量 子 场 论

第 5 章　量子场的相互作用

5.3 节和 5.4 节

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5.3 节 Wick 定理

- 5.2 节借助时序乘积把 S 算符写成紧凑的级数形式

$$S=\sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n}{n!}\int \mathrm{d}^4x_1\cdots\mathrm{d}^4x_n\,\mathsf{T}[\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_n)]$$

- 如何适当地处理级数每一项中的时序乘积 $\mathsf{T}[\mathcal{H}_1(x_1)\cdots\mathcal{H}_1(x_n)]$ 呢?
- 在量子场论中，相互作用哈密顿量密度 $\mathcal{H}_1(x)$ 是由若干个场算符构成的，因而我们需要处理的是多个场算符的时序乘积
- 这不是一个简单问题，接下来将要介绍的 Wick 定理提供了一种简便的处理方法

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- 在相互作用绘景中，实标量场 $\phi(x)$ 的平面波展开式可以分解成两个部分

$$\phi(x)=\phi^{(+)}(x)+\phi^{(-)}(x)$$

- 正能解部分 $\phi^{(+)}(x)\equiv\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}a_{\mathbf{p}}\mathrm{e}^{-\mathrm{i}p\cdot x}$ 包含湮灭算符和 $\mathrm{e}^{-\mathrm{i}p\cdot x}$ 因子
- 负能解部分 $\phi^{(-)}(x)\equiv\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}a_{\mathbf{p}}^{\dagger}\mathrm{e}^{\mathrm{i}p\cdot x}$ 包含产生算符和 $\mathrm{e}^{\mathrm{i}p\cdot x}$ 因子

矢量场和旋量场的正负能解

- 同样，可以把**相互作用绘景**中的**有质量实矢量场 $A^\mu(x)$** 分解为

$$A^\mu(x) = A^{\mu(+)}(x) + A^{\mu(-)}(x)$$

- **正能解**部分为 $A^{\mu(+)}(x) \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \varepsilon^\mu(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} \mathrm{e}^{-\mathrm{i}p\cdot x}$
- **负能解**部分为 $A^{\mu(-)}(x) \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \varepsilon^{\mu *}(\mathbf{p},\lambda) a^\dagger_{\mathbf{p},\lambda} \mathrm{e}^{\mathrm{i}p\cdot x}$

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相互作用绘景中 **Dirac 旋量场 $\psi_a(x)$** 的平面波展开式也具有 **Heisenberg 绘景**中自由场展开式的形式，即

$$\psi_a(x) = \psi_a^{(+)}(x) + \psi_a^{(-)}(x)$$

正能解部分为 $\psi_a^{(+)}(x) \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} u_a(\mathbf{p},\lambda) a_{\mathbf{p},\lambda} \mathrm{e}^{-\mathrm{i}p\cdot x}$

负能解部分为 $\psi_a^{(-)}(x) \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm} v_a(\mathbf{p},\lambda) b^\dagger_{\mathbf{p},\lambda} \mathrm{e}^{\mathrm{i}p\cdot x}$

注意：各类量子场的**正能解**部分只包含**湮灭算符**，而**负能解**部分只包含**产生算符**

正规乘积

- 引入**正规乘积** (normal product) 的概念，以 N 为记号，它的作用是将乘积中的**所有湮灭算符**移动到**所有产生算符**的**右边**，形成**正规排序** (normal order)
- 正规排序对于**多个产生**（或湮灭）**算符之间谁先谁后**没有规定，每种排列都**等价**
- 移动算符的过程中如果涉及**奇数次费米子算符**之间的**交换**，就应该额外添加一个**负号**，这个规定与费米子产生湮灭算符的**反对易性**相匹配

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- 对于**标量场**的产生湮灭算符，有 $\mathrm{N}(a_{\mathbf{p}}a_{\mathbf{q}}^{\dagger}a_{\mathbf{k}}a_{\mathbf{l}}^{\dagger}) = a_{\mathbf{q}}^{\dagger}a_{\mathbf{l}}^{\dagger}a_{\mathbf{p}}a_{\mathbf{k}} = a_{\mathbf{l}}^{\dagger}a_{\mathbf{q}}^{\dagger}a_{\mathbf{p}}a_{\mathbf{k}} = a_{\mathbf{l}}^{\dagger}a_{\mathbf{q}}^{\dagger}a_{\mathbf{k}}a_{\mathbf{p}}$
- 第二步写出两个产生算符的另一种次序，第三步写出两个湮灭算符的另一种次序，而**对易关系** $[a_{\mathbf{q}}^{\dagger}, a_{\mathbf{l}}^{\dagger}] = [a_{\mathbf{p}}, a_{\mathbf{k}}] = 0$ 保证这些表达式是**等价**的

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- 对于**旋量场**的产生湮灭算符，有 $\mathrm{N}(b_{\mathbf{p},\lambda_1}a_{\mathbf{q},\lambda_2}^{\dagger}a_{\mathbf{k},\lambda_3}b_{\mathbf{l},\lambda_4}^{\dagger}) = -a_{\mathbf{q},\lambda_2}^{\dagger}b_{\mathbf{l},\lambda_4}^{\dagger}b_{\mathbf{p},\lambda_1}a_{\mathbf{k},\lambda_3}$
$$= b_{\mathbf{l},\lambda_4}^{\dagger}a_{\mathbf{q},\lambda_2}^{\dagger}b_{\mathbf{p},\lambda_1}a_{\mathbf{k},\lambda_3} = -b_{\mathbf{l},\lambda_4}^{\dagger}a_{\mathbf{q},\lambda_2}^{\dagger}a_{\mathbf{k},\lambda_3}b_{\mathbf{p},\lambda_1}$$
- 第二步写出两个产生算符的另一种次序，与第一步相差一个**负号**
- 第三步写出两个湮灭算符的另一种次序，与第二步相差一个**负号**
- **反对易关系** $\{a_{\mathbf{q},\lambda_2}^{\dagger}, b_{\mathbf{l},\lambda_4}^{\dagger}\} = \{b_{\mathbf{p},\lambda_1}, a_{\mathbf{k},\lambda_3}\} = 0$ 保证这些表达式是**等价**的

正规乘积的真空期待值

- 正能解包含湮灭算符，负能解包含产生算符，正规排序中正能解处于负能解右边
- 两个标量场算符的正规乘积为

$$\mathrm{N}[\phi(x)\phi(y)] = \phi^{(-)}(x)\phi^{(-)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(+)}(x)\phi^{(+)}(y) + \phi^{(-)}(y)\phi^{(+)}(x)$$

- 最后一项中 $\phi^{(+)}(x)$ 被正规排序移动到 $\phi^{(-)}(y)$ 的右边
- 两个旋量场算符的正规乘积为

$$\mathrm{N}[\psi_a(x)\psi_b(y)] = \psi_a^{(-)}(x)\psi_b^{(-)}(y)+\psi_a^{(-)}(x)\psi_b^{(+)}(y)+\psi_a^{(+)}(x)\psi_b^{(+)}(y)-\psi_b^{(-)}(y)\psi_a^{(+)}(x)$$

- 最后一项中 $\psi_a^{(+)}(x)$ 被正规排序移动到 $\psi_b^{(-)}(y)$ 的右边，并出现一个负号

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- 最后一项中 $\psi_a^{(+)}(x)$ 被正规排序移动到 $\psi_b^{(-)}(y)$ 的右边，并出现一个负号
- 湮灭算符对真空态 $|0\rangle$ 的作用为零，如 $a_{\mathbf{p}}|0\rangle = 0$，相应地，$\langle 0| a_{\mathbf{p}}^\dagger = 0$
- 因此，对一组产生湮灭算符的任意乘积取正规排序之后，真空期待值为零，即

$$\langle 0|\,\mathsf{N}\left(\text{产生湮灭算符的乘积}\right)|0\rangle = 0$$

一般场算符的正负能解

用**统一记号** $\Phi_a(x)$ 代表**一般场算符**，它可以是标量场 $\phi(x)$ 或 $\phi^\dagger(x)$，也可以是矢量场 $A^\mu(x)$ 的一个分量，还可以是旋量场 $\psi_a(x)$ 或 $\bar{\psi}_a(x)$ 的一个分量

