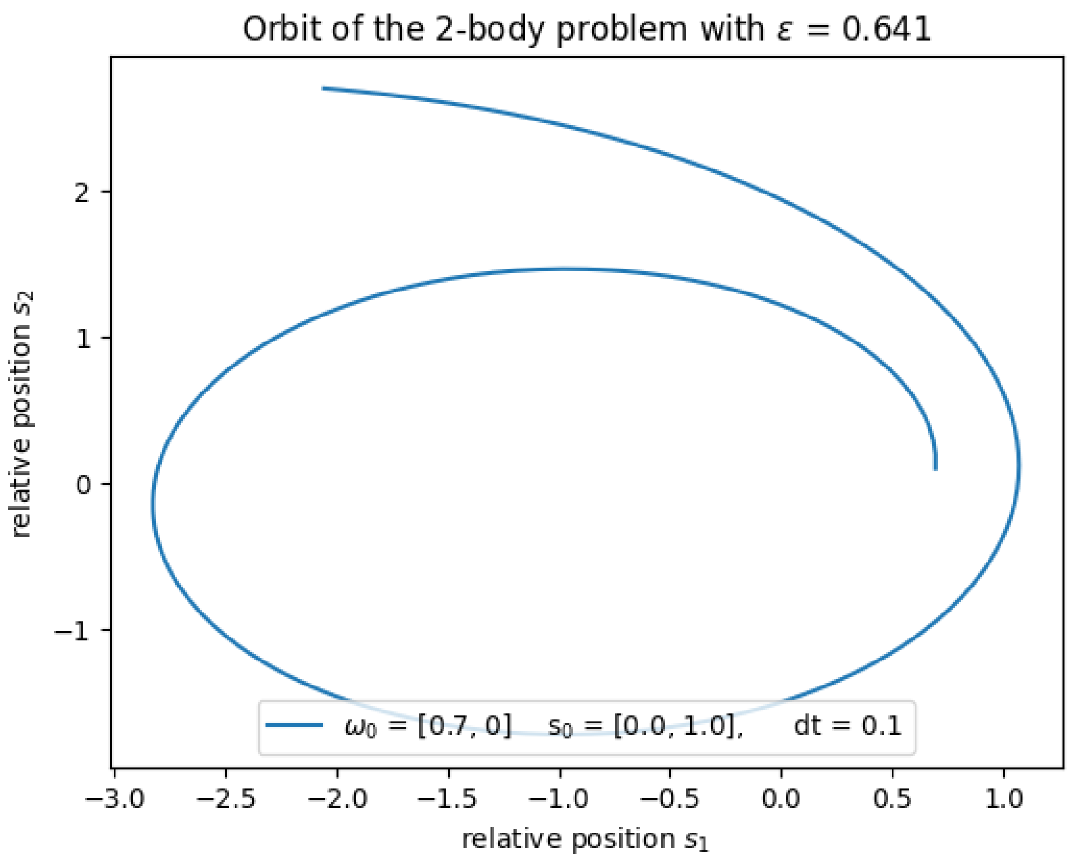
The results are as expected till $dt > 4$, at this point the simulation is so erroneous, that the result of the Euler Method is a linear trajectory, for which the Energy is constant. A constant energy is the ideal case, as it is visible for \$dt→\$ 0, however if the trajectory is erroneously integrated, this result can't be taken as a correct one.

3.2 b) leap-frog algorithm

[ ]:
```
leap_frog([0.0,1.0],[0.7,0], 0.1,30)
```

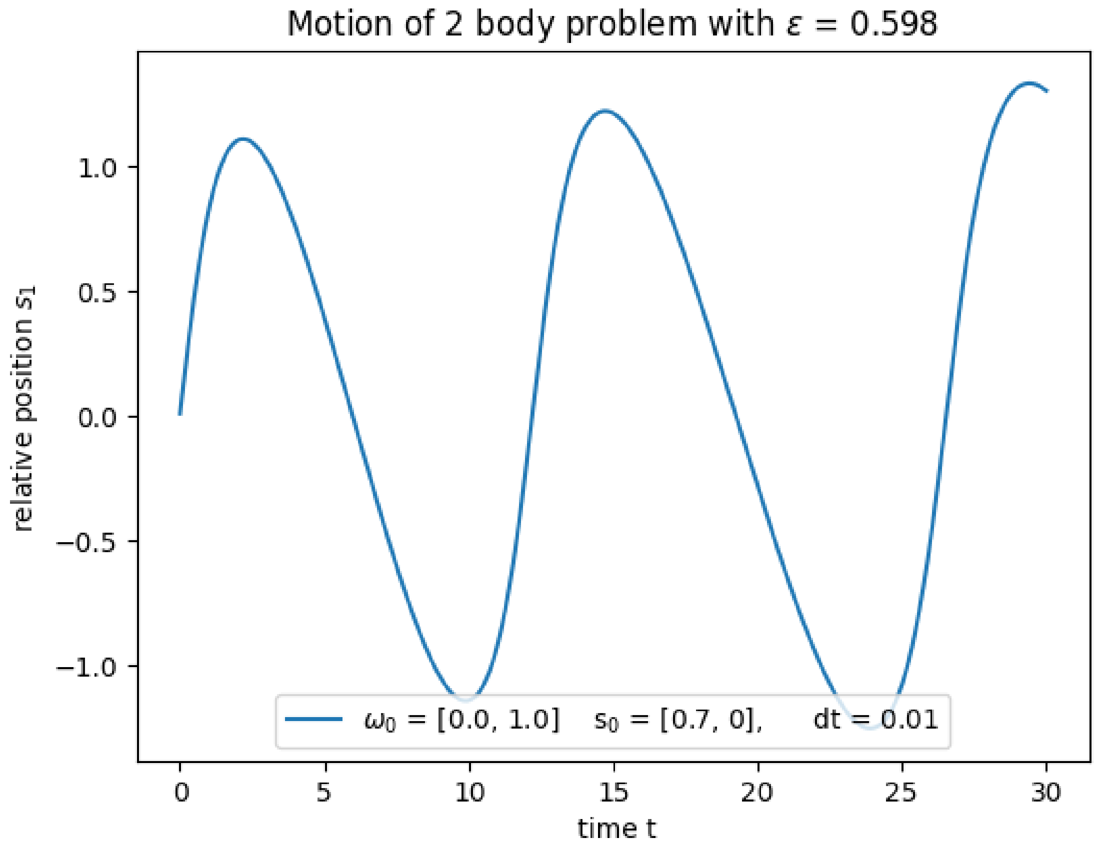
```
3.895588310504747
4.416367359310763
epislon calculated geometrically :0.534
```

