sponses have the *same* subscript, the rules of the symbolic method are no longer adequate.[1] First, it is seen that the subscripts and superscripts of the multiplying (and operational) cofactor cannot be the same. This is not serious in itself because a reexamination of Cederbaum's work reveals that while the subscripts of the cofactors can indeed be obtained from the subscripts of the old responses which remain as new responses, the *superscripts* of the cofactors are obtained from the subscripts of the old sources which remain as new sources. Unfortunately, even taking this into account does not permit application of the original symbolic combination rules. An attempt to partially invert the simple two-terminal-pair z matrix to the chain ($ABCD$) matrix is a convincing test.

The escape from the dilemma turns out to be almost anticlimatically simple.

Statement of the Extension

To arrange a system of linear equations for application of an all-inclusive symbolic method for partial inversion, *it is first necessary to distinguish the subscripts of the old sources from those of the old responses* by, say, primes. The second step is to form in ascending order the superscripts of the multiplying and operational cofactors from the subscripts (primed) of the *old sources* which remain as *new sources*. The subscripts of the cofactors are found as in the original Cederbaum method. The new responses and new sources can be arranged in any desired order. The index pairs are formed in exactly the same way as in (2) with due regard for the primes. The minus signs appear in the same kind of entries as in (2). An example of the extended partial inversion is given in (5).

$$F_{12}^{3'4'}\begin{bmatrix} y_1 \\ y_2 \\ x_{5'} \\ x_{2'} \\ x_{1'} \end{bmatrix} = \begin{bmatrix} -\begin{Bmatrix}1\\3\end{Bmatrix} & \begin{Bmatrix}1\\4'\end{Bmatrix} & \begin{Bmatrix}1\\3'\end{Bmatrix} & -\begin{Bmatrix}1\\5\end{Bmatrix} & -\begin{Bmatrix}1\\4\end{Bmatrix} & \\ -\begin{Bmatrix}2\\3\end{Bmatrix} & \begin{Bmatrix}2\\4'\end{Bmatrix} & \begin{Bmatrix}2\\3'\end{Bmatrix} & -\begin{Bmatrix}2\\5\end{Bmatrix} & -\begin{Bmatrix}2\\4\end{Bmatrix} & \\ \begin{Bmatrix}5'\\3\end{Bmatrix} & \begin{Bmatrix}5'\\4'\end{Bmatrix} & \begin{Bmatrix}5'\\3'\end{Bmatrix} & \begin{Bmatrix}5'\\5\end{Bmatrix} & \begin{Bmatrix}5'\\4\end{Bmatrix} & \{F_{12}^{3'4'}\} \\ \begin{Bmatrix}2'\\3\end{Bmatrix} & \begin{Bmatrix}2'\\4'\end{Bmatrix} & \begin{Bmatrix}2'\\3'\end{Bmatrix} & \begin{Bmatrix}2'\\5\end{Bmatrix} & \begin{Bmatrix}2'\\4\end{Bmatrix} & \\ \begin{Bmatrix}1'\\3\end{Bmatrix} & \begin{Bmatrix}1'\\4'\end{Bmatrix} & \begin{Bmatrix}1'\\3'\end{Bmatrix} & \begin{Bmatrix}1'\\5\end{Bmatrix} & \begin{Bmatrix}1'\\4\end{Bmatrix} & \end{bmatrix} \begin{bmatrix} y_3 \\ x_{4'} \\ x_{3'} \\ y_5 \\ y_4 \end{bmatrix} \tag{5}$$

Note that after the symbolic operation is complete, the primes may be discarded, if desired.

The proof of the validity of the extension is straightforward, but extremely long, and space does not permit its inclusion here. However, no new concepts are required since Cederbaum's original work[1] provides the basis for the proof. A more complete (unpublished) version of the extension which outlines the method of proof is available from the author upon request.