比如，$\Phi_a(x)\Phi_b(x)\Phi_c(x)$ 可以表示 $\phi(x)\phi(x)\phi(x)$，也可以表示 $A_\mu(x)\bar{\psi}_a(x)\psi_b(x)$

后者不是 Lorentz 不变的，但可以利用 Dirac 矩阵线性地组合出 **Lorentz 不变量**

$$A_\mu(x)\bar{\psi}_a(x)(\gamma^\mu)_{ab}\psi_b(x) = A_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x)$$

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令 $\Phi_a(x) = \Phi_a^{(+)}(x) + \Phi_a^{(-)}(x)$，分解为**正能解** $\Phi_a^{(+)}(x)$ 和**负能解** $\Phi_a^{(-)}(x)$，则

$$\Phi_a(x)\Phi_b(y) = \Phi_a^{(-)}(x)\Phi_b^{(-)}(y) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(-)}(y)$$

由于正能解包含**湮灭算符**，负能解包含**产生算符**，有

$$\Phi_a^{(+)}(x)\,|0\rangle = 0, \quad \langle 0|\,\Phi_a^{(-)}(x) = 0,$$

从而推出

$$\langle 0|\,\Phi_a(x)\Phi_b(y)\,|0\rangle = \langle 0|\,\Phi_a^{(+)}(x)\Phi_b^{(-)}(y)\,|0\rangle$$

两个一般场算符的正规乘积

- 将 $\Phi_a(x)$ 与 $\Phi_b(y)$ 的**正规乘积**分解为

$$\mathsf{N}[\Phi_a(x)\Phi_b(y)] = \Phi_a^{(-)}(x)\Phi_b^{(-)}(y) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(+)}(y) + \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x)$$

- 因子 $\epsilon_{ab} = \pm 1$ 考虑了**费米子算符**的**反对易性**
- 若 $\Phi_a(x)$ 和 $\Phi_b(y)$ **都是费米子算符**，则 $\epsilon_{ab} = -1$；**其余情况** $\epsilon_{ab} = +1$

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🚄 利用 ϵ_{ab} 记号，依照两个产生（湮灭）算符的对易性或反对易性交换等式右边第一项（第三项）的两个场算符，得到

$$\begin{aligned}
&\mathsf{N}[\Phi_a(x)\Phi_b(y)] \\
&= \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \epsilon_{ab}\Phi_b^{(+)}(y)\Phi_a^{(+)}(x) + \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x) \\
&= \epsilon_{ab}[\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \epsilon_{ab}\Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_b^{(+)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(-)}(y)\Phi_a^{(+)}(x)]
\end{aligned}$$

即

$$\boxed{\mathsf{N}[\Phi_a(x)\Phi_b(y)] = \epsilon_{ab}\,\mathsf{N}[\Phi_b(y)\Phi_a(x)]}$$

👉 交换两个场算符之后，正规乘积只相差一个由费米子算符反对易性导致的符号

两个一般场算符的时序乘积

- 另一方面，将 $\Phi_a(x)$ 与 $\Phi_b(y)$ 的**时序乘积**化为

$$\begin{aligned}\mathsf{T}[\Phi_a(x)\Phi_b(y)] &= \Phi_a(x)\Phi_b(y)\theta(x^0-y^0)+\epsilon_{ab}\Phi_b(y)\Phi_a(x)\theta(y^0-x^0)\\ &= \epsilon_{ab}[\epsilon_{ab}\Phi_a(x)\Phi_b(y)\theta(x^0-y^0)+\Phi_b(y)\Phi_a(x)\theta(y^0-x^0)]\end{aligned}$$

- 交换两个场算符后，**时序乘积**也只相差一个由费米子算符的**反对易性**导致的**符号**

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- 交换两个场算符后，**时序乘积**也只相差一个由费米子算符的**反对易性**导致的**符号**

$$\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \epsilon_{ab}\,\mathsf{T}[\Phi_b(y)\Phi_a(x)]$$

- **当 $x^0 \geq y^0$ 时**，$\Phi_a(x)$ 与 $\Phi_b(y)$ 的时序乘积为

$$\begin{aligned}&\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \Phi_a(x)\Phi_b(y)\\ &= \Phi_a^{(-)}(x)\Phi_b^{(-)}(y)+\Phi_a^{(-)}(x)\Phi_b^{(+)}(y)+\Phi_a^{(+)}(x)\Phi_b^{(+)}(y)+\Phi_a^{(+)}(x)\Phi_b^{(-)}(y)\end{aligned}$$

- 把最后一项改写成

$$\begin{aligned}\Phi_a^{(+)}(x)\Phi_b^{(-)}(y) &= \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x)+\Phi_a^{(+)}(x)\Phi_b^{(-)}(y)-\epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x)\\ &= \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x)+[\Phi_a^{(+)}(x),\Phi_b^{(-)}(y)]_{\mp}\end{aligned}$$

- 这里 $[\cdot,\cdot]_- = [\cdot,\cdot]$ 代表**对易子**，$[\cdot,\cdot]_+ = \{\cdot,\cdot\}$ 代表**反对易子**
- $\mp$ **号**仅当 $\Phi_a(x)$ 和 $\Phi_b(y)$ **都是费米子算符**时取**正号**，其余情况取**负号**

$x^0 \geq y^0$ 时的 $\mathsf{T}[\Phi_a(x)\Phi_b(y)]$

🐏 于是推出

$$
\begin{aligned}
\mathsf{T}[\Phi_a(x)\Phi_b(y)] &= \Phi_a^{(-)}(x)\Phi_b^{(-)}(y) + \Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + \Phi_a^{(+)}(x)\Phi_b^{(+)}(y) \\
&\quad + \epsilon_{ab}\Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp} \\
&= \mathsf{N}[\Phi_a(x)\Phi_b(y)] + [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}
\end{aligned}
$$

🚃 $[\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}$ 必定是一个 **c 数**，因为 $\Phi_a^{(+)}(x)$ 中**湮灭算符**与 $\Phi_b^{(-)}(y)$ 中**产生算符**的**对易子**或**反对易子**并不是算符，而是 c 数

$x^0 \geq y^0$ 时的 $\mathsf{T}[\Phi_a(x)\Phi_b(y)]$

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🚃 由 $\Phi_a^{(+)}(x)\,|0\rangle = 0$ 和 $\langle 0|\,\Phi_a(x)\Phi_b(y)\,|0\rangle = \langle 0|\,\Phi_a^{(+)}(x)\Phi_b^{(-)}(y)\,|0\rangle$ 得

$$\begin{aligned}[\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_\mp &= \langle 0|\,[\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_\mp\,|0\rangle = \langle 0|\,\Phi_a^{(+)}(x)\Phi_b^{(-)}(y)\,|0\rangle \\ &= \langle 0|\,\Phi_a(x)\Phi_b(y)\,|0\rangle = \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle\end{aligned}$$

🚇 因此**当** $x^0 \geq y^0$ **时**，有 $\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y)] + \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle$