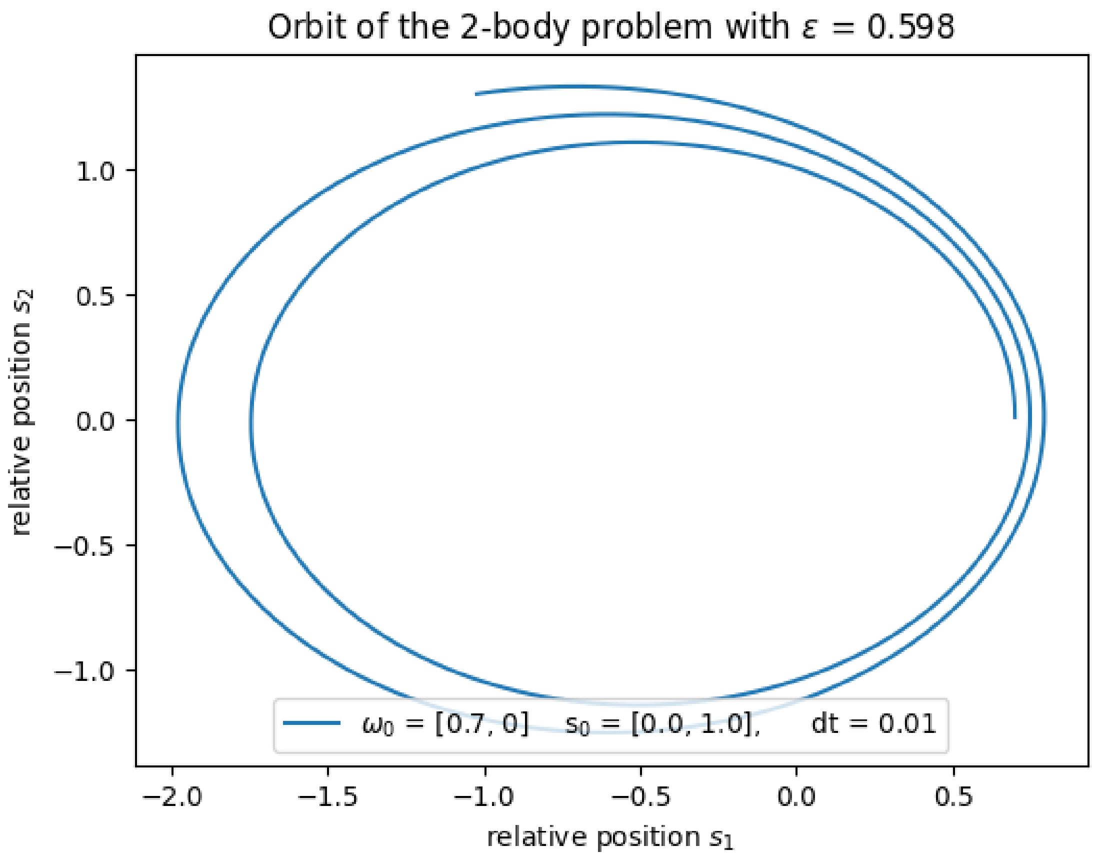

Motion of 2 body problem with ε = 0.641
relative position s_1
time t
ω_0 = [0.0, 1.0] s_0 = [0.7, 0],   dt = 0.1
2
1
0
−1
0
5
10
15
20
25
30



[ ]:
```
leap_frog([0.0,1.0],[0.7,0], 0.01,30)
```

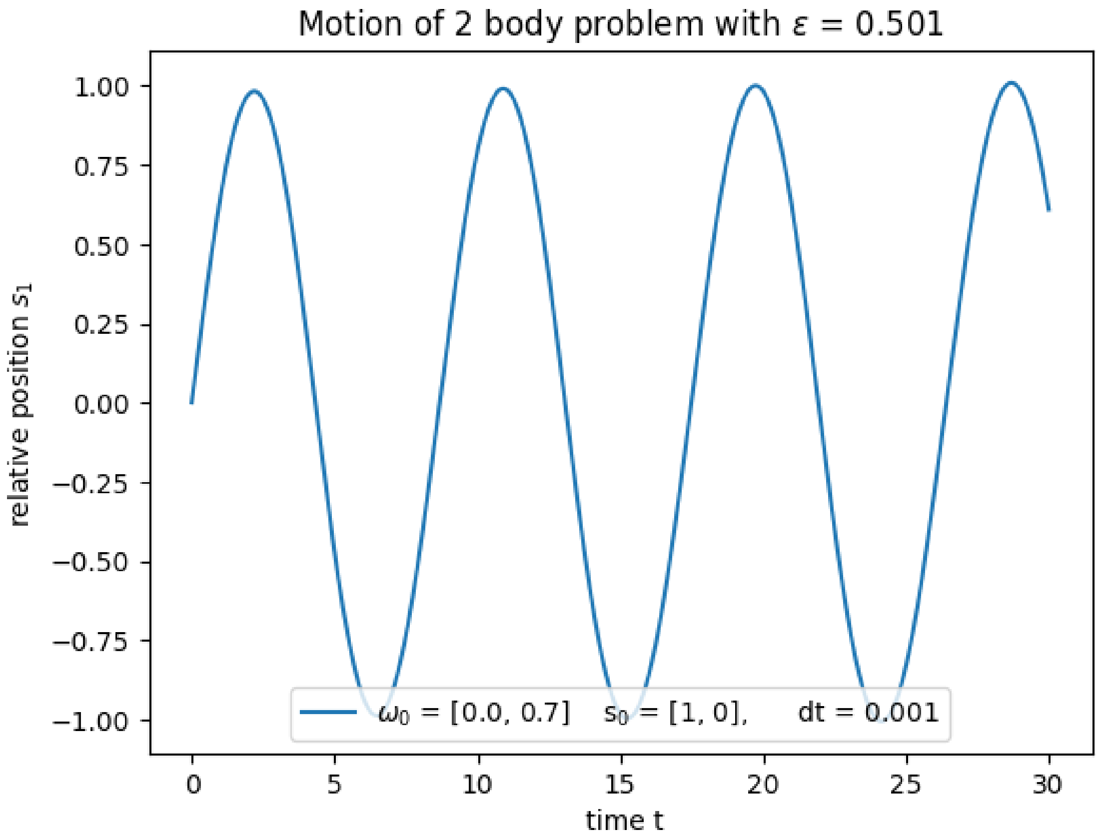
```
2.773665244342849
2.5827428157153465
epislon calculated geometrically :0.365
```