An Equivalent Circuit for the Salient-Pole Synchronous Machine

C. B. GRAY

***Abstract*—A time-phasor diagram is generally used to display amplitude and phase relationships between currents and voltages existing in an electrical network. However, although the simple equivalent circuit corresponding to steady-state operation of the round rotor synchronous machine and its associated phasor diagram are very familiar, the phasor diagram of the salient-pole machine is presented in many undergraduate texts without any attempt to specify the displayed quantities on a related circuit diagram.**

An equivalent circuit for the unsaturated salient-pole synchronous machine operated under balanced, steady-state conditions is derived from a consideration of the voltage and current equations implicit in the phasor diagram. Further equations are developed such that a knowledge of the terminal loading conditions of the machine, together with its armature resistance and synchronous reactances, enables the equivalent circuit parameters to be calculated. The unexcited synchronous reluctance machine is considered as a special case.

Manuscript received November 1, 1967.
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Introduction

It is well known that under balanced, steady-state conditions the unsaturated cylindrical rotor synchronous machine can be simply represented by an equivalent circuit comprising a voltage source $\boldsymbol{E}$, the EMF per phase induced by the field current, in series with synchronous impedance $\boldsymbol{Z_s}$, which accounts for the armature reaction effect and the leakage reactance and resistance per phase of the machine.

In the two-reaction theory of the salient-pole machine the armature MMF, assumed sinusoidally distributed, is resolved into two components in space-quadrature. The direct-axis component, algebraically added to the field MMF, acts on a magnetic circuit with assumed constant permeance of relatively high value; the quadrature-axis component acts on a magnetic circuit with assumed constant permeance of lower value. Alignment torque is developed by the interaction between the quadrature-axis component of the armature MMF and the field MMF. The direct-axis component of armature MMF influences the field excitation required for the production of the necessary resultant flux and interacts with the quadrature-axis component of armature MMF to produce reluctance torque.

The direct-axis armature reaction effect is accounted for by defining a quantity X_d, the direct-axis synchronous reactance per phase, which includes the leakage reactance of the armature phase winding. Similarly, X_q, the quadrature-axis synchronous reactance per phase, accounts for the quadrature-axis armature reaction effect and also includes leakage reactance.

The Phasor Diagram of the Salient-Pole Synchronous Machine

The phasor diagram of the salient-pole synchronous machine may be constructed using X_d and X_q and fictional components $\boldsymbol{I_d}$ and $\boldsymbol{I_q}$ of the armature current $\boldsymbol{I}$, defined respectively as the direct-axis and quadrature-axis components of armature current per phase. The terminal voltage per phase $\boldsymbol{V}$ is derived as the phasor sum of $\boldsymbol{E}$, the EMF induced by the field winding, $j\boldsymbol{I_d}X_d$, $j\boldsymbol{I_q}X_q$, and $\boldsymbol{I}r_a$, where r_a is the armature resistance per phase.

$\boldsymbol{I_d}$ and $\boldsymbol{I_q}$ represent the currents which would be required to flow in the reference phase in order to produce, in conjunction with equivalent currents flowing in the other phases, MMF's directed respectively along the direct-axis and quadrature-axis of the machine.

The phasor diagram of Fig. 1 corresponding to balanced, steady-state operation satisfies simultaneously the voltage equation

$$\boldsymbol{V} = \boldsymbol{E} + j\boldsymbol{I_d}X_d + j\boldsymbol{I_q}X_q + \boldsymbol{I}r_a$$

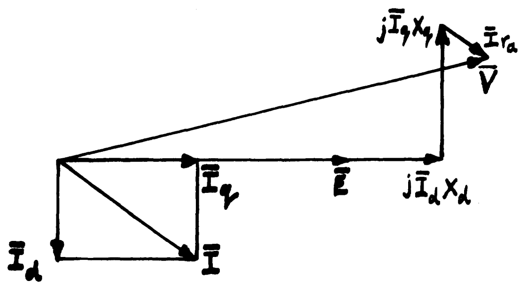

Fig. 1. Phasor diagram of salient-pole synchronous machine.

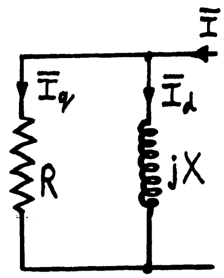

Fig. 2. Parallel combination of resistance and reactance.

and the current equation

$$I = I_d + I_q.$$

According to the mode of operation of the machine, I_q is either in phase with or in antiphase to E, corresponding to whether the machine develops a motoring or a generating torque, whereas I_d either lags or leads E by 90°.

