量 子 场 论

第 8 章 分立对称性和 Majorana 旋量场

8.1 节至 8.3 节

余钊焕

中山大学物理学院

https://yzhxxzxy.github.io

更新日期：2022 年 10 月 11 日

第 8 章 分立对称性和 Majorana 旋量场

相对于**连续变换**，不能用连续变化的参数描述的对称变换称为**分立** (discrete) **变换**，也称为**离散变换**

如果系统作用量在一种分立变换下不变，则系统具有相应的**分立对称性** (discrete symmetry)

1.3 节末提到的**宇称变换**和**时间反演变换**是两种重要的分立变换

本章讨论量子场的分立变换和分立对称性，顺便介绍 **Majorana 旋量**

8.1 节 标量场的分立变换 8.1.1 小节 标量场的 P 变换

- 1.3 节提到，时空坐标 $x^\mu = (t, \mathbf{x})$ 的**宇称变换**为

$$\mathcal{P}^\mu{}_\nu = (\mathcal{P}^{-1})^\mu{}_\nu = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

- 这是一种**非固有保时向** Lorentz 变换
- 将 x^μ 变换为 $x'^\mu = \mathcal{P}^\mu{}_\nu x^\nu = (\mathcal{P}x)^\mu = (t, -\mathbf{x})$，将 ∂_μ 变换为 $\partial'_\mu = (\mathcal{P}^{-1})^\nu{}_\mu \partial_\nu$
- 将四维动量 $p^\mu = (E, \mathbf{p})$ 变换为 $p'^\mu = (\mathcal{P}p)^\mu = (E, -\mathbf{p})$
- 但同时**保持时空体积元不变**，$\mathrm{d}^4x' = |\det(\mathcal{P})|\,\mathrm{d}^4x = \mathrm{d}^4x$

8.1 节 标量场的分立变换 8.1.1 小节 标量场的 P 变换

☀️ 1.3 节提到，时空坐标 $x^\mu = (t, \mathbf{x})$ 的**宇称变换**为

$$\mathcal{P}^\mu{}_\nu = (\mathcal{P}^{-1})^\mu{}_\nu = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

🏚️ 这是一种**非固有保时向** Lorentz 变换

🪨 将 x^μ 变换为 $x'^\mu = \mathcal{P}^\mu{}_\nu x^\nu = (\mathcal{P}x)^\mu = (t, -\mathbf{x})$，将 ∂_μ 变换为 $\partial'_\mu = (\mathcal{P}^{-1})^\nu{}_\mu \partial_\nu$

🪵 将四维动量 $p^\mu = (E, \mathbf{p})$ 变换为 $p'^\mu = (\mathcal{P}p)^\mu = (E, -\mathbf{p})$

🧱 但同时**保持时空体积元不变**，$\mathrm{d}^4x' = |\det(\mathcal{P})|\,\mathrm{d}^4x = \mathrm{d}^4x$

🏗️ 如果系统的**作用量** $S = \int \mathrm{d}^4x\,\mathcal{L}(x)$ 在宇称变换下**不变**，则**运动方程**的形式也在宇称变换下**不变**，此时称系统是**宇称守恒**的，即具有**空间反射对称性**

🏘️ 在宇称守恒的理论中，拉氏量 $\mathcal{L}(x)$ 的宇称变换应当满足 $\mathcal{L}'(x') = \mathcal{L}(x)$，从而使得 $S' = \int \mathrm{d}^4x'\,\mathcal{L}'(x') = \int \mathrm{d}^4x\,\mathcal{L}(x) = S$

P 变换

☁️ 类似于 3.1.1 小节的讨论，在**宇称守恒**的**量子理论**中，**宇称变换**在 Hilbert 空间中诱导出态矢 $|\Psi\rangle$ 的线性幺正变换 $|\Psi'\rangle = U(\mathcal{P})|\Psi\rangle = P|\Psi\rangle$，称为 ***P* 变换**

🛖 $P \equiv U(\mathcal{P})$ 是一个**线性幺正算符**，根据**同态关系**和 $\mathcal{P}^{-1} = \mathcal{P}$，有

$$P^{-1} = U^{-1}(\mathcal{P}) = U(\mathcal{P}^{-1}) = U(\mathcal{P}) = P$$

🏛️ 即 P 是**自身的逆变换算符**

🏠 从而 $P^2 = P^{-1}P = 1$，故相继两次 P 变换的作用等同于恒等变换

🏡 由 P 的**幺正性**得 $P^\dagger = P^{-1} = P$，因而 P 是**厄米**算符

P 变换

☁️ 类似于 3.1.1 小节的讨论，在宇称守恒的量子理论中，宇称变换在 Hilbert 空间中诱导出态矢 $|\Psi\rangle$ 的线性幺正变换 $|\Psi'\rangle = U(\mathcal{P})|\Psi\rangle = P|\Psi\rangle$，称为 **$P$ 变换**

🛖 $P \equiv U(\mathcal{P})$ 是一个线性幺正算符，根据同态关系和 $\mathcal{P}^{-1} = \mathcal{P}$，有

$$P^{-1} = U^{-1}(\mathcal{P}) = U(\mathcal{P}^{-1}) = U(\mathcal{P}) = P$$

🏛️ 即 P 是自身的逆变换算符

🏠 从而 $P^2 = P^{-1}P = 1$，故相继两次 P 变换的作用等同于恒等变换

🏡 由 P 的幺正性得 $P^\dagger = P^{-1} = P$，因而 P 是厄米算符

🏢 由量子场构成的拉氏量在宇称变换下不变意味着 $\mathcal{L}'(x') = P^{-1}\mathcal{L}(x')P = \mathcal{L}(x)$

🏥 作替换 $x' \to x$，$x \to \mathcal{P}^{-1}x = \mathcal{P}x$，将这个式子等价地写成

$$P^{-1}\mathcal{L}(x)P = +\mathcal{L}(\mathcal{P}x)$$

🏦 使拉氏量满足上式，也就是说，让拉氏量具有偶宇称（指上式右边的 $+$ 号），就能得到宇称守恒的量子场论

Poincaré 生成元算符的 P 变换

现在讨论量子 Poincaré 变换 $U(\Lambda, a)$ 的**生成元算符** $J^{\mu\nu}$ 和 P^{μ} 在**宇称守恒**理论中的 P 变换

将**同态关系** $U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)$ 用于宇称变换 $\mathcal{P}$，得

$$P^{-1}U(\Lambda, a)P = U^{-1}(\mathcal{P})U(\Lambda, a)U(\mathcal{P}) = U^{-1}(\mathcal{P})U(\Lambda\mathcal{P}, a) = U(\mathcal{P}^{-1}\Lambda\mathcal{P}, \mathcal{P}^{-1}a)$$

相应的**无穷小变换**为 $P^{-1}U(\mathbf{1}+\omega, \varepsilon)P = U(\mathbf{1}+\mathcal{P}^{-1}\omega\mathcal{P}, \mathcal{P}^{-1}\varepsilon)$，类似于 3.1.1 小节得到的 $U^{-1}(\Lambda, a)U(\mathbf{1}+\omega, \varepsilon)U(\Lambda, a) = U(\mathbf{1}+\Lambda^{-1}\omega\Lambda, \Lambda^{-1}\omega a + \Lambda^{-1}\varepsilon)$

Poincaré 生成元算符的 P 变换

现在讨论量子 Poincaré 变换 $U(\Lambda, a)$ 的**生成元算符** $J^{\mu\nu}$ 和 P^μ 在**宇称守恒**理论中的 P 变换

将**同态关系** $U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)$ 用于宇称变换 $\mathcal{P}$，得

$$P^{-1}U(\Lambda, a)P = U^{-1}(\mathcal{P})U(\Lambda, a)U(\mathcal{P}) = U^{-1}(\mathcal{P})U(\Lambda\mathcal{P}, a) = U(\mathcal{P}^{-1}\Lambda\mathcal{P}, \mathcal{P}^{-1}a)$$

相应的**无穷小变换**为 $P^{-1}U(\mathbf{1}+\omega, \varepsilon)P = U(\mathbf{1}+\mathcal{P}^{-1}\omega\mathcal{P}, \mathcal{P}^{-1}\varepsilon)$，类似于 3.1.1 小节得到的 $U^{-1}(\Lambda, a)U(\mathbf{1}+\omega, \varepsilon)U(\Lambda, a) = U(\mathbf{1}+\Lambda^{-1}\omega\Lambda, \Lambda^{-1}\omega a + \Lambda^{-1}\varepsilon)$

由此推出类似于 $U^{-1}(\Lambda)J^{\mu\nu}U(\Lambda) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma J^{\rho\sigma}$ 和 $U^{-1}(\Lambda)P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu$ 的 **P 变换规则**

$$P^{-1}J^{\mu\nu}P = \mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma J^{\rho\sigma}, \quad P^{-1}P^\mu P = \mathcal{P}^\mu{}_\nu P^\nu$$

从而，$J^i \equiv \varepsilon^{ijk}J^{jk}/2$ 和 $K^i \equiv J^{0i}$ 在 P 变换下变成

$$P^{-1}J^iP = \frac{1}{2}\varepsilon^{ijk}P^{-1}J^{jk}P = \frac{1}{2}\varepsilon^{ijk}\mathcal{P}^j{}_l\mathcal{P}^k{}_m J^{lm} = \frac{1}{2}\varepsilon^{ijk}(-\delta^j{}_l)(-\delta^k{}_m)J^{lm} = +J^i$$

$$P^{-1}K^iP = P^{-1}J^{0i}P = \mathcal{P}^0{}_0\mathcal{P}^i{}_j J^{0j} = -\delta^i{}_j J^{0j} = -K^i$$

总角动量、动量和哈密顿量的 P 变换

- 写成三维矢量的形式，得

$$P^{-1}\mathbf{J}P = +\mathbf{J}, \quad P^{-1}\mathbf{K}P = -\mathbf{K}$$

- **总角动量算符 $\mathbf{J}$** 具有**偶宇称**，体现了它作为**三维轴矢量**的变换性质
- 因此**角动量**在宇称变换下**不变**

总角动量、动量和哈密顿量的 P 变换

- 写成三维矢量的形式，得

$$P^{-1}\mathbf{J}P = +\mathbf{J}, \quad P^{-1}\mathbf{K}P = -\mathbf{K}$$

- **总角动量算符 $\mathbf{J}$** 具有**偶宇称**，体现了它作为**三维轴矢量**的变换性质
- 因此**角动量**在宇称变换下**不变**
- 将四维动量算符分解为 $P^\mu = (H, \mathbf{P})$，则 $P^{-1}P^\mu P = \mathcal{P}^\mu{}_\nu P^\nu$ 分解成

$$P^{-1}HP = +H, \quad P^{-1}\mathbf{P}P = -\mathbf{P}$$

- **动量算符 $\mathbf{P}$** 具有**奇宇称**，体现了它作为**三维极矢量**的变换性质
- 因而**动量方向**在宇称变换下**反转**
- **哈密顿量算符 H** 的宇称为**偶**，而 $P^{-1}HP = +H$ 等价于 $[H, P] = 0$
- 这在量子理论中意味着**宇称算符 P** 是一个**守恒量**

宇称守恒和宇称破坏

- 通常希望一个物理理论包含比较多的对称性，从而具有比较优良的性质
- 为了得到一个**宇称守恒**的量子场论，首先，需要让拉氏量中的**自由部分**在宇称变换下不变，这是容易做到的
- 其次，还得要求拉氏量中的**相互作用部分**也在宇称变换下不变，这一点**并不平庸**
- 尽管宇称在**电磁相互作用**和**强相互作用**中**守恒**，在**弱相互作用**中却遭到**极大破坏**

宇称守恒和宇称破坏

- 通常希望一个物理理论包含比较多的对称性，从而具有比较优良的性质
- 为了得到一个**宇称守恒**的量子场论，首先，需要让拉氏量中的**自由部分**在宇称变换下不变，这是容易做到的
- 其次，还得要求拉氏量中的**相互作用部分**也在宇称变换下不变，这一点**并不平庸**
- 尽管宇称在**电磁相互作用**和**强相互作用**中**守恒**，在**弱相互作用**中却遭到**极大破坏**
- 这个出乎意料的**现象**是 1956 年**李政道**和**杨振宁**对实验结果作理论分析时发现的
- 随后由**吴健雄**在实验中确证
- 李政道和杨振宁因此获得 1957 年诺贝尔物理学奖
- 为了分析量子场论中的宇称守恒情况，我们需要知道各种**量子场**在 P **变换**下的性质

李政道 (1926–)

杨振宁 (1922–)

吴健雄 (1912–1997)

复标量场产生湮灭算符的 P 变换

- 下面先讨论自由的复标量场 $\phi(x)$，拉氏量为 $\mathcal{L} = (\partial^\mu \phi^\dagger)\partial_\mu \phi - m^2 \phi^\dagger \phi$
- $\phi(x)$ 的运动方程是 Klein-Gordon 方程 $(\partial^2 + m^2)\phi(x) = 0$
- 宇称变换翻转动量方向，P 变换对正标量玻色子 ϕ 单粒子态 $\left|\mathbf{p}^+\right\rangle = \sqrt{2E_\mathbf{p}}\, a_\mathbf{p}^\dagger \left|0\right\rangle$ 的作用为 $P\left|\mathbf{p}^+\right\rangle = \eta_P \left|-\mathbf{p}^+\right\rangle$
- η_P 是一个复的相位因子，满足 $|\eta_P| = 1$
- 出现 η_P 的原因是相差一个相位因子的态矢描述相同的物理

复标量场产生湮灭算符的 P 变换

- 下面先讨论**自由**的**复标量场** $\phi(x)$，拉氏量为 $\mathcal{L} = (\partial^\mu \phi^\dagger)\partial_\mu \phi - m^2 \phi^\dagger \phi$
- $\phi(x)$ 的运动方程是 Klein-Gordon 方程 $(\partial^2 + m^2)\phi(x) = 0$
- 宇称变换**翻转**动量方向，P 变换对正标量玻色子 ϕ 单粒子态 $\left|\mathbf{p}^+\right\rangle = \sqrt{2E_\mathbf{p}}\, a_\mathbf{p}^\dagger \left|0\right\rangle$ 的作用为 $P\left|\mathbf{p}^+\right\rangle = \eta_P \left|-\mathbf{p}^+\right\rangle$
- η_P 是一个**复**的**相位因子**，满足 $|\eta_P| = 1$
- 出现 η_P 的原因是**相差一个相位因子**的态矢描述**相同**的物理
- 真空态在 P 变换下不变，故 $P\left|0\right\rangle = \left|0\right\rangle$，从而

$$P\left|\mathbf{p}^+\right\rangle = P^{-1}\left|\mathbf{p}^+\right\rangle = \sqrt{2E_\mathbf{p}}\, P^{-1} a_\mathbf{p}^\dagger P \left|0\right\rangle$$

- 与 $P\left|\mathbf{p}^+\right\rangle = \eta_P \left|-\mathbf{p}^+\right\rangle = \sqrt{2E_\mathbf{p}}\, \eta_P a_{-\mathbf{p}}^\dagger \left|0\right\rangle$ 比较，推出**产生湮灭算符的 P 变换**

$$P^{-1} a_\mathbf{p}^\dagger P = \eta_P a_{-\mathbf{p}}^\dagger, \quad P^{-1} a_\mathbf{p} P = \eta_P^* a_{-\mathbf{p}}$$

- 第二式由第一式取厄米共轭得到：$(P^{-1} a_\mathbf{p}^\dagger P)^\dagger = (P^\dagger a_\mathbf{p}^\dagger P)^\dagger = P^\dagger a_\mathbf{p} P = P^{-1} a_\mathbf{p} P$

复标量场平面波展开式的 P 变换

另一方面，P 变换对反标量玻色子 $\bar{\phi}$ 的单粒子态 $\left|\mathbf{p}^-\right\rangle = \sqrt{2E_\mathbf{p}}\, b_\mathbf{p}^\dagger \left|0\right\rangle$ 的作用为 $P\left|\mathbf{p}^-\right\rangle = \tilde{\eta}_P \left|-\mathbf{p}^-\right\rangle$，出现**另一个相位因子** $\tilde{\eta}_P$，同理推出

$$P^{-1} b_\mathbf{p}^\dagger P = \tilde{\eta}_P b_{-\mathbf{p}}^\dagger, \quad P^{-1} b_\mathbf{p} P = \tilde{\eta}_P^* b_{-\mathbf{p}}$$

复标量场 $\phi(x)$ 的**平面波展开式**在 P 变换下化为

$$\begin{aligned}
P^{-1}\phi(x)P &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \left(P^{-1} a_\mathbf{p} P \mathrm{e}^{-\mathrm{i}p\cdot x} + P^{-1} b_\mathbf{p}^\dagger P \mathrm{e}^{\mathrm{i}p\cdot x} \right) \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \left(\eta_P^* a_{-\mathbf{p}} \mathrm{e}^{-\mathrm{i}p\cdot x} + \tilde{\eta}_P b_{-\mathbf{p}}^\dagger \mathrm{e}^{\mathrm{i}p\cdot x} \right) \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \left[\eta_P^* a_\mathbf{p} \mathrm{e}^{-\mathrm{i}p\cdot(\mathcal{P}x)} + \tilde{\eta}_P b_\mathbf{p}^\dagger \mathrm{e}^{\mathrm{i}p\cdot(\mathcal{P}x)} \right]
\end{aligned}$$

最后一步作了**变量替换** $\mathbf{p} \to -\mathbf{p}$，并利用 $(\mathcal{P}p)\cdot x = p\cdot(\mathcal{P}x)$

复标量场的 P 变换

- 为了**保持 $\phi(x)$ 的运动方程**形式不变，必须要求 η_P 和 $\tilde{\eta}_P$ 满足关系式

$$\eta_P^* = \tilde{\eta}_P$$

- 使得 $P^{-1}\phi(x)P = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left[\eta_P^* a_{\mathbf{p}} \mathrm{e}^{-\mathrm{i}p\cdot(\mathcal{P}x)} + \tilde{\eta}_P b_{\mathbf{p}}^\dagger \mathrm{e}^{\mathrm{i}p\cdot(\mathcal{P}x)}\right] = \eta_P^* \phi(\mathcal{P}x)$

- 故 $\phi(x)$ 和 $\phi^\dagger(x)$ 的 **P 变换**为

$$P^{-1}\phi(x)P = \eta_P^* \phi(\mathcal{P}x), \quad P^{-1}\phi^\dagger(x)P = \eta_P \phi^\dagger(\mathcal{P}x)$$

- 即 $P^{-1}\phi(x)P$ 与 $\phi(\mathcal{P}x)$ **只相差一个常数因子** η_P^*

复标量场的 P 变换

为了**保持 $\phi(x)$ 的运动方程**形式不变，必须要求 η_P 和 $\tilde{\eta}_P$ 满足关系式

$$\eta_P^* = \tilde{\eta}_P$$

使得 $P^{-1}\phi(x)P = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left[\eta_P^* a_{\mathbf{p}}\mathrm{e}^{-\mathrm{i}p\cdot(\mathcal{P}x)} + \tilde{\eta}_P b_{\mathbf{p}}^\dagger \mathrm{e}^{\mathrm{i}p\cdot(\mathcal{P}x)}\right] = \eta_P^*\phi(\mathcal{P}x)$

故 $\phi(x)$ 和 $\phi^\dagger(x)$ 的 **P 变换**为

$$P^{-1}\phi(x)P = \eta_P^*\phi(\mathcal{P}x), \quad P^{-1}\phi^\dagger(x)P = \eta_P\phi^\dagger(\mathcal{P}x)$$

即 $P^{-1}\phi(x)P$ 与 $\phi(\mathcal{P}x)$ **只相差一个常数因子** η_P^*

从而，作宇称变换之后的场 $\phi'(x') = P^{-1}\phi(x')P = \eta_P^*\phi(\mathcal{P}x') = \eta_P^*\phi(x)$ 也满足 **Klein-Gordon 方程**：$(\partial'^2 + m^2)\phi'(x') = \eta_P^*(\partial^2 + m^2)\phi(x) = 0$

这样的话，$b_{\mathbf{p}}^\dagger$ 和 $b_{\mathbf{p}}$ 的 P 变换 $P^{-1}b_{\mathbf{p}}^\dagger P = \tilde{\eta}_P b_{-\mathbf{p}}^\dagger$ 和 $P^{-1}b_{\mathbf{p}}P = \tilde{\eta}_P^* b_{-\mathbf{p}}$ 变成

$$P^{-1}b_{\mathbf{p}}^\dagger P = \eta_P^* b_{-\mathbf{p}}^\dagger, \quad P^{-1}b_{\mathbf{p}}P = \eta_P b_{-\mathbf{p}}$$

拉氏量的 P 变换

- 现在讨论自由复标量场的**拉氏量** $\mathcal{L} = (\partial^\mu \phi^\dagger)\partial_\mu \phi - m^2 \phi^\dagger \phi$ 的宇称变换性质
- 这会涉及时空导数的宇称变换 $\partial'_\mu = (\mathcal{P}^{-1})^\nu{}_\mu \partial_\nu$
- 为便于表述，我们在**形式上**以**时空导数**的**宇称变换**定义 $\boldsymbol{P}$ **变换**，

$$P^{-1}\partial_\mu P \equiv \partial'_\mu = (\mathcal{P}^{-1})^\nu{}_\mu \partial_\nu, \quad P^{-1}\partial^\mu P = \partial'^\mu = \mathcal{P}^\mu{}_\nu \partial^\nu$$

- 于是，拉氏量中**动能项算符** $(\partial^\mu \phi^\dagger)\partial_\mu \phi$ 的 P 变换为

$$\begin{aligned} P^{-1}\partial^\mu \phi^\dagger(x)\partial_\mu \phi(x) P &= P^{-1}\partial^\mu P P^{-1}\phi^\dagger(x) P P^{-1}\partial_\mu P P^{-1}\phi(x) P \\ &= |\eta_P|^2 \mathcal{P}^\mu{}_\nu (\mathcal{P}^{-1})^\rho{}_\mu \partial^\nu \phi^\dagger(\mathcal{P}x)\partial_\rho \phi(\mathcal{P}x) = +\partial^\mu \phi^\dagger(\mathcal{P}x)\partial_\mu \phi(\mathcal{P}x) \end{aligned}$$

拉氏量的 P 变换

- 现在讨论自由复标量场的**拉氏量** $\mathcal{L} = (\partial^\mu \phi^\dagger)\partial_\mu \phi - m^2 \phi^\dagger \phi$ 的宇称变换性质
- 这会涉及时空导数的宇称变换 $\partial'_\mu = (\mathcal{P}^{-1})^\nu{}_\mu \partial_\nu$
- 为便于表述，我们在**形式上**以**时空导数**的**宇称变换**定义 $\boldsymbol{P}$ **变换**，

$$P^{-1}\partial_\mu P \equiv \partial'_\mu = (\mathcal{P}^{-1})^\nu{}_\mu \partial_\nu, \quad P^{-1}\partial^\mu P = \partial'^\mu = \mathcal{P}^\mu{}_\nu \partial^\nu$$

- 于是，拉氏量中**动能项算符** $(\partial^\mu \phi^\dagger)\partial_\mu \phi$ 的 P 变换为

$$\begin{aligned} P^{-1}\partial^\mu \phi^\dagger(x)\partial_\mu \phi(x) P &= P^{-1}\partial^\mu P P^{-1}\phi^\dagger(x) P P^{-1}\partial_\mu P P^{-1}\phi(x) P \\ &= |\eta_P|^2 \mathcal{P}^\mu{}_\nu (\mathcal{P}^{-1})^\rho{}_\mu \partial^\nu \phi^\dagger(\mathcal{P}x)\partial_\rho \phi(\mathcal{P}x) = +\partial^\mu \phi^\dagger(\mathcal{P}x)\partial_\mu \phi(\mathcal{P}x) \end{aligned}$$

- **质量项算符** $\phi^\dagger \phi$ 的 P 变换为

$$P^{-1}\phi^\dagger(x)\phi(x)P = P^{-1}\phi^\dagger(x) P P^{-1}\phi(x) P = |\eta_P|^2 \phi^\dagger(\mathcal{P}x)\phi(\mathcal{P}x) = +\phi^\dagger(\mathcal{P}x)\phi(\mathcal{P}x)$$

- 以上两式最右边表达式中的 $+$ 号表明算符 $(\partial^\mu \phi^\dagger)\partial_\mu \phi$ 和 $\phi^\dagger \phi$ 都具有**偶宇称**
- 因此，拉氏量满足 $P^{-1}\mathcal{L}(x)P = +\mathcal{L}(\mathcal{P}x)$，即在宇称变换下**不变**

守恒流算符的 P 变换

- 在 P 变换下，$\mathrm{U}(1)$ **守恒流算符** $\mathrm{i}\phi^\dagger\overleftrightarrow{\partial^\mu}\phi$ 变成

$$P^{-1}\mathrm{i}\phi^\dagger(x)\overleftrightarrow{\partial^\mu}\phi(x)P = \mathrm{i}P^{-1}\phi^\dagger(x)PP^{-1}\overleftrightarrow{\partial^\mu}PP^{-1}\phi(x)P = \mathcal{P}^\mu{}_\nu\mathrm{i}\phi^\dagger(\mathcal{P}x)\overleftrightarrow{\partial^\nu}\phi(\mathcal{P}x)$$

- 符合 $\mathrm{i}\phi^\dagger\overleftrightarrow{\partial^\mu}\phi$ 作为**极矢量**的变换规则

守恒流算符的 P 变换

在 P 变换下，$\mathrm{U}(1)$ 守恒流算符 $\mathrm{i}\phi^\dagger \overleftrightarrow{\partial^\mu} \phi$ 变成

$$P^{-1}\mathrm{i}\phi^\dagger(x)\overleftrightarrow{\partial^\mu}\phi(x)P = \mathrm{i}P^{-1}\phi^\dagger(x)PP^{-1}\overleftrightarrow{\partial^\mu}PP^{-1}\phi(x)P = \mathcal{P}^\mu{}_\nu \mathrm{i}\phi^\dagger(\mathcal{P}x)\overleftrightarrow{\partial^\nu}\phi(\mathcal{P}x)$$

符合 $\mathrm{i}\phi^\dagger \overleftrightarrow{\partial^\mu} \phi$ 作为极矢量的变换规则

在质心系中考虑一对正反标量玻色子 $\phi\bar{\phi}$ 组成的态

$$\left|\phi\bar{\phi}\right\rangle = \int \mathrm{d}^3p\, \Phi(\mathbf{p}) a_{\mathbf{p}}^\dagger b_{-\mathbf{p}}^\dagger \left|0\right\rangle$$

$\Phi(\mathbf{p})$ 是动量空间中的波函数

当轨道角动量的量子数为 L 时，$\Phi(\mathbf{p})$ 正比于球谐函数 $Y_{LM}(\theta,\phi)$，其中 θ 和 ϕ 分别是球坐标系中动量 $\mathbf{p}$ 的极角和方位角

$-\mathbf{p}$ 的极角为 $\pi-\theta$，方位角为 $\pi+\phi$

由 $Y_{LM}(\pi-\theta,\pi+\phi) = (-)^L Y_{LM}(\theta,\phi)$ 得 $\Phi(-\mathbf{p}) = (-)^L\Phi(\mathbf{p})$

轨道宇称和内禀宇称

❄️ 于是，$|\phi\bar{\phi}\rangle$ 的 P 变换为

$$
\begin{aligned}
P\,|\phi\bar{\phi}\rangle &= \int \mathrm{d}^3p\,\Phi(\mathbf{p})P^{-1}a^\dagger_{\mathbf{p}}PP^{-1}b^\dagger_{-\mathbf{p}}P\,|0\rangle = |\eta_P|^2\int \mathrm{d}^3p\,\Phi(\mathbf{p})a^\dagger_{-\mathbf{p}}b^\dagger_{\mathbf{p}}\,|0\rangle \\
&= \int \mathrm{d}^3p\,\Phi(-\mathbf{p})a^\dagger_{\mathbf{p}}b^\dagger_{-\mathbf{p}}\,|0\rangle = (-)^L\int \mathrm{d}^3p\,\Phi(\mathbf{p})a^\dagger_{\mathbf{p}}b^\dagger_{-\mathbf{p}}\,|0\rangle = (-)^L\,|\phi\bar{\phi}\rangle
\end{aligned}
$$

🌉 第三步作了**变量替换 $\mathbf{p}\to-\mathbf{p}$**

🌉 这表明 $|\phi\bar{\phi}\rangle$ 是 P 算符的本征态，本征值 $(-)^L$ 是可观测量，称作 $|\phi\bar{\phi}\rangle$ 的**宇称**

