Introduction to Computational Physics SS2023

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Exercise 8 from June 14, 2023
Return before noon of June 23, 2023

1 Preparation: Volterra-Lotka System

The Jacobi Matrix of the non-trivial fixed point for the Volterra-Lotka system is (see lecture, we choose $\alpha = 0.5$):

$$A = \begin{pmatrix} 0 & -1 \\ 0.5 & 0 \end{pmatrix}$$

Determine the eigenvalues λ_i and eigenvectors $\mathbf{x_i}$ of this matrix. We are interested in the time evolution of

$$\mathbf{v}(\tau) = \exp(A\tau)\mathbf{v}(\mathbf{0}) = \left(\sum_{i=1}^{2} c_i \exp(\lambda_i \tau)\mathbf{x_i} \right)$$

Here we have used that $\mathbf{v}(0)$ can be written as a linear combination $c_1\mathbf{x_1} + c_2\mathbf{x_2}$ using the eigenvectors of A and coefficients c_i (see below and lecture). Find the solution for $\mathbf{v}(\tau)$ for an initial value $\mathbf{v}(\mathbf{0}) = (0.1, 0.1)$. Remember that the fixed point is $\mathbf{u}^\star = (1, 1)$, and $\mathbf{v}$ denotes the deviation from the fixed point. Note that the eigenvectors $\mathbf{x_i}$ are not uniquely defined - there is a free complex scaling factor for every eigenvector. But no matter which scaling for the eigenvector you use, the following steps should always lead to the same result for $\mathbf{v}(0)$.

Hint: Assume that the system of eigenvectors spans our vector space; we need to find coefficients c_i, such that $\mathbf{v}(\mathbf{0}) = c_1\mathbf{x_1} + c_2\mathbf{x_2}$. In vector notation:

$$\mathbf{v}(\mathbf{0}) = EVC \cdot \mathbf{c}$$

where $\mathbf{c} = (c_1, c_2)$ and EVC is the matrix containing all eigenvectors of A. By inverting EVC you can find $\mathbf{c}$:

$$\mathbf{c} = EVC^{-1}\mathbf{v}(\mathbf{0})$$

Double check your result, by making sure that $\mathbf{v}(0) = c_1\mathbf{x_1} + c_2\mathbf{x_2}$, with the c_i computed as above. Plot the solution for $\mathbf{v}(\tau)$ as a function of τ; or you can plot the $u_i(\tau) = u_i^\star + v_i(\tau)$ $(i = 1, 2)$ and compare with the plots published in the old lecture script. Try some other initial values.

2 Homework: Many Species Population Dynamics

Six Populations

In this exercise we study the evolution of 6 populations according to the following equations for population dynamics: 3 predator- (P_i) and 3 prey-species (N_i), all parameters positive, always $i, j = 1, \ldots, 3$:

$$\frac{\mathrm{d}N_i}{\mathrm{d}t} = N_i \left(a_i - N_i - \sum_j b_{ij} P_j \right) \quad ; \quad \frac{\mathrm{d}P_i}{\mathrm{d}t} = P_i \left(\sum_j c_{ij} N_j - d_i \right)$$

The parameters chosen are $a_1 = 11$, $a_2 = 12$, $a_3 = 10$; $d_1 = 8$, $d_2 = 9$, $d_3 = 17$; the parameters bij and c_{ij} are given in matrix form here:

$$b_{ij} = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 3 & 7 \\ 4 & 3 & 2 \end{pmatrix} \quad c_{ij} = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 3 & 3 \\ 7 & 8 & 2 \end{pmatrix}$$

Notice: the unusual feature here in the equations is that the prey populations N_i have a Verhulst style growth limiting factor in their equations, which limits their growth even if there is no predator (model for limited resources even in absence of predators).

Notice 2: please do not try to make the equations dimensionless, just use the numbers given here, and the time t instead of normalized τ.

Stability Analysis

1. (5 points) What are the fixed points for the system of equations given above?
Hint 1: No complicated computations are necessary, the idea is that you should guess the fixed points very easily. Compare our previous examples.
Hint 2: This time there are three fixed points! In addition to our 'usual' ones, there is a third one related to the Verhulst growth limiting factor in the first three equations.

2. (5 points) What is the Jacobi matrix $\mathbf{A}$ at the non-trivial fixed point? (non-trivial FP means here that ALL elements are unequal to zero. There is only one FP with this property.)

3. (10 points) Determine the eigenvalues and eigenvectors λ_i and $\mathbf{v}_i$, $i = 1, \ldots 6$ of $\mathbf{A}$ for this fixed point.
Plot and discuss the solutions for all 6 populations, starting from an initial value of

$$\mathbf{v}(0) = (0.1, 0.1, 0.1, 0.1, 0.1, 0.1)$$

Hint: if you have a well working preparation exercise for the two populations, this one is relatively easy to generalize for six dimensions here. Plot and discuss the time dependent evolution of the six populations What about oscillations, growth or decay? What is wrong about this solution (not in the mathematical sense)?