$x^0 < y^0$ 时的 $\mathsf{T}[\Phi_a(x)\Phi_b(y)]$

🐵 当 $x^0 < y^0$ 时，$\Phi_a(x)$ 与 $\Phi_b(y)$ 的时序乘积变成

$$
\begin{aligned}
&\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \epsilon_{ab}\Phi_b(y)\Phi_a(x) \\
&= \epsilon_{ab}[\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(+)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(+)}(y)\Phi_a^{(-)}(x)] \\
&= \epsilon_{ab}\{\Phi_b^{(-)}(y)\Phi_a^{(-)}(x) + \Phi_b^{(-)}(y)\Phi_a^{(+)}(x) + \Phi_b^{(+)}(y)\Phi_a^{(+)}(x) \\
&\qquad + \epsilon_{ab}\Phi_a^{(-)}(x)\Phi_b^{(+)}(y) + [\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_\mp\} \\
&= \epsilon_{ab}\,\mathsf{N}[\Phi_b(y)\Phi_a(x)] + \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_\mp \\
&= \mathsf{N}[\Phi_a(x)\Phi_b(y)] + \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_\mp
\end{aligned}
$$

🚡 注意到

$$
\begin{aligned}
\epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_\mp &= \epsilon_{ab}\langle 0|\,[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_\mp\,|0\rangle = \epsilon_{ab}\langle 0|\,\Phi_b^{(+)}(y)\Phi_a^{(-)}(x)\,|0\rangle \\
&= \epsilon_{ab}\langle 0|\,\Phi_b(y)\Phi_a(x)\,|0\rangle = \epsilon_{ab}\langle 0|\,\mathsf{T}[\Phi_b(y)\Phi_a(x)]\,|0\rangle \\
&= \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle
\end{aligned}
$$

🚠 因此也有 $\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y)] + \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle$

场算符的缩并

- 于是，**无论 x^0 和 y^0 孰大孰小**，$\Phi_a(x)$ 与 $\Phi_b(y)$ 的时序乘积都可以统一地表达为

$$\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y)] + \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle$$

场算符的缩并

- 于是，**无论 x^0 和 y^0 孰大孰小**，$\Phi_a(x)$ 与 $\Phi_b(y)$ 的时序乘积都可以统一地表达为

$$\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y)] + \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle$$

- 引入**场算符的缩并** (contraction) 概念，将 $\Phi_a(x)$ 与 $\Phi_b(y)$ 的缩并定义为

$$\overbracket{\Phi_a(x)\Phi_b(y)} \equiv \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle = \begin{cases} [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_{\mp}, & x^0 \geq y^0 \\ \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_{\mp}, & x^0 < y^0 \end{cases}$$

- 上式仅当 $\Phi_a^{(+)}(x)$ 中的**湮灭算符**与 $\Phi_b^{(-)}(y)$ 中的**产生算符**属于**同一套产生湮灭算符**时才能得到**非零**结果，**不同类型**的场算符 (如标量场与旋量场) **的缩并为零**

- 两个场算符的缩并是一个 **c 数**，**不会**受到**正规排序 N** 的影响

场算符的缩并

- 于是，**无论 x^0 和 y^0 孰大孰小**，$\Phi_a(x)$ 与 $\Phi_b(y)$ 的时序乘积都可以统一地表达为

$$\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y)] + \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle$$

- 引入**场算符的缩并** (contraction) 概念，将 $\Phi_a(x)$ 与 $\Phi_b(y)$ 的缩并定义为

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- 上式仅当 $\Phi_a^{(+)}(x)$ 中的**湮灭算符**与 $\Phi_b^{(-)}(y)$ 中的**产生算符**属于**同一套产生湮灭算符**时才能得到**非零**结果，**不同类型**的场算符 (如标量场与旋量场) **的缩并为零**
- 两个场算符的缩并是一个 **c 数**，**不会**受到**正规排序 N** 的影响
- 在正规乘积中出现缩并记号时，参与缩并的一对场算符可以**不相邻**
- 为了使它们相邻，需要**适当地交换**场算符，并考虑交换费米子算符时引起的**符号差异**，**约定**这样得到的式子与原先的式子**相等**，例如

$$\mathsf{N}(\Phi_a\Phi_b\Phi_c\Phi_d\Phi_e\Phi_f) = \epsilon_{cd}\epsilon_{ef}\mathsf{N}(\Phi_a\Phi_b\Phi_d\Phi_c\Phi_f\Phi_e) = \epsilon_{cd}\epsilon_{ef}\Phi_b\Phi_d\Phi_c\Phi_f\mathsf{N}(\Phi_a\Phi_e)$$

Wick 定理

🐶 将 $\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y)] + \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle$ 改写为

$$\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y) + \overbracket{\Phi_a(x)\Phi_b(y)}]$$

🚒 可见，两个场算符的**时序乘积**等于它们的**正规乘积**加上它们的**缩并**

Wick 定理

🐶 将 $\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y)] + \langle 0|\,\mathsf{T}[\Phi_a(x)\Phi_b(y)]\,|0\rangle$ 改写为

$$\mathsf{T}[\Phi_a(x)\Phi_b(y)] = \mathsf{N}[\Phi_a(x)\Phi_b(y) + \overbracket{\Phi_a(x)\Phi_b(y)}]$$

🚒 可见，两个场算符的**时序乘积**等于它们的**正规乘积**加上它们的**缩并**

🚋 这个结论可以**推广**成

Gian Wick (1909–1992)

Wick 定理

🚋 一组场算符的**时序乘积**可以分解为它们的**正规乘积**及**所有可能缩并**的正规乘积之和，也就是说，

$$\mathsf{T}[\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)\cdots\Phi_{a_n}(x_n)] = \mathsf{N}\big[\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)\cdots\Phi_{a_n}(x_n) + (\Phi_{a_1}\Phi_{a_2}\cdots\Phi_{a_n}\text{的所有可能缩并})\big]$$

🚐 Wick 定理的**证明**见**讲义 5.3.2 小节选读内容**

Wick 定理应用举例

- Wick 定理中的缩并是在保持各个场算符位置的情况下直接添加缩并线
- 对于四个场算符的情况，有

$$
\begin{aligned}
\mathsf{T}(\Phi_a\Phi_b\Phi_c\Phi_d) = \mathsf{N}\big(&\Phi_a\Phi_b\Phi_c\Phi_d + \overbracket{\Phi_a\Phi_b}\Phi_c\Phi_d + \overbracket{\Phi_a\Phi_b\Phi_c}\Phi_d + \overbracket{\Phi_a\Phi_b\Phi_c\Phi_d} \\
&+ \Phi_a\overbracket{\Phi_b\Phi_c}\Phi_d + \Phi_a\overbracket{\Phi_b\Phi_c\Phi_d} + \Phi_a\Phi_b\overbracket{\Phi_c\Phi_d} \\
&+ \overbracket{\Phi_a\Phi_b}\overbracket{\Phi_c\Phi_d} + \overbracket{\Phi_a\overbracket{\Phi_b\Phi_c}\Phi_d} + \overbracket{\Phi_a\overbracket{\Phi_b\Phi_c}\Phi_d}\big)
\end{aligned}
$$