轨道宇称和内禀宇称

❄️ 于是，$|\phi\bar{\phi}\rangle$ 的 P 变换为

$$
\begin{aligned}
P\,|\phi\bar{\phi}\rangle &= \int \mathrm{d}^3p\,\Phi(\mathbf{p})P^{-1}a^\dagger_{\mathbf{p}}PP^{-1}b^\dagger_{-\mathbf{p}}P\,|0\rangle = |\eta_P|^2\int \mathrm{d}^3p\,\Phi(\mathbf{p})a^\dagger_{-\mathbf{p}}b^\dagger_{\mathbf{p}}\,|0\rangle \\
&= \int \mathrm{d}^3p\,\Phi(-\mathbf{p})a^\dagger_{\mathbf{p}}b^\dagger_{-\mathbf{p}}\,|0\rangle = (-)^L\int \mathrm{d}^3p\,\Phi(\mathbf{p})a^\dagger_{\mathbf{p}}b^\dagger_{-\mathbf{p}}\,|0\rangle = (-)^L\,|\phi\bar{\phi}\rangle
\end{aligned}
$$

- 第三步作了**变量替换 $\mathbf{p}\to-\mathbf{p}$**
- 这表明 $|\phi\bar{\phi}\rangle$ 是 P 算符的本征态，本征值 $(-)^L$ 是可观测量，称作 $|\phi\bar{\phi}\rangle$ 的**宇称**
- 宇称是一种**相乘性量子数** (multiplicative quantum number)
- ➡️ 态矢的总宇称是各部分贡献的宇称之**积**
- 波函数 $\Phi(\mathbf{p})$ 对 $|\phi\bar{\phi}\rangle$ 宇称的贡献为 $(-)^L$，这部分称为**轨道宇称** (orbital parity)
- **扣除轨道宇称**之后，剩下的部分称为**内禀宇称** (intrinsic parity)
- $|\phi\bar{\phi}\rangle$ 的内禀宇称为 $+$，即一对正反标量玻色子的内禀宇称为**偶**

实标量场的 P 变换

接下来讨论**自由**的**实标量场** $\phi(x)$，**拉氏量**为 $\mathcal{L} = \frac{1}{2}(\partial^\mu\phi)\partial_\mu\phi - \frac{1}{2}m^2\phi^2$

比较实标量场与复标量场的平面波展开式，可以看出，实标量场的情况相当于要求以上讨论在 $b_{\mathbf{p}} = a_{\mathbf{p}}$ 的条件下进行，由此推出 $\eta_P = \tilde{\eta}_P = \eta_P^*$

于是，**相位因子** η_P 是**实数**，满足 $\eta_P^2 = |\eta_P|^2 = 1$，故 $\eta_P = \pm 1$

实标量场的 P 变换

接下来讨论自由的实标量场 $\phi(x)$，拉氏量为 $\mathcal{L} = \frac{1}{2}(\partial^\mu\phi)\partial_\mu\phi - \frac{1}{2}m^2\phi^2$

比较实标量场与复标量场的平面波展开式，可以看出，实标量场的情况相当于要求以上讨论在 $b_{\mathbf{p}} = a_{\mathbf{p}}$ 的条件下进行，由此推出 $\eta_P = \tilde{\eta}_P = \eta_P^*$

于是，相位因子 η_P 是实数，满足 $\eta_P^2 = |\eta_P|^2 = 1$，故 $\eta_P = \pm 1$

若 $\eta_P = +1$，则实标量场 $\phi(x)$ 是**狭义**的标量场，宇称为偶，P 变换为

$$P^{-1}\phi(x)P = +\phi(\mathcal{P}x),$$

若 $\eta_P = -1$，则称它是赝标量场，宇称为奇，P 变换为

$$P^{-1}\phi(x)P = -\phi(\mathcal{P}x)$$

无论实标量场 $\phi(x)$ 具有哪种宇称，拉氏量 $\mathcal{L}(x)$ 都在宇称变换下不变

不过，与旋量场发生宇称守恒的相互作用时，宇称不同的实标量场 $\phi(x)$ 具有不同的性质，可以通过实验来分辨

U(1) 整体对称性与宇称变换

- 对于**复标量场** $\phi(x)$，宇称变换**相位因子**的取值实际上是**任意**的
- 如 2.4.3 小节所述，自由的复标量场具有 **U(1) 整体对称性** $\phi'(x) = \mathrm{e}^{\mathrm{i}q\theta}\phi(x)$
- 相应的**守恒荷算符** Q 满足 $[Q, \phi] = -q\phi$（习题 2.4），由此得到多重对易子表达式

$$[\phi, Q^{(1)}] = q\phi, \quad [\phi, Q^{(2)}] = [[\phi, Q^{(1)}], Q] = q[\phi, Q] = q^2\phi, \quad \cdots, \quad [\phi, Q^{(n)}] = q^n\phi$$

- 再利用 $\mathrm{e}^{-A}B\mathrm{e}^{A} = \sum_{n=0}^{\infty}\frac{1}{n!}[B, A^{(n)}]$，推出

$$\mathrm{e}^{\mathrm{i}\theta Q}\phi\mathrm{e}^{-\mathrm{i}\theta Q} = \sum_{n=0}^{\infty}\frac{1}{n!}[\phi, (-\mathrm{i}\theta Q)^{(n)}] = \sum_{n=0}^{\infty}\frac{(-\mathrm{i}\theta)^n}{n!}[\phi, Q^{(n)}] = \sum_{n=0}^{\infty}\frac{(-\mathrm{i}q\theta)^n}{n!}\phi = \mathrm{e}^{-\mathrm{i}q\theta}\phi$$

U(1) 整体对称性与宇称变换

- 对于**复标量场** $\phi(x)$，宇称变换**相位因子**的取值实际上是**任意**的
- 如 2.4.3 小节所述，自由的复标量场具有 **U(1) 整体对称性** $\phi'(x) = \mathrm{e}^{\mathrm{i}q\theta}\phi(x)$
- 相应的**守恒荷算符** Q 满足 $[Q, \phi] = -q\phi$（习题 2.4），由此得到多重对易子表达式

$$[\phi, Q^{(1)}] = q\phi, \quad [\phi, Q^{(2)}] = [[\phi, Q^{(1)}], Q] = q[\phi, Q] = q^2\phi, \quad \cdots, \quad [\phi, Q^{(n)}] = q^n\phi$$

- 再利用 $\mathrm{e}^{-A}B\mathrm{e}^{A} = \sum_{n=0}^{\infty}\frac{1}{n!}[B, A^{(n)}]$，推出

$$\mathrm{e}^{\mathrm{i}\theta Q}\phi\mathrm{e}^{-\mathrm{i}\theta Q} = \sum_{n=0}^{\infty}\frac{1}{n!}[\phi, (-\mathrm{i}\theta Q)^{(n)}] = \sum_{n=0}^{\infty}\frac{(-\mathrm{i}\theta)^n}{n!}[\phi, Q^{(n)}] = \sum_{n=0}^{\infty}\frac{(-\mathrm{i}q\theta)^n}{n!}\phi = \mathrm{e}^{-\mathrm{i}q\theta}\phi$$

- 用 Q 和 P 定义一个新的**幺正算符** $P' \equiv \mathrm{e}^{-\mathrm{i}\theta Q}P$
- 它满足 $P'^{-1} = P'^{\dagger} = P^{\dagger}\mathrm{e}^{\mathrm{i}\theta Q} = P^{-1}\mathrm{e}^{\mathrm{i}\theta Q}$，则复标量场 $\phi(x)$ 的 P' 变换为

$$P'^{-1}\phi(x)P' = P^{-1}\mathrm{e}^{\mathrm{i}\theta Q}\phi(x)\mathrm{e}^{-\mathrm{i}\theta Q}P = \mathrm{e}^{-\mathrm{i}q\theta}P^{-1}\phi(x)P = \mathrm{e}^{-\mathrm{i}q\theta}\eta_P^*\phi(\mathcal{P}x) = \eta_P'^*\phi(\mathcal{P}x)$$

- 其中 $\eta_P' \equiv \mathrm{e}^{\mathrm{i}q\theta}\eta_P$ 是 P' **算符**的**相位因子**

分立变换相位因子的任意性

- 尽管 P' 算符不是厄米的，它的作用与 P 算符的作用在实质上是相同的，能够将 $\phi(x)$ 变换成 $\phi(\mathcal{P}x)$，只相差一个相位因子 η'_P
- 因此可以用 P' **取代** P 作为宇称变换算符，而相位因子 η'_P 将**取代**原来的 η_P
- 由于 $\mathrm{U}(1)$ 变换参数 θ 是任意的，$\eta'_P = \mathrm{e}^{\mathrm{i}q\theta}\eta_P$ 的取值也是任意的

分立变换相位因子的任意性

- 尽管 P' 算符不是厄米的，它的作用与 P 算符的作用在实质上是相同的，能够将 $\phi(x)$ 变换成 $\phi(\mathcal{P}x)$，只相差一个相位因子 η'_P
- 因此可以用 P' **取代** P 作为宇称变换算符，而相位因子 η'_P 将**取代**原来的 η_P
- 由于 $\mathrm{U}(1)$ 变换参数 θ 是任意的，$\eta'_P = \mathrm{e}^{\mathrm{i}q\theta}\eta_P$ 的取值也是任意的

- 另一方面，实标量场 $\phi(x)$ **不具备** $\mathrm{U}(1)$ 整体对称性，因而相应宇称变换相位因子不具有这样的任意性
- 实标量场的 $\mathrm{U}(1)$ 荷 $q = 0$，守恒荷算符 $Q = 0$，因而 $P' \equiv \mathrm{e}^{-\mathrm{i}\theta Q}P$ 与 P 相同

分立变换相位因子的任意性

- 尽管 P' 算符不是厄米的，它的作用与 P 算符的作用在实质上是相同的，能够将 $\phi(x)$ 变换成 $\phi(\mathcal{P}x)$，只相差一个相位因子 η'_P
- 因此可以用 P' **取代** P 作为宇称变换算符，而相位因子 η'_P 将**取代**原来的 η_P
- 由于 $\mathrm{U}(1)$ 变换参数 θ 是任意的，$\eta'_P = \mathrm{e}^{\mathrm{i}q\theta}\eta_P$ 的取值也是任意的
- 另一方面，实标量场 $\phi(x)$ **不具备** $\mathrm{U}(1)$ 整体对称性，因而相应宇称变换相位因子不具有这样的任意性
- 实标量场的 $\mathrm{U}(1)$ 荷 $q = 0$，守恒荷算符 $Q = 0$，因而 $P' \equiv \mathrm{e}^{-\mathrm{i}\theta Q}P$ 与 P 相同
- 这个结论可以推广到其它分立对称性和其它量子场：

> 分立变换的相位因子对于像复标量场这样的复场是任意的，对于像实标量场这样的自共轭场则是确定的

8.1.2 小节 标量场的 T 变换

- 时空坐标的 x^μ 的**时间反演变换**为

$$\mathcal{T}^\mu{}_\nu = (\mathcal{T}^{-1})^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$$

- 这是一种**非固有反时向** Lorentz 变换
- 将 $x^\mu = (t, \mathbf{x})$ 变换为 $x'^\mu = (\mathcal{T}x)^\mu = (-t, \mathbf{x})$
- 将 ∂_μ 变换为 $\partial'_\mu = (\mathcal{T}^{-1})^\nu{}_\mu \partial_\nu$
- 将 $p^\mu = (E, \mathbf{p})$ 变换为 $p'^\mu = (\mathcal{T}p)^\mu = (-E, \mathbf{p})$
- 同时**保持时空体积元不变**，$\mathrm{d}^4x' = |\det(\mathcal{T})|\,\mathrm{d}^4x = \mathrm{d}^4x$
- 于是，只要拉氏量在时间反演变换下**不变**，满足 $\mathcal{L}'(x') = \mathcal{L}(x)$，系统就具有**时间反演对称性**

T 变换

☂️ 在具有**时间反演对称性**的**量子理论**中，时间反演变换在 Hilbert 空间中诱导出态矢 $|\Psi\rangle$ 的 **T 变换** $|\Psi'\rangle = U(\mathcal{T})|\Psi\rangle = T|\Psi\rangle$，其中 $T \equiv U(\mathcal{T})$

🚜 T 是**自身的逆变换算符**，$T^{-1} = T$

🚙 仿照上面关于 P 变换的讨论，根据**同态关系**，量子 Poincaré 变换 $U(\Lambda, a)$ 满足

$$T^{-1}U(\Lambda, a)T = U^{-1}(\mathcal{T})U(\Lambda, a)U(\mathcal{T}) = U(\mathcal{T}^{-1}\Lambda\mathcal{T}, \mathcal{T}^{-1}a)$$

🚒 由此推出 **T 变换规则** $T^{-1}\mathrm{i}J^{\mu\nu}T = \mathrm{i}\mathcal{T}^{\mu}{}_{\rho}\mathcal{T}^{\nu}{}_{\sigma}J^{\rho\sigma}$ 和 $T^{-1}\mathrm{i}P^{\mu}T = \mathrm{i}\mathcal{T}^{\mu}{}_{\nu}P^{\nu}$

🚚 这里保留了推导过程中的**虚数** i

T 变换

☂️ 在具有**时间反演对称性**的**量子理论**中，时间反演变换在 Hilbert 空间中诱导出态矢 $|\Psi\rangle$ 的 **T 变换** $|\Psi'\rangle = U(\mathcal{T})|\Psi\rangle = T|\Psi\rangle$，其中 $T \equiv U(\mathcal{T})$

🚜 T 是**自身的逆变换算符**，$T^{-1} = T$

🚙 仿照上面关于 P 变换的讨论，根据**同态关系**，量子 Poincaré 变换 $U(\Lambda, a)$ 满足

$$T^{-1}U(\Lambda,a)T = U^{-1}(\mathcal{T})U(\Lambda,a)U(\mathcal{T}) = U(\mathcal{T}^{-1}\Lambda\mathcal{T}, \mathcal{T}^{-1}a)$$

🚚 由此推出 **T 变换规则** $T^{-1}\mathrm{i}J^{\mu\nu}T = \mathrm{i}\mathcal{T}^{\mu}{}_{\rho}\mathcal{T}^{\nu}{}_{\sigma}J^{\rho\sigma}$ 和 $T^{-1}\mathrm{i}P^{\mu}T = \mathrm{i}\mathcal{T}^{\mu}{}_{\nu}P^{\nu}$

🚛 这里保留了推导过程中的**虚数** i

⚠️ 如果时间反演算符 T 与宇称算符 P 一样是**线性幺正算符**，那么，

$$T^{-1}HT = T^{-1}P^0T = \mathcal{T}^0{}_0P^0 = -H$$

⛔ 这意味着哈密顿量算符 H 在 T 变换下**发生改变**，导致理论**不具有**时间反演对称性，与我们的假设**矛盾**

🚫 因此，**T 不可能是线性幺正算符**

反线性反幺正算符

☔ 不过，**对称变换**在**量子力学**中不一定要用线性幺正算符表述，也可以用**反线性** (antilinear) **反幺正** (antiunitary) **算符**描述

🚟 一般地，在**反线性反幺正算符** U 的作用下，态矢 $|\Psi_1\rangle$ 和 $|\Psi_2\rangle$ 满足

$$U(a\,|\Psi_1\rangle + b\,|\Psi_2\rangle) = a^* U\,|\Psi_1\rangle + b^* U\,|\Psi_2\rangle\,, \quad \langle U\Psi_1|U\Psi_2\rangle = \langle\Psi_1|\Psi_2\rangle^*$$

🚇 其中 a 和 b 是任意复数，第一个等式代表**反线性**，第二个等式代表**反幺正**

🚄 **反线性**意味着 $-\mathrm{i}\,|\Psi_1\rangle = U^{-1}U(-\mathrm{i}\,|\Psi_1\rangle) = U^{-1}\mathrm{i}U\,|\Psi_1\rangle$，即

$$U^{-1}\mathrm{i}U = -\mathrm{i}$$

🚋 对**任意复数** a，则有 $U^{-1}aU = a^*$

反线性反幺正算符

☔ 不过，对称变换在量子力学中不一定要用线性幺正算符表述，也可以用反线性 (antilinear) 反幺正 (antiunitary) 算符描述

🚉 一般地，在反线性反幺正算符 U 的作用下，态矢 $|\Psi_1\rangle$ 和 $|\Psi_2\rangle$ 满足

$$U(a\,|\Psi_1\rangle + b\,|\Psi_2\rangle) = a^* U\,|\Psi_1\rangle + b^* U\,|\Psi_2\rangle\,, \quad \langle U\Psi_1|U\Psi_2\rangle = \langle\Psi_1|\Psi_2\rangle^*$$

🚇 其中 a 和 b 是任意复数，第一个等式代表反线性，第二个等式代表反幺正

🚄 反线性意味着 $-\mathrm{i}\,|\Psi_1\rangle = U^{-1}U(-\mathrm{i}\,|\Psi_1\rangle) = U^{-1}\mathrm{i}U\,|\Psi_1\rangle$，即

$$U^{-1}\mathrm{i}U = -\mathrm{i}$$

🚇 对任意复数 a，则有 $U^{-1}aU = a^*$

🚋 反线性反幺正算符 U 的厄米共轭算符 $U^\dagger$ 通过下式定义，

$$\langle\Psi_1|U^\dagger\Psi_2\rangle \equiv \langle U\Psi_1|\Psi_2\rangle^* = \langle\Psi_2|U\Psi_1\rangle$$

🚃 因而反幺正意味着 $\langle\Psi_1|U^{-1}U\Psi_2\rangle = \langle\Psi_1|\Psi_2\rangle = \langle U\Psi_1|U\Psi_2\rangle^* = \langle\Psi_1|U^\dagger U\Psi_2\rangle$

🚞 故 $U^{-1} = U^\dagger$，这个等式与线性幺正算符满足的等式相同

T 变换的反线性反幺正性

现在将 T 算符定义为反线性反幺正算符，满足

$$T^{-1}\mathrm{i}T = -\mathrm{i}, \quad T^{-1} = T^{\dagger}$$

那么，$T^{-1}\mathrm{i}J^{\mu\nu}T = \mathrm{i}\mathcal{T}^{\mu}{}_{\rho}\mathcal{T}^{\nu}{}_{\sigma}J^{\rho\sigma}$ 和 $T^{-1}\mathrm{i}P^{\mu}T = \mathrm{i}\mathcal{T}^{\mu}{}_{\nu}P^{\nu}$ 表明生成元算符 $J^{\mu\nu}$ 和 P^{μ} 的 T 变换为

$$T^{-1}J^{\mu\nu}T = -\mathcal{T}^{\mu}{}_{\rho}\mathcal{T}^{\nu}{}_{\sigma}J^{\rho\sigma}, \quad T^{-1}P^{\mu}T = -\mathcal{T}^{\mu}{}_{\nu}P^{\nu}$$

即

$$T^{-1}\mathbf{J}T = -\mathbf{J}, \quad T^{-1}\mathbf{K}T = +\mathbf{K}, \quad T^{-1}HT = +H, \quad T^{-1}\mathbf{P}T = -\mathbf{P}$$

T 变换的反线性反幺正性

现在将 T 算符定义为反线性反幺正算符，满足

$$T^{-1}\mathrm{i}T = -\mathrm{i}, \quad T^{-1} = T^{\dagger}$$

那么，$T^{-1}\mathrm{i}J^{\mu\nu}T = \mathrm{i}\mathcal{T}^{\mu}{}_{\rho}\mathcal{T}^{\nu}{}_{\sigma}J^{\rho\sigma}$ 和 $T^{-1}\mathrm{i}P^{\mu}T = \mathrm{i}\mathcal{T}^{\mu}{}_{\nu}P^{\nu}$ 表明生成元算符 $J^{\mu\nu}$ 和 P^{μ} 的 T 变换为

$$T^{-1}J^{\mu\nu}T = -\mathcal{T}^{\mu}{}_{\rho}\mathcal{T}^{\nu}{}_{\sigma}J^{\rho\sigma}, \quad T^{-1}P^{\mu}T = -\mathcal{T}^{\mu}{}_{\nu}P^{\nu}$$

即

$$T^{-1}\mathbf{J}T = -\mathbf{J}, \quad T^{-1}\mathbf{K}T = +\mathbf{K}, \quad T^{-1}HT = +H, \quad T^{-1}\mathbf{P}T = -\mathbf{P}$$

从而，哈密顿量算符 H 在时间反演变换下不变，满足 $[H, T] = 0$，符合理论具有时间反演对称性的假设

动量算符 $\mathbf{P}$ 和总角动量算符 $\mathbf{J}$ 在 T 变换下分别反转成 $-\mathbf{P}$ 和 $-\mathbf{J}$

时间反演变换相当于逆着时间方向进行观察，因而动量和角动量的方向都会反转

复标量场平面波展开式的 T 变换

- 下面讨论**自由复标量场** $\phi(x)$
- T 变换会**翻转**粒子态对应的**动量方向**，因而产生湮灭算符的 T 变换为

$$T^{-1}a_{\mathbf{p}}^{\dagger}T=\eta_T a_{-\mathbf{p}}^{\dagger},\quad T^{-1}a_{\mathbf{p}}T=\eta_T^{*}a_{-\mathbf{p}},\quad T^{-1}b_{\mathbf{p}}^{\dagger}T=\tilde{\eta}_T b_{-\mathbf{p}}^{\dagger},\quad T^{-1}b_{\mathbf{p}}T=\tilde{\eta}_T^{*}b_{-\mathbf{p}}$$

- η_T 和 $\tilde{\eta}_T$ 是两个模为 1 的**相位因子**

复标量场平面波展开式的 T 变换

- 下面讨论**自由复标量场** $\phi(x)$
- T 变换会**翻转**粒子态对应的**动量方向**，因而产生湮灭算符的 T 变换为

$$T^{-1}a_{\mathbf{p}}^{\dagger}T=\eta_T a_{-\mathbf{p}}^{\dagger},\quad T^{-1}a_{\mathbf{p}}T=\eta_T^{*}a_{-\mathbf{p}},\quad T^{-1}b_{\mathbf{p}}^{\dagger}T=\tilde{\eta}_T b_{-\mathbf{p}}^{\dagger},\quad T^{-1}b_{\mathbf{p}}T=\tilde{\eta}_T^{*}b_{-\mathbf{p}}$$

- η_T 和 $\tilde{\eta}_T$ 是两个模为 1 的**相位因子**
- 复标量场**平面波展开式**的 T 变换为

$$\begin{aligned}T^{-1}\phi(x)T&=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(T^{-1}a_{\mathbf{p}}TT^{-1}\mathrm{e}^{-\mathrm{i}p\cdot x}T+T^{-1}b_{\mathbf{p}}^{\dagger}TT^{-1}\mathrm{e}^{\mathrm{i}p\cdot x}T\right)\\&=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(\eta_T^{*}a_{-\mathbf{p}}\mathrm{e}^{\mathrm{i}p\cdot x}+\tilde{\eta}_T b_{-\mathbf{p}}^{\dagger}\mathrm{e}^{-\mathrm{i}p\cdot x}\right)\\&=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left[\eta_T^{*}a_{\mathbf{p}}\mathrm{e}^{-\mathrm{i}p\cdot(\mathcal{T}x)}+\tilde{\eta}_T b_{\mathbf{p}}^{\dagger}\mathrm{e}^{\mathrm{i}p\cdot(\mathcal{T}x)}\right]\end{aligned}$$

- 最后一步作了**变量替换** $\mathbf{p}\to-\mathbf{p}$，并利用 $(\mathcal{P}p)\cdot x=p\cdot(\mathcal{P}x)=-p\cdot(\mathcal{T}x)$

复标量场的 T 变换

为了保持 $\phi(x)$ 的运动方程形式不变，$T^{-1}\phi(x)T$ 与 $\phi(\mathcal{T}x)$ 最多只能相差一个常数因子，因此必须要求

$$\eta_T^* = \tilde{\eta}_T$$

使得 $\phi(x)$ 和 $\phi^\dagger(x)$ 的 T 变换为

$$T^{-1}\phi(x)T = \eta_T^*\phi(\mathcal{T}x), \quad T^{-1}\phi^\dagger(x)T = \eta_T\phi^\dagger(\mathcal{T}x)$$

复标量场的 T 变换

为了保持 $\phi(x)$ 的运动方程形式不变，$T^{-1}\phi(x)T$ 与 $\phi(\mathcal{T}x)$ 最多只能相差一个常数因子，因此必须要求

$$\eta_T^* = \tilde{\eta}_T$$

使得 $\phi(x)$ 和 $\phi^\dagger(x)$ 的 T 变换为

$$T^{-1}\phi(x)T = \eta_T^*\phi(\mathcal{T}x), \quad T^{-1}\phi^\dagger(x)T = \eta_T\phi^\dagger(\mathcal{T}x)$$

为方便表述，在形式上以时空导数的时间反演变换定义 T 变换，

$$T^{-1}\partial_\mu T \equiv \partial'_\mu = (\mathcal{T}^{-1})^\nu{}_\mu\partial_\nu, \quad T^{-1}\partial^\mu T = \partial'^\mu = \mathcal{T}^\mu{}_\nu\partial^\nu$$

那么算符 $(\partial^\mu\phi^\dagger)\partial_\mu\phi$、$\phi^\dagger\phi$ 和 $\mathrm{i}\phi^\dagger\overleftrightarrow{\partial^\mu}\phi$ 的 T 变换是

$$\begin{aligned}
T^{-1}\partial^\mu\phi^\dagger(x)\partial_\mu\phi(x)T &= |\eta_T|^2\mathcal{T}^\mu{}_\nu(\mathcal{T}^{-1})^\rho{}_\mu\partial^\nu\phi^\dagger(\mathcal{T}x)\partial_\rho\phi(\mathcal{T}x) = +\partial^\mu\phi^\dagger(\mathcal{T}x)\partial_\mu\phi(\mathcal{T}x) \\
T^{-1}\phi^\dagger(x)\phi(x)T &= |\eta_T|^2\phi^\dagger(\mathcal{T}x)\phi(\mathcal{T}x) = +\phi^\dagger(\mathcal{T}x)\phi(\mathcal{T}x) \\
T^{-1}\mathrm{i}\phi^\dagger(x)\overleftrightarrow{\partial^\mu}\phi(x)T &= -|\eta_T|^2\mathcal{T}^\mu{}_\nu\mathrm{i}\phi^\dagger(\mathcal{T}x)\overleftrightarrow{\partial^\nu}\phi(\mathcal{T}x) = -\mathcal{T}^\mu{}_\nu\mathrm{i}\phi^\dagger(\mathcal{T}x)\overleftrightarrow{\partial^\nu}\phi(\mathcal{T}x)
\end{aligned}$$

自由复标量场的拉氏量在时间反演变换下不变，满足 $T^{-1}\mathcal{L}(x)T = +\mathcal{L}(\mathcal{T}x)$

实标量场的 T 变换

- 令 $b_{\mathbf{p}} = a_{\mathbf{p}}$，就得到**自由实标量场** $\phi(x)$ 的情况，此时 $\eta_T = \tilde{\eta}_T = \eta_T^*$
- 故 $\eta_T^2 = |\eta_T|^2 = 1$，有

$$\eta_T = \pm 1$$

- 因此，存在**两类**实标量场
- 一类实标量场具有 $\eta_T = +1$，相应的 T 变换为

$$T^{-1}\phi(x)T = +\phi(\mathcal{T}x)$$

- 另一类实标量场具有 $\eta_T = -1$，相应的 T 变换为

$$T^{-1}\phi(x)T = -\phi(\mathcal{T}x)$$

- 无论哪一类实标量场 $\phi(x)$，**自由拉氏量**都具有**时间反演不变性**

8.1.3 小节 标量场的 C 变换

除了宇称和时间反演，另一种重要的分立变换是**电荷共轭** (charge conjugation)

电荷共轭变换**将正反粒子互相转换**，不只转换正反电荷，也转换所有其它正反 U(1) 守恒荷，但对时空坐标、四维动量、角动量和螺旋度**没有影响**

8.1.3 小节 标量场的 C 变换

除了宇称和时间反演，另一种重要的分立变换是**电荷共轭** (charge conjugation)

电荷共轭变换**将正反粒子互相转换**，不只转换正反电荷，也转换所有其它正反 U(1) 守恒荷，但对时空坐标、四维动量、角动量和螺旋度**没有影响**

在具有**电荷共轭对称性**的**量子理论**中，电荷共轭变换在 Hilbert 空间中诱导出态矢 $|\Psi\rangle$ 的**线性幺正变换**

$$|\Psi'\rangle = C\,|\Psi\rangle$$

称为 **C 变换**

C 是**自身的逆变换算符**，满足

$$C^\dagger = C^{-1} = C$$

因而 C 算符是**厄米**的

复标量场的 C 变换

- 对于自由复标量场 $\phi(x)$，C 变换将正粒子态转化成反粒子态，$C\left|\mathbf{p}^{+}\right\rangle=\eta_C\left|\mathbf{p}^{-}\right\rangle$
- η_C 是相位因子，真空态在 C 变换下不变，满足 $C|0\rangle=|0\rangle$，从而

$$\sqrt{2E_{\mathbf{p}}}\,C^{-1}a_{\mathbf{p}}^{\dagger}C\,|0\rangle=C\sqrt{2E_{\mathbf{p}}}\,a_{\mathbf{p}}^{\dagger}\,|0\rangle=C\left|\mathbf{p}^{+}\right\rangle=\eta_C\left|\mathbf{p}^{-}\right\rangle=\sqrt{2E_{\mathbf{p}}}\,\eta_C b_{\mathbf{p}}^{\dagger}\,|0\rangle$$

$$C^{-1}a_{\mathbf{p}}^{\dagger}C=\eta_C b_{\mathbf{p}}^{\dagger},\quad C^{-1}a_{\mathbf{p}}C=\eta_C^{*}b_{\mathbf{p}}$$