Considering the machine to operate as a synchronous motor at lagging power factor, i.e., absorbing watts and vars from the system to which it is connected, the circuit of Fig. 2 satisfies the necessary relationship between I, I_d, and I_q in Fig. 1.

For motoring action at lagging power factor, R and X will be generally positive quantities. However, the four possible modes of operation of the synchronous machine are accounted for by the four possible combinations of resistance and reactance making no restriction as to sign.

Equivalent Circuit of the Salient-Pole Synchronous Machine

If R and X are chosen so that the flow of current I develops the voltage E across the parallel combination of R and X, then the voltage and current equations above are simultaneously satisfied by the equivalent circuit of Fig. 3.

The induced voltages jI_dX_d and jI_qX_q are proportional to the currents I_d and I_q existing elsewhere in the network of Fig. 3, and hence X_d and X_q have the nature of mutual reactance. These mutual reactances are nonreciprocal, however, and in consequence the equivalent circuit is not directly physically realizable. Nevertheless, it is possible to find particular values of R and X for specified terminal loading conditions, given the machine parameters X_d, X_q, and r_a.

The voltage equations for the two-mesh circuit of Fig. 3 may be written as

$$\begin{bmatrix} V - jI_qX_q - jI_dX_d \\ 0 \end{bmatrix} = \begin{bmatrix} r_a + R & -R \\ -R & R + jX \end{bmatrix} \begin{bmatrix} I_d + I_q \\ I_d \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} V \\ 0 \end{bmatrix} = \begin{bmatrix} r_a + R + jX_q & r_a + jX_d \\ -R & jX \end{bmatrix} \begin{bmatrix} I_q \\ I_d \end{bmatrix}.$$

It may be shown that, regardless of the sign of R and X,

$$\frac{I_q}{I} = \frac{X(X + jR)}{X^2 + R^2}$$

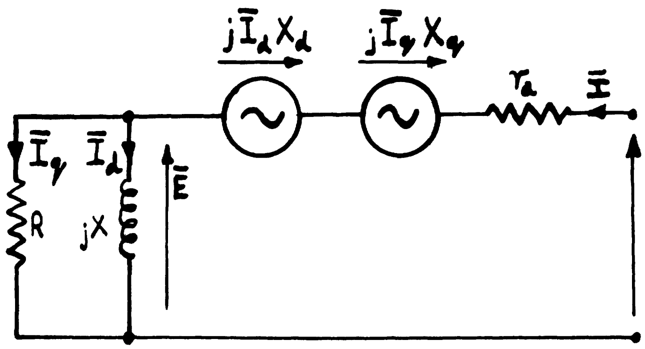

Fig. 3. Equivalent circuit of salient-pole synchronous machine.

and

$$\frac{I_d}{I} = \frac{R(R - jX)}{X^2 + R^2}$$

provided X and R are not both zero. Thus,

$$V = \left[(r_a + R + jX_q) \frac{X(X + jR)}{(X^2 + R^2)} + (r_a + jX_d) \frac{R(R - jX)}{(X^2 + R^2)} \right] I$$

giving

$$V = \frac{1}{R^2 + X^2} \left[(r_a + R)X^2 + r_aR^2 + (X_d - X_q)XR \right.$$
$$\left. + j(X_qX^2 + R^2(X + X_d) \right] I.$$

Multiplying both sides of the above equation by the conjugate of the current I yields an expression for the complex input power at the machine terminals in terms of the parameters of the equivalent circuit and the terminal current. Thus,

$$\text{input real power} = P = \frac{I^2}{R^2 + X^2} \left[r_a(X^2 + R^2) + RX(X + X_d - X_q) \right] \quad (1)$$

$$\text{input reactive voltamperes} = Q = \frac{I^2}{R^2 + X^2} \left[R^2(X_d + X) + X^2X_q \right]. \quad (2)$$

An alternative approach, evaluating the complex input power directly as

$$S = P + jQ = I^2r_a + [E + jI_dX_d + jI_qX_q][I_d + I_q]^*$$

indicates the alignment and reluctance components of torque and yields (1) and (2) after suitable substitution for E.