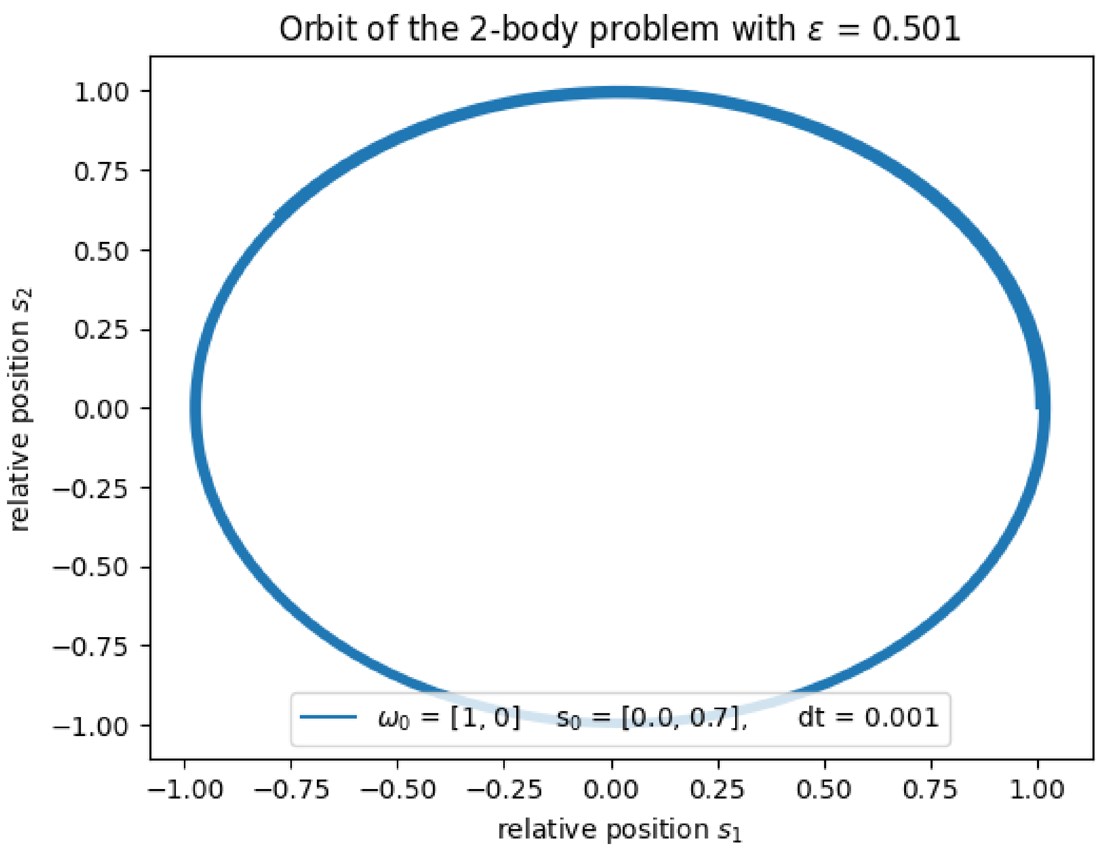

Motion of 2 body problem with ε = 0.598
relative position s_1
time t
ω_0 = [0.0, 1.0] s_0 = [0.7, 0],   dt = 0.01
1.0
0.5
0.0
−0.5
−1.0
0
5
10
15
20
25
30