- 第一个等式等价于 $a_{\mathbf{p}}^{\dagger}=\eta_C C b_{\mathbf{p}}^{\dagger}C^{-1}$，因而 $\eta_C^{*}a_{\mathbf{p}}^{\dagger}=|\eta_C|^2 C b_{\mathbf{p}}^{\dagger}C^{-1}=C b_{\mathbf{p}}^{\dagger}C^{-1}$，即

$$C^{-1}b_{\mathbf{p}}^{\dagger}C=\eta_C^{*}a_{\mathbf{p}}^{\dagger},\quad C^{-1}b_{\mathbf{p}}C=\eta_C a_{\mathbf{p}}$$

复标量场的 C 变换

- 对于**自由复标量场** $\phi(x)$，C 变换将**正粒子态**转化成**反粒子态**，$C\left|\mathbf{p}^{+}\right\rangle=\eta_C\left|\mathbf{p}^{-}\right\rangle$
- η_C 是**相位因子**，真空态在 C 变换下不变，满足 $C|0\rangle=|0\rangle$，从而

$$\sqrt{2E_{\mathbf{p}}}\,C^{-1}a_{\mathbf{p}}^{\dagger}C\,|0\rangle=C\sqrt{2E_{\mathbf{p}}}\,a_{\mathbf{p}}^{\dagger}\,|0\rangle=C\left|\mathbf{p}^{+}\right\rangle=\eta_C\left|\mathbf{p}^{-}\right\rangle=\sqrt{2E_{\mathbf{p}}}\,\eta_C b_{\mathbf{p}}^{\dagger}\,|0\rangle$$

$$C^{-1}a_{\mathbf{p}}^{\dagger}C=\eta_C b_{\mathbf{p}}^{\dagger},\quad C^{-1}a_{\mathbf{p}}C=\eta_C^{*}b_{\mathbf{p}}$$

- 第一个等式等价于 $a_{\mathbf{p}}^{\dagger}=\eta_C C b_{\mathbf{p}}^{\dagger}C^{-1}$，因而 $\eta_C^{*}a_{\mathbf{p}}^{\dagger}=|\eta_C|^2 C b_{\mathbf{p}}^{\dagger}C^{-1}=C b_{\mathbf{p}}^{\dagger}C^{-1}$，即

$$C^{-1}b_{\mathbf{p}}^{\dagger}C=\eta_C^{*}a_{\mathbf{p}}^{\dagger},\quad C^{-1}b_{\mathbf{p}}C=\eta_C a_{\mathbf{p}}$$

- 于是，复标量场 $\phi(x)$ 的 **C 变换**为

$$\begin{aligned}C^{-1}\phi(x)C&=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(C^{-1}a_{\mathbf{p}}C\mathrm{e}^{-\mathrm{i}p\cdot x}+C^{-1}b_{\mathbf{p}}^{\dagger}C\mathrm{e}^{\mathrm{i}p\cdot x}\right)\\&=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(\eta_C^{*}b_{\mathbf{p}}\mathrm{e}^{-\mathrm{i}p\cdot x}+\eta_C^{*}a_{\mathbf{p}}^{\dagger}\mathrm{e}^{\mathrm{i}p\cdot x}\right)=\eta_C^{*}\phi^{\dagger}(x)\end{aligned}$$

- 可见，C 变换使 $\phi(x)$ 与其厄米共轭场 $\phi^{\dagger}(x)$ 相互转换：

$$C^{-1}\phi(x)C=\eta_C^{*}\phi^{\dagger}(x),\quad C^{-1}\phi^{\dagger}(x)C=\eta_C\phi(x)$$

内禀 C 宇称

- 注意到等时对易关系，算符 $(\partial^\mu\phi^\dagger)\partial_\mu\phi$、$\phi^\dagger\phi$ 和 $\mathrm{i}\phi^\dagger\overleftrightarrow{\partial^\mu}\phi$ 的 C 变换是

$$
\begin{aligned}
C^{-1}\partial^\mu\phi^\dagger(x)\partial_\mu\phi(x)C &= |\eta_C|^2\partial^\mu\phi(x)\partial_\mu\phi^\dagger(x) = +\partial^\mu\phi^\dagger(x)\partial_\mu\phi(x),\\
C^{-1}\phi^\dagger(x)\phi(x)C &= |\eta_C|^2\phi(x)\phi^\dagger(x) = +\phi^\dagger(x)\phi(x),\\
C^{-1}\mathrm{i}\phi^\dagger(x)\overleftrightarrow{\partial^\mu}\phi(x)C &= |\eta_C|^2\mathrm{i}\phi(x)\overleftrightarrow{\partial^\mu}\phi^\dagger(x) = -\mathrm{i}\phi^\dagger(x)\overleftrightarrow{\partial^\mu}\phi(x)
\end{aligned}
$$

- 因此，自由复标量场的拉氏量在电荷共轭变换下不变，满足 $C^{-1}\mathcal{L}(x)C = +\mathcal{L}(x)$

内禀 C 宇称

- 注意到等时对易关系，算符 $(\partial^\mu\phi^\dagger)\partial_\mu\phi$、$\phi^\dagger\phi$ 和 $\mathrm{i}\phi^\dagger\overleftrightarrow{\partial^\mu}\phi$ 的 C 变换是

$$
\begin{aligned}
C^{-1}\partial^\mu\phi^\dagger(x)\partial_\mu\phi(x)C &= |\eta_C|^2\partial^\mu\phi(x)\partial_\mu\phi^\dagger(x) = +\partial^\mu\phi^\dagger(x)\partial_\mu\phi(x),\\
C^{-1}\phi^\dagger(x)\phi(x)C &= |\eta_C|^2\phi(x)\phi^\dagger(x) = +\phi^\dagger(x)\phi(x),\\
C^{-1}\mathrm{i}\phi^\dagger(x)\overleftrightarrow{\partial^\mu}\phi(x)C &= |\eta_C|^2\mathrm{i}\phi(x)\overleftrightarrow{\partial^\mu}\phi^\dagger(x) = -\mathrm{i}\phi^\dagger(x)\overleftrightarrow{\partial^\mu}\phi(x)
\end{aligned}
$$

- 因此，自由复标量场的**拉氏量**在电荷共轭变换下**不变**，满足 $C^{-1}\mathcal{L}(x)C = +\mathcal{L}(x)$
- 对**正反标量玻色子态** $|\phi\bar{\phi}\rangle = \int \mathrm{d}^3p\,\Phi(\mathbf{p})a^\dagger_{\mathbf{p}}b^\dagger_{-\mathbf{p}}|0\rangle$ 作 C 变换，得

$$
C|\phi\bar{\phi}\rangle = \int \mathrm{d}^3p\,\Phi(\mathbf{p})C^{-1}a^\dagger_{\mathbf{p}}CC^{-1}b^\dagger_{-\mathbf{p}}C|0\rangle = |\eta_C|^2\int \mathrm{d}^3p\,\Phi(\mathbf{p})b^\dagger_{\mathbf{p}}a^\dagger_{-\mathbf{p}}|0\rangle
$$

$[a^\dagger_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] = 0$ ➡ $= \int \mathrm{d}^3p\,\Phi(\mathbf{p})a^\dagger_{-\mathbf{p}}b^\dagger_{\mathbf{p}}|0\rangle = \int \mathrm{d}^3p\,\Phi(-\mathbf{p})a^\dagger_{\mathbf{p}}b^\dagger_{-\mathbf{p}}|0\rangle$ ⬅ **变量替换 $\mathbf{p}\to-\mathbf{p}$**

$\Phi(-\mathbf{p}) = (-)^L\Phi(\mathbf{p})$ ➡ $= (-)^L\int \mathrm{d}^3p\,\Phi(\mathbf{p})a^\dagger_{\mathbf{p}}b^\dagger_{-\mathbf{p}}|0\rangle = (-)^L|\phi\bar{\phi}\rangle$

内禀 C 宇称

- 注意到等时对易关系，算符 $(\partial^\mu\phi^\dagger)\partial_\mu\phi$、$\phi^\dagger\phi$ 和 $\mathrm{i}\phi^\dagger\overleftrightarrow{\partial^\mu}\phi$ 的 C 变换是

$$
\begin{aligned}
C^{-1}\partial^\mu\phi^\dagger(x)\partial_\mu\phi(x)C &= |\eta_C|^2\partial^\mu\phi(x)\partial_\mu\phi^\dagger(x) = +\partial^\mu\phi^\dagger(x)\partial_\mu\phi(x),\\
C^{-1}\phi^\dagger(x)\phi(x)C &= |\eta_C|^2\phi(x)\phi^\dagger(x) = +\phi^\dagger(x)\phi(x),\\
C^{-1}\mathrm{i}\phi^\dagger(x)\overleftrightarrow{\partial^\mu}\phi(x)C &= |\eta_C|^2\mathrm{i}\phi(x)\overleftrightarrow{\partial^\mu}\phi^\dagger(x) = -\mathrm{i}\phi^\dagger(x)\overleftrightarrow{\partial^\mu}\phi(x)
\end{aligned}
$$

- 因此，自由复标量场的**拉氏量**在电荷共轭变换下**不变**，满足 $C^{-1}\mathcal{L}(x)C = +\mathcal{L}(x)$
- 对**正反标量玻色子态** $|\phi\bar{\phi}\rangle = \int \mathrm{d}^3p\,\Phi(\mathbf{p})a^\dagger_{\mathbf{p}}b^\dagger_{-\mathbf{p}}|0\rangle$ 作 C 变换，得

$$
\begin{aligned}
C|\phi\bar{\phi}\rangle &= \int \mathrm{d}^3p\,\Phi(\mathbf{p})C^{-1}a^\dagger_{\mathbf{p}}CC^{-1}b^\dagger_{-\mathbf{p}}C|0\rangle = |\eta_C|^2\int \mathrm{d}^3p\,\Phi(\mathbf{p})b^\dagger_{\mathbf{p}}a^\dagger_{-\mathbf{p}}|0\rangle\\
&= \int \mathrm{d}^3p\,\Phi(\mathbf{p})a^\dagger_{-\mathbf{p}}b^\dagger_{\mathbf{p}}|0\rangle = \int \mathrm{d}^3p\,\Phi(-\mathbf{p})a^\dagger_{\mathbf{p}}b^\dagger_{-\mathbf{p}}|0\rangle\\
&= (-)^L\int \mathrm{d}^3p\,\Phi(\mathbf{p})a^\dagger_{\mathbf{p}}b^\dagger_{-\mathbf{p}}|0\rangle = (-)^L|\phi\bar{\phi}\rangle
\end{aligned}
$$

- $|\phi\bar{\phi}\rangle$ 是 C 算符的本征态，**本征值** $(-)^L$ 是可观测量，称为 **C 宇称** (C-parity)
- 扣除**波函数** $\Phi(\mathbf{p})$ 贡献的 $(-)^L$ 因子之后，剩下的 C 宇称为 $+$
- 即一对正反标量玻色子的**内禀 C 宇称**为**偶**

实标量场的 C 变换

- 对于**自由实标量场**，$\phi^\dagger(x) = \phi(x)$
- $C^{-1}\phi(x)C = \eta_C^*\phi^\dagger(x)$ 和 $C^{-1}\phi^\dagger(x)C = \eta_C\phi(x)$ 表明 $\eta_C = \eta_C^*$
- 故 $\eta_C^2 = |\eta_C|^2 = 1$，有 $\eta_C = \pm 1$
- 这是 C 宇称的两种取值，也就是说，存在两类实标量场
- 一类实标量场具有**偶**的 C 宇称，$\eta_C = +1$，相应的 C 变换为

$$C^{-1}\phi(x)C = +\phi(x)$$

- 另一类实标量场具有**奇**的 C 宇称，$\eta_C = -1$，相应的 C 变换为

$$C^{-1}\phi(x)C = -\phi(x)$$

- 无论实标量场 $\phi(x)$ 具有哪种 C 宇称，**自由拉氏量**都具有**电荷共轭不变性**

8.2 节 旋量场的分立变换 8.2.1 小节 旋量场的 C 变换

🐱 现在讨论**自由 Dirac 旋量场** $\psi(x)$ 的 ***C* 变换**

🎵 C 变换在**互换正反粒子**的同时，**不改变**动量 $\mathbf{p}$ 和螺旋度 $\lambda = \pm$，因此将正费米子态 $\left|\mathbf{p}^+, \lambda\right\rangle = \sqrt{2E_\mathbf{p}}\, a^\dagger_{\mathbf{p},\lambda} \left|0\right\rangle$ 转化成反费米子态 $\left|\mathbf{p}^-, \lambda\right\rangle = \sqrt{2E_\mathbf{p}}\, b^\dagger_{\mathbf{p},\lambda} \left|0\right\rangle$，

$$C \left|\mathbf{p}^+, \lambda\right\rangle = \zeta_C \left|\mathbf{p}^-, \lambda\right\rangle$$

🎶 ζ_C 是**相位因子**

8.2 节 旋量场的分立变换 8.2.1 小节 旋量场的 C 变换

现在讨论**自由 Dirac 旋量场** $\psi(x)$ 的 **C 变换**

C 变换在**互换正反粒子**的同时，**不改变**动量 $\mathbf{p}$ 和螺旋度 $\lambda=\pm$，因此将正费米子态 $\left|\mathbf{p}^+,\lambda\right\rangle=\sqrt{2E_\mathbf{p}}\,a^\dagger_{\mathbf{p},\lambda}\left|0\right\rangle$ 转化成反费米子态 $\left|\mathbf{p}^-,\lambda\right\rangle=\sqrt{2E_\mathbf{p}}\,b^\dagger_{\mathbf{p},\lambda}\left|0\right\rangle$，

$$C\left|\mathbf{p}^+,\lambda\right\rangle=\zeta_C\left|\mathbf{p}^-,\lambda\right\rangle$$

ζ_C 是**相位因子**，由此推出

$$C^{-1}a^\dagger_{\mathbf{p},\lambda}C=\zeta_C b^\dagger_{\mathbf{p},\lambda},\quad C^{-1}a_{\mathbf{p},\lambda}C=\zeta_C^* b_{\mathbf{p},\lambda},\quad C^{-1}b^\dagger_{\mathbf{p},\lambda}C=\zeta_C^* a^\dagger_{\mathbf{p},\lambda},\quad C^{-1}b_{\mathbf{p},\lambda}C=\zeta_C a_{\mathbf{p},\lambda}$$

$\psi(x)$ **平面波展开式**的 C 变换为

$$\begin{aligned}C^{-1}\psi(x)C&=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_\lambda\left[u(\mathbf{p},\lambda)C^{-1}a_{\mathbf{p},\lambda}C\mathrm{e}^{-\mathrm{i}p\cdot x}+v(\mathbf{p},\lambda)C^{-1}b^\dagger_{\mathbf{p},\lambda}C\mathrm{e}^{\mathrm{i}p\cdot x}\right]\\&=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_\lambda\left[\zeta_C^*\,u(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}+\zeta_C^*\,v(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}\right]\end{aligned}$$

螺旋态关系式

为了得到 $\psi(x)$ 的**电荷共轭场**，需要探讨旋量系数 $u(\mathbf{p},\lambda)$ 和 $v(\mathbf{p},\lambda)$ 之间的关系

在 **Weyl 表象**中，对 $\bar{u}(\mathbf{p},\lambda)$ 和 $\bar{v}(\mathbf{p},\lambda)$ 进行**转置**，得

$$\bar{u}^{\mathrm{T}}(\mathbf{p},\lambda)=\begin{pmatrix}\omega_\lambda(\mathbf{p})\xi^*_\lambda(\mathbf{p})\\ \omega_{-\lambda}(\mathbf{p})\xi^*_\lambda(\mathbf{p})\end{pmatrix},\quad \bar{v}^{\mathrm{T}}(\mathbf{p},\lambda)=\begin{pmatrix}-\lambda\omega_{-\lambda}(\mathbf{p})\xi^*_{-\lambda}(\mathbf{p})\\ \lambda\omega_\lambda(\mathbf{p})\xi^*_{-\lambda}(\mathbf{p})\end{pmatrix}$$

另一方面，按照螺旋态的具体形式，$\mathrm{i}\sigma^2$ 对 $\xi^*_\lambda(\mathbf{p})$ 的作用为

$$\mathrm{i}\sigma^2\xi^*_+(\mathbf{p}) = N\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\begin{pmatrix}|\mathbf{p}|+p^3\\ p^1-\mathrm{i}p^2\end{pmatrix}=N\begin{pmatrix}p^1-\mathrm{i}p^2\\ -(|\mathbf{p}|+p^3)\end{pmatrix}=-\xi_-(\mathbf{p}) \tag{1}$$

$$\mathrm{i}\sigma^2\xi^*_-(\mathbf{p}) = N\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\begin{pmatrix}-p^1-\mathrm{i}p^2\\ |\mathbf{p}|+p^3\end{pmatrix}=N\begin{pmatrix}|\mathbf{p}|+p^3\\ p^1+\mathrm{i}p^2\end{pmatrix}=+\xi_+(\mathbf{p}) \tag{2}$$

其中归一化因子 $N\equiv[2|\mathbf{p}|(|\mathbf{p}|+p^3)]^{-1/2}$，归纳起来，得

$$\mathrm{i}\sigma^2\xi^*_\lambda(\mathbf{p})=-\lambda\xi_{-\lambda}(\mathbf{p}),\quad \mathrm{i}\sigma^2\xi^*_{-\lambda}(\mathbf{p})=\lambda\xi_\lambda(\mathbf{p})$$

电荷共轭场

引入旋量空间中的**电荷共轭矩阵**

$$\mathcal{C} \equiv \mathrm{i}\gamma^0\gamma^2 = \mathrm{i}\begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix}\begin{pmatrix} & \sigma^2 \\ -\sigma^2 & \end{pmatrix} = \begin{pmatrix} -\mathrm{i}\sigma^2 & \\ & \mathrm{i}\sigma^2 \end{pmatrix}$$

就可以导出平面波旋量系数的关系式

$$\mathcal{C}\bar{u}^{\mathrm{T}}(\mathbf{p},\lambda) = \begin{pmatrix} -\omega_\lambda(\mathbf{p})\mathrm{i}\sigma^2\xi^*_\lambda(\mathbf{p}) \\ \omega_{-\lambda}(\mathbf{p})\mathrm{i}\sigma^2\xi^*_\lambda(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \lambda\omega_\lambda(\mathbf{p})\xi_{-\lambda}(\mathbf{p}) \\ -\lambda\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = v(\mathbf{p},\lambda)$$

$$\mathcal{C}\bar{v}^{\mathrm{T}}(\mathbf{p},\lambda) = \begin{pmatrix} \lambda\omega_{-\lambda}(\mathbf{p})\mathrm{i}\sigma^2\xi^*_{-\lambda}(\mathbf{p}) \\ \lambda\omega_\lambda(\mathbf{p})\mathrm{i}\sigma^2\xi^*_{-\lambda}(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p}) \end{pmatrix} = u(\mathbf{p},\lambda)$$

电荷共轭场

引入旋量空间中的**电荷共轭矩阵**

$$\mathcal{C} \equiv \mathrm{i}\gamma^0\gamma^2 = \mathrm{i}\begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix}\begin{pmatrix} & \sigma^2 \\ -\sigma^2 & \end{pmatrix} = \begin{pmatrix} -\mathrm{i}\sigma^2 & \\ & \mathrm{i}\sigma^2 \end{pmatrix}$$

就可以导出平面波旋量系数的关系式

$$\mathcal{C}\bar{u}^{\mathrm{T}}(\mathbf{p},\lambda) = \begin{pmatrix} -\omega_\lambda(\mathbf{p})\mathrm{i}\sigma^2\xi^*_\lambda(\mathbf{p}) \\ \omega_{-\lambda}(\mathbf{p})\mathrm{i}\sigma^2\xi^*_\lambda(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \lambda\omega_\lambda(\mathbf{p})\xi_{-\lambda}(\mathbf{p}) \\ -\lambda\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = v(\mathbf{p},\lambda)$$

$$\mathcal{C}\bar{v}^{\mathrm{T}}(\mathbf{p},\lambda) = \begin{pmatrix} \lambda\omega_{-\lambda}(\mathbf{p})\mathrm{i}\sigma^2\xi^*_{-\lambda}(\mathbf{p}) \\ \lambda\omega_\lambda(\mathbf{p})\mathrm{i}\sigma^2\xi^*_{-\lambda}(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p}) \end{pmatrix} = u(\mathbf{p},\lambda)$$

于是，$\psi(x)$ 的 C 变换化为

$$C^{-1}\psi(x)C = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_\lambda\left[\zeta_C^*\, u(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x} + \zeta_C^*\, v(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}\right]$$

$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_\lambda\left[\zeta_C^*\, \mathcal{C}\bar{v}^{\mathrm{T}}(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x} + \zeta_C^*\, \mathcal{C}\bar{u}^{\mathrm{T}}(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}\right] = \zeta_C^*\psi^{\mathrm{C}}(x)$$

这里 $\psi^{\mathrm{C}}(x) \equiv \mathcal{C}\bar{\psi}^{\mathrm{T}}(x)$ 便是 $\psi(x)$ 的**电荷共轭场**

电荷共轭矩阵的性质

- 现在研究**电荷共轭矩阵 $\mathcal{C}$** 的性质
- 利用 $(\sigma^2)^{\mathrm{T}} = -\sigma^2$、$\gamma^0$ 的厄米性、γ^2 的反厄米性和 $\gamma^2\gamma^0 = -\gamma^0\gamma^2$，有

$$\mathcal{C}^{\mathrm{T}} = \begin{pmatrix} -\mathrm{i}(\sigma^2)^{\mathrm{T}} & \\ & \mathrm{i}(\sigma^2)^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \mathrm{i}\sigma^2 & \\ & -\mathrm{i}\sigma^2 \end{pmatrix} = -\mathcal{C}$$

$$\mathcal{C}^{\dagger} = -\mathrm{i}(\gamma^2)^{\dagger}(\gamma^0)^{\dagger} = \mathrm{i}\gamma^2\gamma^0 = -\mathrm{i}\gamma^0\gamma^2 = -\mathcal{C},$$

$$\mathcal{C}^{\dagger}\mathcal{C} = \gamma^0\gamma^2\gamma^0\gamma^2 = -(\gamma^0)^2(\gamma^2)^2 = \mathbf{1}$$

- 可见，$\mathcal{C}$ 是**幺正矩阵**，满足

$$\mathcal{C}^{\mathrm{T}} = \mathcal{C}^{\dagger} = \mathcal{C}^{-1} = -\mathcal{C}$$

电荷共轭矩阵的性质

现在研究**电荷共轭矩阵** $\mathcal{C}$ 的性质

利用 $(\sigma^2)^{\mathrm{T}} = -\sigma^2$、$\gamma^0$ 的厄米性、γ^2 的反厄米性和 $\gamma^2\gamma^0 = -\gamma^0\gamma^2$，有

$$\mathcal{C}^{\mathrm{T}} = \begin{pmatrix} -\mathrm{i}(\sigma^2)^{\mathrm{T}} & \\ & \mathrm{i}(\sigma^2)^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \mathrm{i}\sigma^2 & \\ & -\mathrm{i}\sigma^2 \end{pmatrix} = -\mathcal{C}$$

$$\mathcal{C}^{\dagger} = -\mathrm{i}(\gamma^2)^{\dagger}(\gamma^0)^{\dagger} = \mathrm{i}\gamma^2\gamma^0 = -\mathrm{i}\gamma^0\gamma^2 = -\mathcal{C},$$

$$\mathcal{C}^{\dagger}\mathcal{C} = \gamma^0\gamma^2\gamma^0\gamma^2 = -(\gamma^0)^2(\gamma^2)^2 = \mathbf{1}$$

可见，$\mathcal{C}$ 是**幺正矩阵**，满足

$$\mathcal{C}^{\mathrm{T}} = \mathcal{C}^{\dagger} = \mathcal{C}^{-1} = -\mathcal{C}$$

Pauli 矩阵满足 $\sigma^2\sigma^1\sigma^2 = \mathrm{i}\sigma^2\sigma^3 = -\sigma^1 = -(\sigma^1)^{\mathrm{T}}$、$\sigma^2\sigma^2\sigma^2 = \sigma^2 = -(\sigma^2)^{\mathrm{T}}$ 和 $\sigma^2\sigma^3\sigma^2 = \mathrm{i}\sigma^1\sigma^2 = -\sigma^3 = -(\sigma^3)^{\mathrm{T}}$，归纳得到 $\sigma^2\mathbf{1}\sigma^2 = \mathbf{1}^{\mathrm{T}}$ 和 $\sigma^2\boldsymbol{\sigma}\sigma^2 = -\boldsymbol{\sigma}^{\mathrm{T}}$，因此

$$\sigma^2\sigma^\mu\sigma^2 = (\bar{\sigma}^\mu)^{\mathrm{T}}, \quad \sigma^2\bar{\sigma}^\mu\sigma^2 = (\sigma^\mu)^{\mathrm{T}}$$

Dirac 矩阵的电荷共轭变换

利用 $\sigma^2\sigma^\mu\sigma^2 = (\bar{\sigma}^\mu)^{\mathrm{T}}$ 和 $\sigma^2\bar{\sigma}^\mu\sigma^2 = (\sigma^\mu)^{\mathrm{T}}$ 推出

$$\mathcal{C}^{-1}\gamma^\mu\mathcal{C} = \begin{pmatrix} \mathrm{i}\sigma^2 & \\ & -\mathrm{i}\sigma^2 \end{pmatrix} \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} \begin{pmatrix} -\mathrm{i}\sigma^2 & \\ & \mathrm{i}\sigma^2 \end{pmatrix} = -\begin{pmatrix} & \sigma^2\sigma^\mu\sigma^2 \\ \sigma^2\bar{\sigma}^\mu\sigma^2 & \end{pmatrix}$$

$$= -\begin{pmatrix} & (\bar{\sigma}^\mu)^{\mathrm{T}} \\ (\sigma^\mu)^{\mathrm{T}} & \end{pmatrix}$$

$$\mathcal{C}^{-1}\gamma^5\mathcal{C} = \begin{pmatrix} \mathrm{i}\sigma^2 & \\ & -\mathrm{i}\sigma^2 \end{pmatrix} \begin{pmatrix} -\mathbf{1} & \\ & \mathbf{1} \end{pmatrix} \begin{pmatrix} -\mathrm{i}\sigma^2 & \\ & \mathrm{i}\sigma^2 \end{pmatrix} = \begin{pmatrix} -(\sigma^2)^2 & \\ & (\sigma^2)^2 \end{pmatrix} = \begin{pmatrix} -\mathbf{1} & \\ & \mathbf{1} \end{pmatrix}$$

即得 γ^μ 和 γ^5 关于 $\mathcal{C}$ 的**相似变换性质**

$$\mathcal{C}^{-1}\gamma^\mu\mathcal{C} = -(\gamma^\mu)^{\mathrm{T}}, \quad \mathcal{C}^{-1}\gamma^5\mathcal{C} = \gamma^5$$

由于 $\mathcal{C}^{-1} = -\mathcal{C}$，这两个式子等价于

$$\mathcal{C}^{-1}(\gamma^\mu)^{\mathrm{T}}\mathcal{C} = -\gamma^\mu, \quad \mathcal{C}^{-1}(\gamma^5)^{\mathrm{T}}\mathcal{C} = \gamma^5$$

$\psi^{\mathrm{C}}(x)$ 的运动方程

- 如果 **Dirac 旋量场** $\psi(x)$ 携带**电荷** Q，相应的运动方程是

$$[\gamma^\mu(\mathrm{i}\partial_\mu - QeA_\mu) - m]\psi = 0$$

- 对上式取厄米共轭，再右乘 γ^0，得

$$0 = \psi^\dagger[(\gamma^\mu)^\dagger(-\mathrm{i}\partial_\mu - QeA_\mu) - m]\gamma^0 = \bar{\psi}[-\gamma^\mu(\mathrm{i}\partial_\mu + QeA_\mu) - m]$$

- **转置**，推出 $[-(\gamma^\mu)^{\mathrm{T}}(\mathrm{i}\partial_\mu + QeA_\mu) - m]\bar{\psi}^{\mathrm{T}} = 0$