- 由于 $\langle 0|\,\mathsf{N}\,(\text{产生湮灭算符的乘积})\,|0\rangle = 0$，上式的真空期待值为

$$
\begin{aligned}
&\langle 0|\,\mathsf{T}(\Phi_a\Phi_b\Phi_c\Phi_d)\,|0\rangle \\
=\ &\langle 0|\,\mathsf{N}(\overbracket{\Phi_a\Phi_b}\overbracket{\Phi_c\Phi_d} + \overbracket{\Phi_a\overbracket{\Phi_b\Phi_c}\Phi_d} + \overbracket{\Phi_a\overbracket{\Phi_b\Phi_c}\Phi_d})\,|0\rangle \\
=\ &\mathsf{N}(\overbracket{\Phi_a\Phi_b}\overbracket{\Phi_c\Phi_d} + \epsilon_{bc}\overbracket{\Phi_a\Phi_c}\overbracket{\Phi_b\Phi_d} + \epsilon_{cd}\epsilon_{bd}\overbracket{\Phi_a\Phi_d}\overbracket{\Phi_b\Phi_c}) \\
=\ &\langle 0|\,\mathsf{T}(\Phi_a\Phi_b)\,|0\rangle\,\langle 0|\,\mathsf{T}(\Phi_c\Phi_d)\,|0\rangle + \epsilon_{bc}\,\langle 0|\,\mathsf{T}(\Phi_a\Phi_c)\,|0\rangle\,\langle 0|\,\mathsf{T}(\Phi_b\Phi_d)\,|0\rangle \\
&+ \epsilon_{cd}\epsilon_{bd}\,\langle 0|\,\mathsf{T}(\Phi_a\Phi_d)\,|0\rangle\,\langle 0|\,\mathsf{T}(\Phi_b\Phi_c)\,|0\rangle
\end{aligned}
$$

5.4 节 Feynman 传播子

- 在应用 **Wick 定理**时，**两个场算符的缩并**是一种基本要素
- 上一节已经指出，仅当参与缩并的场算符中含有**同一套产生湮灭算符**时，缩并的结果才**不为零**
- **Feynman 传播子** (propagator) 就是这样的**非零缩并**
- 本节将导出 Feynman 传播子的**显式**结果

Richard Feynman
(1918–1988)

5.4.1 小节 实标量场的 Feynman 传播子

- 实标量场 $\phi(x)$ 的 Feynman 传播子 $D_{\mathrm{F}}(x-y)$ 定义为

$$D_{\mathrm{F}}(x-y) \equiv \overbracket{\phi(x)\phi(y)} = \langle 0|\, \mathsf{T}[\phi(x)\phi(y)]\, |0\rangle$$

- 根据正负能解展开式，当 $x^0 > y^0$ 时，有

$$\begin{aligned}
&\langle 0|\, \mathsf{T}[\phi(x)\phi(y)]\, |0\rangle = \langle 0|\, \phi(x)\phi(y)\, |0\rangle = \langle 0|\, \phi^{(+)}(x)\phi^{(-)}(y)\, |0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \langle 0|\, a_{\mathbf{p}}\mathrm{e}^{-\mathrm{i}p\cdot x} a_{\mathbf{q}}^{\dagger}\mathrm{e}^{\mathrm{i}q\cdot y}\, |0\rangle = \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot x-q\cdot y)}}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \langle 0|\, [a_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}]\, |0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot x-q\cdot y)}}{(2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\, \delta^{(3)}(\mathbf{p}-\mathbf{q}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{2E_{\mathbf{p}}} \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\, \frac{\mathrm{e}^{-\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}
\end{aligned}$$

- 第四步用到产生湮灭算符的对易关系
- 借助复变函数的知识，可以将最后一行中的因子 $\dfrac{\mathrm{e}^{-\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}$ 化为一维积分

复平面上的曲线积分

- 将 p^0 视作复变量，在 p^0 的复平面上考虑函数 $\dfrac{\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})}$ 的曲线积分
- 这个函数具有两个一阶极点，$p^0=\pm E_{\mathbf{p}}$，均位于实轴上
- 右下图画出 p^0 复平面上的两条积分路径
- 路径 Γ_{F} 在两个极点处分别通过一个半径无穷小的半圆绕过极点，当 $R\to\infty$ 时，Γ_{F} 将从 $p^0=-\infty$ 一直延伸到 $p^0=+\infty$
- 将 Γ_{F} 与下半平面上的半圆弧 $\Gamma_R^{(-)}$ 组成围线 $C_{\mathrm{F}}^{(-)}=\Gamma_{\mathrm{F}}+\Gamma_R^{(-)}$，曲线方向沿顺时针方向，即反方向

复平面上的曲线积分

- 将 p^0 视作复变量，在 p^0 的复平面上考虑函数 $\dfrac{\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0-E_\mathbf{p})(p^0+E_\mathbf{p})}$ 的曲线积分
- 这个函数具有两个一阶极点，$p^0=\pm E_\mathbf{p}$，均位于实轴上
- 右下图画出 p^0 复平面上的两条积分路径
- 路径 Γ_F 在两个极点处分别通过一个半径无穷小的半圆绕过极点，当 $R\to\infty$ 时，Γ_F 将从 $p^0=-\infty$ 一直延伸到 $p^0=+\infty$
- 将 Γ_F 与下半平面上的半圆弧 $\Gamma_R^{(-)}$ 组成围线 $C_\mathrm{F}^{(-)}=\Gamma_\mathrm{F}+\Gamma_R^{(-)}$，曲线方向沿顺时针方向，即反方向
- 由于 $x^0-y^0>0$，$\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}$ 中的因子 $\mathrm{e}^{\mathrm{Im}(p^0)(x^0-y^0)}$ 在下半平面 $(\mathrm{Im}\,p^0<0)$ 随着 R 增大而指数衰减
- 根据复分析的 Jordan 引理，有

$$\lim_{R\to\infty}\int_{\Gamma_R^{(-)}}\mathrm{d}p^0\,\frac{\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0-E_\mathbf{p})(p^0+E_\mathbf{p})}=0$$

沿 Γ_{F} 的积分

当 $R \to \infty$ 时，通过留数定理计算相应的积分主值，得

$$\int_{\Gamma_{\mathrm{F}}} \mathrm{d}p^0 \, \frac{\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})} = \int_{C_{\mathrm{F}}^{(-)}=\Gamma_{\mathrm{F}}+\Gamma_R^{(-)}} \mathrm{d}p^0 \, \frac{\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})}$$

$$= -2\pi\mathrm{i} \operatorname*{Res}_{p^0=E_{\mathbf{p}}} \frac{\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})} = -2\pi\mathrm{i} \, \frac{\mathrm{e}^{-\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}$$

沿 Γ_{F} 的积分

- 当 $R \to \infty$ 时，通过留数定理计算相应的积分主值，得

$$\int_{\Gamma_{\mathrm{F}}} \mathrm{d}p^0 \, \frac{\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})} = \int_{C_{\mathrm{F}}^{(-)}=\Gamma_{\mathrm{F}}+\Gamma_R^{(-)}} \mathrm{d}p^0 \, \frac{\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})}$$

$$= -2\pi\mathrm{i} \operatorname*{Res}_{p^0=E_{\mathbf{p}}} \frac{\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})} = -2\pi\mathrm{i} \, \frac{\mathrm{e}^{-\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}$$

- 利用

$$\begin{aligned}(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}}) &= (p^0)^2 - E_{\mathbf{p}}^2 \\ &= (p^0)^2 - |\mathbf{p}|^2 - m^2 \\ &= p^2 - m^2\end{aligned}$$

- 进一步得到

$$\frac{\mathrm{e}^{-\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} = -\frac{1}{2\pi\mathrm{i}} \int_{\Gamma_{\mathrm{F}}} \mathrm{d}p^0 \, \frac{\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})} = \int_{\Gamma_{\mathrm{F}}} \frac{\mathrm{d}p^0}{2\pi} \, \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{p^2-m^2}$$