[ ]:
```
leap_frog([0.0,0.7],[1,0], 0.001,30)
```

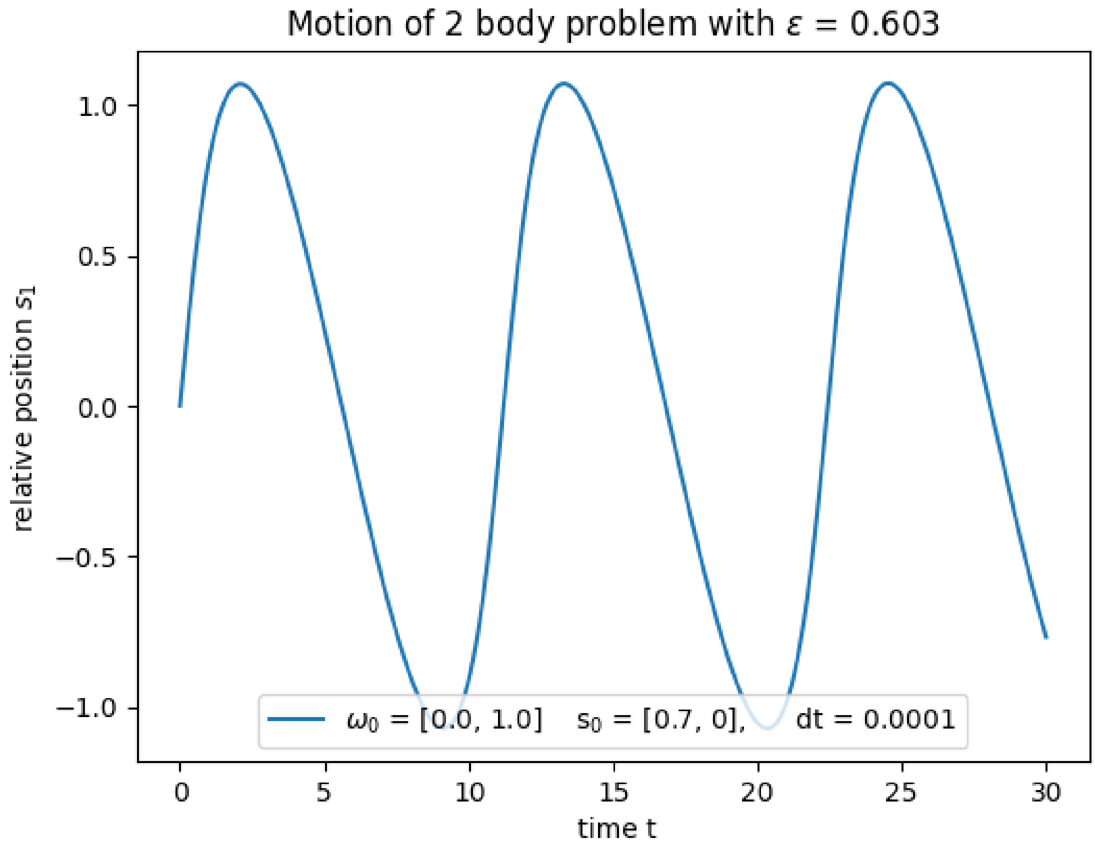
```
2.010055015594763
2.0139096011602406
epislon calculated geometrically :0.062
```

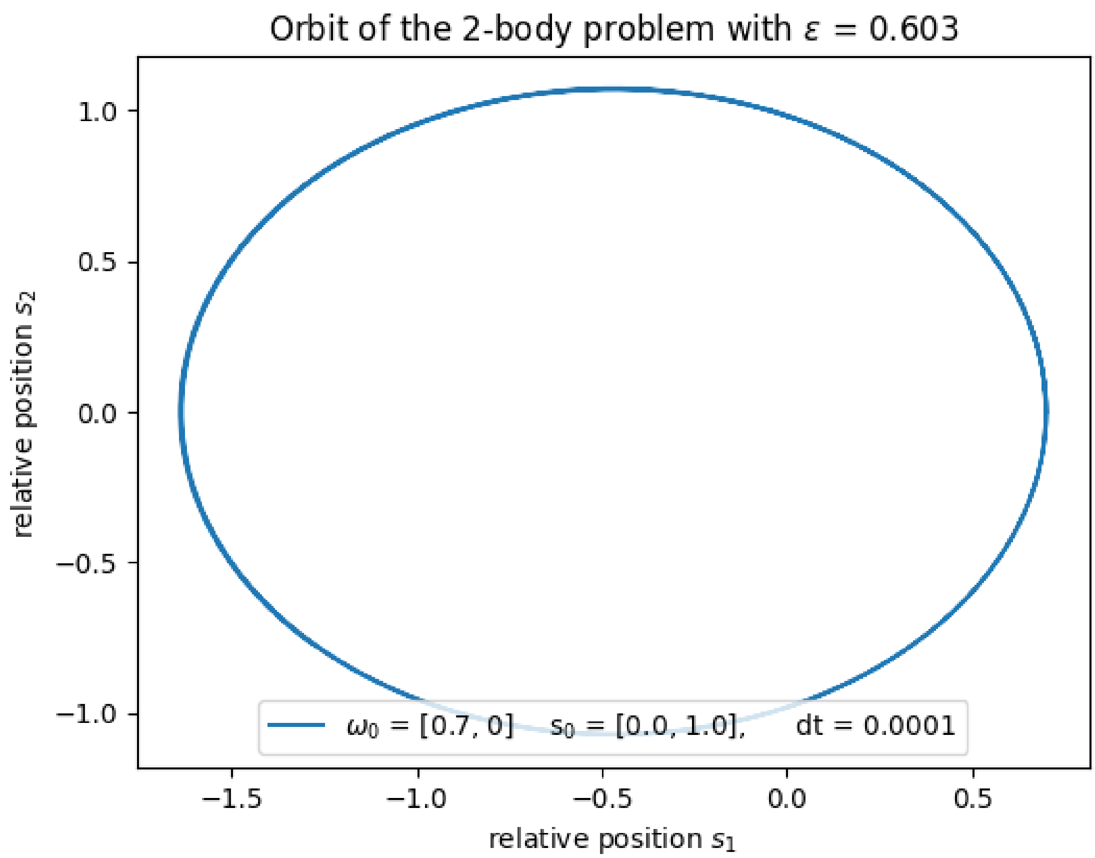

Motion of 2 body problem with ε = 0.501
relative position s_1
time t
ω_0 = [0.0, 0.7] s_0 = [1, 0],   dt = 0.001
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0
5
10
15
20
25
30