$\psi^{\mathrm{C}}(x)$ 的运动方程

如果 **Dirac 旋量场** $\psi(x)$ 携带**电荷** Q，相应的运动方程是

$$[\gamma^\mu(\mathrm{i}\partial_\mu - QeA_\mu) - m]\psi = 0$$

对上式取厄米共轭，再右乘 γ^0，得

$$0 = \psi^\dagger[(\gamma^\mu)^\dagger(-\mathrm{i}\partial_\mu - QeA_\mu) - m]\gamma^0 = \bar{\psi}[-\gamma^\mu(\mathrm{i}\partial_\mu + QeA_\mu) - m]$$

转置，推出 $[-(\gamma^\mu)^{\mathrm{T}}(\mathrm{i}\partial_\mu + QeA_\mu) - m]\bar{\psi}^{\mathrm{T}} = 0$

利用 $\mathcal{C}^{-1}\gamma^\mu\mathcal{C} = -(\gamma^\mu)^{\mathrm{T}}$，将上式化为

$$0 = [\mathcal{C}^{-1}\gamma^\mu\mathcal{C}(\mathrm{i}\partial_\mu + QeA_\mu) - m\,\mathcal{C}^{-1}\mathcal{C}]\bar{\psi}^{\mathrm{T}} = \mathcal{C}^{-1}[\gamma^\mu(\mathrm{i}\partial_\mu + QeA_\mu) - m]\mathcal{C}\bar{\psi}^{\mathrm{T}}$$

从而得到**电荷共轭场** $\psi^{\mathrm{C}}(x)$ 的**运动方程**

$$[\gamma^\mu(\mathrm{i}\partial_\mu + QeA_\mu) - m]\psi^{\mathrm{C}} = 0$$

对比 $\psi(x)$ **的运动方程**，可以看出 $\psi^{\mathrm{C}}(x)$ 确实携带**相反**的**电荷** $-Q$

同理，$\psi^{\mathrm{C}}(x)$ 携带的**任何 U(1) 守恒荷**都与 $\psi(x)$ **相反**

$\psi(x)$ 和 $\bar{\psi}(x)$ 的 C 变换

- 根据 $\mathcal{C}^{\mathrm{T}} = \mathcal{C}^{\dagger} = \mathcal{C}^{-1} = -\mathcal{C}$ 和 $\mathcal{C}^{-1}(\gamma^{\mu})^{\mathrm{T}}\mathcal{C} = -\gamma^{\mu}$
- **电荷共轭场** $\psi^{\mathrm{C}}(x) = \mathcal{C}\bar{\psi}^{\mathrm{T}}(x)$ 的 **Dirac 共轭**为

$$\begin{aligned}\bar{\psi}^{\mathrm{C}} &= (\psi^{\mathrm{C}})^{\dagger}\gamma^{0} = (\mathcal{C}\bar{\psi}^{\mathrm{T}})^{\dagger}\gamma^{0} = [(\gamma^{0})^{\mathrm{T}}(\bar{\psi}\mathcal{C}^{\mathrm{T}})^{\dagger}]^{\mathrm{T}} = [(\gamma^{0})^{\mathrm{T}}(\psi^{\dagger}\gamma^{0}\mathcal{C}^{\dagger})^{\dagger}]^{\mathrm{T}} = [(\gamma^{0})^{\mathrm{T}}\mathcal{C}\gamma^{0}\psi]^{\mathrm{T}} \\ &= [\mathcal{C}\mathcal{C}^{-1}(\gamma^{0})^{\mathrm{T}}\mathcal{C}\gamma^{0}\psi]^{\mathrm{T}} = -(\mathcal{C}\gamma^{0}\gamma^{0}\psi)^{\mathrm{T}} = (\mathcal{C}^{\mathrm{T}}\psi)^{\mathrm{T}}\end{aligned}$$

- 即

$$\bar{\psi}^{\mathrm{C}}(x) = \psi^{\mathrm{T}}(x)\,\mathcal{C}$$

$\psi(x)$ 和 $\bar{\psi}(x)$ 的 C 变换

根据 $\mathcal{C}^{\mathrm{T}} = \mathcal{C}^{\dagger} = \mathcal{C}^{-1} = -\mathcal{C}$ 和 $\mathcal{C}^{-1}(\gamma^{\mu})^{\mathrm{T}}\mathcal{C} = -\gamma^{\mu}$

电荷共轭场 $\psi^{\mathrm{C}}(x) = \mathcal{C}\bar{\psi}^{\mathrm{T}}(x)$ 的 **Dirac 共轭**为

$$\begin{aligned}\bar{\psi}^{\mathrm{C}} &= (\psi^{\mathrm{C}})^{\dagger}\gamma^{0} = (\mathcal{C}\bar{\psi}^{\mathrm{T}})^{\dagger}\gamma^{0} = [(\gamma^{0})^{\mathrm{T}}(\bar{\psi}\mathcal{C}^{\mathrm{T}})^{\dagger}]^{\mathrm{T}} = [(\gamma^{0})^{\mathrm{T}}(\psi^{\dagger}\gamma^{0}\mathcal{C}^{\dagger})^{\dagger}]^{\mathrm{T}} = [(\gamma^{0})^{\mathrm{T}}\mathcal{C}\gamma^{0}\psi]^{\mathrm{T}} \\ &= [\mathcal{C}\mathcal{C}^{-1}(\gamma^{0})^{\mathrm{T}}\mathcal{C}\gamma^{0}\psi]^{\mathrm{T}} = -(\mathcal{C}\gamma^{0}\gamma^{0}\psi)^{\mathrm{T}} = (\mathcal{C}^{\mathrm{T}}\psi)^{\mathrm{T}}\end{aligned}$$

即

$$\bar{\psi}^{\mathrm{C}}(x) = \psi^{\mathrm{T}}(x)\,\mathcal{C}$$

于是 $C^{-1}\bar{\psi}C = C^{\dagger}\psi^{\dagger}C\gamma^{0} = (C^{-1}\psi C)^{\dagger}\gamma^{0} = (\zeta_C^{*}\psi^{\mathrm{C}})^{\dagger}\gamma^{0} = \zeta_C\bar{\psi}^{\mathrm{C}} = \zeta_C\psi^{\mathrm{T}}\mathcal{C}$

也就是说，Dirac 旋量场 $\psi(x)$ 及其 Dirac 共轭场 $\bar{\psi}(x)$ 的 C 变换是

$$C^{-1}\psi(x)C = \zeta_C^{*}\psi^{\mathrm{C}}(x) = \zeta_C^{*}\,\mathcal{C}\bar{\psi}^{\mathrm{T}}(x), \quad C^{-1}\bar{\psi}(x)C = \zeta_C\bar{\psi}^{\mathrm{C}}(x) = \zeta_C\psi^{\mathrm{T}}(x)\,\mathcal{C}$$

仿照 $\bar{\psi}^{\mathrm{C}}(x) = \psi^{\mathrm{T}}(x)\,\mathcal{C}$ 的推导，由 $\mathcal{C}\bar{u}^{\mathrm{T}}(\mathbf{p},\lambda) = v(\mathbf{p},\lambda)$ 和 $\mathcal{C}\bar{v}^{\mathrm{T}}(\mathbf{p},\lambda) = u(\mathbf{p},\lambda)$ 得

$$\bar{v}(\mathbf{p},\lambda) = u^{\mathrm{T}}(\mathbf{p},\lambda)\,\mathcal{C}, \quad \bar{u}(\mathbf{p},\lambda) = v^{\mathrm{T}}(\mathbf{p},\lambda)\,\mathcal{C}$$

一般旋量双线性型的 C 变换

🐻 考虑**一般**旋量双线性型 $\bar{\psi}(x)\Gamma\psi(x)$，其中 Γ 是**旋量空间**中的**任意 4×4 矩阵**，则

$$C^{-1}\bar{\psi}\Gamma\psi C = C^{-1}\bar{\psi}C\Gamma C^{-1}\psi C = |\zeta_C|^2\psi^{\mathrm{T}}\mathcal{C}\Gamma\mathcal{C}\bar{\psi}^{\mathrm{T}} = \psi^{\mathrm{T}}\mathcal{C}\Gamma\mathcal{C}\bar{\psi}^{\mathrm{T}} = -\bar{\psi}\mathcal{C}^{\mathrm{T}}\Gamma^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\psi$$

📹 最后一步进行了**转置**，需要注意的是，**转置两个旋量场**会**交换**两者的位置，为了与**反对易关系**相匹配，必须引进一个额外的**负号**

一般旋量双线性型的 C 变换

🐻 考虑**一般**旋量双线性型 $\bar{\psi}(x)\Gamma\psi(x)$，其中 Γ 是**旋量空间**中的**任意 4×4 矩阵**，则

$$C^{-1}\bar{\psi}\Gamma\psi C = C^{-1}\bar{\psi}C\Gamma C^{-1}\psi C = |\zeta_C|^2\psi^{\mathrm{T}}\mathcal{C}\Gamma\mathcal{C}\bar{\psi}^{\mathrm{T}} = \psi^{\mathrm{T}}\mathcal{C}\Gamma\mathcal{C}\bar{\psi}^{\mathrm{T}} = -\bar{\psi}\mathcal{C}^{\mathrm{T}}\Gamma^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\psi$$

📹 最后一步进行了**转置**，需要注意的是，**转置两个旋量场**会**交换**两者的位置，为了与**反对易关系**相匹配，必须引进一个额外的**负号**

📷 从而，由 $\mathcal{C}^{\mathrm{T}} = \mathcal{C}^{-1} = -\mathcal{C}$ 得到 $\bar{\psi}\Gamma\psi$ 的 C 变换

$$C^{-1}\bar{\psi}(x)\Gamma\psi(x)C = \bar{\psi}(x)\mathcal{C}^{-1}\Gamma^{\mathrm{T}}\mathcal{C}\psi(x) = \bar{\psi}(x)\Gamma^{\mathrm{C}}\psi(x)$$

📸 其中 $\Gamma^{\mathrm{C}} \equiv \mathcal{C}^{-1}\Gamma^{\mathrm{T}}\mathcal{C}$ 可视为矩阵 Γ 的**电荷共轭变换**

一般旋量双线性型的 C 变换

🐻 考虑一般旋量双线性型 $\bar{\psi}(x)\Gamma\psi(x)$，其中 Γ 是旋量空间中的任意 4×4 矩阵，则

$$C^{-1}\bar{\psi}\Gamma\psi C = C^{-1}\bar{\psi}C\Gamma C^{-1}\psi C = |\zeta_C|^2\psi^{\mathrm{T}}\mathcal{C}\Gamma\mathcal{C}\bar{\psi}^{\mathrm{T}} = \psi^{\mathrm{T}}\mathcal{C}\Gamma\mathcal{C}\bar{\psi}^{\mathrm{T}} = -\bar{\psi}\mathcal{C}^{\mathrm{T}}\Gamma^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\psi$$

📹 最后一步进行了转置，需要注意的是，转置两个旋量场会**交换**两者的位置，为了与反对易关系相匹配，必须引进一个额外的负号

📷 从而，由 $\mathcal{C}^{\mathrm{T}} = \mathcal{C}^{-1} = -\mathcal{C}$ 得到 $\bar{\psi}\Gamma\psi$ 的 C 变换

$$C^{-1}\bar{\psi}(x)\Gamma\psi(x)C = \bar{\psi}(x)\mathcal{C}^{-1}\Gamma^{\mathrm{T}}\mathcal{C}\psi(x) = \bar{\psi}(x)\Gamma^{\mathrm{C}}\psi(x)$$

📸 其中 $\Gamma^{\mathrm{C}} \equiv \mathcal{C}^{-1}\Gamma^{\mathrm{T}}\mathcal{C}$ 可视为矩阵 Γ 的电荷共轭变换

💿 根据 $\mathcal{C}^{-1}(\gamma^\mu)^{\mathrm{T}}\mathcal{C} = -\gamma^\mu$ 和 $\mathcal{C}^{-1}(\gamma^5)^{\mathrm{T}}\mathcal{C} = \gamma^5$，有

$$\begin{aligned}
\mathbf{1}^{\mathrm{C}} &= \mathcal{C}^{-1}\mathbf{1}^{\mathrm{T}}\mathcal{C} = +\mathbf{1}, & (\mathrm{i}\gamma^5)^{\mathrm{C}} &= \mathcal{C}^{-1}(\mathrm{i}\gamma^5)^{\mathrm{T}}\mathcal{C} = +\mathrm{i}\gamma^5 \\
(\gamma^\mu)^{\mathrm{C}} &= \mathcal{C}^{-1}(\gamma^\mu)^{\mathrm{T}}\mathcal{C} = -\gamma^\mu, & (\gamma^\mu\gamma^5)^{\mathrm{C}} &= \mathcal{C}^{-1}(\gamma^\mu\gamma^5)^{\mathrm{T}}\mathcal{C} = +\gamma^\mu\gamma^5 \\
(\gamma^\mu\gamma^\nu)^{\mathrm{C}} &= \mathcal{C}^{-1}(\gamma^\mu\gamma^\nu)^{\mathrm{T}}\mathcal{C} = +\gamma^\nu\gamma^\mu, & (\sigma^{\mu\nu})^{\mathrm{C}} &= \mathcal{C}^{-1}(\sigma^{\mu\nu})^{\mathrm{T}}\mathcal{C} = -\sigma^{\mu\nu}
\end{aligned}$$

旋量双线性型的 C 变换

🦄 于是，各种旋量双线性型的 C 变换为

$$
\begin{aligned}
C^{-1}\bar{\psi}(x)\psi(x)C &= +\bar{\psi}(x)\psi(x) \\
C^{-1}\bar{\psi}(x)\mathrm{i}\gamma^5\psi(x)C &= +\bar{\psi}(x)\mathrm{i}\gamma^5\psi(x), \\
C^{-1}\bar{\psi}(x)\gamma^\mu\psi(x)C &= -\bar{\psi}(x)\gamma^\mu\psi(x) \\
C^{-1}\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)C &= +\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x) \\
C^{-1}\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)C &= -\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)
\end{aligned}
$$

旋量双线性型的 C 变换

- 于是，各种旋量双线性型的 **C 变换**为

$$
\begin{aligned}
C^{-1}\bar{\psi}(x)\psi(x)C &= +\bar{\psi}(x)\psi(x) \\
C^{-1}\bar{\psi}(x)\mathrm{i}\gamma^5\psi(x)C &= +\bar{\psi}(x)\mathrm{i}\gamma^5\psi(x), \\
C^{-1}\bar{\psi}(x)\gamma^\mu\psi(x)C &= -\bar{\psi}(x)\gamma^\mu\psi(x) \\
C^{-1}\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)C &= +\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x) \\
C^{-1}\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)C &= -\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)
\end{aligned}
$$

- 拉氏量中的**动能项算符** $i\bar{\psi}\gamma^\mu\partial_\mu\psi$ 在 **C 变换**下化为

$$
\begin{aligned}
C^{-1}\mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi C &= \mathrm{i}\psi^{\mathrm{T}}\mathcal{C}\gamma^\mu\mathcal{C}\partial_\mu\bar{\psi}^{\mathrm{T}} = -\mathrm{i}\psi^{\mathrm{T}}\mathcal{C}^{-1}\gamma^\mu\mathcal{C}\partial_\mu\bar{\psi}^{\mathrm{T}} = \mathrm{i}\psi^{\mathrm{T}}(\gamma^\mu)^{\mathrm{T}}\partial_\mu\bar{\psi}^{\mathrm{T}} \\
&= -\mathrm{i}(\partial_\mu\bar{\psi})\gamma^\mu\psi = -\mathrm{i}\partial_\mu(\bar{\psi}\gamma^\mu\psi) + \mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi
\end{aligned}
$$

- 最后的表达式中第一项是**全散度**，全时空积分后对作用量没有贡献，可以**丢弃**
- 因而上式表明动能项算符在 C 变换下**不变**
- 自由 Dirac 旋量场的**拉氏量** $\mathcal{L} = \mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$ 具有**电荷共轭不变性**

手征旋量场算符的 C 变换

对于 7.3 节的**手征旋量场** $\psi_{\mathrm{L}}(x)$ 和 $\psi_{\mathrm{R}}(x)$，算符 $\bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}} = \bar{\psi}P_{\mathrm{L}}\psi = \bar{\psi}(1-\gamma^5)\psi/2$ 和 $\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}} = \bar{\psi}P_{\mathrm{R}}\psi = \bar{\psi}(1+\gamma^5)\psi/2$ 在 C 变换下**不变**，满足

$$
\begin{aligned}
C^{-1}\bar{\psi}_{\mathrm{R}}(x)\psi_{\mathrm{L}}(x)C &= +\bar{\psi}_{\mathrm{R}}(x)\psi_{\mathrm{L}}(x), \\
C^{-1}\bar{\psi}_{\mathrm{L}}(x)\psi_{\mathrm{R}}(x)C &= +\bar{\psi}_{\mathrm{L}}(x)\psi_{\mathrm{R}}(x)
\end{aligned}
$$

手征旋量场算符的 C 变换

对于 7.3 节的**手征旋量场** $\psi_{\mathrm{L}}(x)$ 和 $\psi_{\mathrm{R}}(x)$，算符 $\bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}} = \bar{\psi}P_{\mathrm{L}}\psi = \bar{\psi}(1-\gamma^5)\psi/2$ 和 $\bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}} = \bar{\psi}P_{\mathrm{R}}\psi = \bar{\psi}(1+\gamma^5)\psi/2$ 在 C 变换下**不变**，满足

$$C^{-1}\bar{\psi}_{\mathrm{R}}(x)\psi_{\mathrm{L}}(x)C = +\bar{\psi}_{\mathrm{R}}(x)\psi_{\mathrm{L}}(x),$$
$$C^{-1}\bar{\psi}_{\mathrm{L}}(x)\psi_{\mathrm{R}}(x)C = +\bar{\psi}_{\mathrm{L}}(x)\psi_{\mathrm{R}}(x)$$

另一方面，**左手流算符** $\bar{\psi}_{\mathrm{L}}\gamma^\mu\psi_{\mathrm{L}} = \bar{\psi}\gamma^\mu P_{\mathrm{L}}\psi = \frac{1}{2}\bar{\psi}\gamma^\mu(1-\gamma^5)\psi$ 和

右手流算符 $\bar{\psi}_{\mathrm{R}}\gamma^\mu\psi_{\mathrm{R}} = \bar{\psi}\gamma^\mu P_{\mathrm{R}}\psi = \frac{1}{2}\bar{\psi}\gamma^\mu(1+\gamma^5)\psi$ 的 C 变换为

$$C^{-1}\bar{\psi}_{\mathrm{L}}(x)\gamma^\mu\psi_{\mathrm{L}}(x)C = -\bar{\psi}_{\mathrm{R}}(x)\gamma^\mu\psi_{\mathrm{R}}(x)$$
$$C^{-1}\bar{\psi}_{\mathrm{R}}(x)\gamma^\mu\psi_{\mathrm{R}}(x)C = -\bar{\psi}_{\mathrm{L}}(x)\gamma^\mu\psi_{\mathrm{L}}(x)$$

两者在 C 变换下**相互转化**，并出现一个**负号**

在**弱相互作用**中，**左手流算符**和**右手流算符**参与**不同**的规范相互作用，因而**电荷共轭对称性**遭到破坏

正反费米子系统的 C 变换

在**质心系**中考虑**一对正反费米子** $\psi\bar{\psi}$ 组成的系统，当**轨道角动量**和**总自旋角动量**的量子数分别为 L 和 S 时，态矢表达为

$$\left|\psi\bar{\psi}\right\rangle_{L,S} = \sum_{\lambda\lambda'} \int \mathrm{d}^3 p\, \Phi(\mathbf{p},\lambda,\lambda') a^\dagger_{\mathbf{p},\lambda} b^\dagger_{-\mathbf{p},\lambda'} \left|0\right\rangle$$

波函数 $\Phi(\mathbf{p},\lambda,\lambda')$ 满足 $\Phi(-\mathbf{p},\lambda',\lambda) = (-)^L (-)^{S+1} \Phi(\mathbf{p},\lambda,\lambda')$

$(-)^{S+1}$ 因子来自**交换** λ 和 λ' 时 **Clebsch-Gordan 系数**的性质

正反费米子系统的 C 变换

在质心系中考虑一对正反费米子 $\psi\bar{\psi}$ 组成的系统，当轨道角动量和总自旋角动量的量子数分别为 L 和 S 时，态矢表达为

$$\left|\psi\bar{\psi}\right\rangle_{L,S} = \sum_{\lambda\lambda'}\int \mathrm{d}^3p\,\Phi(\mathbf{p},\lambda,\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{-\mathbf{p},\lambda'}\left|0\right\rangle$$

波函数 $\Phi(\mathbf{p},\lambda,\lambda')$ 满足 $\Phi(-\mathbf{p},\lambda',\lambda) = (-)^L(-)^{S+1}\Phi(\mathbf{p},\lambda,\lambda')$

$(-)^{S+1}$ 因子来自交换 λ 和 λ' 时 Clebsch-Gordan 系数的性质

$\left|\psi\bar{\psi}\right\rangle_{L,S}$ 的 C 变换为

$$\begin{aligned}
C\left|\psi\bar{\psi}\right\rangle_{L,S} &= \sum_{\lambda\lambda'}\int \mathrm{d}^3p\,\Phi(\mathbf{p},\lambda,\lambda')C^{-1}a^\dagger_{\mathbf{p},\lambda}CC^{-1}b^\dagger_{-\mathbf{p},\lambda'}C\left|0\right\rangle \\
&= \left|\zeta_C\right|^2\sum_{\lambda\lambda'}\int \mathrm{d}^3p\,\Phi(\mathbf{p},\lambda,\lambda')b^\dagger_{\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda'}\left|0\right\rangle \\
&= -\sum_{\lambda\lambda'}\int \mathrm{d}^3p\,\Phi(\mathbf{p},\lambda,\lambda')a^\dagger_{-\mathbf{p},\lambda'}b^\dagger_{\mathbf{p},\lambda}\left|0\right\rangle
\end{aligned}$$

最后一步利用反对易关系 $\{a_{\mathbf{p},\lambda}, b^\dagger_{\mathbf{q},\lambda'}\} = 0$ 交换了两个产生算符

正反费米子系统的内禀 C 宇称

🐻 作变量替换 $\mathbf{p} \to -\mathbf{p}$ 和 $\lambda \leftrightarrow \lambda'$，利用 $\Phi(-\mathbf{p}, \lambda', \lambda) = (-)^{L+S+1}\Phi(\mathbf{p}, \lambda, \lambda')$，得

$$
\begin{aligned}
C\left|\psi\bar{\psi}\right\rangle_{L,S} &= -\sum_{\lambda\lambda'}\int \mathrm{d}^3p\,\Phi(\mathbf{p},\lambda,\lambda')a^\dagger_{-\mathbf{p},\lambda'}b^\dagger_{\mathbf{p},\lambda}\left|0\right\rangle \\
&= -\sum_{\lambda\lambda'}\int \mathrm{d}^3p\,\Phi(-\mathbf{p},\lambda',\lambda)a^\dagger_{\mathbf{p},\lambda}b^\dagger_{-\mathbf{p},\lambda'}\left|0\right\rangle \\
&= (-)^{L+S}\sum_{\lambda\lambda'}\int \mathrm{d}^3p\,\Phi(\mathbf{p},\lambda,\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{-\mathbf{p},\lambda'}\left|0\right\rangle \\
&= (-)^{L+S}\left|\psi\bar{\psi}\right\rangle_{L,S}
\end{aligned}
$$

正反费米子系统的内禀 C 宇称

🐻‍❄️ 作变量替换 $\mathbf{p} \to -\mathbf{p}$ 和 $\lambda \leftrightarrow \lambda'$，利用 $\Phi(-\mathbf{p}, \lambda', \lambda) = (-)^{L+S+1}\Phi(\mathbf{p}, \lambda, \lambda')$，得

$$
\begin{aligned}
C\left|\psi\bar{\psi}\right\rangle_{L,S} &= -\sum_{\lambda\lambda'}\int \mathrm{d}^3p\,\Phi(\mathbf{p},\lambda,\lambda')a^\dagger_{-\mathbf{p},\lambda'}b^\dagger_{\mathbf{p},\lambda}\left|0\right\rangle \\
&= -\sum_{\lambda\lambda'}\int \mathrm{d}^3p\,\Phi(-\mathbf{p},\lambda',\lambda)a^\dagger_{\mathbf{p},\lambda}b^\dagger_{-\mathbf{p},\lambda'}\left|0\right\rangle \\
&= (-)^{L+S}\sum_{\lambda\lambda'}\int \mathrm{d}^3p\,\Phi(\mathbf{p},\lambda,\lambda')a^\dagger_{\mathbf{p},\lambda}b^\dagger_{-\mathbf{p},\lambda'}\left|0\right\rangle \\
&= (-)^{L+S}\left|\psi\bar{\psi}\right\rangle_{L,S}
\end{aligned}
$$

📞 可见，$\left|\psi\bar{\psi}\right\rangle_{L,S}$ 的 **C 宇称**为 $(-)^{L+S}$

☎️ **扣除**波函数 $\Phi(\mathbf{p},\lambda,\lambda')$ 对 C 宇称的贡献 $(-)^{L+S+1}$ 之后

📱 发现一对正反费米子的**内禀 C 宇称**为**奇**

8.2.2 小节 Majorana 旋量场

🐼 如果 $\psi(x)$ 与它的电荷共轭场相同，$\psi(x) = \psi^{\mathrm{C}}(x)$，即满足**自共轭条件**

$$\psi(x) = \mathcal{C}\bar{\psi}^{\mathrm{T}}(x)$$

🍜 那么，$\psi(x)$ 就是一种**纯中性**的场，**不携带**任何 $\mathrm{U}(1)$ 守恒荷，称为 **Majorana 旋量场**，上式称为 **Majorana 条件**

🥚 于是，Majorana 旋量场满足 $\bar{\psi}(x) = \bar{\psi}^{\mathrm{C}}(x)$

Ettore Majorana
(1906–?)

8.2.2 小节 Majorana 旋量场

🐼 如果 $\psi(x)$ 与它的电荷共轭场相同，$\psi(x)=\psi^{\mathrm{C}}(x)$，即满足**自共轭条件**

$$\psi(x)=\mathcal{C}\bar{\psi}^{\mathrm{T}}(x)$$

🍜 那么，$\psi(x)$ 就是一种**纯中性**的场，**不携带**任何 $\mathrm{U}(1)$ 守恒荷，称为 **Majorana 旋量场**，上式称为 **Majorana 条件**

Ettore Majorana (1906–?)

