Introduction to Computational Physics SS2023

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Exercise 7 from June 7, 2023
Return before noon of June 16, 2023

1 Preparation: Population dynamics

The simple equation of growth of a population, as proposed by Malthus, has been improved by Verhulst including a growth limiting term, which represents the finite amount of resources available:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\Big(1 - \frac{N}{K}\Big) \tag{1.1}$$

Here the definitions of r, N, and K are as in the lecture (growth rate, population number, growth limiting number). By dimensional analysis we can renormalize the variables, such that the equation becomes dimensionless and does not depend on the parameters anymore ($\tau = rt$, $n = N/K$) and we get:

$$\frac{\mathrm{d}n}{\mathrm{d}\tau} = n(1 - n) \tag{1.2}$$

Solve this non-linear quadratic differential equation and discuss the solutions as function of the initial value $n_0 = n(\tau = 0)$. Try for fun also values $n_0 < 0$, $n_0 > 1$, even though they are not very realistic.
Can you find the analytic solution for $n(\tau)$?

2 Homework: Population dynamics

In this exercise we study the following equation for population dynamics:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN(1 - N/K) - \frac{BN^2}{A^2 + N^2} \tag{2.3}$$

where all parameters r, K, A and B are positive. It is a more complex example, in which the growth behaviour depends on whether N is smaller or larger than a critical population size A. To make the equation dimensionless choose the following normalizations:

$$\tau = \frac{B}{A}t\ ;\ n = \frac{N}{A}\ ;\ \alpha = \frac{rA}{B}\ ;\ \beta = \frac{A}{K} \tag{2.4}$$

We choose $\beta = 1/8$; then there is only one remaining free parameter left: α, it should be varied between 0.2 and 0.8.

- (7 points) Write down the normalized ordinary differential equation; find its fixed points $n^\star$ (FP). Hint: In total there are four FP's, but you can divide out a factor of n to take away the trivial solution $n^\star = 0$; you need to find the roots of the remaining expression $F(n)$; you can find its roots (the FP's) by plotting $F(n)$ for several α values (at least ten values, or more as needed), or you can transform $F(n)$ to a cubic polynomial and find its roots.

- (7 points) Depending on the value of α there are one or three real FP's $n^\star$ (here we have not counted the trivial FP $n^\star = 0$). What are the limiting values of the remaining free parameter to switch from one to three real roots or vice versa? You can solve this by plotting $F(n)$ again, by bracketing the switching point with α values close to it.

- (6 points) For at least ten α values between 0.2 and 0.8 find out whether the real FP's are stable or not.