[ ]:
```
leap_frog([0.0,1.0],[0.7,0], 0.0001,30)
```

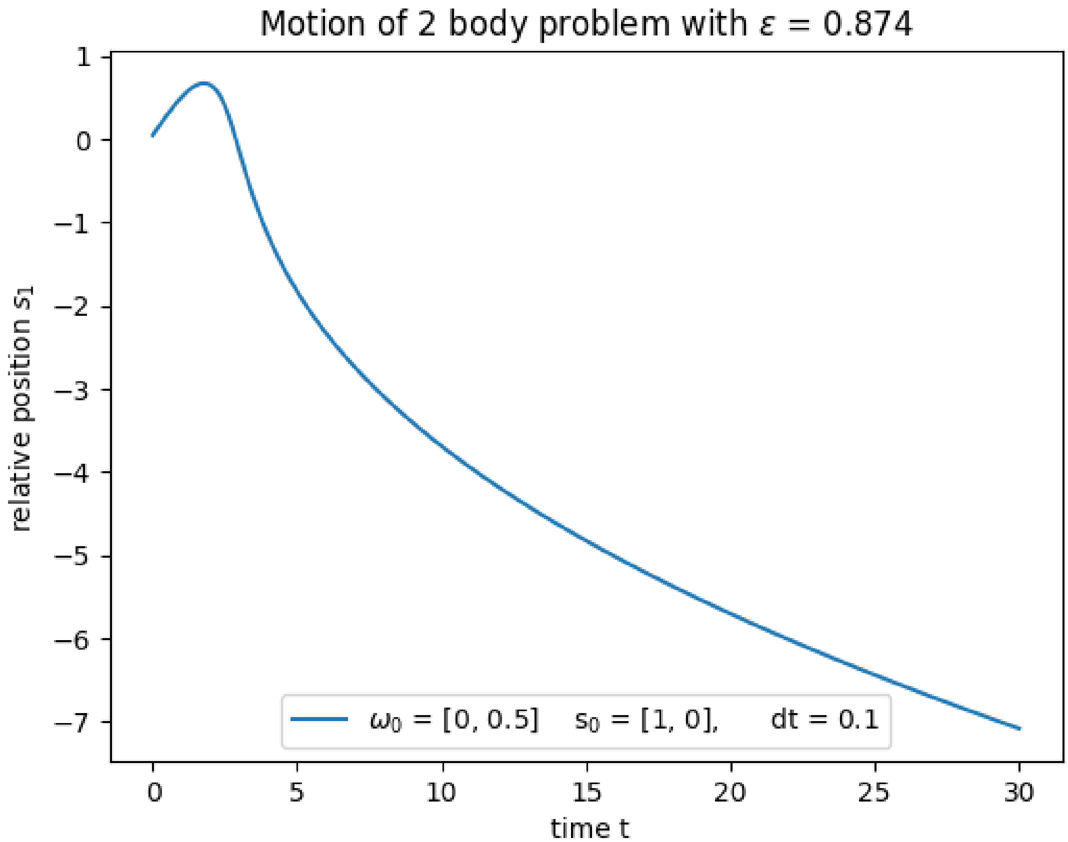
```
2.3399526917437283
2.1430118545307777
epislon calculated geometrically :0.402
```


Motion of 2 body problem with ε = 0.603
relative position s_1
time t
ω_0 = [0.0, 1.0] s_0 = [0.7, 0],   dt = 0.0001
1.0
0.5
0.0
−0.5
−1.0
0
5
10
15
20
25
30

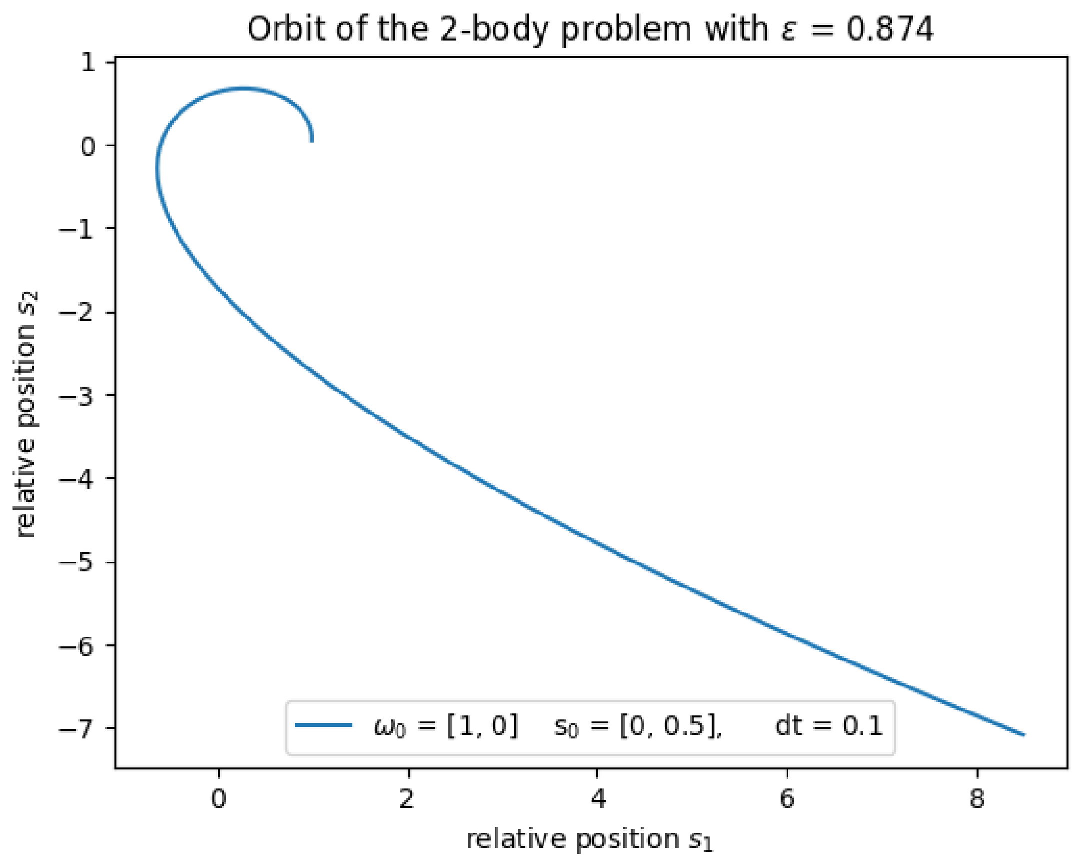



```
[ ]: leap_frog([0,1/2],[1,0], 0.1,30)
```

```
9.115157067317028
7.758732521682457
epislon calculated geometrically :0.525
```