沿实轴积分

- 如右下图所示，如果将左边极点沿正虚轴方向移动一个**无穷小量** $\eta > 0$，右边极点沿负虚轴方向同样移动无穷小量 η，则**沿实轴积分**将等价于原来**沿 Γ_{F} 积分**
- 此时**极点**位置为 $p^0 = \pm(E_{\mathbf{p}} - \mathrm{i}\eta)$，被积函数中的**分母** $p^2 - m^2$ 应改成

$$
\begin{aligned}
[p^0 - (E_{\mathbf{p}} - \mathrm{i}\eta)][p^0 + (E_{\mathbf{p}} - \mathrm{i}\eta)] &= (p^0)^2 - (E_{\mathbf{p}} - \mathrm{i}\eta)^2 \\
&= (p^0)^2 - E_{\mathbf{p}}^2 + 2\mathrm{i}\eta E_{\mathbf{p}} + \eta^2 \simeq p^2 - m^2 + \mathrm{i}\epsilon
\end{aligned}
$$

- 最后一步忽略了 η 的**二阶小量**，而 $\epsilon \equiv 2\eta E_{\mathbf{p}} > 0$ 也是无穷小量，于是得到

$$
\begin{aligned}
\frac{\mathrm{e}^{-\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} &= \int_{\Gamma_{\mathrm{F}}} \frac{\mathrm{d}p^0}{2\pi} \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{p^2 - m^2} \\
&= \int \frac{\mathrm{d}p^0}{2\pi} \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{[p^0 - (E_{\mathbf{p}} - \mathrm{i}\eta)][p^0 + (E_{\mathbf{p}} - \mathrm{i}\eta)]} \\
&= \int \frac{\mathrm{d}p^0}{2\pi} \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{p^2 - m^2 + \mathrm{i}\epsilon}
\end{aligned}
$$

$x^0 > y^0$ 时的 $\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle$

- 将以上 $\dfrac{\mathrm{e}^{-\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} = \displaystyle\int \frac{\mathrm{d}p^0}{2\pi}\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{p^2-m^2+\mathrm{i}\epsilon}$

代入 $x^0 > y^0$ 时 $\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle$ 的表达式，立即推出

$$
\begin{aligned}
\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{\mathrm{e}^{-\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \frac{\mathrm{d}p^0}{2\pi}\,\frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{p^2-m^2+\mathrm{i}\epsilon} \\
&= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2-m^2+\mathrm{i}\epsilon}
\end{aligned}
$$

$x^0 < y^0$ 的情况

- 当 $x^0 < y^0$ 时，时间排序将改变 $\phi(x)$ 和 $\phi(y)$ 的次序，有

$$\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle = \langle 0|\,\phi(y)\phi(x)\,|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{e}^{-\mathrm{i}p\cdot(y-x)}}{2E_{\mathbf{p}}} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}p\cdot(x-y)}}{2E_{\mathbf{p}}}$$

$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{\mathrm{e}^{\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{\mathrm{e}^{\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}$$

- 最后一步把积分变量 $\mathbf{p}$ 替换成 $-\mathbf{p}$

$x^0 < y^0$ 的情况

- 当 $x^0 < y^0$ 时，时间排序将改变 $\phi(x)$ 和 $\phi(y)$ 的次序，有

$$\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle = \langle 0|\,\phi(y)\phi(x)\,|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{e}^{-\mathrm{i}p\cdot(y-x)}}{2E_{\mathbf{p}}} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}p\cdot(x-y)}}{2E_{\mathbf{p}}}$$

$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{\mathrm{e}^{\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{\mathrm{e}^{\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}$$

- 最后一步把积分变量 $\mathbf{p}$ 替换成 $-\mathbf{p}$

- 将 Γ_{F} 与上半平面上的半圆弧 $\Gamma_R^{(+)}$ 组成围线 $C_{\mathrm{F}}^{(+)} = \Gamma_{\mathrm{F}} + \Gamma_R^{(+)}$，曲线方向沿逆时针方向，即正方向

- 由于 $x^0 - y^0 < 0$，$\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}$ 中的因子 $\mathrm{e}^{\mathrm{Im}(p^0)(x^0-y^0)}$ 在上半平面 $(\mathrm{Im}\,p^0 > 0)$ 随着 R 增大而指数衰减，由 Jordan 引理得

$$\lim_{R\to\infty}\int_{\Gamma_R^{(+)}} \mathrm{d}p^0\,\frac{\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})} = 0$$

$x^0 < y^0$ 时的 $\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle$

- 从而在 $R \to \infty$ 时推出

$$\int_{\Gamma_{\mathrm{F}}} \mathrm{d}p^0\, \frac{\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})} = \int_{C_{\mathrm{F}}^{(+)}=\Gamma_{\mathrm{F}}+\Gamma_R^{(+)}} \mathrm{d}p^0\, \frac{\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})}$$

$$= 2\pi\mathrm{i} \operatorname*{Res}_{p^0=-E_{\mathbf{p}}} \frac{\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})} = -2\pi\mathrm{i}\, \frac{\mathrm{e}^{\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}$$

$x^0 < y^0$ 时的 $\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle$

- 从而在 $R \to \infty$ 时推出

$$\int_{\Gamma_{\mathrm{F}}} \mathrm{d}p^0\, \frac{\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})} = \int_{C_{\mathrm{F}}^{(+)}=\Gamma_{\mathrm{F}}+\Gamma_R^{(+)}} \mathrm{d}p^0\, \frac{\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})}$$

$$= 2\pi\mathrm{i} \operatorname*{Res}_{p^0=-E_{\mathbf{p}}} \frac{\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0-E_{\mathbf{p}})(p^0+E_{\mathbf{p}})} = -2\pi\mathrm{i}\, \frac{\mathrm{e}^{\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}$$

- 故

$$\frac{\mathrm{e}^{\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} = -\frac{1}{2\pi\mathrm{i}} \int_{\Gamma_{\mathrm{F}}} \mathrm{d}p^0\, \frac{\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{p^2-m^2}$$

$$= \int \frac{\mathrm{d}p^0}{2\pi}\, \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{p^2-m^2+\mathrm{i}\epsilon}$$

- 代入 $x^0 < y^0$ 时 $\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle$ 的表达式，即得

$$\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\, \frac{\mathrm{e}^{\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\, \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2-m^2+\mathrm{i}\epsilon}$$

实标量场的 Feynman 传播子

- 可见，无论 $x^0 > y^0$ 还是 $x^0 < y^0$，$\langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle$ 的表达式都是**一样**的
- 因此，**实标量场的 Feynman 传播子**总可以表达为

$$D_{\mathrm{F}}(x-y) = \langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

实标量场的 Feynman 传播子

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- 因此，**实标量场的 Feynman 传播子**总可以表达为

$$D_{\mathrm{F}}(x-y) = \langle 0|\,\mathsf{T}[\phi(x)\phi(y)]\,|0\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

- 它是 Lorentz 不变的，而且是一个**偶函数**，

$$D_{\mathrm{F}}(y-x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}p\cdot(x-y)}}{p^2 - m^2 + \mathrm{i}\epsilon} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2 - m^2 + \mathrm{i}\epsilon} = D_{\mathrm{F}}(x-y)$$

- 第二步作了**变量替换** $p^\mu \to -p^\mu$
- 可见，$\overbracket{\phi(y)\phi(x)} = \overbracket{\phi(x)\phi(y)}$

5.4.2 小节 复标量场的 Feynman 传播子

在相互作用绘景中，将复标量场 $\phi(x)$ 的平面波展开式分解为正负能解部分，得

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x), \qquad \phi^\dagger(x) = \phi^{\dagger(+)}(x) + \phi^{\dagger(-)}(x)$$

$$\phi^{(+)}(x) \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} a_{\mathbf{p}} \mathrm{e}^{-\mathrm{i}p\cdot x}, \qquad \phi^{\dagger(+)}(x) \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} b_{\mathbf{p}} \mathrm{e}^{-\mathrm{i}p\cdot x}$$

$$\phi^{(-)}(x) \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} b_{\mathbf{p}}^\dagger \mathrm{e}^{\mathrm{i}p\cdot x}, \qquad \phi^{\dagger(-)}(x) \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} a_{\mathbf{p}}^\dagger \mathrm{e}^{\mathrm{i}p\cdot x}$$

