The material from this lesson is drawn from "Linear Algebra and its Application", 5th Edition, by Lay, "A First Course in Linear Model Theory" by Ravishanker and Dey, and from Dr. Steve MacEachern's Spring 2014 course "STAT 6860: Foundations of the Linear Model" at The Ohio State University.

I. Linear Transformations

Recall that a linear transformation f is a function that preserves additivity and scalar multiplication. That is, given a linear transformation f, vectors $\mathbf{x}$ and $\mathbf{y}$, and a scalar k, then:

$$\begin{aligned} f(\mathbf{x}+\mathbf{y}) &= f(\mathbf{x})+f(\mathbf{y}) \\ f(k\mathbf{x}) &= k \times f(\mathbf{x}) \end{aligned}$$

More importantly, we can express every linear transformation $f(\mathbf{x})$ in the form of multiplying a matrix $\mathbf{A}$ times the vector we are interested in transforming $\mathbf{x}$.

To generate the matrix $\mathbf{A}$ that corresponds to the desired transformation f, we partition the identity matrix $\mathbf{I}_n$ into its columns $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ and apply the transformation f to each unit vector:

$$\begin{aligned} \mathbf{I}_n &= [\mathbf{e}_1\ \mathbf{e}_2\ \cdots\ \mathbf{e}_n] \\ \Rightarrow \mathbf{A} &= [f(\mathbf{e}_1)\ f(\mathbf{e}_2)\ \cdots\ f(\mathbf{e_n})] \end{aligned}$$

II. Eigenvalues and Eigenvectors

For *some* matrices $\mathbf{A}$, there are **nonzero** vectors $\mathbf{x}$ such that $\mathbf{Ax}$ is just a scalar multiple of $\mathbf{x}$. (We most frequently write this $\mathbf{Ax} = \lambda\mathbf{x}$.) The vectors $\mathbf{x}$ and the scalars λ for which this equation holds true are called **eigenvectors** and **eigenvalues**.

Consider the matrix $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ and the vectors $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$$\begin{aligned} \mathbf{Av} &= \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} (3 \times 2) + (-2 \times 1) \\ (1 \times 2) + (0 \times 1) \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 2 \end{bmatrix} \\ &= 2\mathbf{v} \end{aligned}$$

Because $\mathbf{Av} = \lambda\mathbf{v}$, the vector $\mathbf{v}$ is an eigenvector of the matrix $\mathbf{A}$. The corresponding eigenvalue is $\lambda = 2$. (We say that $\lambda = 2$ is an eigenvalue of $\mathbf{A}$.)

$$\begin{aligned} \mathbf{Au} &= \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} (3 \times -1) + (-2 \times 1) \\ (1 \times -1) + (0 \times 1) \end{bmatrix} \\ &= \begin{bmatrix} -5 \\ -1 \end{bmatrix} \end{aligned}$$

Because $\mathbf{Au} \neq \lambda\mathbf{u}$, the vector $\mathbf{u}$ is NOT an eigenvector of the matrix $\mathbf{A}$. There is no eigenvalue here.

Consider the equation $\mathbf{Ax} = \lambda\mathbf{x}$:

$$\begin{aligned} \mathbf{Ax} &= \lambda\mathbf{x} \\ \Rightarrow \mathbf{Ax} - \lambda\mathbf{x} &= \mathbf{0} \\ \Rightarrow \mathbf{Ax} - \lambda\mathbf{Ix} &= \mathbf{0} \\ \Rightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} &= \mathbf{0} \end{aligned}$$

There is a way where we can identify all eigenvectors $\lambda_1,\ldots,\lambda_k$ of our matrix $\mathbf{A}$.

A scalar λ is an eigenvalue of an $n \times n$ matrix $\mathbf{A}$ if and only if λ satisfies the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, where $\det(\cdot)$ indicates the determinant.

While we will usually leave calculating the determinant to a CAS (computer algebra system), there are two types of matrices we will explore calculating the determinant by hand:

- A 2×2 matrix of the form $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has determinant $\det(\mathbf{A}) = ad - bc$.
- A triangular matrix $\mathbf{A}$ will have determinant $\det(\mathbf{A}) = \prod_{i=1}^{n} a_{ii}$. (That is, the determinant will equal the product of the elements on the diagonal.)

Let's find all eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= 0 \\
\Rightarrow 0 &= \det(\mathbf{A} - \lambda\mathbf{I}) \\
&= \det\left(\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}\right) \\
&= (2-\lambda)(-6-\lambda) - (3)(3) \\
&= -12 + \lambda^2 + 4\lambda - 9 \\
&= \lambda^2 + 4\lambda - 21 \\
&= (\lambda+7)(\lambda-3) \\
\Rightarrow \lambda &= -7, +3
\end{aligned}$$

Thus, $\lambda = -7$ and $\lambda = 3$ are the two eigenvalues of $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

Now that we have the eigenvalues of $\mathbf{A}$, we can find their corresponding eigenvectors. Recall that we showed above that $\mathbf{Ax} = \lambda\mathbf{x}$ implies that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$. Let's find the eigenvector associated with $\lambda = -7$.

$$\begin{aligned}
\mathbf{0} &= (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} \\
\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \left(\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \left(\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 9x_1 + 3x_2 \\ 3x_1 + 1x_2 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} -3x_2 \\ -1x_2 \end{bmatrix} &= \begin{bmatrix} 9x_1 \\ 3x_1 \end{bmatrix} \\
\Rightarrow -3x_1 &= x_2 \\
\Rightarrow \mathbf{x} &= \begin{bmatrix} x_1 \\ -3x_1 \end{bmatrix}
\end{aligned}$$

Taking $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ and plugging -7 in for λ, we see that $\mathbf{x}^T = [x_1 \quad -3x_1]$ is an eigenvector for $\mathbf{A}$ that corresponds to the eigenvalue $\lambda = -7$. (We could plug in any value for x_1 here.)

Let's check to see if this actually works!

$$\begin{aligned}
\mathbf{A}\mathbf{x} &= \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ -3x_1 \end{bmatrix} \\
&= \begin{bmatrix} 2x_1 - 9x_1 \\ 3x_1 + 18x_1 \end{bmatrix} \\
&= \begin{bmatrix} -7x_1 \\ 21x_1 \end{bmatrix} \\
&= -7 \begin{bmatrix} x_1 \\ -3x_1 \end{bmatrix} \\
&= \lambda\mathbf{x}
\end{aligned}$$

This clearly does work.

On your own, find an eigenvector $\mathbf{x}$ associated with $\lambda = 3$, then show that your chosen $\mathbf{x}$ satisfies $\mathbf{A}\mathbf{x} = 3\mathbf{x}$.

III. Applications of Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors have many different applications. While we will go through a few in particular in later weeks, we mention some here:

- Eigenvectors and eigenvalues are used in the diagonalization of matrices, which has incredible computational benefits including applications in solving systems of linear equations and Monte Carlo simulations.
- Principal component analysis is a method by which potentially linearly dependent vectors can be linearly transformed and combined into vectors which are orthogonal to one another.
- Systems of differential equations, a set of equations relating functions to their derivatives, can be solved using eigenvectors and eigenvalues.
- When using matrices to describe the spread of infectious diseases, the basic reproductive number R_0 (indicating the average number of people that a standard infectious person will infect) corresponds to the largest eigenvalue of the "next generation matrix."
- For a matrix $\mathbf{A}$, its determinant can be calculated as $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$. (Note that if any of the eigenvalues are zero, then the determinant is zero, implying that $\mathbf{A}$ is not invertible.)