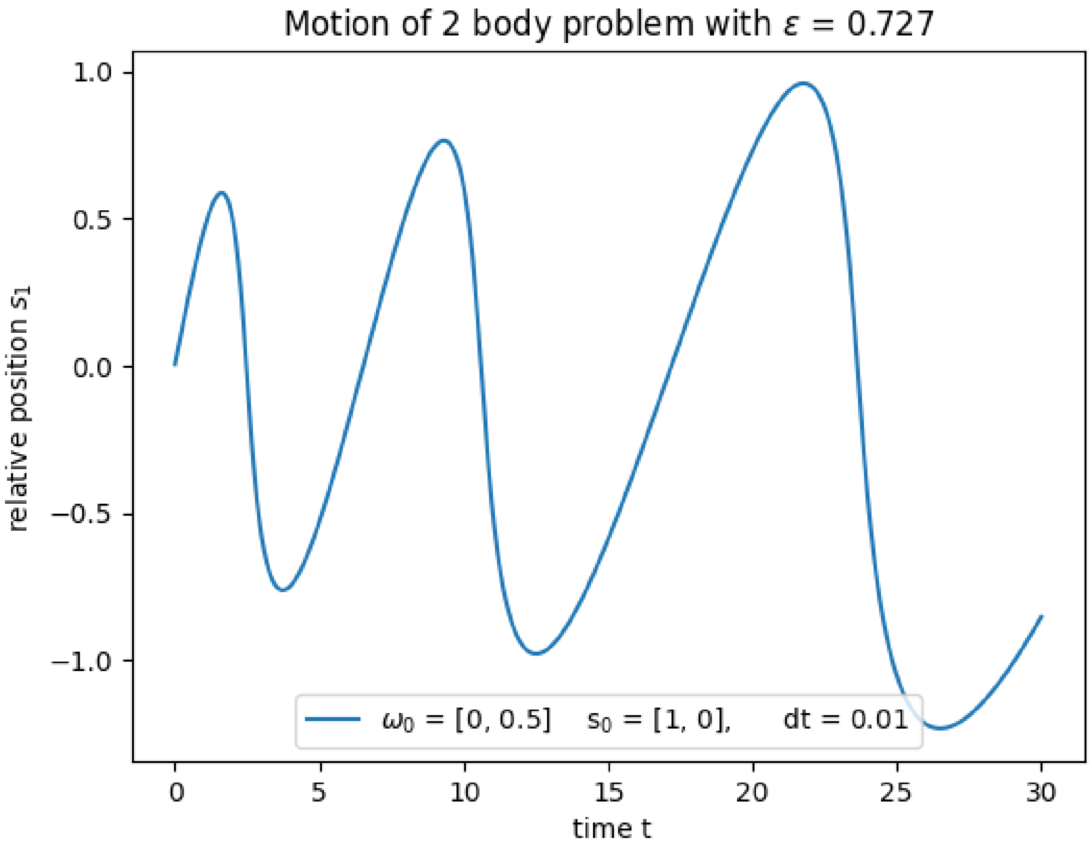


Motion of 2 body problem with ε = 0.874
relative position s_1
time t
ω_0 = [0, 0.5] s_0 = [1, 0], dt = 0.1
1
0
−1
−2
−3
−4
−5
−6
−7
0
5
10
15
20
25
30

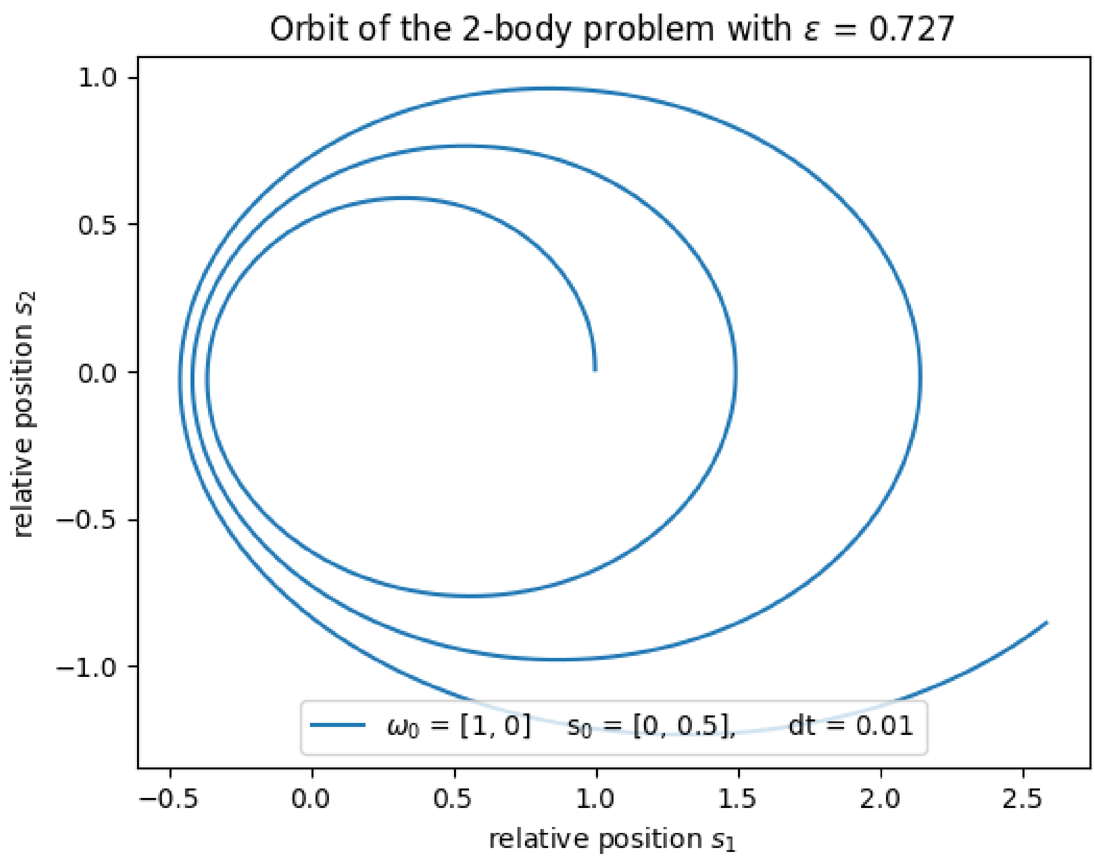



```
[ ]: leap_frog([0,1/2],[1,0], 0.01,30)
```

```
3.0464735953711406
2.191182322829405
epislon calculated geometrically :0.695
```