Evaluation of Circuit Parameters R and X

Solution of (1) and (2) enables the equivalent circuit parameters R and X to be determined for a machine with known values of X_d, X_q, and r_a, operating under specified load conditions.

It is convenient to define two quantities:

$$A = \frac{P}{I^2} - r_a$$

$$B = \frac{Q}{I^2}.$$

Thus (1) and (2) become

$$A(R^2 + X^2) = RX(X + X_d - X_q) \quad (3)$$

$$B(R^2 + X^2) = R^2(X + X_d) + X^2X_q. \quad (4)$$

Investigation of the possible roots of (3) and (4), as shown in the Appendix, yields as the only nonzero solutions

$$X = B - X_d + \frac{A^2}{B - X_q}, \qquad R = \frac{(B - X_d)(B - X_q)}{A} + A.$$

Hence the parameters of the equivalent circuit may be determined for any condition of load.

The Unexcited Machine

Equations (3) and (4) are also apparently satisfied by $X=0$, $R=0$, but for this condition (3) and (4) are invalid. If either X or R or both equal zero, $\boldsymbol{E}=0$ and consequently the machine must be operating with unexcited field. It becomes necessary, therefore, to investigate the simplified equivalent circuit of Fig. 4 satisfying simultaneously the voltage equation

$$\boldsymbol{V} = j\boldsymbol{I}_d X_d + j\boldsymbol{I}_q X_q + \boldsymbol{I} r_a$$

and the current equation

$$\boldsymbol{I} = \boldsymbol{I}_d + \boldsymbol{I}_q.$$

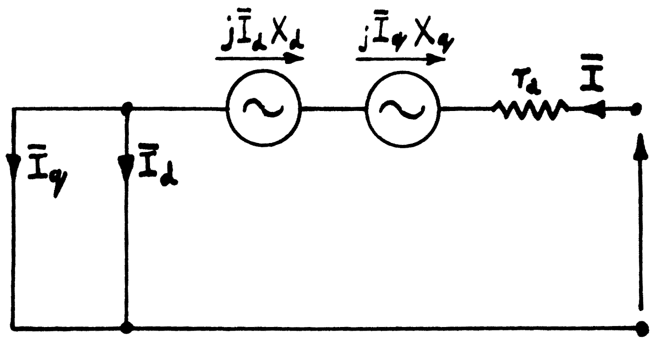

Fig. 4. Equivalent circuit of unexcited salient-pole synchronous machine.