🥚 于是，Majorana 旋量场满足 $\bar{\psi}(x)=\bar{\psi}^{\mathrm{C}}(x)$

🍳 根据 $\bar{\psi}^{\mathrm{C}}(x)=\psi^{\mathrm{T}}(x)\,\mathcal{C}$，Majorana 条件等价于 $\bar{\psi}(x)=\psi^{\mathrm{T}}(x)\,\mathcal{C}$

🥘 这个等式右边没有出现 $\psi^{\dagger}(x)$，而出现 $\psi^{\mathrm{T}}(x)$，表明 $\bar{\psi}(x)$ 与 $\psi(x)$ **线性相关**

🥗 因此，$\bar{\psi}(x)$ **并不是独立**于 $\psi(x)$ 的场变量，这一点与 Dirac 旋量场**不同**

🍱 根据前面的讨论，$\psi^{\mathrm{C}}(x)$ 的平面波展开式为

$$\psi^{\mathrm{C}}(x)=\zeta_C C^{-1}\psi(x)C=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda}\left[u(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}+v(\mathbf{p},\lambda)a^{\dagger}_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}\right]$$

Majorana 费米子

- 将上式与 $\psi(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_\lambda\left[u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}+v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}\right]$ 比较
- 可知 **Majorana 条件** $\psi(x)=\psi^{\mathrm{C}}(x)$ 意味着 $b_{\mathbf{p},\lambda}=a_{\mathbf{p},\lambda}$
- 因此，**Majorana 旋量场** $\psi(x)$ 的平面波展开式是

$$\psi(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm}\left[u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}+v(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}\right]$$

- **产生湮灭算符**满足**反对易关系**

$$\{a_{\mathbf{p},\lambda},a^\dagger_{\mathbf{q},\lambda'}\}=(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}),\quad \{a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}\}=\{a^\dagger_{\mathbf{p},\lambda},a^\dagger_{\mathbf{q},\lambda'}\}=0$$

Majorana 费米子

- 将上式与 $\psi(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_\lambda\left[u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}+v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}\right]$ 比较
- 可知 **Majorana 条件** $\psi(x)=\psi^{\mathrm{C}}(x)$ 意味着 $b_{\mathbf{p},\lambda}=a_{\mathbf{p},\lambda}$
- 因此，**Majorana 旋量场** $\psi(x)$ 的平面波展开式是

$$\psi(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm}\left[u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}+v(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}\right]$$

- **产生湮灭算符**满足**反对易关系**

$$\{a_{\mathbf{p},\lambda},a^\dagger_{\mathbf{q},\lambda'}\}=(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}),\quad \{a_{\mathbf{p},\lambda},a_{\mathbf{q},\lambda'}\}=\{a^\dagger_{\mathbf{p},\lambda},a^\dagger_{\mathbf{q},\lambda'}\}=0$$

- 类似于实标量场，Majorana 旋量场描述一种**纯中性费米子**
- 因而正费米子与反费米子相同，称为 **Majorana 费米子**
- $C^{-1}a_{\mathbf{p},\lambda}C=\zeta_C^*b_{\mathbf{p},\lambda}$、$C^{-1}b_{\mathbf{p},\lambda}C=\zeta_Ca_{\mathbf{p},\lambda}$ 和 $b_{\mathbf{p},\lambda}=a_{\mathbf{p},\lambda}$ 表明 $\zeta_C=\zeta_C^*$
- 故 $\zeta_C=\pm1$，也就是说，Majorana 旋量场的 C **宇称**要么为**偶**，要么为**奇**

Majorana 旋量场双线性型的 C 变换

- 对于 **Majorana 旋量场**，$C^{-1}\psi(x)C = \zeta_C^*\psi^{\mathrm{C}}(x)$ 和 $C^{-1}\bar{\psi}(x)C = \zeta_C\bar{\psi}^{\mathrm{C}}(x)$ 化为

$$C^{-1}\psi(x)C = \zeta_C\psi(x), \quad C^{-1}\bar{\psi}(x)C = \zeta_C\bar{\psi}(x)$$

- 在 C 变换下，由 Majorana 旋量场组成的**一般**旋量双线性型 $\bar{\psi}(x)\Gamma\psi(x)$ 变成

$$C^{-1}\bar{\psi}(x)\Gamma\psi(x)C = C^{-1}\bar{\psi}(x)C\Gamma C^{-1}\psi(x)C = \zeta_C^2\,\bar{\psi}(x)\Gamma\psi(x) = +\bar{\psi}(x)\Gamma\psi(x)$$

- 即**非平庸**的 $\bar{\psi}\Gamma\psi$ 算符的 $\boldsymbol{C}$ **宇称**必须为**偶**

Majorana 旋量场双线性型的 C 变换

- 对于 **Majorana 旋量场**，$C^{-1}\psi(x)C = \zeta_C^*\psi^{\mathrm{C}}(x)$ 和 $C^{-1}\bar{\psi}(x)C = \zeta_C\bar{\psi}^{\mathrm{C}}(x)$ 化为

$$C^{-1}\psi(x)C = \zeta_C\psi(x), \quad C^{-1}\bar{\psi}(x)C = \zeta_C\bar{\psi}(x)$$

- 在 C 变换下，由 Majorana 旋量场组成的**一般**旋量双线性型 $\bar{\psi}(x)\Gamma\psi(x)$ 变成

$$C^{-1}\bar{\psi}(x)\Gamma\psi(x)C = C^{-1}\bar{\psi}(x)C\Gamma C^{-1}\psi(x)C = \zeta_C^2\,\bar{\psi}(x)\Gamma\psi(x) = +\bar{\psi}(x)\Gamma\psi(x)$$

- 即**非平庸**的 $\bar{\psi}\Gamma\psi$ 算符的 C **宇称**必须为**偶**
- 这表明 Majorana 旋量场**不能**构成 C 宇称为**奇**的算符 $\bar{\psi}\gamma^\mu\psi$ 和 $\bar{\psi}\sigma^{\mu\nu}\psi$，即

$$\bar{\psi}(x)\gamma^\mu\psi(x) = 0, \quad \bar{\psi}(x)\sigma^{\mu\nu}\psi(x) = 0$$

- 由 Majorana 旋量场构成的**非平庸**双线性型则具有与 Dirac 旋量场**相同**的 C **变换规则**

自由 Majorana 旋量场拉氏量

- **自由 Majorana 旋量场** $\psi(x)$ 的 **Lorentz 不变拉氏量**为

$$\mathcal{L} = \frac{\mathrm{i}}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{1}{2}m\bar{\psi}\psi = \frac{\mathrm{i}}{2}\psi^{\mathrm{T}}\mathcal{C}\gamma^\mu\partial_\mu\psi - \frac{1}{2}m\psi^{\mathrm{T}}\mathcal{C}\psi$$

- 应用 Euler-Lagrange 方程求经典运动方程时，拉氏量中 ψ^{T} 扮演的角色跟 ψ **相同**
- 如果将**前后**两个旋量场分别标记为 ψ_1 和 ψ_2，动量项算符可化为

$$\begin{aligned}\psi_1^{\mathrm{T}}\mathcal{C}\gamma^\mu\partial_\mu\psi_2 &= (\psi_1^{\mathrm{T}}\mathcal{C}\gamma^\mu\partial_\mu\psi_2)^{\mathrm{T}} = -(\partial_\mu\psi_2^{\mathrm{T}})(\gamma^\mu)^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\psi_1 \\ &= (\partial_\mu\psi_2^{\mathrm{T}})\,\mathcal{C}\mathcal{C}^{-1}(\gamma^\mu)^{\mathrm{T}}\mathcal{C}\psi_1 = -(\partial_\mu\psi_2^{\mathrm{T}})\,\mathcal{C}\gamma^\mu\psi_1\end{aligned}$$

自由 Majorana 旋量场拉氏量

- **自由 Majorana 旋量场** $\psi(x)$ 的 **Lorentz 不变拉氏量**为

$$\mathcal{L} = \frac{\mathrm{i}}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{1}{2}m\bar{\psi}\psi = \frac{\mathrm{i}}{2}\psi^{\mathrm{T}}\mathcal{C}\gamma^\mu\partial_\mu\psi - \frac{1}{2}m\psi^{\mathrm{T}}\mathcal{C}\psi$$

- 应用 Euler-Lagrange 方程求经典运动方程时，拉氏量中 ψ^{T} 扮演的角色跟 ψ **相同**
- 如果将**前后**两个旋量场分别标记为 ψ_1 和 ψ_2，动量项算符可化为

$$\begin{aligned}\psi_1^{\mathrm{T}}\mathcal{C}\gamma^\mu\partial_\mu\psi_2 &= (\psi_1^{\mathrm{T}}\mathcal{C}\gamma^\mu\partial_\mu\psi_2)^{\mathrm{T}} = -(\partial_\mu\psi_2^{\mathrm{T}})(\gamma^\mu)^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\psi_1 \\ &= (\partial_\mu\psi_2^{\mathrm{T}})\,\mathcal{C}\mathcal{C}^{-1}(\gamma^\mu)^{\mathrm{T}}\mathcal{C}\psi_1 = -(\partial_\mu\psi_2^{\mathrm{T}})\,\mathcal{C}\gamma^\mu\psi_1\end{aligned}$$

- 质量项算符可化为 $\psi_1^{\mathrm{T}}\mathcal{C}\psi_2 = (\psi_1^{\mathrm{T}}\mathcal{C}\psi_2)^{\mathrm{T}} = -\psi_2^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\psi_1 = \psi_2^{\mathrm{T}}\mathcal{C}\psi_1$，则

$$\frac{\partial\mathcal{L}}{\partial\psi_1} = -\frac{\mathrm{i}}{2}(\partial_\mu\psi_2^{\mathrm{T}})\,\mathcal{C}\gamma^\mu - \frac{1}{2}m\psi_2^{\mathrm{T}}\mathcal{C}, \quad \frac{\partial\mathcal{L}}{\partial\psi_2} = -\frac{1}{2}m\psi_1^{\mathrm{T}}\mathcal{C}$$

- ψ_1 和 ψ_2 **都是** ψ，因而

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \frac{\mathrm{i}}{2}\psi^{\mathrm{T}}\mathcal{C}\gamma^\mu, \quad \frac{\partial\mathcal{L}}{\partial\psi} = \frac{\partial\mathcal{L}}{\partial\psi_1} + \frac{\partial\mathcal{L}}{\partial\psi_2} = -\frac{\mathrm{i}}{2}(\partial_\mu\psi^{\mathrm{T}})\,\mathcal{C}\gamma^\mu - m\psi^{\mathrm{T}}\mathcal{C}$$

自由 Majorana 旋量场运动方程

- 根据 $\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \dfrac{\mathrm{i}}{2}\,\psi^{\mathrm{T}}\mathcal{C}\gamma^\mu$ 和 $\dfrac{\partial\mathcal{L}}{\partial\psi} = -\dfrac{\mathrm{i}}{2}(\partial_\mu\psi^{\mathrm{T}})\,\mathcal{C}\gamma^\mu - m\psi^{\mathrm{T}}\mathcal{C}$
- **Euler-Lagrange 方程**给出 $0 = \partial_\mu\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} - \dfrac{\partial\mathcal{L}}{\partial\psi} = \mathrm{i}(\partial_\mu\psi^{\mathrm{T}})\,\mathcal{C}\gamma^\mu + m\psi^{\mathrm{T}}\mathcal{C}$
- 对上式转置，并利用 $(\gamma^\mu)^{\mathrm{T}}\mathcal{C}^{\mathrm{T}} = -\mathcal{C}\mathcal{C}^{-1}(\gamma^\mu)^{\mathrm{T}}\mathcal{C} = \mathcal{C}\gamma^\mu = -\mathcal{C}^{\mathrm{T}}\gamma^\mu$，推出

$$0 = \mathrm{i}(\gamma^\mu)^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\partial_\mu\psi + m\,\mathcal{C}^{\mathrm{T}}\psi = \mathcal{C}^{\mathrm{T}}(-\mathrm{i}\gamma^\mu\partial_\mu\psi + m\psi)$$

自由 Majorana 旋量场运动方程

- 根据 $\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}=\dfrac{\mathrm{i}}{2}\psi^{\mathrm{T}}\mathcal{C}\gamma^\mu$ 和 $\dfrac{\partial\mathcal{L}}{\partial\psi}=-\dfrac{\mathrm{i}}{2}(\partial_\mu\psi^{\mathrm{T}})\,\mathcal{C}\gamma^\mu-m\psi^{\mathrm{T}}\mathcal{C}$
- **Euler-Lagrange 方程**给出 $0=\partial_\mu\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}-\dfrac{\partial\mathcal{L}}{\partial\psi}=\mathrm{i}(\partial_\mu\psi^{\mathrm{T}})\,\mathcal{C}\gamma^\mu+m\psi^{\mathrm{T}}\mathcal{C}$
- 对上式转置，并利用 $(\gamma^\mu)^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}=-\mathcal{C}\mathcal{C}^{-1}(\gamma^\mu)^{\mathrm{T}}\mathcal{C}=\mathcal{C}\gamma^\mu=-\mathcal{C}^{\mathrm{T}}\gamma^\mu$，推出

$$0=\mathrm{i}(\gamma^\mu)^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}\partial_\mu\psi+m\,\mathcal{C}^{\mathrm{T}}\psi=\mathcal{C}^{\mathrm{T}}(-\mathrm{i}\gamma^\mu\partial_\mu\psi+m\psi)$$

- 可见，自由的 Majorana 旋量场 $\psi(x)$ 也满足 **Dirac 方程**

$$(\mathrm{i}\gamma^\mu\partial_\mu-m)\psi(x)=0$$

- 这与上述**平面波展开式**

$$\psi(x)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm}\left[u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}+v(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}\right]$$

相容

8.2.3 小节 旋量场的 P 变换

- 下面讨论**自由 Dirac 旋量场** $\psi(x)$ 的 P **变换**
- 宇称变换**反转**动量方向，但保持角动量**不变**，因而会**翻转**螺旋度 λ 的符号
- 平面波旋量系数 $u(\mathbf{p},\lambda)$ 和 $v(\mathbf{p},\lambda)$ 都是用**螺旋态** $\xi_\lambda(\mathbf{p})$ 表达出来的
- 而**宇称变换**联系着螺旋态 $\xi_\lambda(\mathbf{p})$ 和 $\xi_{-\lambda}(-\mathbf{p})$

8.2.3 小节 旋量场的 P 变换

- 下面讨论**自由 Dirac 旋量场** $\psi(x)$ 的 P **变换**
- 宇称变换**反转**动量方向，但保持角动量**不变**，因而会**翻转**螺旋度 λ 的符号
- 平面波旋量系数 $u(\mathbf{p},\lambda)$ 和 $v(\mathbf{p},\lambda)$ 都是用**螺旋态** $\xi_\lambda(\mathbf{p})$ 表达出来的
- 而**宇称变换**联系着螺旋态 $\xi_\lambda(\mathbf{p})$ 和 $\xi_{-\lambda}(-\mathbf{p})$
- 4.4.2 小节推出的 $(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_{-\lambda}(-\mathbf{p})=\lambda\xi_{-\lambda}(-\mathbf{p})$ 表明，$\xi_{-\lambda}(-\mathbf{p})$ 和 $\xi_\lambda(\mathbf{p})$ 都是本征值方程 $(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})\xi_\lambda(\mathbf{p})=\lambda\xi_\lambda(\mathbf{p})$ 的归一化解，因而两者至多相差一个**相位因子**
- 这个相位因子的具体形式与螺旋态 $\xi_\lambda(\mathbf{p})$ 的约定有关
- 将螺旋态 $\xi_+(\mathbf{p})$ 化为

$$\xi_+(\mathbf{p})=\frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix}|\mathbf{p}|+p^3\\ p^1+\mathrm{i}p^2\end{pmatrix}=\frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix}\frac{|\mathbf{p}|+p^3}{p^1-\mathrm{i}p^2}(p^1-\mathrm{i}p^2)\\ \frac{p^1+\mathrm{i}p^2}{|\mathbf{p}|-p^3}(|\mathbf{p}|-p^3)\end{pmatrix}$$

$$=\frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix}\frac{(|\mathbf{p}|+p^3)(p^1+\mathrm{i}p^2)}{(p^1)^2+(p^2)^2}(p^1-\mathrm{i}p^2)\\ \frac{p^1+\mathrm{i}p^2}{|\mathbf{p}|-p^3}(|\mathbf{p}|-p^3)\end{pmatrix}$$

螺旋态的宇称变换相位因子

🐴 利用 $(|\mathbf{p}|+p^3)(|\mathbf{p}|-p^3)=|\mathbf{p}|^2-(p^3)^2=(p^1)^2+(p^2)^2$，推出

$$
\begin{aligned}
\xi_+(\mathbf{p}) &= \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix}\frac{(|\mathbf{p}|+p^3)(p^1+\mathrm{i}p^2)}{(p^1)^2+(p^2)^2}(p^1-\mathrm{i}p^2)\\ \frac{p^1+\mathrm{i}p^2}{|\mathbf{p}|-p^3}(|\mathbf{p}|-p^3)\end{pmatrix}\\
&= \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\frac{p^1+\mathrm{i}p^2}{|\mathbf{p}|-p^3}\begin{pmatrix}p^1-\mathrm{i}p^2\\ |\mathbf{p}|-p^3\end{pmatrix}\\
&= \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|-p^3)}}\frac{p^1+\mathrm{i}p^2}{\sqrt{(p^1)^2+(p^2)^2}}\begin{pmatrix}p^1-\mathrm{i}p^2\\ |\mathbf{p}|-p^3\end{pmatrix}=\kappa_{\mathbf{p},+}\xi_-(-\mathbf{p})
\end{aligned}
$$

螺旋态的宇称变换相位因子

- 利用 $(|\mathbf{p}|+p^3)(|\mathbf{p}|-p^3)=|\mathbf{p}|^2-(p^3)^2=(p^1)^2+(p^2)^2$，推出

$$\begin{aligned}\xi_+(\mathbf{p}) &= \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\begin{pmatrix}\frac{(|\mathbf{p}|+p^3)(p^1+\mathrm{i}p^2)}{(p^1)^2+(p^2)^2}(p^1-\mathrm{i}p^2)\\ \frac{p^1+\mathrm{i}p^2}{|\mathbf{p}|-p^3}(|\mathbf{p}|-p^3)\end{pmatrix}\\ &= \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p^3)}}\frac{p^1+\mathrm{i}p^2}{|\mathbf{p}|-p^3}\begin{pmatrix}p^1-\mathrm{i}p^2\\ |\mathbf{p}|-p^3\end{pmatrix}\\ &= \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|-p^3)}}\frac{p^1+\mathrm{i}p^2}{\sqrt{(p^1)^2+(p^2)^2}}\begin{pmatrix}p^1-\mathrm{i}p^2\\ |\mathbf{p}|-p^3\end{pmatrix}=\kappa_{\mathbf{p},+}\xi_-(-\mathbf{p})\end{aligned}$$

- 其中 $\kappa_{\mathbf{p},+}$ 是相位因子 $\kappa_{\mathbf{p},\lambda}\equiv\dfrac{\lambda p^1+\mathrm{i}p^2}{\sqrt{(p^1)^2+(p^2)^2}}$ 在 $\lambda=+$ 时的特例
- 再由 $|\kappa_{\mathbf{p},\lambda}|^2=\kappa^*_{\mathbf{p},+}\kappa_{\mathbf{p},+}=1$ 推出 $\xi_-(-\mathbf{p})=\kappa^*_{\mathbf{p},+}\xi_+(\mathbf{p})$，故

$$\xi_-(\mathbf{p})=\kappa^*_{-\mathbf{p},+}\xi_+(-\mathbf{p})=\frac{-p^1+\mathrm{i}p^2}{\sqrt{(p^1)^2+(p^2)^2}}\xi_+(-\mathbf{p})=\kappa_{\mathbf{p},-}\xi_+(-\mathbf{p})$$

- 归纳起来，有 $\xi_\lambda(\mathbf{p})=\kappa_{\mathbf{p},\lambda}\xi_{-\lambda}(-\mathbf{p})$，可见 $\kappa_{\mathbf{p},\lambda}$ 就是想要得到的**相位因子**

$\gamma^0 u(\mathbf{p},\lambda)$ 和 $\gamma^0 v(\mathbf{p},\lambda)$

由于 $\xi_\lambda(\mathbf{p}) = \kappa_{\mathbf{p},\lambda}\xi_{-\lambda}(-\mathbf{p})$，**Weyl 表象**中的 $u(\mathbf{p},\lambda)$ 和 $v(\mathbf{p},\lambda)$ 满足

$$\gamma^0 u(\mathbf{p},\lambda) = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p}) \\ \omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p}) \end{pmatrix} = \kappa_{\mathbf{p},\lambda}\begin{pmatrix} \omega_\lambda(-\mathbf{p})\xi_{-\lambda}(-\mathbf{p}) \\ \omega_{-\lambda}(-\mathbf{p})\xi_{-\lambda}(-\mathbf{p}) \end{pmatrix}$$

$$\gamma^0 v(\mathbf{p},\lambda) = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} -\lambda\omega_\lambda(\mathbf{p})\xi_{-\lambda}(\mathbf{p}) \\ \lambda\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \lambda\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}(\mathbf{p}) \\ -\lambda\omega_\lambda(\mathbf{p})\xi_{-\lambda}(\mathbf{p}) \end{pmatrix}$$

$$= \kappa_{\mathbf{p},-\lambda}\begin{pmatrix} \lambda\omega_{-\lambda}(-\mathbf{p})\xi_\lambda(-\mathbf{p}) \\ -\lambda\omega_\lambda(-\mathbf{p})\xi_\lambda(-\mathbf{p}) \end{pmatrix}$$

注意到

$$\kappa_{\mathbf{p},-\lambda} = \frac{-\lambda p^1 + \mathrm{i}p^2}{\sqrt{(p^1)^2+(p^2)^2}} = -\frac{\lambda p^1 - \mathrm{i}p^2}{\sqrt{(p^1)^2+(p^2)^2}} = -\kappa^*_{\mathbf{p},\lambda}$$

有

$$\gamma^0 u(\mathbf{p},\lambda) = \kappa_{\mathbf{p},\lambda} u(-\mathbf{p},-\lambda), \quad \gamma^0 v(\mathbf{p},\lambda) = -\kappa^*_{\mathbf{p},\lambda} v(-\mathbf{p},-\lambda)$$

产生湮灭算符的 P 变换

- 为得到自洽的宇称变换结果，将正反费米子态 $|\mathbf{p}^+,\lambda\rangle$ 和 $|\mathbf{p}^-,\lambda\rangle$ 的 P 变换设为

$$P\,|\mathbf{p}^+,\lambda\rangle = \zeta_P \kappa^*_{-\mathbf{p},-\lambda}\,|-\mathbf{p}^+,-\lambda\rangle, \quad P\,|\mathbf{p}^-,\lambda\rangle = \tilde{\zeta}_P \kappa^*_{-\mathbf{p},-\lambda}\,|-\mathbf{p}^-,-\lambda\rangle$$

- 其中 ζ_P 和 $\tilde{\zeta}_P$ 是两个相位因子，从而推出 $a^\dagger_{\mathbf{p},\lambda}$ 和 $a_{\mathbf{p},\lambda}$ 的 P 变换

$$P^{-1}a^\dagger_{\mathbf{p},\lambda}P = \zeta_P \kappa^*_{-\mathbf{p},-\lambda} a^\dagger_{-\mathbf{p},-\lambda}, \quad P^{-1}a_{\mathbf{p},\lambda}P = \zeta^*_P \kappa_{-\mathbf{p},-\lambda} a_{-\mathbf{p},-\lambda}$$

- 以及 $b^\dagger_{\mathbf{p},\lambda}$ 和 $b_{\mathbf{p},\lambda}$ 的 P 变换

$$P^{-1}b^\dagger_{\mathbf{p},\lambda}P = \tilde{\zeta}_P \kappa^*_{-\mathbf{p},-\lambda} b^\dagger_{-\mathbf{p},-\lambda}, \quad P^{-1}b_{\mathbf{p},\lambda}P = \tilde{\zeta}^*_P \kappa_{-\mathbf{p},-\lambda} b_{-\mathbf{p},-\lambda}$$

产生湮灭算符的 P 变换

为得到自洽的宇称变换结果，将正反费米子态 $|\mathbf{p}^+,\lambda\rangle$ 和 $|\mathbf{p}^-,\lambda\rangle$ 的 P 变换设为

$$P\,|\mathbf{p}^+,\lambda\rangle = \zeta_P \kappa^*_{-\mathbf{p},-\lambda}\,|-\mathbf{p}^+,-\lambda\rangle, \quad P\,|\mathbf{p}^-,\lambda\rangle = \tilde{\zeta}_P \kappa^*_{-\mathbf{p},-\lambda}\,|-\mathbf{p}^-,-\lambda\rangle$$

其中 ζ_P 和 $\tilde{\zeta}_P$ 是两个相位因子，从而推出 $a^\dagger_{\mathbf{p},\lambda}$ 和 $a_{\mathbf{p},\lambda}$ 的 P 变换

$$P^{-1}a^\dagger_{\mathbf{p},\lambda}P = \zeta_P \kappa^*_{-\mathbf{p},-\lambda} a^\dagger_{-\mathbf{p},-\lambda}, \quad P^{-1}a_{\mathbf{p},\lambda}P = \zeta^*_P \kappa_{-\mathbf{p},-\lambda} a_{-\mathbf{p},-\lambda}$$

以及 $b^\dagger_{\mathbf{p},\lambda}$ 和 $b_{\mathbf{p},\lambda}$ 的 P 变换

$$P^{-1}b^\dagger_{\mathbf{p},\lambda}P = \tilde{\zeta}_P \kappa^*_{-\mathbf{p},-\lambda} b^\dagger_{-\mathbf{p},-\lambda}, \quad P^{-1}b_{\mathbf{p},\lambda}P = \tilde{\zeta}^*_P \kappa_{-\mathbf{p},-\lambda} b_{-\mathbf{p},-\lambda}$$

Dirac 旋量场平面波展开式的 P 变换为

$$P^{-1}\psi(x)P = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_\lambda \left[u(\mathbf{p},\lambda)P^{-1}a_{\mathbf{p},\lambda}P\mathrm{e}^{-\mathrm{i}p\cdot x} + v(\mathbf{p},\lambda)P^{-1}b^\dagger_{\mathbf{p},\lambda}P\mathrm{e}^{\mathrm{i}p\cdot x}\right]$$

$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_\lambda \left[\zeta^*_P \kappa_{-\mathbf{p},-\lambda} u(\mathbf{p},\lambda) a_{-\mathbf{p},-\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x} + \tilde{\zeta}_P \kappa^*_{-\mathbf{p},-\lambda} v(\mathbf{p},\lambda) b^\dagger_{-\mathbf{p},-\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}\right]$$

$\psi(x)$ 的 P 变换

作变量替换 $\mathbf{p} \to -\mathbf{p}$ 和 $\lambda \to -\lambda$，得

$$
\begin{aligned}
& P^{-1}\psi(x)P \\
= & \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_\lambda \left[\zeta_P^* \kappa_{-\mathbf{p},-\lambda} u(\mathbf{p},\lambda) a_{-\mathbf{p},-\lambda} \mathrm{e}^{-\mathrm{i}p\cdot x} + \tilde{\zeta}_P \kappa_{-\mathbf{p},-\lambda}^* v(\mathbf{p},\lambda) b_{-\mathbf{p},-\lambda}^\dagger \mathrm{e}^{\mathrm{i}p\cdot x}\right] \\
= & \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_\lambda \left[\zeta_P^* \kappa_{\mathbf{p},\lambda} u(-\mathbf{p},-\lambda) a_{\mathbf{p},\lambda} \mathrm{e}^{-\mathrm{i}p\cdot(\mathcal{P}x)} + \tilde{\zeta}_P \kappa_{\mathbf{p},\lambda}^* v(-\mathbf{p},-\lambda) b_{\mathbf{p},\lambda}^\dagger \mathrm{e}^{\mathrm{i}p\cdot(\mathcal{P}x)}\right] \\
= & \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_\lambda \left[\zeta_P^* \gamma^0 u(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} \mathrm{e}^{-\mathrm{i}p\cdot(\mathcal{P}x)} - \tilde{\zeta}_P \gamma^0 v(\mathbf{p},\lambda) b_{\mathbf{p},\lambda}^\dagger \mathrm{e}^{\mathrm{i}p\cdot(\mathcal{P}x)}\right]
\end{aligned}
$$

最后一步用到 $\gamma^0 u(\mathbf{p},\lambda) = \kappa_{\mathbf{p},\lambda} u(-\mathbf{p},-\lambda)$ 和 $\gamma^0 v(\mathbf{p},\lambda) = -\kappa_{\mathbf{p},\lambda}^* v(-\mathbf{p},-\lambda)$

$\psi(x)$ 的 P 变换

- 作变量替换 $\mathbf{p} \to -\mathbf{p}$ 和 $\lambda \to -\lambda$，得

$$
\begin{aligned}
& P^{-1}\psi(x)P \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_\lambda \left[\zeta_P^* \kappa_{-\mathbf{p},-\lambda} u(\mathbf{p},\lambda) a_{-\mathbf{p},-\lambda} \mathrm{e}^{-\mathrm{i}p\cdot x} + \tilde{\zeta}_P \kappa_{-\mathbf{p},-\lambda}^* v(\mathbf{p},\lambda) b_{-\mathbf{p},-\lambda}^\dagger \mathrm{e}^{\mathrm{i}p\cdot x}\right] \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_\lambda \left[\zeta_P^* \kappa_{\mathbf{p},\lambda} u(-\mathbf{p},-\lambda) a_{\mathbf{p},\lambda} \mathrm{e}^{-\mathrm{i}p\cdot(\mathcal{P}x)} + \tilde{\zeta}_P \kappa_{\mathbf{p},\lambda}^* v(-\mathbf{p},-\lambda) b_{\mathbf{p},\lambda}^\dagger \mathrm{e}^{\mathrm{i}p\cdot(\mathcal{P}x)}\right] \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_\lambda \left[\zeta_P^* \gamma^0 u(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} \mathrm{e}^{-\mathrm{i}p\cdot(\mathcal{P}x)} - \tilde{\zeta}_P \gamma^0 v(\mathbf{p},\lambda) b_{\mathbf{p},\lambda}^\dagger \mathrm{e}^{\mathrm{i}p\cdot(\mathcal{P}x)}\right]
\end{aligned}
$$