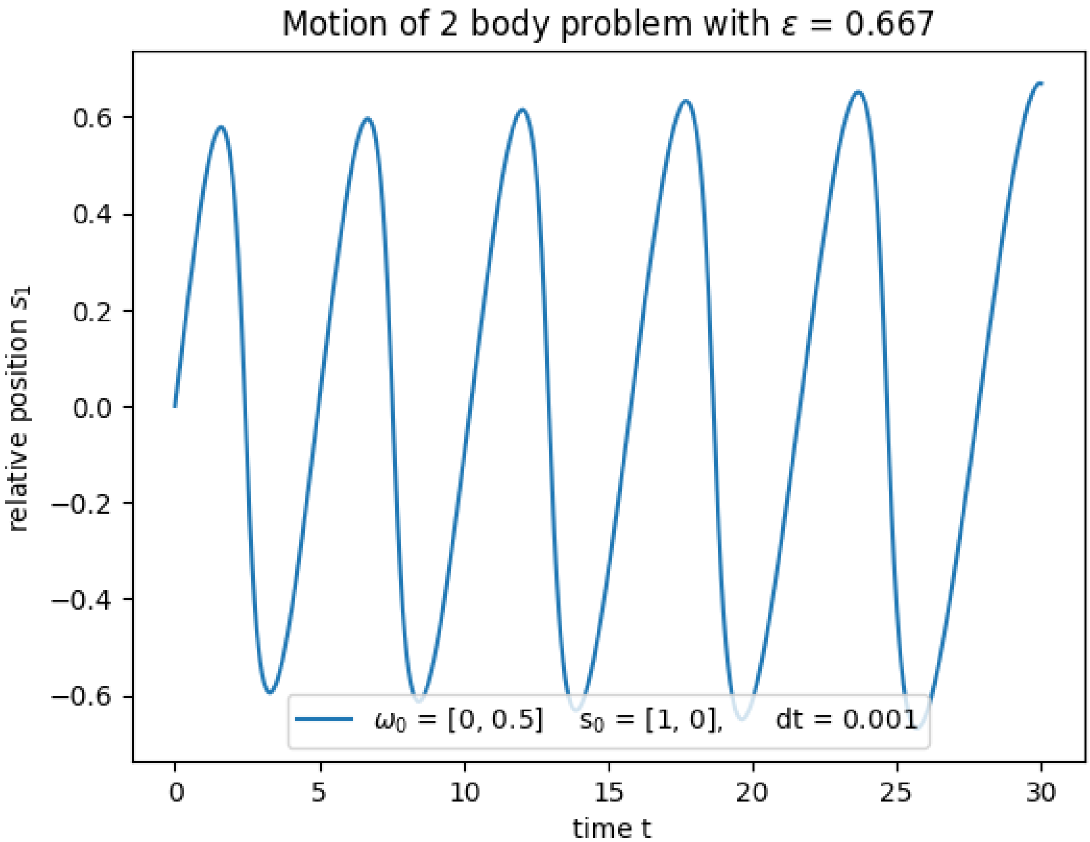


Motion of 2 body problem with ε = 0.727
relative position s_1
time t
ω_0 = [0, 0.5] s_0 = [1, 0],   dt = 0.01
1.0
0.5
0.0
−0.5
−1.0
0
5
10
15
20
25
30

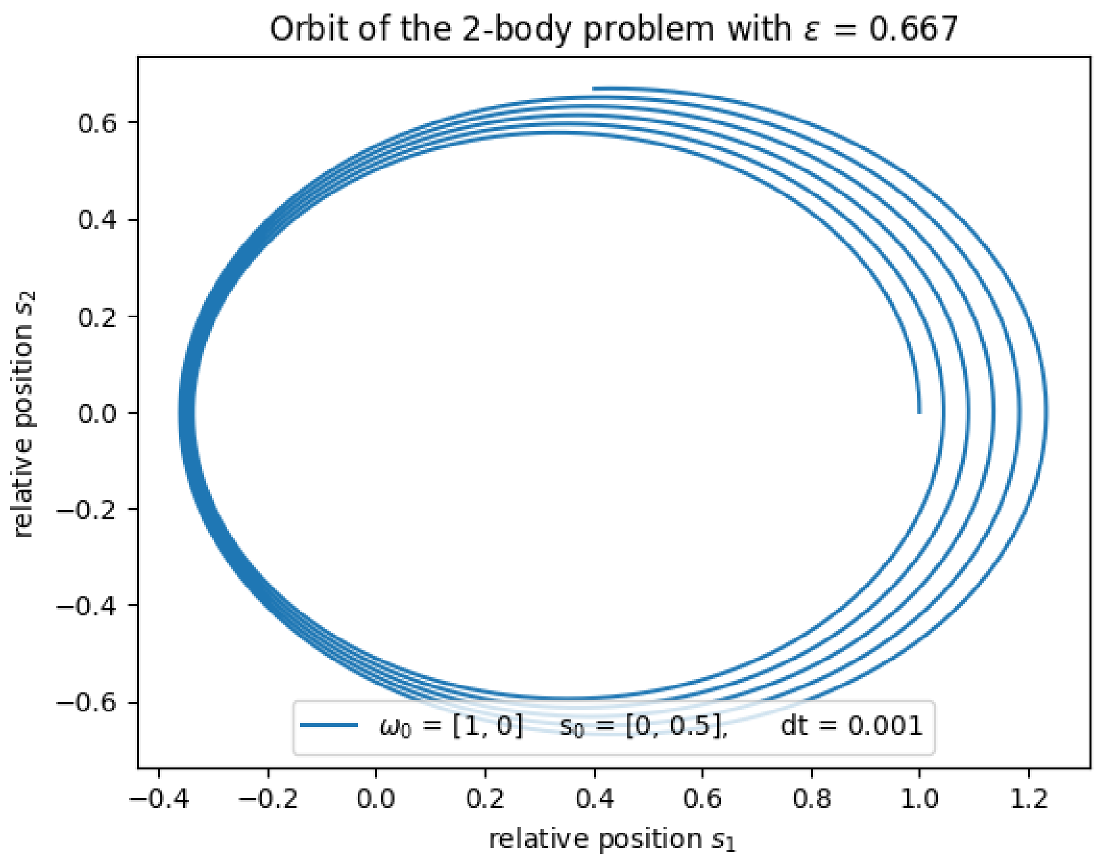


[ ]:
```
leap_frog([0,1/2],[1,0], 0.001,30)
```

```
1.5934798580931757
1.3383089210857317
epislon calculated geometrically :0.543
```