注意场算符缩并的定义是 $\overbracket{\Phi_a(x)\Phi_b(y)} = \begin{cases} [\Phi_a^{(+)}(x), \Phi_b^{(-)}(y)]_\mp, & x^0 \geq y^0 \\ \epsilon_{ab}[\Phi_b^{(+)}(y), \Phi_a^{(-)}(x)]_\mp, & x^0 < y^0 \end{cases}$

由于 $(a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger)$ 和 $(b_{\mathbf{p}}, b_{\mathbf{p}}^\dagger)$ 是两套相互独立的产生湮灭算符，有

$$\overbracket{\phi(x)\phi(y)} = \langle 0| \mathsf{T}[\phi(x)\phi(y)] |0\rangle = 0, \qquad \overbracket{\phi^\dagger(x)\phi^\dagger(y)} = \langle 0| \mathsf{T}[\phi^\dagger(x)\phi^\dagger(y)] |0\rangle = 0$$

复标量场的 Feynman 传播子是非平庸的缩并，定义为

$$D_{\mathrm{F}}(x-y) \equiv \overbracket{\phi(x)\phi^\dagger(y)} = \langle 0| \mathsf{T}[\phi(x)\phi^\dagger(y)] |0\rangle$$

$\langle 0|\,\mathsf{T}[\phi(x)\phi^\dagger(y)]\,|0\rangle$

- 类似于实标量场的计算，利用**产生湮灭算符的对易关系**，得到

$$\begin{aligned}
\langle 0|\,\phi(x)\phi^\dagger(y)\,|0\rangle &= \langle 0|\,\phi^{(+)}(x)\phi^{\dagger(-)}(y)\,|0\rangle = \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\,\langle 0|\,a_{\mathbf{p}}\mathrm{e}^{-\mathrm{i}p\cdot x}a_{\mathbf{q}}^\dagger\mathrm{e}^{\mathrm{i}q\cdot y}\,|0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot x-q\cdot y)}}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\,\langle 0|\,[a_{\mathbf{p}},a_{\mathbf{q}}^\dagger]\,|0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot x-q\cdot y)}}{(2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\,\delta^{(3)}(\mathbf{p}-\mathbf{q}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{2E_{\mathbf{p}}}
\end{aligned}$$

$$\begin{aligned}
\langle 0|\,\phi^\dagger(y)\phi(x)\,|0\rangle &= \langle 0|\,\phi^{\dagger(+)}(y)\phi^{(-)}(x)\,|0\rangle = \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\,\langle 0|\,b_{\mathbf{p}}\mathrm{e}^{-\mathrm{i}p\cdot y}b_{\mathbf{q}}^\dagger\mathrm{e}^{\mathrm{i}q\cdot x}\,|0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot y-q\cdot x)}}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\,\langle 0|\,[b_{\mathbf{p}},b_{\mathbf{q}}^\dagger]\,|0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot y-q\cdot x)}}{(2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\,\delta^{(3)}(\mathbf{p}-\mathbf{q}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}p\cdot(x-y)}}{2E_{\mathbf{p}}}
\end{aligned}$$

$$\begin{aligned}
\langle 0|\,\mathsf{T}[\phi(x)\phi^\dagger(y)]\,|0\rangle &= \theta(x^0-y^0)\,\langle 0|\,\phi(x)\phi^\dagger(y)\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,\phi^\dagger(y)\phi(x)\,|0\rangle \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} + \theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]
\end{aligned}$$

四维动量积分

归纳上面实标量场计算中的相关结果，有

$$\theta(x^0-y^0)\frac{\mathrm{e}^{-\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}+\theta(y^0-x^0)\frac{\mathrm{e}^{\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}=\int\frac{\mathrm{d}p^0}{2\pi}\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{p^2-m^2+\mathrm{i}\epsilon}$$

其中 $\epsilon>0$ 是一个无穷小量，进而推出

$$\begin{aligned}&\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]\\=&\int\frac{\mathrm{d}^3p}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\left[\theta(x^0-y^0)\frac{\mathrm{e}^{-\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}+\theta(y^0-x^0)\frac{\mathrm{e}^{\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}\right]\\=&\int\frac{\mathrm{d}^3p}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\int\frac{\mathrm{d}p^0}{2\pi}\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{p^2-m^2+\mathrm{i}\epsilon}=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2-m^2+\mathrm{i}\epsilon}\end{aligned}$$

第二步将方括号第二项积分变量 $\mathbf{p}$ **替换成** $-\mathbf{p}$，从而把 $\mathrm{e}^{\mathrm{i}p\cdot(x-y)}$ 中的 $\mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}$ 因子变成 $\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}$ **因子**

最后一步将三维动量积分和 p^0 积分合成**四维动量积分**

复标量场的 Feynman 传播子

于是，**复标量场的 Feynman 传播子**表达为

$$
\begin{aligned}
D_{\mathrm{F}}(x-y) &= \langle 0|\,\mathsf{T}[\phi(x)\phi^{\dagger}(y)]\,|0\rangle \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}] \\
&= \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}
\end{aligned}
$$

可见，复标量场与**实标量场**具有**相同**形式的 Feynman 传播子

此外，由 $\mathsf{T}[\Phi_a(x)\Phi_b(y)]=\epsilon_{ab}\,\mathsf{T}[\Phi_b(y)\Phi_a(x)]$ 和 $D_{\mathrm{F}}(x-y)$ 的**偶函数**性质得

$$
\overbracket{\phi^{\dagger}(x)\phi(y)} = \langle 0|\,\mathsf{T}[\phi^{\dagger}(x)\phi(y)]\,|0\rangle = \langle 0|\,\mathsf{T}[\phi(y)\phi^{\dagger}(x)]\,|0\rangle = D_{\mathrm{F}}(y-x) = D_{\mathrm{F}}(x-y)
$$

也就是说，$\overbracket{\phi^{\dagger}(x)\phi(y)}$ **与** $\overbracket{\phi(x)\phi^{\dagger}(y)}$ **相等**

5.4.3 小节 有质量矢量场的 Feynman 传播子

有质量实矢量场 $A^\mu(x)$ 的 Feynman 传播子 $\Delta_{\mathrm{F}}^{\mu\nu}(x-y)$ 定义为

$$\Delta_{\mathrm{F}}^{\mu\nu}(x-y) \equiv \overbracket{A^\mu(x)A^\nu(y)} = \langle 0|\,\mathsf{T}[A^\mu(x)A^\nu(y)]\,|0\rangle$$

根据平面波展开式、产生湮灭算符的对易关系和极化矢量求和关系，有

$$\begin{aligned}
&\langle 0|\,A^\mu(x)A^\nu(y)\,|0\rangle = \langle 0|\,A^{\mu(+)}(x)A^{\nu(-)}(y)\,|0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \sum_{\lambda\lambda'} \langle 0|\,\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}\varepsilon^{\nu*}(\mathbf{q},\lambda')a^\dagger_{\mathbf{q},\lambda'}\mathrm{e}^{\mathrm{i}q\cdot y}\,|0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot x-q\cdot y)}}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \sum_{\lambda\lambda'} \varepsilon^\mu(\mathbf{p},\lambda)\varepsilon^{\nu*}(\mathbf{q},\lambda')\,\langle 0|\,[a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}]\,|0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot x-q\cdot y)}}{(2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \sum_{\lambda\lambda'} \varepsilon^\mu(\mathbf{p},\lambda)\varepsilon^{\nu*}(\mathbf{q},\lambda')\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{(2\pi)^3 2E_{\mathbf{p}}} \sum_{\lambda} \varepsilon^\mu(\mathbf{p},\lambda)\varepsilon^{\nu*}(\mathbf{p},\lambda) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{m^2}\right)\frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{2E_{\mathbf{p}}}
\end{aligned}$$