- 最后一步用到 $\gamma^0 u(\mathbf{p},\lambda) = \kappa_{\mathbf{p},\lambda} u(-\mathbf{p},-\lambda)$ 和 $\gamma^0 v(\mathbf{p},\lambda) = -\kappa_{\mathbf{p},\lambda}^* v(-\mathbf{p},-\lambda)$
- 为了保持 $\psi(x)$ 的运动方程形式不变，必须要求 $\zeta_P^* = -\tilde{\zeta}_P$，使 $b_{\mathbf{p},\lambda}^\dagger$ 和 $b_{\mathbf{p},\lambda}$ 的 P 变换化为 $P^{-1} b_{\mathbf{p},\lambda}^\dagger P = -\zeta_P^* \kappa_{-\mathbf{p},-\lambda}^* b_{-\mathbf{p},-\lambda}^\dagger$ 和 $P^{-1} b_{\mathbf{p},\lambda} P = -\zeta_P \kappa_{-\mathbf{p},-\lambda} b_{-\mathbf{p},-\lambda}$
- 从而 $\psi(x)$ 的 P 变换为 $P^{-1}\psi(x)P = D(\mathcal{P})\psi(\mathcal{P}x)$
- 其中 $D(\mathcal{P}) \equiv \zeta_P^* \gamma^0$ 是旋量空间中的宇称变换矩阵

$\bar{\psi}(x)$ 的 P 变换

- **宇称变换矩阵** $D(\mathcal{P}) = \zeta_P^* \gamma^0$ 在 4.1 节中写作 $D(\mathcal{P}) = \zeta\gamma^0$，比较得 $\zeta_P^* = \zeta$
- 现在，作宇称变换后，$\psi'(x') = P^{-1}\psi(x')P = D(\mathcal{P})\psi(\mathcal{P}x') = D(\mathcal{P})\psi(x)$ **也满足 Dirac 方程**，

$$
\begin{aligned}
(\mathrm{i}\gamma^\mu\partial'_\mu - m)\psi'(x') &= [\mathrm{i}\gamma^\mu(\mathcal{P}^{-1})^\nu{}_\mu\partial_\nu - m]D(\mathcal{P})\psi(x) \\
&= D(\mathcal{P})[\mathrm{i}D^{-1}(\mathcal{P})\gamma^\mu D(\mathcal{P})(\mathcal{P}^{-1})^\nu{}_\mu\partial_\nu - m]\psi(x) \\
&= D(\mathcal{P})[\mathrm{i}\mathcal{P}^\mu{}_\rho\gamma^\rho(\mathcal{P}^{-1})^\nu{}_\mu\partial_\nu - m]\psi(x) = D(\mathcal{P})(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi(x) = 0
\end{aligned}
$$

- 第三步用到 $D^{-1}(\mathcal{P})\gamma^\mu D(\mathcal{P}) = \mathcal{P}^\mu{}_\nu\gamma^\nu$

$\bar{\psi}(x)$ 的 P 变换

- 宇称变换矩阵 $D(\mathcal{P}) = \zeta_P^* \gamma^0$ 在 4.1 节中写作 $D(\mathcal{P}) = \zeta\gamma^0$，比较得 $\zeta_P^* = \zeta$
- 现在，作宇称变换后，$\psi'(x') = P^{-1}\psi(x')P = D(\mathcal{P})\psi(\mathcal{P}x') = D(\mathcal{P})\psi(x)$ 也满足 Dirac 方程，

$$
\begin{aligned}
&(\mathrm{i}\gamma^\mu\partial'_\mu - m)\psi'(x') = [\mathrm{i}\gamma^\mu(\mathcal{P}^{-1})^\nu{}_\mu\partial_\nu - m]D(\mathcal{P})\psi(x) \\
=\ & D(\mathcal{P})[\mathrm{i}D^{-1}(\mathcal{P})\gamma^\mu D(\mathcal{P})(\mathcal{P}^{-1})^\nu{}_\mu\partial_\nu - m]\psi(x) \\
=\ & D(\mathcal{P})[\mathrm{i}\mathcal{P}^\mu{}_\rho\gamma^\rho(\mathcal{P}^{-1})^\nu{}_\mu\partial_\nu - m]\psi(x) = D(\mathcal{P})(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi(x) = 0
\end{aligned}
$$

- 第三步用到 $D^{-1}(\mathcal{P})\gamma^\mu D(\mathcal{P}) = \mathcal{P}^\mu{}_\nu\gamma^\nu$
- 宇称变换矩阵 $D(\mathcal{P})$ 是**幺正**的，满足 $D^{-1}(\mathcal{P}) = D^\dagger(\mathcal{P}) = \zeta_P\gamma^0$
- 由 $P^{-1}\psi^\dagger(x)P = [P^{-1}\psi(x)P]^\dagger = [\zeta_P^*\gamma^0\psi(\mathcal{P}x)]^\dagger = \zeta_P\psi^\dagger(\mathcal{P}x)\gamma^0$ 得

$$
P^{-1}\bar{\psi}(x)P = P^{-1}\psi^\dagger(x)P\,\gamma^0 = \zeta_P\psi^\dagger(\mathcal{P}x)\gamma^0\gamma^0 = \bar{\psi}(\mathcal{P}x)\zeta_P\gamma^0
$$

- 即 $\bar{\psi}(x)$ 的 P 变换为

$$
P^{-1}\bar{\psi}(x)P = \bar{\psi}(\mathcal{P}x)D^{-1}(\mathcal{P})
$$

旋量双线性型的 P 变换

- 从而，一般旋量双线性型 $\bar{\psi}(x)\Gamma\psi(x)$ 的 P 变换为

$$P^{-1}\bar{\psi}(x)\Gamma\psi(x)P = P^{-1}\bar{\psi}(x)P\Gamma P^{-1}\psi(x)P = \bar{\psi}(\mathcal{P}x)D^{-1}(\mathcal{P})\Gamma D(\mathcal{P})\psi(\mathcal{P}x)$$

- 利用在 4.1 节推导出来的结果，$D^{-1}(\mathcal{P})\mathbf{1}D(\mathcal{P}) = +\mathbf{1}$

$$D^{-1}(\mathcal{P})\mathrm{i}\gamma^5 D(\mathcal{P}) = -\mathrm{i}\gamma^5, \qquad D^{-1}(\mathcal{P})\gamma^\mu D(\mathcal{P}) = \mathcal{P}^\mu{}_\nu\gamma^\nu$$

$$D^{-1}(\mathcal{P})\gamma^\mu\gamma^5 D(\mathcal{P}) = -\mathcal{P}^\mu{}_\nu\gamma^\nu\gamma^5, \quad D^{-1}(\mathcal{P})\sigma^{\mu\nu}D(\mathcal{P}) = \mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma\sigma^{\rho\sigma}$$

旋量双线性型的 P 变换

- 从而，一般旋量双线性型 $\bar{\psi}(x)\Gamma\psi(x)$ 的 P 变换为

$$P^{-1}\bar{\psi}(x)\Gamma\psi(x)P = P^{-1}\bar{\psi}(x)P\Gamma P^{-1}\psi(x)P = \bar{\psi}(\mathcal{P}x)D^{-1}(\mathcal{P})\Gamma D(\mathcal{P})\psi(\mathcal{P}x)$$

- 利用在 4.1 节推导出来的结果，$D^{-1}(\mathcal{P})\mathbf{1}D(\mathcal{P}) = +\mathbf{1}$

$$D^{-1}(\mathcal{P})\mathrm{i}\gamma^5 D(\mathcal{P}) = -\mathrm{i}\gamma^5, \qquad D^{-1}(\mathcal{P})\gamma^\mu D(\mathcal{P}) = \mathcal{P}^\mu{}_\nu\gamma^\nu$$

$$D^{-1}(\mathcal{P})\gamma^\mu\gamma^5 D(\mathcal{P}) = -\mathcal{P}^\mu{}_\nu\gamma^\nu\gamma^5, \quad D^{-1}(\mathcal{P})\sigma^{\mu\nu}D(\mathcal{P}) = \mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma\sigma^{\rho\sigma}$$

- 得到

$$\begin{aligned}
P^{-1}\bar{\psi}(x)\psi(x)P &= +\bar{\psi}(\mathcal{P}x)\psi(\mathcal{P}x)\\
P^{-1}\bar{\psi}(x)\mathrm{i}\gamma^5\psi(x)P &= -\bar{\psi}(\mathcal{P}x)\mathrm{i}\gamma^5\psi(\mathcal{P}x)\\
P^{-1}\bar{\psi}(x)\gamma^\mu\psi(x)P &= \mathcal{P}^\mu{}_\nu\bar{\psi}(\mathcal{P}x)\gamma^\nu\psi(\mathcal{P}x)\\
P^{-1}\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)P &= -\mathcal{P}^\mu{}_\nu\bar{\psi}(\mathcal{P}x)\gamma^\nu\gamma^5\psi(\mathcal{P}x)\\
P^{-1}\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)P &= \mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma\bar{\psi}(\mathcal{P}x)\sigma^{\rho\sigma}\psi(\mathcal{P}x)
\end{aligned}$$

- 因此，$\bar{\psi}\psi$ 是狭义的标量算符，$\bar{\psi}\mathrm{i}\gamma^5\psi$ 是赝标量算符
- $\bar{\psi}\gamma^\mu\psi$ 是极矢量算符，$\bar{\psi}\gamma^\mu\gamma^5\psi$ 是轴矢量算符

手征旋量双线性型的 P 变换

- 进一步推出

$$
\begin{aligned}
P^{-1}\bar{\psi}_{\mathrm{R}}(x)\psi_{\mathrm{L}}(x)P &= \bar{\psi}_{\mathrm{L}}(\mathcal{P}x)\psi_{\mathrm{R}}(\mathcal{P}x) \\
P^{-1}\bar{\psi}_{\mathrm{L}}(x)\psi_{\mathrm{R}}(x)P &= \bar{\psi}_{\mathrm{R}}(\mathcal{P}x)\psi_{\mathrm{L}}(\mathcal{P}x) \\
P^{-1}\bar{\psi}_{\mathrm{L}}(x)\gamma^{\mu}\psi_{\mathrm{L}}(x)P &= \mathcal{P}^{\mu}{}_{\nu}\bar{\psi}_{\mathrm{R}}(\mathcal{P}x)\gamma^{\nu}\psi_{\mathrm{R}}(\mathcal{P}x) \\
P^{-1}\bar{\psi}_{\mathrm{R}}(x)\gamma^{\mu}\psi_{\mathrm{R}}(x)P &= \mathcal{P}^{\mu}{}_{\nu}\bar{\psi}_{\mathrm{L}}(\mathcal{P}x)\gamma^{\nu}\psi_{\mathrm{L}}(\mathcal{P}x)
\end{aligned}
$$

- **左手流算符** $\bar{\psi}_{\mathrm{L}}\gamma^{\mu}\psi_{\mathrm{L}}$ 和**右手流算符** $\bar{\psi}_{\mathrm{R}}\gamma^{\mu}\psi_{\mathrm{R}}$ 在 P 变换下**相互转化**
- 在**弱相互作用**中，**左手流算符**和**右手流算符**参与**不同**的规范相互作用，因而**空间反射对称性遭到破坏**

手征旋量双线性型的 P 变换

- 进一步推出

$$
\begin{aligned}
P^{-1}\bar{\psi}_{\mathrm{R}}(x)\psi_{\mathrm{L}}(x)P &= \bar{\psi}_{\mathrm{L}}(\mathcal{P}x)\psi_{\mathrm{R}}(\mathcal{P}x) \\
P^{-1}\bar{\psi}_{\mathrm{L}}(x)\psi_{\mathrm{R}}(x)P &= \bar{\psi}_{\mathrm{R}}(\mathcal{P}x)\psi_{\mathrm{L}}(\mathcal{P}x) \\
P^{-1}\bar{\psi}_{\mathrm{L}}(x)\gamma^{\mu}\psi_{\mathrm{L}}(x)P &= \mathcal{P}^{\mu}{}_{\nu}\bar{\psi}_{\mathrm{R}}(\mathcal{P}x)\gamma^{\nu}\psi_{\mathrm{R}}(\mathcal{P}x) \\
P^{-1}\bar{\psi}_{\mathrm{R}}(x)\gamma^{\mu}\psi_{\mathrm{R}}(x)P &= \mathcal{P}^{\mu}{}_{\nu}\bar{\psi}_{\mathrm{L}}(\mathcal{P}x)\gamma^{\nu}\psi_{\mathrm{L}}(\mathcal{P}x)
\end{aligned}
$$

- **左手流算符** $\bar{\psi}_{\mathrm{L}}\gamma^{\mu}\psi_{\mathrm{L}}$ 和**右手流算符** $\bar{\psi}_{\mathrm{R}}\gamma^{\mu}\psi_{\mathrm{R}}$ 在 P 变换下**相互转化**
- 在**弱相互作用**中，**左手流算符**和**右手流算符**参与**不同**的规范相互作用，因而**空间反射对称性遭到破坏**
- **动能项算符** $\mathrm{i}\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi$ 的 P 变换为

$$
\begin{aligned}
P^{-1}\mathrm{i}\bar{\psi}(x)\gamma^{\mu}\partial_{\mu}\psi(x)P &= \mathrm{i}\bar{\psi}(\mathcal{P}x)D^{-1}(\mathcal{P})\gamma^{\mu}D(\mathcal{P})(\mathcal{P}^{-1})^{\nu}{}_{\mu}\partial_{\nu}\psi(\mathcal{P}x) \\
&= \mathrm{i}\bar{\psi}(\mathcal{P}x)\mathcal{P}^{\mu}{}_{\rho}\gamma^{\rho}(\mathcal{P}^{-1})^{\nu}{}_{\mu}\partial_{\nu}\psi(\mathcal{P}x) = +\mathrm{i}\bar{\psi}(\mathcal{P}x)\gamma^{\mu}\partial_{\mu}\psi(\mathcal{P}x)
\end{aligned}
$$

- 因此，自由 Dirac 旋量场的**拉氏量**在宇称变换下**不变**

正反费米子态的 P 变换

- 在质心系中考虑一对正反费米子 $\psi\bar{\psi}$ 组成的系统
- 设两个粒子的螺旋度相反，轨道角动量量子数为 L 的态矢为

$$|\psi\bar{\psi}\rangle = \sum_{\lambda}\int \mathrm{d}^3p\,\Phi(\mathbf{p})a^{\dagger}_{\mathbf{p},\lambda}b^{\dagger}_{-\mathbf{p},-\lambda}|0\rangle\,,\quad \Phi(-\mathbf{p}) = (-)^{L}\Phi(\mathbf{p})$$

- 相应的 P 变换是

$$\begin{aligned}
P|\psi\bar{\psi}\rangle &= \sum_{\lambda}\int \mathrm{d}^3p\,\Phi(\mathbf{p})P^{-1}a^{\dagger}_{\mathbf{p},\lambda}PP^{-1}b^{\dagger}_{-\mathbf{p},-\lambda}P|0\rangle \\
&= -|\zeta_P|^2\sum_{\lambda}\int \mathrm{d}^3p\,\Phi(\mathbf{p})\kappa^{*}_{-\mathbf{p},-\lambda}\kappa^{*}_{\mathbf{p},\lambda}a^{\dagger}_{-\mathbf{p},-\lambda}b^{\dagger}_{\mathbf{p},\lambda}|0\rangle \\
&= -\sum_{\lambda}\int \mathrm{d}^3p\,\Phi(-\mathbf{p})\kappa^{*}_{\mathbf{p},\lambda}\kappa^{*}_{-\mathbf{p},-\lambda}a^{\dagger}_{\mathbf{p},\lambda}b^{\dagger}_{-\mathbf{p},-\lambda}|0\rangle
\end{aligned}$$

- 最后一步作了变量替换 $\mathbf{p}\to-\mathbf{p}$ 和 $\lambda\to-\lambda$

内禀宇称

- 利用 $\kappa^*_{\mathbf{p},\lambda}\kappa^*_{-\mathbf{p},-\lambda} = \dfrac{\lambda p^1 - \mathrm{i}p^2}{\sqrt{(p^1)^2+(p^2)^2}}\dfrac{\lambda p^1 + \mathrm{i}p^2}{\sqrt{(p^1)^2+(p^2)^2}} = 1$，推出

$$
\begin{aligned}
P\left|\psi\bar{\psi}\right\rangle &= -\sum_\lambda \int \mathrm{d}^3p\, \Phi(-\mathbf{p})\kappa^*_{\mathbf{p},\lambda}\kappa^*_{-\mathbf{p},-\lambda} a^\dagger_{\mathbf{p},\lambda} b^\dagger_{-\mathbf{p},-\lambda}\left|0\right\rangle \\
&= -(-)^L \sum_\lambda \int \mathrm{d}^3p\, \Phi(\mathbf{p}) a^\dagger_{\mathbf{p},\lambda} b^\dagger_{-\mathbf{p},-\lambda}\left|0\right\rangle = (-)^{L+1}\left|\psi\bar{\psi}\right\rangle
\end{aligned}
$$

- 可见，$\left|\psi\bar{\psi}\right\rangle$ 的**总宇称**为 $(-)^{L+1}$，包含**轨道宇称** $(-)^L$ 和**内禀宇称** $-$
- 也就是说，一对正反费米子的**内禀宇称**为**奇**

内禀宇称

利用 $\kappa^*_{\mathbf{p},\lambda}\kappa^*_{-\mathbf{p},-\lambda} = \dfrac{\lambda p^1 - \mathrm{i}p^2}{\sqrt{(p^1)^2+(p^2)^2}}\dfrac{\lambda p^1 + \mathrm{i}p^2}{\sqrt{(p^1)^2+(p^2)^2}} = 1$，推出

$$
\begin{aligned}
P\left|\psi\bar{\psi}\right\rangle &= -\sum_\lambda \int \mathrm{d}^3p\, \Phi(-\mathbf{p})\kappa^*_{\mathbf{p},\lambda}\kappa^*_{-\mathbf{p},-\lambda} a^\dagger_{\mathbf{p},\lambda} b^\dagger_{-\mathbf{p},-\lambda}\left|0\right\rangle \\
&= -(-)^L \sum_\lambda \int \mathrm{d}^3p\, \Phi(\mathbf{p}) a^\dagger_{\mathbf{p},\lambda} b^\dagger_{-\mathbf{p},-\lambda}\left|0\right\rangle = (-)^{L+1}\left|\psi\bar{\psi}\right\rangle
\end{aligned}
$$

可见，$\left|\psi\bar{\psi}\right\rangle$ 的**总宇称**为 $(-)^{L+1}$，包含**轨道宇称** $(-)^L$ 和**内禀宇称** $-$

也就是说，一对正反费米子的**内禀宇称**为**奇**

对于**自由**的 **Majorana 旋量场** $\psi(x)$，$b_{\mathbf{p},\lambda} = a_{\mathbf{p},\lambda}$，则 $\tilde{\zeta}_P = \zeta_P$

从而 $\zeta_P^* = -\tilde{\zeta}_P = -\zeta_P$，因此

$$\zeta_P = \pm\mathrm{i}$$

故 Majorana 旋量场的**宇称**是**虚数**

8.2.4 小节 旋量场的 T 变换

- 下面讨论**自由 Dirac 旋量场** $\psi(x)$ 的 $\boldsymbol{T}$ **变换**
- 时间反演变换同时**反转动量**和**角动量**的方向，因而会**保持螺旋度** λ **不变**
- 另一方面，T 变换是**反幺正**的，将一个复数变换成它的**复共轭**
- 因此 $\boldsymbol{T}$ **变换**联系着螺旋态 $\xi_\lambda(\mathbf{p})$ 和 $\xi_\lambda^*(-\mathbf{p})$，它们存在 8.2.1 小节推出的关系

$$\mathrm{i}\sigma^2\xi_\lambda^*(\mathbf{p}) = -\lambda\xi_{-\lambda}(\mathbf{p}), \quad \mathrm{i}\sigma^2\xi_{-\lambda}^*(\mathbf{p}) = \lambda\xi_\lambda(\mathbf{p})$$

8.2.4 小节 旋量场的 T 变换

- 下面讨论**自由 Dirac 旋量场** $\psi(x)$ 的 **T 变换**
- 时间反演变换同时**反转动量**和**角动量**的方向，因而会**保持螺旋度 λ 不变**
- 另一方面，T 变换是**反幺正**的，将一个复数变换成它的**复共轭**
- 因此 **T 变换**联系着螺旋态 $\xi_\lambda(\mathbf{p})$ 和 $\xi^*_\lambda(-\mathbf{p})$，它们存在 8.2.1 小节推出的关系

$$\mathrm{i}\sigma^2\xi^*_\lambda(\mathbf{p}) = -\lambda\xi_{-\lambda}(\mathbf{p}), \quad \mathrm{i}\sigma^2\xi^*_{-\lambda}(\mathbf{p}) = \lambda\xi_\lambda(\mathbf{p})$$

- 对第一式取复共轭，利用 $\xi_\lambda(\mathbf{p}) = \kappa_{\mathbf{p},\lambda}\xi_{-\lambda}(-\mathbf{p})$，得

$$\mathrm{i}\sigma^2\xi_\lambda(\mathbf{p}) = -\lambda\xi^*_{-\lambda}(\mathbf{p}) = -\lambda\kappa^*_{\mathbf{p},-\lambda}\xi^*_\lambda(-\mathbf{p})$$

- 故

$$\mathrm{i}\sigma^2\xi_{-\lambda}(\mathbf{p}) = \lambda\kappa^*_{\mathbf{p},\lambda}\xi^*_{-\lambda}(-\mathbf{p})$$

- $\mathrm{i}\sigma^2$ 出现在矩阵 $\mathcal{C}\gamma^5 = \begin{pmatrix} -\mathrm{i}\sigma^2 & \\ & \mathrm{i}\sigma^2 \end{pmatrix}\begin{pmatrix} -\mathbf{1} & \\ & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathrm{i}\sigma^2 & \\ & \mathrm{i}\sigma^2 \end{pmatrix}$ 中

产生湮灭算符的 T 变换

- $\mathcal{C}\gamma^5$ 对平面波旋量系数的作用是

$$\mathcal{C}\gamma^5 u(\mathbf{p},\lambda) = \begin{pmatrix} \mathrm{i}\sigma^2 & \\ & \mathrm{i}\sigma^2 \end{pmatrix} \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p}) \end{pmatrix} = -\lambda\kappa^*_{\mathbf{p},-\lambda} \begin{pmatrix} \omega_{-\lambda}(-\mathbf{p})\xi^*_\lambda(-\mathbf{p}) \\ \omega_\lambda(-\mathbf{p})\xi^*_\lambda(-\mathbf{p}) \end{pmatrix}$$

$$\mathcal{C}\gamma^5 v(\mathbf{p},\lambda) = \begin{pmatrix} \mathrm{i}\sigma^2 & \\ & \mathrm{i}\sigma^2 \end{pmatrix} \begin{pmatrix} \lambda\omega_\lambda(\mathbf{p})\xi_{-\lambda}(\mathbf{p}) \\ -\lambda\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = \lambda\kappa^*_{\mathbf{p},\lambda} \begin{pmatrix} \lambda\omega_\lambda(-\mathbf{p})\xi^*_{-\lambda}(-\mathbf{p}) \\ -\lambda\omega_{-\lambda}(-\mathbf{p})\xi^*_{-\lambda}(-\mathbf{p}) \end{pmatrix}$$

- 由 $\kappa_{\mathbf{p},-\lambda} = -\kappa^*_{\mathbf{p},\lambda}$ 得 $\kappa^*_{\mathbf{p},-\lambda} = -\kappa_{\mathbf{p},\lambda}$，从而推出

$$\mathcal{C}\gamma^5 u(\mathbf{p},\lambda) = \lambda\kappa_{\mathbf{p},\lambda} u^*(-\mathbf{p},\lambda), \quad \mathcal{C}\gamma^5 v(\mathbf{p},\lambda) = \lambda\kappa^*_{\mathbf{p},\lambda} v^*(-\mathbf{p},\lambda)$$

产生湮灭算符的 T 变换

- $\mathcal{C}\gamma^5$ 对平面波旋量系数的作用是

$$\mathcal{C}\gamma^5 u(\mathbf{p},\lambda) = \begin{pmatrix} \mathrm{i}\sigma^2 & \\ & \mathrm{i}\sigma^2 \end{pmatrix} \begin{pmatrix} \omega_{-\lambda}(\mathbf{p})\xi_\lambda(\mathbf{p}) \\ \omega_\lambda(\mathbf{p})\xi_\lambda(\mathbf{p}) \end{pmatrix} = -\lambda\kappa^*_{\mathbf{p},-\lambda} \begin{pmatrix} \omega_{-\lambda}(-\mathbf{p})\xi^*_\lambda(-\mathbf{p}) \\ \omega_\lambda(-\mathbf{p})\xi^*_\lambda(-\mathbf{p}) \end{pmatrix}$$

$$\mathcal{C}\gamma^5 v(\mathbf{p},\lambda) = \begin{pmatrix} \mathrm{i}\sigma^2 & \\ & \mathrm{i}\sigma^2 \end{pmatrix} \begin{pmatrix} \lambda\omega_\lambda(\mathbf{p})\xi_{-\lambda}(\mathbf{p}) \\ -\lambda\omega_{-\lambda}(\mathbf{p})\xi_{-\lambda}(\mathbf{p}) \end{pmatrix} = \lambda\kappa^*_{\mathbf{p},\lambda} \begin{pmatrix} \lambda\omega_\lambda(-\mathbf{p})\xi^*_{-\lambda}(-\mathbf{p}) \\ -\lambda\omega_{-\lambda}(-\mathbf{p})\xi^*_{-\lambda}(-\mathbf{p}) \end{pmatrix}$$

- 由 $\kappa_{\mathbf{p},-\lambda} = -\kappa^*_{\mathbf{p},\lambda}$ 得 $\kappa^*_{\mathbf{p},-\lambda} = -\kappa_{\mathbf{p},\lambda}$，从而推出

$$\mathcal{C}\gamma^5 u(\mathbf{p},\lambda) = \lambda\kappa_{\mathbf{p},\lambda} u^*(-\mathbf{p},\lambda), \quad \mathcal{C}\gamma^5 v(\mathbf{p},\lambda) = \lambda\kappa^*_{\mathbf{p},\lambda} v^*(-\mathbf{p},\lambda)$$

- 为得到**自洽时间反演变换**结果，设正反费米子态 $|\mathbf{p}^+,\lambda\rangle$ 和 $|\mathbf{p}^-,\lambda\rangle$ 的 T **变换**为

$$T|\mathbf{p}^+,\lambda\rangle = \zeta_T\lambda\kappa^*_{-\mathbf{p},\lambda}|-\mathbf{p}^+,-\lambda\rangle, \quad T|\mathbf{p}^-,\lambda\rangle = \tilde{\zeta}_T\lambda\kappa^*_{-\mathbf{p},\lambda}|-\mathbf{p}^-,-\lambda\rangle$$

- 其中 ζ_T 和 $\tilde{\zeta}_T$ 是两个**相位因子**，由此得到 $a^\dagger_{\mathbf{p},\lambda}$、$a_{\mathbf{p},\lambda}$、$b^\dagger_{\mathbf{p},\lambda}$ 和 $b_{\mathbf{p},\lambda}$ 的 T 变换

$$T^{-1}a^\dagger_{\mathbf{p},\lambda}T = \zeta_T\lambda\kappa^*_{-\mathbf{p},\lambda}a^\dagger_{-\mathbf{p},\lambda}, \quad T^{-1}a_{\mathbf{p},\lambda}T = \zeta^*_T\lambda\kappa_{-\mathbf{p},\lambda}a_{-\mathbf{p},\lambda}$$

$$T^{-1}b^\dagger_{\mathbf{p},\lambda}T = \tilde{\zeta}_T\lambda\kappa^*_{-\mathbf{p},\lambda}b^\dagger_{-\mathbf{p},\lambda}, \quad T^{-1}b_{\mathbf{p},\lambda}T = \tilde{\zeta}^*_T\lambda\kappa_{-\mathbf{p},\lambda}b_{-\mathbf{p},\lambda}$$

平面波展开式的 T 变换

注意到 $T^{-1}\mathrm{i}T=-\mathrm{i}$，Dirac 旋量场**平面波展开式**的 T 变换是

$$
\begin{aligned}
&T^{-1}\psi(x)T\\
=&\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_\lambda T^{-1}\left[u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}+v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}\right]T\\
=&\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_\lambda\left[\zeta_T^*\lambda\kappa_{-\mathbf{p},\lambda}u^*(\mathbf{p},\lambda)a_{-\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}+\tilde{\zeta}_T\lambda\kappa^*_{-\mathbf{p},\lambda}v^*(\mathbf{p},\lambda)b^\dagger_{-\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}\right]\\
=&\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_\lambda\left[\zeta_T^*\lambda\kappa_{\mathbf{p},\lambda}u^*(-\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot(\mathcal{T}x)}+\tilde{\zeta}_T\lambda\kappa^*_{\mathbf{p},\lambda}v^*(-\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot(\mathcal{T}x)}\right]\\
=&\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_\lambda\left[\zeta_T^*\,\mathcal{C}\gamma^5u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot(\mathcal{T}x)}+\tilde{\zeta}_T\,\mathcal{C}\gamma^5v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot(\mathcal{T}x)}\right]
\end{aligned}
$$