Motion of 2 body problem with ε = 0.667
relative position s_1
time t
ω_0 = [0, 0.5] s_0 = [1, 0],   dt = 0.001
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
0
5
10
15
20
25
30



[ ]:
```
leap_frog([0,1/2],[1,0], 0.0001,30)
```

[ ]:
```
def leap_frog_0(w0,s0,h,e):
    #the function will take the initial velocity
    # initial position, the whished step size
    # and the desired upper bondary for the integral
    G = 1 #grativational constant
    m = 1 #mass of both bodies
    N = int(e/h) #time steps

    s = np.zeros((N,2))
    w = np.zeros((N,2))
    LRL = np.zeros((N,2))

    w[0] = w0
    s[0] = s0

    #First the programm will compute the s_1/2 step using the
    #euler forward methode
```

```
    a = 0.5*h/np.linalg.norm(s0)**3
    wn = w[0] - h/2*s[0]/np.linalg.norm(s[0])
    s[0] = s0 + h*wn

    #leap-frog algorith
    for i in range(N-1):
        w[i + 1] = w[i]-h/2*s[i]/np.linalg.norm(s[i])**3
        s[i + 1] = s[i] + w[i]*h
    return w

def energy_lf(w0,s0,dt,e):
    # dt should be a list with initila and final ime step for which the␣
 ↪difference of energy has
    # to be computed
    # w0 is the initial velocity, s0, initila position
    # and e the time span in which the motion will take place

    dt = np.arange(dt[0], dt[-1], 0.1) #list of dt to be consider
    E = np.array([])
    #computiong the error in the energy
    for i in dt:
        w = leap_frog_0(w0,s0,i,e)
        print(w)
        print()
        e_i = 1/2*np.dot(w[0],w[0]) #energy at beginning of motion
        e_f = 1/2*np.dot(w[-1],w[-1]) #energy at end of motion
        E = np.append(E, abs(e_f-e_i))

    #plotting error as funktion of dt
    plt.loglog(dt,E)
    print(dt)
    print()
    print(E)
```

[ ]:
```
energy_lf([0,1/np.sqrt(2)],[1,0], [0.0001,6],10)
```

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 4.05210791e-02 3.64571980e-02 1.71017854e-02 1.51516436e-02
 1.34697032e-02 1.20134938e-02 1.07480243e-02 9.64436034e-03
 8.67848588e-03 7.83039088e-03 7.08333933e-03 6.42328064e-03
 5.83837469e-03 5.31860774e-03 4.85548053e-03 4.44175428e-03
 4.07124300e-03 3.73864319e-03 0.00000000e+00 0.00000000e+00
 0.00000000e+00 0.00000000e+00 0.00000000e+00 0.00000000e+00
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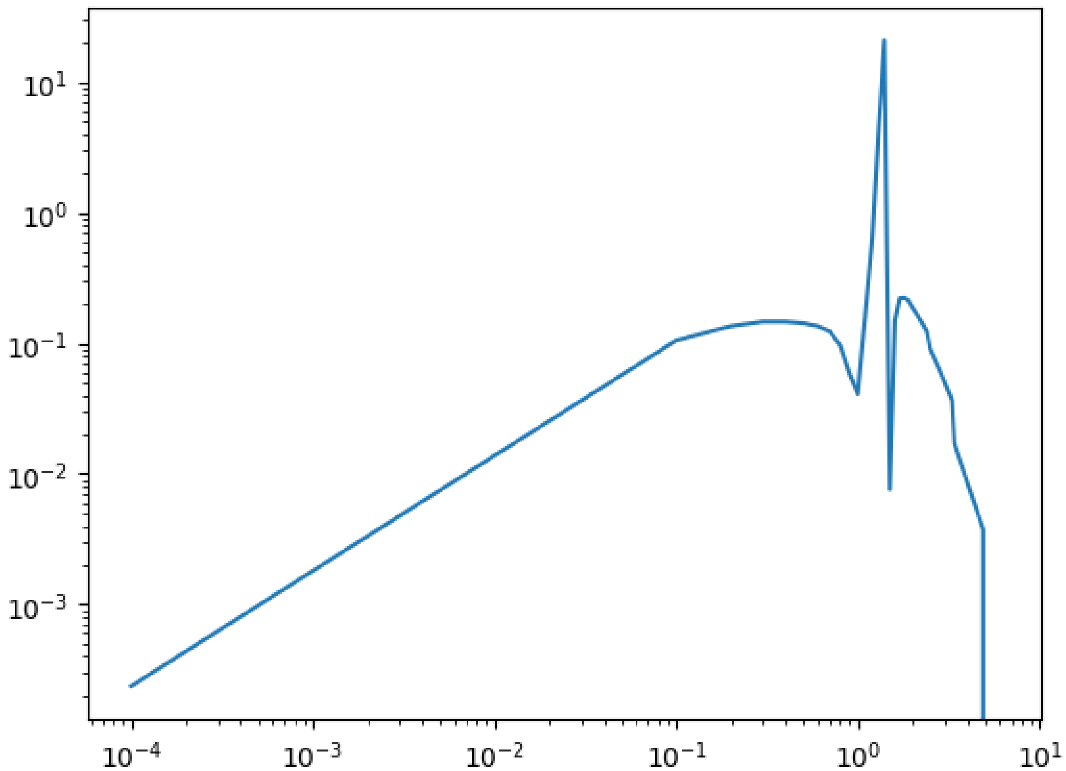

10^1
10^0
10^{-1}
10^{-2}
10^{-3}
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10^{-2}
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