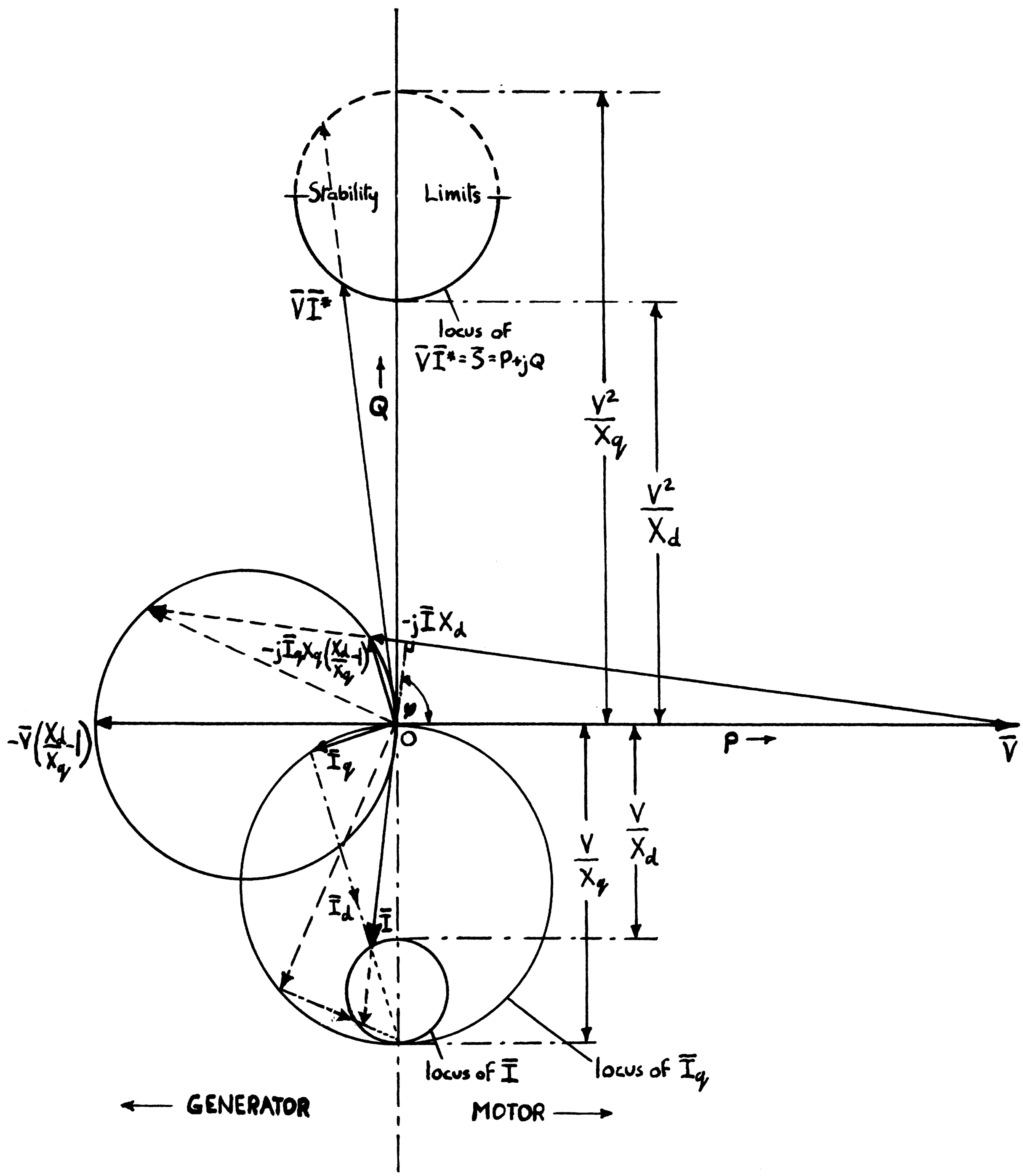

Fig. 5. Phasor diagram and power chart at constant terminal voltage for unexcited salient-pole synchronous machine with armature resistance neglected. Generating condition illustrated with lagging power factor cos φ.

With the same terminology as above, the following relationships may be established for the unexcited machine:

$$P - I^2 r_a = I_d I_q (X_d - X_q) \tag{5}$$

$$Q = I_d^2 X_d + I_q^2 X_q \tag{6}$$

in which a positive value of I_d corresponds to $\boldsymbol{I_d}$ lagging $\boldsymbol{I_q}$ by 90°, and $\boldsymbol{I_q}$ is taken as the reference phasor so that I_q has positive values only.

Equation (6) requires that Q must have a positive value, confirming the inability of the unexcited machine to supply reactive voltamperes to the system to which it is connected. Equation (5) justifies the ability of the unexcited machine to develop torque corresponding to motor or generator action with respectively a positive or negative sign for I_d.

If armature resistance is neglected, (5) and (6) may be readily solved for I_q and I_d in terms of the machine parameters and terminal loading conditions, giving

$$I_q^2 = \frac{1}{2X_q}\left[Q \pm \sqrt{Q^2 - 4X_d X_q\left(\frac{P}{X_d - X_q}\right)^2}\right]$$

$$I_d = \frac{P}{I_q(X_d - X_q)}.$$

Since I_q^2 and consequently the expression

$$\left[Q \pm \sqrt{Q^2 - 4X_d X_q\left(\frac{P}{X_d - X_q}\right)^2}\right]$$

must have positive values, Q must be positive and

$$Q^2 \geq 4X_d X_q\left(\frac{P}{X_d - X_q}\right)^2$$

to give generally two values of I_q and I_d for each possible condition of load; unless

$$Q^2 = 4X_d X_q\left(\frac{P}{X_d - X_q}\right)^2$$

corresponding to the maximum load power factor condition for which

$$I_q = \sqrt{\frac{Q}{2X_q}}, \qquad I_d = \sqrt{\frac{Q}{2X_d}}.$$

For the special case of $P=0$, two values of $\boldsymbol{I}$ exist, corresponding respectively to

$$1)\ I_q = 0, \qquad I_i = I_d = \sqrt{\frac{Q}{X_d}} = \frac{V_i}{X_d}$$

$$2)\ I_d = 0, \qquad I_{ii} = I_q = \sqrt{\frac{Q}{X_q}} = \frac{V_{ii}}{X_q}.$$