第三步作了**变量替换** $\mathbf{p}\to-\mathbf{p}$

第四步用到 $\mathcal{C}\gamma^5u(\mathbf{p},\lambda)=\lambda\kappa_{\mathbf{p},\lambda}u^*(-\mathbf{p},\lambda)$ 和 $\mathcal{C}\gamma^5v(\mathbf{p},\lambda)=\lambda\kappa^*_{\mathbf{p},\lambda}v^*(-\mathbf{p},\lambda)$

$\psi(x)$ 的 T 变换

为了让 $\psi(x)$ 的**运动方程**具有**时间反演对称性**，

$$T^{-1}\psi(x)T = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda}\Big[\zeta_T^*\,\mathcal{C}\gamma^5 u(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot(\mathcal{T}x)} + \tilde{\zeta}_T\,\mathcal{C}\gamma^5 v(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot(\mathcal{T}x)}\Big]$$

与 $\psi(\mathcal{T}x)$ 最多只能**相差一个常数矩阵**，因而必须要求

$$\zeta_T^* = \tilde{\zeta}_T$$

使得 $T^{-1}b^\dagger_{\mathbf{p},\lambda}T = \tilde{\zeta}_T\lambda\kappa^*_{-\mathbf{p},\lambda}b^\dagger_{-\mathbf{p},\lambda}$ 和 $T^{-1}b_{\mathbf{p},\lambda}T = \tilde{\zeta}_T^*\lambda\kappa_{-\mathbf{p},\lambda}b_{-\mathbf{p},\lambda}$ 化为

$$T^{-1}b^\dagger_{\mathbf{p},\lambda}T = \zeta_T^*\lambda\kappa^*_{-\mathbf{p},\lambda}b^\dagger_{-\mathbf{p},\lambda},\quad T^{-1}b_{\mathbf{p},\lambda}T = \zeta_T\lambda\kappa_{-\mathbf{p},\lambda}b_{-\mathbf{p},\lambda}$$

从而 $\psi(x)$ 的 **T 变换**为

$$T^{-1}\psi(x)T = D(\mathcal{T})\psi(\mathcal{T}x)$$

其中 $D(\mathcal{T}) \equiv \zeta_T^*\,\mathcal{C}\gamma^5$ 是旋量空间中的**时间反演矩阵**

$\bar{\psi}(x)$ 的 T 变换

- 由 $D^\dagger(\mathcal{T})D(\mathcal{T}) = \zeta_T\gamma^5\mathcal{C}^\dagger\zeta_T^*\mathcal{C}\gamma^5 = |\zeta_T|^2\gamma^5\mathcal{C}^{-1}\mathcal{C}\gamma^5 = \mathbf{1}$ 可知
- 时间反演矩阵 $D(\mathcal{T})$ 是**幺正**的，满足 $D^{-1}(\mathcal{T}) = D^\dagger(\mathcal{T}) = \zeta_T\gamma^5\mathcal{C}^{-1}$
- 对 $T^{-1}\psi(x)T = D(\mathcal{T})\psi(\mathcal{T}x)$ 取厄米共轭，得

$$T^{-1}\psi^\dagger(x)T = \psi^\dagger(\mathcal{T}x)D^\dagger(\mathcal{T}) = \psi^\dagger(\mathcal{T}x)\zeta_T\gamma^5\mathcal{C}^{-1}$$

$\bar{\psi}(x)$ 的 T 变换

- 由 $D^\dagger(\mathcal{T})D(\mathcal{T}) = \zeta_T\gamma^5\mathcal{C}^\dagger\zeta_T^*\mathcal{C}\gamma^5 = |\zeta_T|^2\gamma^5\mathcal{C}^{-1}\mathcal{C}\gamma^5 = \mathbf{1}$ 可知
- 时间反演矩阵 $D(\mathcal{T})$ 是**幺正**的，满足 $D^{-1}(\mathcal{T}) = D^\dagger(\mathcal{T}) = \zeta_T\gamma^5\mathcal{C}^{-1}$
- 对 $T^{-1}\psi(x)T = D(\mathcal{T})\psi(\mathcal{T}x)$ 取厄米共轭，得

$$T^{-1}\psi^\dagger(x)T = \psi^\dagger(\mathcal{T}x)D^\dagger(\mathcal{T}) = \psi^\dagger(\mathcal{T}x)\zeta_T\gamma^5\mathcal{C}^{-1}$$

- 由 γ^0 的**厄米性**有 $(\gamma^0)^* = (\gamma^0)^{\mathrm{T}}$，由 $\mathcal{C}^{-1}(\gamma^\mu)^{\mathrm{T}}\mathcal{C} = -\gamma^\mu$ 得 $\mathcal{C}^{-1}(\gamma^0)^{\mathrm{T}} = -\gamma^0\mathcal{C}^{-1}$
- 再利用 $\gamma^5\gamma^0 = -\gamma^0\gamma^5$ 推出

$$T^{-1}\bar{\psi}(x)T = T^{-1}\psi^\dagger(x)T(\gamma^0)^* = \psi^\dagger(\mathcal{T}x)\zeta_T\gamma^5\mathcal{C}^{-1}(\gamma^0)^{\mathrm{T}} = \psi^\dagger(\mathcal{T}x)\gamma^0\zeta_T\gamma^5\mathcal{C}^{-1}$$

- 即 $\bar{\psi}(x)$ 的 T 变换为 $T^{-1}\bar{\psi}(x)T = \bar{\psi}(\mathcal{T}x)D^{-1}(\mathcal{T})$

$\bar{\psi}(x)$ 的 T 变换

- 由 $D^\dagger(\mathcal{T})D(\mathcal{T}) = \zeta_T\gamma^5\mathcal{C}^\dagger\zeta_T^*\mathcal{C}\gamma^5 = |\zeta_T|^2\gamma^5\mathcal{C}^{-1}\mathcal{C}\gamma^5 = \mathbf{1}$ 可知
- 时间反演矩阵 $D(\mathcal{T})$ 是**幺正**的，满足 $D^{-1}(\mathcal{T}) = D^\dagger(\mathcal{T}) = \zeta_T\gamma^5\mathcal{C}^{-1}$
- 对 $T^{-1}\psi(x)T = D(\mathcal{T})\psi(\mathcal{T}x)$ 取厄米共轭，得

$$T^{-1}\psi^\dagger(x)T = \psi^\dagger(\mathcal{T}x)D^\dagger(\mathcal{T}) = \psi^\dagger(\mathcal{T}x)\zeta_T\gamma^5\mathcal{C}^{-1}$$

- 由 γ^0 的**厄米性**有 $(\gamma^0)^* = (\gamma^0)^{\mathrm{T}}$，由 $\mathcal{C}^{-1}(\gamma^\mu)^{\mathrm{T}}\mathcal{C} = -\gamma^\mu$ 得 $\mathcal{C}^{-1}(\gamma^0)^{\mathrm{T}} = -\gamma^0\mathcal{C}^{-1}$
- 再利用 $\gamma^5\gamma^0 = -\gamma^0\gamma^5$ 推出

$$T^{-1}\bar{\psi}(x)T = T^{-1}\psi^\dagger(x)T(\gamma^0)^* = \psi^\dagger(\mathcal{T}x)\zeta_T\gamma^5\mathcal{C}^{-1}(\gamma^0)^{\mathrm{T}} = \psi^\dagger(\mathcal{T}x)\gamma^0\zeta_T\gamma^5\mathcal{C}^{-1}$$

- 即 $\bar{\psi}(x)$ 的 T 变换为 $T^{-1}\bar{\psi}(x)T = \bar{\psi}(\mathcal{T}x)D^{-1}(\mathcal{T})$
- 据此，**一般**旋量双线性型 $\bar{\psi}(x)\Gamma\psi(x)$ 的 T 变换是

$$T^{-1}\bar{\psi}(x)\Gamma\psi(x)T = T^{-1}\bar{\psi}(x)TT^{-1}\Gamma TT^{-1}\psi(x)T = \bar{\psi}(\mathcal{T}x)D^{-1}(\mathcal{T})\Gamma^*D(\mathcal{T})\psi(\mathcal{T}x)$$

- 即问题归结为计算 $D^{-1}(\mathcal{T})\Gamma^*D(\mathcal{T})$

旋量空间中矩阵的 T 变换

- 根据 γ^5 的厄米性、γ^i 的反厄米性、$\mathcal{C}^{-1}(\gamma^\mu)^{\mathrm{T}}\mathcal{C} = -\gamma^\mu$ 和 $\mathcal{C}^{-1}(\gamma^5)^{\mathrm{T}}\mathcal{C} = \gamma^5$，有

$$
\begin{aligned}
D^{-1}(\mathcal{T})(\gamma^5)^* D(\mathcal{T}) &= \gamma^5\mathcal{C}^{-1}(\gamma^5)^{\mathrm{T}}\mathcal{C}\gamma^5 = (\gamma^5)^3 = \gamma^5 \\
D^{-1}(\mathcal{T})(\gamma^0)^* D(\mathcal{T}) &= \gamma^5\mathcal{C}^{-1}(\gamma^0)^{\mathrm{T}}\mathcal{C}\gamma^5 = -\gamma^5\gamma^0\gamma^5 = \gamma^0 \\
D^{-1}(\mathcal{T})(\gamma^i)^* D(\mathcal{T}) &= -\gamma^5\mathcal{C}^{-1}(\gamma^i)^{\mathrm{T}}\mathcal{C}\gamma^5 = \gamma^5\gamma^i\gamma^5 = -\gamma^i
\end{aligned}
$$

旋量空间中矩阵的 T 变换

根据 γ^5 的厄米性、γ^i 的反厄米性、$\mathcal{C}^{-1}(\gamma^\mu)^{\mathrm{T}}\mathcal{C} = -\gamma^\mu$ 和 $\mathcal{C}^{-1}(\gamma^5)^{\mathrm{T}}\mathcal{C} = \gamma^5$，有

$$
\begin{aligned}
D^{-1}(\mathcal{T})(\gamma^5)^* D(\mathcal{T}) &= \gamma^5\mathcal{C}^{-1}(\gamma^5)^{\mathrm{T}}\mathcal{C}\gamma^5 = (\gamma^5)^3 = \gamma^5 \\
D^{-1}(\mathcal{T})(\gamma^0)^* D(\mathcal{T}) &= \gamma^5\mathcal{C}^{-1}(\gamma^0)^{\mathrm{T}}\mathcal{C}\gamma^5 = -\gamma^5\gamma^0\gamma^5 = \gamma^0 \\
D^{-1}(\mathcal{T})(\gamma^i)^* D(\mathcal{T}) &= -\gamma^5\mathcal{C}^{-1}(\gamma^i)^{\mathrm{T}}\mathcal{C}\gamma^5 = \gamma^5\gamma^i\gamma^5 = -\gamma^i
\end{aligned}
$$

进而得到

$$
\begin{aligned}
D^{-1}(\mathcal{T})\mathbf{1}^* D(\mathcal{T}) &= +\mathbf{1} \\
D^{-1}(\mathcal{T})(\mathrm{i}\gamma^5)^* D(\mathcal{T}) &= -\mathrm{i}\gamma^5 \\
D^{-1}(\mathcal{T})(\gamma^\mu)^* D(\mathcal{T}) &= -\mathcal{T}^\mu{}_\nu\gamma^\nu \\
D^{-1}(\mathcal{T})(\gamma^\mu\gamma^5)^* D(\mathcal{T}) &= D^{-1}(\mathcal{T})(\gamma^\mu)^* D(\mathcal{T}) D^{-1}(\mathcal{T})(\gamma^5)^* D(\mathcal{T}) = -\mathcal{T}^\mu{}_\nu\gamma^\nu\gamma^5 \\
D^{-1}(\mathcal{T})(\sigma^{\mu\nu})^* D(\mathcal{T}) &= -\frac{\mathrm{i}}{2} D^{-1}(\mathcal{T})[(\gamma^\mu)^*(\gamma^\nu)^* - (\gamma^\nu)^*(\gamma^\mu)^*]D(\mathcal{T}) \\
&= -\frac{\mathrm{i}}{2}\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\mu) = -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma\sigma^{\rho\sigma}
\end{aligned}
$$

旋量双线性型的 T 变换

- 于是，各种**旋量双线性型**的 T 变换为

$$
\begin{aligned}
T^{-1}\bar{\psi}(x)\psi(x)T &= +\bar{\psi}(\mathcal{T}x)\psi(\mathcal{T}x) \\
T^{-1}\bar{\psi}(x)\mathrm{i}\gamma^5\psi(x)T &= -\bar{\psi}(\mathcal{T}x)\mathrm{i}\gamma^5\psi(\mathcal{T}x) \\
T^{-1}\bar{\psi}(x)\gamma^\mu\psi(x)T &= -\mathcal{T}^\mu{}_\nu\bar{\psi}(\mathcal{T}x)\gamma^\nu\psi(\mathcal{T}x) \\
T^{-1}\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)T &= -\mathcal{T}^\mu{}_\nu\bar{\psi}(\mathcal{T}x)\gamma^\nu\gamma^5\psi(\mathcal{T}x) \\
T^{-1}\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)T &= -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma\bar{\psi}(\mathcal{T}x)\sigma^{\rho\sigma}\psi(\mathcal{T}x) \\
T^{-1}\bar{\psi}_{\mathrm{R}}(x)\psi_{\mathrm{L}}(x)T &= +\bar{\psi}_{\mathrm{R}}(\mathcal{T}x)\psi_{\mathrm{L}}(\mathcal{T}x) \\
T^{-1}\bar{\psi}_{\mathrm{L}}(x)\psi_{\mathrm{R}}(x)T &= +\bar{\psi}_{\mathrm{L}}(\mathcal{T}x)\psi_{\mathrm{R}}(\mathcal{T}x) \\
T^{-1}\bar{\psi}_{\mathrm{L}}(x)\gamma^\mu\psi_{\mathrm{L}}(x)T &= -\mathcal{T}^\mu{}_\nu\bar{\psi}_{\mathrm{L}}(\mathcal{T}x)\gamma^\nu\psi_{\mathrm{L}}(\mathcal{T}x) \\
T^{-1}\bar{\psi}_{\mathrm{R}}(x)\gamma^\mu\psi_{\mathrm{R}}(x)T &= -\mathcal{T}^\mu{}_\nu\bar{\psi}_{\mathrm{R}}(\mathcal{T}x)\gamma^\nu\psi_{\mathrm{R}}(\mathcal{T}x)
\end{aligned}
$$

拉氏量和运动方程的时间反演对称性

- **动能项算符** $\mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi$ 的 T 变换为

$$
\begin{aligned}
T^{-1}\mathrm{i}\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x)T &= -\mathrm{i}\bar{\psi}(\mathcal{T}x)D^{-1}(\mathcal{T})(\gamma^\mu)^*D(\mathcal{T})(\mathcal{T}^{-1})^\nu{}_\mu\partial_\nu\psi(\mathcal{T}x) \\
&= \mathrm{i}\bar{\psi}(\mathcal{T}x)\mathcal{T}^\mu{}_\rho\gamma^\rho(\mathcal{T}^{-1})^\nu{}_\mu\partial_\nu\psi(\mathcal{T}x) = +\mathrm{i}\bar{\psi}(\mathcal{T}x)\gamma^\mu\partial_\mu\psi(\mathcal{T}x)
\end{aligned}
$$

- 因此，自由 Dirac 旋量场的**拉氏量**在时间反演变换下**不变**

拉氏量和运动方程的时间反演对称性

- **动能项算符** $\mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi$ 的 T 变换为

$$
\begin{aligned}
T^{-1}\mathrm{i}\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x)T &= -\mathrm{i}\bar{\psi}(\mathcal{T}x)D^{-1}(\mathcal{T})(\gamma^\mu)^*D(\mathcal{T})(\mathcal{T}^{-1})^\nu{}_\mu\partial_\nu\psi(\mathcal{T}x) \\
&= \mathrm{i}\bar{\psi}(\mathcal{T}x)\mathcal{T}^\mu{}_\rho\gamma^\rho(\mathcal{T}^{-1})^\nu{}_\mu\partial_\nu\psi(\mathcal{T}x) = +\mathrm{i}\bar{\psi}(\mathcal{T}x)\gamma^\mu\partial_\mu\psi(\mathcal{T}x)
\end{aligned}
$$

- 因此，自由 Dirac 旋量场的**拉氏量**在时间反演变换下**不变**

- 由于 T 变换是**反线性**的，**Dirac 方程** $(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi(x) = 0$ 的**时间反演对称性**意味着运动方程 $[-\mathrm{i}(\gamma^\mu)^*\partial'_\mu - m]\psi'(x') = 0$ 成立，这一点在下面的推导中得到验证：

$$
\begin{aligned}
[-\mathrm{i}(\gamma^\mu)^*\partial'_\mu - m]\psi'(x') &= [-\mathrm{i}(\gamma^\mu)^*(\mathcal{T}^{-1})^\nu{}_\mu\partial_\nu - m]T^{-1}\psi(x')T \\
&= D(\mathcal{T})[-\mathrm{i}D^{-1}(\mathcal{T})(\gamma^\mu)^*D(\mathcal{T})(\mathcal{T}^{-1})^\nu{}_\mu\partial_\nu - m]\psi(\mathcal{T}x') \\
&= D(\mathcal{T})[\mathrm{i}\mathcal{T}^\mu{}_\rho\gamma^\rho(\mathcal{T}^{-1})^\nu{}_\mu\partial_\nu - m]\psi(x) \\
&= D(\mathcal{T})(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi(x) = 0
\end{aligned}
$$

拉氏量和运动方程的时间反演对称性

- 动能项算符 $\mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi$ 的 T 变换为

$$
\begin{aligned}
T^{-1}\mathrm{i}\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x)T &= -\mathrm{i}\bar{\psi}(\mathcal{T}x)D^{-1}(\mathcal{T})(\gamma^\mu)^*D(\mathcal{T})(\mathcal{T}^{-1})^\nu{}_\mu\partial_\nu\psi(\mathcal{T}x) \\
&= \mathrm{i}\bar{\psi}(\mathcal{T}x)\mathcal{T}^\mu{}_\rho\gamma^\rho(\mathcal{T}^{-1})^\nu{}_\mu\partial_\nu\psi(\mathcal{T}x) = +\mathrm{i}\bar{\psi}(\mathcal{T}x)\gamma^\mu\partial_\mu\psi(\mathcal{T}x)
\end{aligned}
$$

- 因此，自由 Dirac 旋量场的拉氏量在时间反演变换下不变

- 由于 T 变换是反线性的，**Dirac 方程** $(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi(x) = 0$ 的时间反演对称性意味着运动方程 $[-\mathrm{i}(\gamma^\mu)^*\partial'_\mu - m]\psi'(x') = 0$ 成立，这一点在下面的推导中得到验证：

$$
\begin{aligned}
[-\mathrm{i}(\gamma^\mu)^*\partial'_\mu - m]\psi'(x') &= [-\mathrm{i}(\gamma^\mu)^*(\mathcal{T}^{-1})^\nu{}_\mu\partial_\nu - m]T^{-1}\psi(x')T \\
&= D(\mathcal{T})[-\mathrm{i}D^{-1}(\mathcal{T})(\gamma^\mu)^*D(\mathcal{T})(\mathcal{T}^{-1})^\nu{}_\mu\partial_\nu - m]\psi(\mathcal{T}x') \\
&= D(\mathcal{T})[\mathrm{i}\mathcal{T}^\mu{}_\rho\gamma^\rho(\mathcal{T}^{-1})^\nu{}_\mu\partial_\nu - m]\psi(x) \\
&= D(\mathcal{T})(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi(x) = 0
\end{aligned}
$$

- 对于自由的 **Majorana 旋量场** $\psi(x)$，$b_{\mathbf{p},\lambda} = a_{\mathbf{p},\lambda}$，则 $\zeta_T = \tilde{\zeta}_T = \zeta_T^*$，故

$$\zeta_T = \pm 1$$

8.3 节 矢量场的分立变换

8.3.1 节 有质量矢量场的 C、P、T 变换

- 接下来讨论**自由有质量复矢量场** $A^\mu(x)$ 的 C、P、T 变换
- **拉氏量**为 $\mathcal{L} = -\frac{1}{2} F^\dagger_{\mu\nu} F^{\mu\nu} + m^2 A^\dagger_\mu A^\mu$
- 用**极化矢量** $\varepsilon^\mu(\mathbf{p}, \lambda)$ $(\lambda = \pm, 0)$ 表达的平面波展开式为

$$A^\mu(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} \mathrm{e}^{-\mathrm{i}p\cdot x} + \varepsilon^{\mu *}(\mathbf{p}, \lambda) b^\dagger_{\mathbf{p},\lambda} \mathrm{e}^{\mathrm{i}p\cdot x} \right]$$

8.3 节 矢量场的分立变换

8.3.1 节 有质量矢量场的 C、P、T 变换

- 接下来讨论**自由有质量复矢量场** $A^\mu(x)$ 的 C、P、T 变换
- **拉氏量**为 $\mathcal{L} = -\frac{1}{2} F^\dagger_{\mu\nu} F^{\mu\nu} + m^2 A^\dagger_\mu A^\mu$
- 用**极化矢量** $\varepsilon^\mu(\mathbf{p}, \lambda)$ $(\lambda = \pm, 0)$ 表达的平面波展开式为

$$A^\mu(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_{\lambda=\pm,0} \left[\varepsilon^\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} \mathrm{e}^{-\mathrm{i}p\cdot x} + \varepsilon^{\mu *}(\mathbf{p}, \lambda) b^\dagger_{\mathbf{p},\lambda} \mathrm{e}^{\mathrm{i}p\cdot x} \right]$$

- **宇称变换**联系着 $\varepsilon^\mu(\mathbf{p}, \lambda)$ 和 $\varepsilon^\mu(-\mathbf{p}, -\lambda)$，根据极化矢量的具体形式，有

$$\begin{aligned}
\varepsilon^\mu(-\mathbf{p}, +) &= \frac{1}{\sqrt{2}|\mathbf{p}||\mathbf{p}_\mathrm{T}|} \begin{pmatrix} 0 \\ -p^1 p^3 - \mathrm{i}p^2|\mathbf{p}| \\ -p^2 p^3 + \mathrm{i}p^1|\mathbf{p}| \\ |\mathbf{p}_\mathrm{T}|^2 \end{pmatrix} = -\frac{1}{\sqrt{2}|\mathbf{p}||\mathbf{p}_\mathrm{T}|} \begin{pmatrix} 0 \\ p^1 p^3 + \mathrm{i}p^2|\mathbf{p}| \\ p^2 p^3 - \mathrm{i}p^1|\mathbf{p}| \\ -|\mathbf{p}_\mathrm{T}|^2 \end{pmatrix} \\
&= -\varepsilon^\mu(\mathbf{p}, -) = \mathcal{P}^\mu{}_\nu \varepsilon^\nu(\mathbf{p}, -) \\
\varepsilon^\mu(-\mathbf{p}, -) &= -\varepsilon^\mu(\mathbf{p}, +) = \mathcal{P}^\mu{}_\nu \varepsilon^\nu(\mathbf{p}, +)
\end{aligned}$$

宇称变换与极化矢量

此外还有

$$\varepsilon^\mu(-\mathbf{p},0)=\frac{1}{m|\mathbf{p}|}\begin{pmatrix}|\mathbf{p}|^2\\-p^0p^1\\-p^0p^2\\-p^0p^3\end{pmatrix}=\frac{1}{m|\mathbf{p}|}\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}\begin{pmatrix}|\mathbf{p}|^2\\p^0p^1\\p^0p^2\\p^0p^3\end{pmatrix}=\mathcal{P}^\mu{}_\nu\varepsilon^\nu(\mathbf{p},0)$$

因此宇称变换与极化矢量的关系为

$$\varepsilon^\mu(-\mathbf{p},-\lambda)=\mathcal{P}^\mu{}_\nu\varepsilon^\nu(\mathbf{p},\lambda)$$

宇称变换与极化矢量

此外还有

$$\varepsilon^\mu(-\mathbf{p},0)=\frac{1}{m|\mathbf{p}|}\begin{pmatrix}|\mathbf{p}|^2\\-p^0p^1\\-p^0p^2\\-p^0p^3\end{pmatrix}=\frac{1}{m|\mathbf{p}|}\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}\begin{pmatrix}|\mathbf{p}|^2\\p^0p^1\\p^0p^2\\p^0p^3\end{pmatrix}=\mathcal{P}^\mu{}_\nu\varepsilon^\nu(\mathbf{p},0)$$

因此宇称变换与极化矢量的关系为

$$\varepsilon^\mu(-\mathbf{p},-\lambda)=\mathcal{P}^\mu{}_\nu\varepsilon^\nu(\mathbf{p},\lambda)$$

时间反演变换联系着 $\varepsilon^\mu(\mathbf{p},\lambda)$ 和 $\varepsilon^{\mu*}(-\mathbf{p},\lambda)$，有

$$\varepsilon^{\mu*}(-\mathbf{p},+)=\frac{1}{\sqrt{2}|\mathbf{p}||\mathbf{p}_\mathrm{T}|}\begin{pmatrix}0\\-p^1p^3+\mathrm{i}p^2|\mathbf{p}|\\-p^2p^3-\mathrm{i}p^1|\mathbf{p}|\\|\mathbf{p}_\mathrm{T}|^2\end{pmatrix}=\mathcal{T}^\mu{}_\nu\varepsilon^\nu(\mathbf{p},+)$$

时间反演变换与极化矢量

🍏 还有

$$\varepsilon^{\mu *}(-\mathbf{p},-) = \frac{1}{\sqrt{2}|\mathbf{p}||\mathbf{p}_{\mathrm{T}}|}\begin{pmatrix} 0 \\ p^1p^3 + \mathrm{i}p^2|\mathbf{p}| \\ p^2p^3 - \mathrm{i}p^1|\mathbf{p}| \\ -|\mathbf{p}_{\mathrm{T}}|^2 \end{pmatrix} = \mathcal{T}^{\mu}{}_{\nu}\varepsilon^{\nu}(\mathbf{p},-)$$

$$\varepsilon^{\mu *}(-\mathbf{p},0) = \frac{1}{m|\mathbf{p}|}\begin{pmatrix} |\mathbf{p}|^2 \\ -p^0p^1 \\ -p^0p^2 \\ -p^0p^3 \end{pmatrix} = \frac{1}{m|\mathbf{p}|}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} |\mathbf{p}|^2 \\ p^0p^1 \\ p^0p^2 \\ p^0p^3 \end{pmatrix} = -\mathcal{T}^{\mu}{}_{\nu}\varepsilon^{\nu}(\mathbf{p},0)$$

⚒️ 以此推出**时间反演变换**与**极化矢量**的关系

$$(-)^{1+\lambda}\varepsilon^{\mu *}(-\mathbf{p},\lambda) = \mathcal{T}^{\mu}{}_{\nu}\varepsilon^{\nu}(\mathbf{p},\lambda)$$

产生湮灭算符的分立变换

- 类似于标量场和旋量场的 C、P、T 变换规律，$C^{-1}A^\mu(x)C$、$P^{-1}A^\mu(x)P$ 和 $T^{-1}A^\mu(x)T$ 应当分别正比于 $A^{\mu\dagger}(x)$、$\mathcal{P}^\mu{}_\nu A^\nu(\mathcal{P}x)$ 和 $\mathcal{T}^\mu{}_\nu A^\nu(\mathcal{T}x)$
- 为此，将**产生湮灭算符**的 C、P、T 变换表达成

$$
\begin{aligned}
C^{-1}a^\dagger_{\mathbf{p},\lambda}C &= \xi_C b^\dagger_{\mathbf{p},\lambda}, & C^{-1}a_{\mathbf{p},\lambda}C &= \xi_C^* b_{\mathbf{p},\lambda} \\
C^{-1}b^\dagger_{\mathbf{p},\lambda}C &= \xi_C^* a^\dagger_{\mathbf{p},\lambda}, & C^{-1}b_{\mathbf{p},\lambda}C &= \xi_C a_{\mathbf{p},\lambda} \\
P^{-1}a^\dagger_{\mathbf{p},\lambda}P &= \xi_P a^\dagger_{-\mathbf{p},-\lambda}, & P^{-1}a_{\mathbf{p},\lambda}P &= \xi_P^* a_{-\mathbf{p},-\lambda} \\
P^{-1}b^\dagger_{\mathbf{p},\lambda}P &= \xi_P^* b^\dagger_{-\mathbf{p},-\lambda}, & P^{-1}b_{\mathbf{p},\lambda}P &= \xi_P b_{-\mathbf{p},-\lambda} \\
T^{-1}a^\dagger_{\mathbf{p},\lambda}T &= \xi_T(-)^{1+\lambda}a^\dagger_{-\mathbf{p},\lambda}, & T^{-1}a_{\mathbf{p},\lambda}T &= \xi_T^*(-)^{1+\lambda}a_{-\mathbf{p},\lambda} \\
T^{-1}b^\dagger_{\mathbf{p},\lambda}T &= \xi_T^*(-)^{1+\lambda}b^\dagger_{-\mathbf{p},\lambda}, & T^{-1}b_{\mathbf{p},\lambda}T &= \xi_T(-)^{1+\lambda}b_{-\mathbf{p},\lambda}
\end{aligned}
$$