$\langle 0|\,\mathsf{T}[A^\mu(x)A^\nu(y)]\,|0\rangle$

由上述 $\langle 0|\,A^\mu(x)A^\nu(y)\,|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{m^2}\right)\frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{2E_{\mathbf{p}}}$ 得到

$$\begin{aligned}\langle 0|\,A^\nu(y)A^\mu(x)\,|0\rangle &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(-g^{\nu\mu}+\frac{p^\nu p^\mu}{m^2}\right)\frac{\mathrm{e}^{-\mathrm{i}p\cdot(y-x)}}{2E_{\mathbf{p}}}\\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{m^2}\right)\frac{\mathrm{e}^{\mathrm{i}p\cdot(x-y)}}{2E_{\mathbf{p}}}\end{aligned}$$

从而推出 Feynman 传播子表达式

$$\begin{aligned}&\Delta_{\mathrm{F}}^{\mu\nu}(x-y) = \langle 0|\,\mathsf{T}[A^\mu(x)A^\nu(y)]\,|0\rangle\\ &= \theta(x^0-y^0)\,\langle 0|\,A^\mu(x)A^\nu(y)\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,A^\nu(y)A^\mu(x)\,|0\rangle\\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{m^2}\right)\frac{1}{2E_{\mathbf{p}}}[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]\end{aligned}$$

最后一行圆括号中的项 $p^\mu p^\nu/m^2$ 与 p^0 有关，因此不能直接应用前面的

$$\theta(x^0-y^0)\,\frac{\mathrm{e}^{-\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}+\theta(y^0-x^0)\,\frac{\mathrm{e}^{\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} = \int \frac{\mathrm{d}p^0}{2\pi}\,\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{p^2-m^2+\mathrm{i}\epsilon}$$

合成四维动量积分

求导运算

- 为了得到简洁的表达式，需要将 $p^\mu p^\nu/m^2$ 转换为**求导运算**
- 记 $\partial_x^\mu \equiv \partial/\partial x_\mu$，利用阶跃函数与 δ 函数的关系 $\theta'(x)=\delta(x)$，推出

$$
\begin{aligned}
&\partial_x^\mu\partial_x^\nu[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]\\
=\ &\partial_x^\mu[-\mathrm{i}p^\nu\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+g^{\nu0}\delta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\mathrm{i}p^\nu\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}\\
&\quad -g^{\nu0}\delta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]\\
=\ &-p^\mu p^\nu\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}-\mathrm{i}g^{\mu0}p^\nu\delta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}-\mathrm{i}p^\mu g^{\nu0}\delta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}\\
&+g^{\mu0}g^{\nu0}\partial_x^0\delta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}-p^\mu p^\nu\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}-\mathrm{i}g^{\mu0}p^\nu\delta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}\\
&-\mathrm{i}p^\mu g^{\nu0}\delta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}+g^{\mu0}g^{\nu0}\partial_x^0\delta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}\\
=\ &-p^\mu p^\nu[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]\\
&-\mathrm{i}(g^{\mu0}p^\nu+g^{\nu0}p^\mu)\delta(x^0-y^0)[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]\\
&+g^{\mu0}g^{\nu0}\partial_x^0\delta(x^0-y^0)[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}-\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]
\end{aligned}
$$

分解

从而将包含 $p^\mu p^\nu/m^2$ 的因子化为

$$\begin{aligned}
&\frac{p^\mu p^\nu}{m^2}\left[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}\right]\\
=&-\frac{\partial_x^\mu\partial_x^\nu}{m^2}\left[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}\right]\\
&-\frac{\mathrm{i}}{m^2}(g^{\mu0}p^\nu+g^{\nu0}p^\mu)\delta(x^0-y^0)[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]\\
&+\frac{g^{\mu0}g^{\nu0}}{m^2}\partial_x^0\delta(x^0-y^0)[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}-\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]
\end{aligned}$$

由此将 Feynman 传播子**分解**成 $\Delta_{\mathrm{F}}^{\mu\nu}(x-y)=f_1^{\mu\nu}(x,y)+f_2^{\mu\nu}(x,y)+f_3^{\mu\nu}(x,y)$

$$f_1^{\mu\nu}(x,y)\equiv-\left(g^{\mu\nu}+\frac{\partial_x^\mu\partial_x^\nu}{m^2}\right)\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}\right]$$

$$f_2^{\mu\nu}(x,y)\equiv-\frac{\mathrm{i}}{m^2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(g^{\mu0}p^\nu+g^{\nu0}p^\mu)\delta(x^0-y^0)[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]$$

$$f_3^{\mu\nu}(x,y)\equiv\frac{g^{\mu0}g^{\nu0}}{m^2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\partial_x^0\delta(x^0-y^0)[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}-\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]$$

$f_1^{\mu\nu}(x,y)$ 和 $f_2^{\mu\nu}(x,y)$

将第一项化为

$$f_1^{\mu\nu}(x,y) = -\left(g^{\mu\nu} + \frac{\partial_x^\mu \partial_x^\nu}{m^2}\right) \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left[\theta(x^0 - y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} + \theta(y^0 - x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}\right]$$

$$= -\left(g^{\mu\nu} + \frac{\partial_x^\mu \partial_x^\nu}{m^2}\right) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2 - m^2 + \mathrm{i}\epsilon} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{-\mathrm{i}(g^{\mu\nu} - p^\mu p^\nu / m^2)}{p^2 - m^2 + \mathrm{i}\epsilon} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

$\delta(x^0 - y^0)$ 只在 $x^0 - y^0 = 0$ 处非零，该处有 $\mathrm{e}^{-\mathrm{i}E_{\mathbf{p}}(x^0-y^0)} = \mathrm{e}^{\mathrm{i}E_{\mathbf{p}}(x^0-y^0)} = 1$，故

$$f_2^{i0}(x,y) = f_2^{0i}(x,y) = -\frac{\mathrm{i}}{m^2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{p^i}{2E_{\mathbf{p}}} \delta(x^0 - y^0)[\mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}] = 0$$

上式中被积函数是关于 $\mathbf{p}$ 的奇函数，因而对整个三维动量空间积分为零

利用 Fourier 变换公式导出

$$\begin{aligned} f_2^{00}(x,y) &= -\frac{\mathrm{i}}{m^2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{2p^0}{2E_{\mathbf{p}}} \delta(x^0 - y^0)[\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}] \\ &= -\frac{2\mathrm{i}}{m^2} \delta(x^0 - y^0)\delta^{(3)}(\mathbf{x} - \mathbf{y}) = -\frac{2\mathrm{i}}{m^2} \delta^{(4)}(x - y) \end{aligned}$$

归纳得到第二项表达式 $f_2^{\mu\nu}(x,y) = -\dfrac{2\mathrm{i}}{m^2} g^{\mu 0} g^{\nu 0} \delta^{(4)}(x - y)$

$f_3^{\mu\nu}(x,y)$

另一方面，根据 δ 函数的导数的定义，有

$$\int \mathrm{d}x\, f(x)\delta'(x-a) = -f'(a) = -\int \mathrm{d}x\, f'(x)\delta(x-a)$$