It is the case, therefore, that if acceptable values of P and Q are specified at the terminals of the unexcited machine, generally two combinations of terminal voltage and current are theoretically possible. This result may be deduced from Fig. 5 which illustrates the loci of the phasors $j\boldsymbol{I}X_d$, $\boldsymbol{I_q}$, and $\boldsymbol{I}$ together with terminal power for the unexcited salient-pole machine with negligible armature resistance, a given terminal voltage, and varying conditions of load.[1] This diagram indicates directly that, for given values of $\boldsymbol{V}$ and terminal power factor (less than the maximum possible value), two values of $\boldsymbol{I}$ and hence P and Q exist. It follows that if P and Q are single-valued quantities, two values of $\boldsymbol{I}$ and its components $\boldsymbol{I_d}$ and $\boldsymbol{I_q}$ are possible, together with two corresponding values of $\boldsymbol{V}$.

Conclusions

While the practical value of the equivalent circuit derived here is limited, the circuit does serve to complement the well-known phasor diagram. As an example of its use, the terminal power/load angle relationship may be determined in a straightforward manner with little complexity resulting from the retention of the armature resistance parameter, whereas this quantity is usually neglected for convenience in textbook derivations from the phasor diagram.

Appendix

Investigation of Possible Roots of Equations (3) and (4)

$$A(R^2 + X^2) = RX(X + X_d - X_q) \tag{3}$$

$$B(R^2 + X^2) = R^2(X + X_d) + X^2 X_q \tag{4}$$

From (4),

$$R = \pm X\sqrt{\frac{(X_q - B)}{(B - X_d - X)}}. \tag{7}$$

Substituting (7) into (3) gives

$$X^2 A\left[\frac{X_q - X_d - X}{B - X_d - X}\right] = \pm X^2(X_q - X_d - X)\sqrt{\frac{(X_q - B)}{(B - X_d - X)}}. \tag{8}$$

$X^2=0$ is a solution of (8) giving $X=0$, $R=0$, as possible roots. $X_q - X_d - X = 0$ is a further solution giving $X = X_q - X_d$, and from (7)

$$R^2 = \frac{(X_q - X_d)^2(X_q - B)}{(B - X_q)} = -(X_q - X_d)^2.$$

This solution is not possible for the equations as formulated. Equation (8) therefore becomes

$$\frac{A}{B - X_d - X} = \pm\sqrt{\frac{(X_q - B)}{(B - X_d - X)}}. \tag{9}$$

Hence,

$$A = \pm\sqrt{(X_q - B)(B - X_d - X)}$$

$$\frac{A^2}{X_q - B} = B - X_d - X$$

giving

$$X = B - X_d + \frac{A^2}{B - X_q}. \tag{10}$$

Substituting (10) into (7),

$$R = \pm\left(B - X_d + \frac{A^2}{B - X_q}\right)\cdot\frac{(X_q - B)}{A}$$

$$= \pm\left[\frac{(B - X_d)(B - X_q)}{A} + A\right]. \tag{11}$$

Substitution in (3) of the alternative expressions for R given in (11) shows that the only valid solution for R is

$$R = \frac{(B - X_d)(B - X_q)}{A} + A.$$

[1] J. H. Walker, "Operating characteristics of salient-pole machines," *Proc. IEE* (London), vol. 100, pt. 2, pp. 13–24, 1953.

An Algorithm for the Lowpass to Bandpass Transformation

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***Abstract*—An algorithm is described which is used to determine the coefficients of the denominator polynomial of a bandpass network function. The polynomial is related to the denominator polynomial of a lowpass function by the lowpass to bandpass transformation. The algorithm is readily implemented on the digital computer.**

Manuscript received August 28, 1967. This research was supported in part by NASA under Grant NGR-03-002-136.

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