- 其中 ξ_C、ξ_P 和 ξ_T 是相位因子

平面波展开式的分立变换

从而推出平面波展开式的分立变换

$$
\begin{aligned}
& C^{-1}A^{\mu}(x)C \\
= & \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda}[\varepsilon^{\mu}(\mathbf{p},\lambda)C^{-1}a_{\mathbf{p},\lambda}C\mathrm{e}^{-\mathrm{i}p\cdot x}+\varepsilon^{\mu *}(\mathbf{p},\lambda)C^{-1}b^{\dagger}_{\mathbf{p},\lambda}C\mathrm{e}^{\mathrm{i}p\cdot x}] \\
= & \xi_C^{*}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda}[\varepsilon^{\mu}(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}+\varepsilon^{\mu *}(\mathbf{p},\lambda)a^{\dagger}_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}] \\
& P^{-1}A^{\mu}(x)P \\
= & \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda}[\varepsilon^{\mu}(\mathbf{p},\lambda)P^{-1}a_{\mathbf{p},\lambda}P\mathrm{e}^{-\mathrm{i}p\cdot x}+\varepsilon^{\mu *}(\mathbf{p},\lambda)P^{-1}b^{\dagger}_{\mathbf{p},\lambda}P\mathrm{e}^{\mathrm{i}p\cdot x}] \\
= & \xi_P^{*}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda}[\varepsilon^{\mu}(\mathbf{p},\lambda)a_{-\mathbf{p},-\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}+\varepsilon^{\mu *}(\mathbf{p},\lambda)b^{\dagger}_{-\mathbf{p},-\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}] \\
= & \xi_P^{*}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda}[\varepsilon^{\mu}(-\mathbf{p},-\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot(\mathcal{P}x)}+\varepsilon^{\mu *}(-\mathbf{p},-\lambda)b^{\dagger}_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot(\mathcal{P}x)}] \\
= & \xi_P^{*}\mathcal{P}^{\mu}{}_{\nu}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda}[\varepsilon^{\nu}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot(\mathcal{P}x)}+\varepsilon^{\nu *}(\mathbf{p},\lambda)b^{\dagger}_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot(\mathcal{P}x)}]
\end{aligned}
$$

$A^\mu(x)$ 的分立变换

以及

$$
\begin{aligned}
& T^{-1}A^\mu(x)T \\
=& \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_\lambda T^{-1}[\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}+\varepsilon^{\mu*}(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}]T \\
=& \xi_T^*\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_\lambda[(-)^{1+\lambda}\varepsilon^{\mu*}(\mathbf{p},\lambda)a_{-\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}+(-)^{1+\lambda}\varepsilon^{\mu}(\mathbf{p},\lambda)b^\dagger_{-\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}] \\
=& \xi_T^*\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_\lambda[(-)^{1+\lambda}\varepsilon^{\mu*}(-\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot(\mathcal{T}x)}+(-)^{1+\lambda}\varepsilon^{\mu}(-\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot(\mathcal{T}x)}] \\
=& \xi_T^*\mathcal{T}^\mu{}_\nu\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_\lambda[\varepsilon^{\nu}(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot(\mathcal{T}x)}+\varepsilon^{\nu*}(\mathbf{p},\lambda)b^\dagger_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot(\mathcal{T}x)}]
\end{aligned}
$$

因此，**有质量复矢量场** $A^\mu(x)$ 的**分立变换**为

$$
\begin{aligned}
C^{-1}A^\mu(x)C &= \xi_C^*A^{\mu\dagger}(x), & C^{-1}A^{\mu\dagger}(x)C &= \xi_C A^\mu(x) \\
P^{-1}A^\mu(x)P &= \xi_P^*\mathcal{P}^\mu{}_\nu A^\nu(\mathcal{P}x), & P^{-1}A^{\mu\dagger}(x)P &= \xi_P\mathcal{P}^\mu{}_\nu A^{\nu\dagger}(\mathcal{P}x) \\
T^{-1}A^\mu(x)T &= \xi_T^*\mathcal{T}^\mu{}_\nu A^\nu(\mathcal{T}x), & T^{-1}A^{\mu\dagger}(x)T &= \xi_T\mathcal{T}^\mu{}_\nu A^{\nu\dagger}(\mathcal{T}x)
\end{aligned}
$$

场强张量的分立变换

- 根据**形式**上的表达式 $P^{-1}\partial^\mu P = \partial'^\mu = \mathcal{P}^\mu{}_\nu \partial^\nu$ 和 $T^{-1}\partial^\mu T = \partial'^\mu = \mathcal{T}^\mu{}_\nu \partial^\nu$
- **场强张量** $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ 的 C、P、T 变换为

$$
\begin{aligned}
C^{-1}F^{\mu\nu}(x)C &= \xi_C^*[\partial^\mu A^{\nu\dagger}(x) - \partial^\nu A^{\mu\dagger}(x)] = \xi_C^* F^{\mu\nu\dagger}(x) \\
P^{-1}F^{\mu\nu}(x)P &= \xi_P^* \mathcal{P}^\mu{}_\rho \mathcal{P}^\nu{}_\sigma[\partial^\rho A^\sigma(\mathcal{P}x) - \partial^\sigma A^\rho(\mathcal{P}x)] = \xi_P^* \mathcal{P}^\mu{}_\rho \mathcal{P}^\nu{}_\sigma F^{\rho\sigma}(\mathcal{P}x) \\
T^{-1}F^{\mu\nu}(x)T &= \xi_T^* \mathcal{T}^\mu{}_\rho \mathcal{T}^\nu{}_\sigma[\partial^\rho A^\sigma(\mathcal{T}x) - \partial^\sigma A^\rho(\mathcal{T}x)] = \xi_T^* \mathcal{T}^\mu{}_\rho \mathcal{T}^\nu{}_\sigma F^{\rho\sigma}(\mathcal{T}x)
\end{aligned}
$$

场强张量的分立变换

- 根据**形式**上的表达式 $P^{-1}\partial^\mu P = \partial'^\mu = \mathcal{P}^\mu{}_\nu \partial^\nu$ 和 $T^{-1}\partial^\mu T = \partial'^\mu = \mathcal{T}^\mu{}_\nu \partial^\nu$
- **场强张量** $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ 的 C、P、T 变换为

$$
\begin{aligned}
C^{-1}F^{\mu\nu}(x)C &= \xi_C^*[\partial^\mu A^{\nu\dagger}(x) - \partial^\nu A^{\mu\dagger}(x)] = \xi_C^* F^{\mu\nu\dagger}(x) \\
P^{-1}F^{\mu\nu}(x)P &= \xi_P^* \mathcal{P}^\mu{}_\rho \mathcal{P}^\nu{}_\sigma[\partial^\rho A^\sigma(\mathcal{P}x) - \partial^\sigma A^\rho(\mathcal{P}x)] = \xi_P^* \mathcal{P}^\mu{}_\rho \mathcal{P}^\nu{}_\sigma F^{\rho\sigma}(\mathcal{P}x) \\
T^{-1}F^{\mu\nu}(x)T &= \xi_T^* \mathcal{T}^\mu{}_\rho \mathcal{T}^\nu{}_\sigma[\partial^\rho A^\sigma(\mathcal{T}x) - \partial^\sigma A^\rho(\mathcal{T}x)] = \xi_T^* \mathcal{T}^\mu{}_\rho \mathcal{T}^\nu{}_\sigma F^{\rho\sigma}(\mathcal{T}x)
\end{aligned}
$$

- **动能项算符** $F^\dagger_{\mu\nu}F^{\mu\nu}$ 的 C、P、T 变换为

$$
\begin{aligned}
C^{-1}F^\dagger_{\mu\nu}(x)F^{\mu\nu}(x)C &= |\xi_C|^2 F_{\mu\nu}(x)F^{\mu\nu\dagger}(x) = +F^\dagger_{\mu\nu}(x)F^{\mu\nu}(x) \\
P^{-1}F^\dagger_{\mu\nu}(x)F^{\mu\nu}(x)P &= |\xi_P|^2 F^\dagger_{\alpha\beta}(\mathcal{P}x)(\mathcal{P}^{-1})^\alpha{}_\mu(\mathcal{P}^{-1})^\beta{}_\nu \mathcal{P}^\mu{}_\rho \mathcal{P}^\nu{}_\sigma F^{\rho\sigma}(\mathcal{P}x) \\
&= +F^\dagger_{\mu\nu}(\mathcal{P}x)F^{\mu\nu}(\mathcal{P}x) \\
T^{-1}F^\dagger_{\mu\nu}(x)F^{\mu\nu}(x)T &= |\xi_T|^2 F^\dagger_{\alpha\beta}(\mathcal{T}x)(\mathcal{T}^{-1})^\alpha{}_\mu(\mathcal{T}^{-1})^\beta{}_\nu \mathcal{T}^\mu{}_\rho \mathcal{T}^\nu{}_\sigma F^{\rho\sigma}(\mathcal{T}x) \\
&= +F^\dagger_{\mu\nu}(\mathcal{T}x)F^{\mu\nu}(\mathcal{T}x)
\end{aligned}
$$

对偶场强张量的分立变换

- 由 $\varepsilon^{\mu\nu\rho\sigma} = -\mathcal{P}^{\mu}{}_{\alpha}\mathcal{P}^{\nu}{}_{\beta}\mathcal{P}^{\rho}{}_{\gamma}\mathcal{P}^{\sigma}{}_{\delta}\varepsilon^{\alpha\beta\gamma\delta}$ 得

$$\begin{aligned}\varepsilon^{\mu\nu\rho\sigma}(\mathcal{P}^{-1})^{\alpha}{}_{\rho}(\mathcal{P}^{-1})^{\beta}{}_{\sigma} &= -\mathcal{P}^{\mu}{}_{\gamma}\mathcal{P}^{\nu}{}_{\delta}\mathcal{P}^{\rho}{}_{\tau}\mathcal{P}^{\sigma}{}_{\phi}\varepsilon^{\gamma\delta\tau\phi}(\mathcal{P}^{-1})^{\alpha}{}_{\rho}(\mathcal{P}^{-1})^{\beta}{}_{\sigma}\\ &= -\mathcal{P}^{\mu}{}_{\gamma}\mathcal{P}^{\nu}{}_{\delta}\delta^{\alpha}{}_{\tau}\delta^{\beta}{}_{\phi}\varepsilon^{\gamma\delta\tau\phi} = -\mathcal{P}^{\mu}{}_{\rho}\mathcal{P}^{\nu}{}_{\sigma}\varepsilon^{\rho\sigma\alpha\beta}\end{aligned}$$

- 同理有 $\varepsilon^{\mu\nu\rho\sigma}(\mathcal{T}^{-1})^{\alpha}{}_{\rho}(\mathcal{T}^{-1})^{\beta}{}_{\sigma} = -\mathcal{T}^{\mu}{}_{\rho}\mathcal{T}^{\nu}{}_{\sigma}\varepsilon^{\rho\sigma\alpha\beta}$

对偶场强张量的分立变换

由 $\varepsilon^{\mu\nu\rho\sigma} = -\mathcal{P}^{\mu}{}_{\alpha}\mathcal{P}^{\nu}{}_{\beta}\mathcal{P}^{\rho}{}_{\gamma}\mathcal{P}^{\sigma}{}_{\delta}\varepsilon^{\alpha\beta\gamma\delta}$ 得

$$\begin{aligned}\varepsilon^{\mu\nu\rho\sigma}(\mathcal{P}^{-1})^{\alpha}{}_{\rho}(\mathcal{P}^{-1})^{\beta}{}_{\sigma} &= -\mathcal{P}^{\mu}{}_{\gamma}\mathcal{P}^{\nu}{}_{\delta}\mathcal{P}^{\rho}{}_{\tau}\mathcal{P}^{\sigma}{}_{\phi}\varepsilon^{\gamma\delta\tau\phi}(\mathcal{P}^{-1})^{\alpha}{}_{\rho}(\mathcal{P}^{-1})^{\beta}{}_{\sigma}\\ &= -\mathcal{P}^{\mu}{}_{\gamma}\mathcal{P}^{\nu}{}_{\delta}\delta^{\alpha}{}_{\tau}\delta^{\beta}{}_{\phi}\varepsilon^{\gamma\delta\tau\phi} = -\mathcal{P}^{\mu}{}_{\rho}\mathcal{P}^{\nu}{}_{\sigma}\varepsilon^{\rho\sigma\alpha\beta}\end{aligned}$$

同理有 $\varepsilon^{\mu\nu\rho\sigma}(\mathcal{T}^{-1})^{\alpha}{}_{\rho}(\mathcal{T}^{-1})^{\beta}{}_{\sigma} = -\mathcal{T}^{\mu}{}_{\rho}\mathcal{T}^{\nu}{}_{\sigma}\varepsilon^{\rho\sigma\alpha\beta}$

从而，**对偶场强张量** $\tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}/2$ 的 C、P、T 变换是

$$\begin{aligned}C^{-1}\tilde{F}^{\mu\nu}(x)C &= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}C^{-1}F_{\rho\sigma}(x)C = \xi_C^*\tilde{F}^{\mu\nu\dagger}(x)\\ P^{-1}\tilde{F}^{\mu\nu}(x)P &= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}P^{-1}F_{\rho\sigma}(x)P = \frac{1}{2}\xi_P^*\varepsilon^{\mu\nu\rho\sigma}F_{\alpha\beta}(\mathcal{P}x)(\mathcal{P}^{-1})^{\alpha}{}_{\rho}(\mathcal{P}^{-1})^{\beta}{}_{\sigma}\\ &= -\frac{1}{2}\xi_P^*\mathcal{P}^{\mu}{}_{\rho}\mathcal{P}^{\nu}{}_{\sigma}\varepsilon^{\rho\sigma\alpha\beta}F_{\alpha\beta}(\mathcal{P}x) = -\xi_P^*\mathcal{P}^{\mu}{}_{\rho}\mathcal{P}^{\nu}{}_{\sigma}\tilde{F}^{\rho\sigma}(\mathcal{P}x)\\ T^{-1}\tilde{F}^{\mu\nu}(x)T &= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}T^{-1}F_{\rho\sigma}(x)T = \frac{1}{2}\xi_T^*\varepsilon^{\mu\nu\rho\sigma}F_{\alpha\beta}(\mathcal{T}x)(\mathcal{T}^{-1})^{\alpha}{}_{\rho}(\mathcal{T}^{-1})^{\beta}{}_{\sigma}\\ &= -\frac{1}{2}\xi_T^*\mathcal{T}^{\mu}{}_{\rho}\mathcal{T}^{\nu}{}_{\sigma}\varepsilon^{\rho\sigma\alpha\beta}F_{\alpha\beta}(\mathcal{T}x) = -\xi_T^*\mathcal{T}^{\mu}{}_{\rho}\mathcal{T}^{\nu}{}_{\sigma}\tilde{F}^{\rho\sigma}(\mathcal{T}x)\end{aligned}$$

可见，$\tilde{F}^{\mu\nu}$ 的 P、T 变换性质与 $F^{\mu\nu}$ 不同，相差一个**负号**

有质量实矢量场的分立变换

🍆 **质量项算符** $A_\mu^\dagger A^\mu$ 的 C、P、T 变换是

$$
\begin{aligned}
C^{-1}A_\mu^\dagger(x)A^\mu(x)C &= |\xi_C|^2 A_\mu(x)A^{\mu\dagger}(x) = +A_\mu^\dagger(x)A^\mu(x) \\
P^{-1}A_\mu^\dagger(x)A^\mu(x)P &= |\xi_P|^2 A_\rho^\dagger(\mathcal{P}x)(\mathcal{P}^{-1})^\rho{}_\mu \mathcal{P}^\mu{}_\nu A^\nu(\mathcal{P}x) = +A_\mu^\dagger(\mathcal{P}x)A^\mu(\mathcal{P}x) \\
T^{-1}A_\mu^\dagger(x)A^\mu(x)T &= |\xi_T|^2 A_\rho^\dagger(\mathcal{T}x)(\mathcal{T}^{-1})^\rho{}_\mu \mathcal{T}^\mu{}_\nu A^\nu(\mathcal{P}x) = +A_\mu^\dagger(\mathcal{T}x)A^\mu(\mathcal{T}x)
\end{aligned}
$$

⚖️ 因此，由**动能项**和**质量项**组成的自由有质量复矢量场**拉氏量**在 C、P、T 变换下都是**不变**的

有质量实矢量场的分立变换

🍆 **质量项算符** $A^\dagger_\mu A^\mu$ 的 C、P、T 变换是

$$
\begin{aligned}
C^{-1}A^\dagger_\mu(x)A^\mu(x)C &= |\xi_C|^2 A_\mu(x)A^{\mu\dagger}(x) = +A^\dagger_\mu(x)A^\mu(x) \\
P^{-1}A^\dagger_\mu(x)A^\mu(x)P &= |\xi_P|^2 A^\dagger_\rho(\mathcal{P}x)(\mathcal{P}^{-1})^\rho{}_\mu \mathcal{P}^\mu{}_\nu A^\nu(\mathcal{P}x) = +A^\dagger_\mu(\mathcal{P}x)A^\mu(\mathcal{P}x) \\
T^{-1}A^\dagger_\mu(x)A^\mu(x)T &= |\xi_T|^2 A^\dagger_\rho(\mathcal{T}x)(\mathcal{T}^{-1})^\rho{}_\mu \mathcal{T}^\mu{}_\nu A^\nu(\mathcal{P}x) = +A^\dagger_\mu(\mathcal{T}x)A^\mu(\mathcal{T}x)
\end{aligned}
$$

⚖️ 因此，由**动能项**和**质量项**组成的自由有质量复矢量场**拉氏量**在 C、P、T 变换下都是**不变**的

🍅 对于**自由有质量实矢量场**，$A^{\mu\dagger}(x) = A^\mu(x)$，故 $\xi_C = \xi_C^*$、$\xi_P = \xi_P^*$、$\xi_T = \xi_T^*$，

$$\xi_C = \pm 1, \quad \xi_P = \pm 1, \quad \xi_T = \pm 1$$

🧵 当 $\xi_P = +1$ 时，$A^\mu(x)$ 是**极矢量场**，满足 $P^{-1}A^\mu(x)P = \mathcal{P}^\mu{}_\nu A^\nu(\mathcal{P}x)$

🧶 当 $\xi_P = -1$ 时，$A^\mu(x)$ 是**轴矢量场**，满足 $P^{-1}A^\mu(x)P = -\mathcal{P}^\mu{}_\nu A^\nu(\mathcal{P}x)$

🧲 无论 ξ_C、ξ_P 和 ξ_T 取 ± 1 中的哪些值，自由有质量实矢量场的**拉氏量**在 C、P、T 变换下也都是**不变**的

8.3.2 节 电磁场的 C、P、T 变换

相比于有质量矢量场，**无质量**矢量场的物理极化矢量只有 $\varepsilon^\mu(\mathbf{p},+)$ 和 $\varepsilon^\mu(\mathbf{p},-)$ 两个，但 C、P、T 变换性质是类似的

自由无质量复矢量场 $A^\mu(x)$ 的 C、P、T 变换为（习题 8.3）

$$
\begin{aligned}
C^{-1}A^\mu(x)C &= \xi_C^* A^{\mu\dagger}(x), & C^{-1}A^{\mu\dagger}(x)C &= \xi_C A^\mu(x) \\
P^{-1}A^\mu(x)P &= \xi_P^* \mathcal{P}^\mu{}_\nu A^\nu(\mathcal{P}x), & P^{-1}A^{\mu\dagger}(x)P &= \xi_P \mathcal{P}^\mu{}_\nu A^{\nu\dagger}(\mathcal{P}x) \\
T^{-1}A^\mu(x)T &= \xi_T^* \mathcal{T}^\mu{}_\nu A^\nu(\mathcal{T}x), & T^{-1}A^{\mu\dagger}(x)T &= \xi_T \mathcal{T}^\mu{}_\nu A^{\nu\dagger}(\mathcal{T}x)
\end{aligned}
$$

对于**自由无质量实矢量场**，ξ_C、ξ_P 和 ξ_T 的取值只能是 ± 1

8.3.2 节 电磁场的 C、P、T 变换

相比于有质量矢量场，**无质量**矢量场的物理极化矢量只有 $\varepsilon^\mu(\mathbf{p},+)$ 和 $\varepsilon^\mu(\mathbf{p},-)$ 两个，但 C、P、T 变换性质是类似的

自由无质量复矢量场 $A^\mu(x)$ 的 C、P、T 变换为（习题 8.3）

$$
\begin{aligned}
C^{-1}A^\mu(x)C &= \xi_C^* A^{\mu\dagger}(x), & C^{-1}A^{\mu\dagger}(x)C &= \xi_C A^\mu(x) \\
P^{-1}A^\mu(x)P &= \xi_P^* \mathcal{P}^\mu{}_\nu A^\nu(\mathcal{P}x), & P^{-1}A^{\mu\dagger}(x)P &= \xi_P \mathcal{P}^\mu{}_\nu A^{\nu\dagger}(\mathcal{P}x) \\
T^{-1}A^\mu(x)T &= \xi_T^* \mathcal{T}^\mu{}_\nu A^\nu(\mathcal{T}x), & T^{-1}A^{\mu\dagger}(x)T &= \xi_T \mathcal{T}^\mu{}_\nu A^{\nu\dagger}(\mathcal{T}x)
\end{aligned}
$$

对于**自由无质量实矢量场**，ξ_C、ξ_P 和 ξ_T 的取值只能是 ± 1

电磁场是一种无质量实矢量场，可通过 **QED 相互作用**确定 ξ_C、ξ_P 和 ξ_T 的取值

在**相互作用绘景**中，参与相互作用的量子场与自由场具有形式**相同**的平面波展开式，因此前面推导出来的量子场 C、P、T 变换性质照样**成立**

于是，理论的 C、P、T 对称性由拉氏量中**相互作用项**的 C、P、T 变换性质决定

电磁场的分立变换

在 **QED 理论**中，**相互作用项**的算符形式为 $A_\mu\bar{\psi}\gamma^\mu\psi$

根据 $\bar{\psi}\gamma^\mu\psi$ **算符**的 C 变换 $C^{-1}\bar{\psi}(x)\gamma^\mu\psi(x)C = -\bar{\psi}(x)\gamma^\mu\psi(x)$

P 变换 $P^{-1}\bar{\psi}(x)\gamma^\mu\psi(x)P = \mathcal{P}^\mu{}_\nu\bar{\psi}(\mathcal{P}x)\gamma^\nu\psi(\mathcal{P}x)$

T 变换 $T^{-1}\bar{\psi}(x)\gamma^\mu\psi(x)T = -\mathcal{T}^\mu{}_\nu\bar{\psi}(\mathcal{T}x)\gamma^\nu\psi(\mathcal{T}x)$

电磁场 $A^\mu(x)$ 的 C、P、T 变换必须是

$$
\begin{aligned}
C^{-1}A^\mu(x)C &= -A^\mu(x) \\
P^{-1}A^\mu(x)P &= \mathcal{P}^\mu{}_\nu A^\nu(\mathcal{P}x) \\
T^{-1}A^\mu(x)T &= -\mathcal{T}^\mu{}_\nu A^\nu(\mathcal{T}x)
\end{aligned}
$$

才能使 QED 相互作用项分别在 C、P、T 变换下**不变**

电磁场的分立变换

在 **QED 理论**中，**相互作用项**的算符形式为 $A_\mu\bar{\psi}\gamma^\mu\psi$

根据 $\bar{\psi}\gamma^\mu\psi$ **算符**的 C 变换 $C^{-1}\bar{\psi}(x)\gamma^\mu\psi(x)C = -\bar{\psi}(x)\gamma^\mu\psi(x)$

P 变换 $P^{-1}\bar{\psi}(x)\gamma^\mu\psi(x)P = \mathcal{P}^\mu{}_\nu\bar{\psi}(\mathcal{P}x)\gamma^\nu\psi(\mathcal{P}x)$

T 变换 $T^{-1}\bar{\psi}(x)\gamma^\mu\psi(x)T = -\mathcal{T}^\mu{}_\nu\bar{\psi}(\mathcal{T}x)\gamma^\nu\psi(\mathcal{T}x)$

电磁场 $A^\mu(x)$ 的 C、P、T 变换必须是

$$
\begin{aligned}
C^{-1}A^\mu(x)C &= -A^\mu(x) \\
P^{-1}A^\mu(x)P &= \mathcal{P}^\mu{}_\nu A^\nu(\mathcal{P}x) \\
T^{-1}A^\mu(x)T &= -\mathcal{T}^\mu{}_\nu A^\nu(\mathcal{T}x)
\end{aligned}
$$

才能使 QED 相互作用项分别在 C、P、T 变换下**不变**

也就是说，必须取 $\xi_C = -1$, $\xi_P = +1$, $\xi_T = -1$

这样的取法是做得到的，因此说 QED 理论同时具有 C、P、T 对称性，即具有**电荷共轭对称性**、**空间反射对称性**和**时间反演对称性**

从而，电磁场是 C **宇称为奇**的**极**矢量场

电磁场强张量的分立变换

电磁场的**场强张量** $F^{\mu\nu}$ 和**对偶场强张量** $\tilde{F}^{\mu\nu}$ 的 C、P、T 变换是

$$
\begin{aligned}
C^{-1}F^{\mu\nu}(x)C &= -F^{\mu\nu}(x) \\
P^{-1}F^{\mu\nu}(x)P &= \mathcal{P}^{\mu}{}_{\rho}\mathcal{P}^{\nu}{}_{\sigma}F^{\rho\sigma}(\mathcal{P}x) \\
T^{-1}F^{\mu\nu}(x)T &= -\mathcal{T}^{\mu}{}_{\rho}\mathcal{T}^{\nu}{}_{\sigma}F^{\rho\sigma}(\mathcal{T}x) \\
C^{-1}\tilde{F}^{\mu\nu}(x)C &= -\tilde{F}^{\mu\nu}(x) \\
P^{-1}\tilde{F}^{\mu\nu}(x)P &= -\mathcal{P}^{\mu}{}_{\rho}\mathcal{P}^{\nu}{}_{\sigma}\tilde{F}^{\rho\sigma}(\mathcal{P}x) \\
T^{-1}\tilde{F}^{\mu\nu}(x)T &= \mathcal{T}^{\mu}{}_{\rho}\mathcal{T}^{\nu}{}_{\sigma}\tilde{F}^{\rho\sigma}(\mathcal{T}x)
\end{aligned}
$$

- $F^{\mu\nu}$ 和 $\tilde{F}^{\mu\nu}$ 的 C **宇称**均为**奇**
- $F^{\mu\nu}$ 与 $\tilde{F}^{\mu\nu}$ 的 P、T 变换都相差一个**负号**
- 就 P **变换**性质而言，$F^{\mu\nu}$ 是**狭义的张量**，而 $\tilde{F}^{\mu\nu}$ 是**赝张量**

$F^{\mu\nu}F_{\mu\nu}$ 和 $\tilde{F}^{\mu\nu}F_{\mu\nu}$ 的分立变换

于是，算符 $F^{\mu\nu}F_{\mu\nu}$ 和 $\tilde{F}^{\mu\nu}F_{\mu\nu}$ 的 C、P、T 变换为

$$
\begin{aligned}
C^{-1}F^{\mu\nu}(x)F_{\mu\nu}(x)C &= +F^{\mu\nu}(x)F_{\mu\nu}(x) \\
P^{-1}F^{\mu\nu}(x)F_{\mu\nu}(x)P &= +F^{\mu\nu}(\mathcal{P}x)F_{\mu\nu}(\mathcal{P}x) \\
T^{-1}F^{\mu\nu}(x)F_{\mu\nu}(x)T &= +F^{\mu\nu}(\mathcal{T}x)F_{\mu\nu}(\mathcal{T}x) \\
C^{-1}\tilde{F}^{\mu\nu}(x)F_{\mu\nu}(x)C &= +\tilde{F}^{\mu\nu}(x)F_{\mu\nu}(x) \\
P^{-1}\tilde{F}^{\mu\nu}(x)F_{\mu\nu}(x)P &= -\tilde{F}^{\mu\nu}(\mathcal{P}x)F_{\mu\nu}(\mathcal{P}x) \\
T^{-1}\tilde{F}^{\mu\nu}(x)F_{\mu\nu}(x)T &= -\tilde{F}^{\mu\nu}(\mathcal{T}x)F_{\mu\nu}(\mathcal{T}x)
\end{aligned}
$$

- $F^{\mu\nu}F_{\mu\nu}$ 的 C 宇称和宇称均为偶
- $\tilde{F}^{\mu\nu}F_{\mu\nu}$ 的 C 宇称为偶，宇称为奇