因而对第三项中的被积函数可作替换

$$\partial_x^0\delta(x^0-y^0)[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}p\cdot(x-y)}] \to -\delta(x^0-y^0)\partial_x^0[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}p\cdot(x-y)}]$$

由此将第三项化为

$$\begin{aligned}
f_3^{\mu\nu}(x,y) &= \frac{g^{\mu0}g^{\nu0}}{m^2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\,\partial_x^0\delta(x^0-y^0)[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}p\cdot(x-y)}] \\
&= -\frac{g^{\mu0}g^{\nu0}}{m^2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\,\delta(x^0-y^0)\partial_x^0[\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}p\cdot(x-y)}] \\
&= -\frac{g^{\mu0}g^{\nu0}}{m^2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\,\delta(x^0-y^0)[-\mathrm{i}p^0\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - \mathrm{i}p^0\mathrm{e}^{\mathrm{i}p\cdot(x-y)}] \\
&= \frac{\mathrm{i}}{2m^2}g^{\mu0}g^{\nu0}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\delta(x^0-y^0)[\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}] \\
&= \frac{\mathrm{i}}{m^2}g^{\mu0}g^{\nu0}\delta(x^0-y^0)\delta^{(3)}(\mathbf{x}-\mathbf{y}) = \frac{\mathrm{i}}{m^2}g^{\mu0}g^{\nu0}\delta^{(4)}(x-y)
\end{aligned}$$

有质量矢量场的 Feynman 传播子

综合起来，**有质量矢量场 Feynman 传播子**的表达式为

$$\begin{aligned}\Delta_{\mathrm{F}}^{\mu\nu}(x-y) &= f_1^{\mu\nu}(x,y)+f_2^{\mu\nu}(x,y)+f_3^{\mu\nu}(x,y)\\ &= \int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{-\mathrm{i}(g^{\mu\nu}-p^\mu p^\nu/m^2)}{p^2-m^2+\mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}-\frac{\mathrm{i}}{m^2}g^{\mu0}g^{\nu0}\delta^{(4)}(x-y)\end{aligned}$$

第一项是 **Lorentz 协变**的，但第二项是**非协变**的

幸好，6.4 节将证明这个非协变项在微扰论中的贡献被**相互作用哈密顿量密度**中**非协变项** $\mathcal{H}_{J^0}^{\mathrm{I}}=\frac{g^2}{2m^2}(J^{\mathrm{I},0})^2$ 的贡献**精确抵消**，从而**理论是 Lorentz 协变的**

因此，在实际计算中可以**只保留协变项**

$$\Delta_{\mathrm{F}}^{\mu\nu}(x-y)=\langle 0|\,\mathsf{T}[A^\mu(x)A^\nu(y)]\,|0\rangle\to\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{-\mathrm{i}(g^{\mu\nu}-p^\mu p^\nu/m^2)}{p^2-m^2+\mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

5.4.4 小节 无质量矢量场的 Feynman 传播子

- 无质量实矢量场 $A^\mu(x)$ 的 Feynman 传播子依赖于规范的选择
- 这里取 Feynman 规范 $(\xi = 1)$
- 在相互作用绘景中，将无质量实矢量场 $A^\mu(x)$ 的平面波展开式分解为

$$\text{正能解}\ A^{\mu(+)}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma) b_{\mathbf{p},\sigma} \mathrm{e}^{-\mathrm{i}p\cdot x}$$

$$\text{负能解}\ A^{\mu(-)}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma) b^\dagger_{\mathbf{p},\sigma} \mathrm{e}^{\mathrm{i}p\cdot x}$$

- 相应的 Feynman 传播子定义为

$$\Delta_{\mathrm{F}}^{\mu\nu}(x-y) \equiv \overbracket{A^\mu(x)A^\nu(y)} = \langle 0|\, \mathrm{T}[A^\mu(x)A^\nu(y)]\, |0\rangle$$

$\langle 0| A^\mu(x)A^\nu(y)|0\rangle$

根据产生湮灭算符的对易关系和极化矢量的完备性关系，得到

$$
\begin{aligned}
&\langle 0| A^\mu(x)A^\nu(y)|0\rangle = \langle 0| A^{\mu(+)}(x)A^{\nu(-)}(y)|0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \sum_{\sigma\sigma'} \langle 0| e^\mu(\mathbf{p},\sigma) b_{\mathbf{p},\sigma} \mathrm{e}^{-\mathrm{i}p\cdot x} e^\nu(\mathbf{q},\sigma') b^\dagger_{\mathbf{q},\sigma'} \mathrm{e}^{\mathrm{i}q\cdot y}|0\rangle \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot x-q\cdot y)}}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \sum_{\sigma\sigma'} e^\mu(\mathbf{p},\sigma)e^\nu(\mathbf{q},\sigma')\langle 0|[b_{\mathbf{p},\sigma}, b^\dagger_{\mathbf{q},\sigma'}]|0\rangle \\
&= -\int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot x-q\cdot y)}}{(2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \sum_{\sigma\sigma'} e^\mu(\mathbf{p},\sigma)e^\nu(\mathbf{q},\sigma') g_{\sigma\sigma'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&= -\int \frac{\mathrm{d}^3p\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{(2\pi)^3 2E_{\mathbf{p}}} \sum_{\sigma} g_{\sigma\sigma} e^\mu(\mathbf{p},\sigma)e^\nu(\mathbf{p},\sigma) = -g^{\mu\nu}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{2E_{\mathbf{p}}}
\end{aligned}
$$

进而推出

$$
\langle 0| A^\nu(y)A^\mu(x)|0\rangle = -g^{\nu\mu}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{e}^{-\mathrm{i}p\cdot(y-x)}}{2E_{\mathbf{p}}} = -g^{\mu\nu}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}p\cdot(x-y)}}{2E_{\mathbf{p}}}
$$

无质量矢量场的 Feynman 传播子

从而得到

$$
\begin{aligned}
\Delta_{\mathrm{F}}^{\mu\nu}(x-y) &= \langle 0|\,\mathsf{T}[A^{\mu}(x)A^{\nu}(y)]\,|0\rangle \\
&= \theta(x^0-y^0)\,\langle 0|\,A^{\mu}(x)A^{\nu}(y)\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,A^{\nu}(y)A^{\mu}(x)\,|0\rangle \\
&= -g^{\mu\nu}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} + \theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]
\end{aligned}
$$

当质量 $m=0$ 时，有

$$
\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} + \theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}] = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2+\mathrm{i}\epsilon}
$$

于是，**Feynman 规范**下**无质量矢量场的 Feynman 传播子**可以表达为

$$
\Delta_{\mathrm{F}}^{\mu\nu}(x-y) = \langle 0|\,\mathsf{T}[A^{\mu}(x)A^{\nu}(y)]\,|0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{-\mathrm{i}g^{\mu\nu}}{p^2+\mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}
$$

5.4.5 小节 Dirac 旋量场的 Feynman 传播子

Dirac 旋量场 $\psi_a(x)$ 的 Feynman 传播子 $S_{\mathrm{F},ab}(x-y)$ 定义为

$$S_{\mathrm{F},ab}(x-y) \equiv \overbrace{\psi_a(x)\bar{\psi}_b(y)} = \langle 0|\,\mathsf{T}[\psi_a(x)\bar{\psi}_b(y)]\,|0\rangle$$

在**相互作用绘景**中，将 $\psi_a(x)$ 和 $\bar{\psi}_a(x)$ 的平面波展开式分解为**正负**能解部分，得

$$\psi_a(x) = \psi_a^{(+)}(x) + \psi_a^{(-)}(x), \quad \bar{\psi}_a(x) = \bar{\psi}_a^{(+)}(x) + \bar{\psi}_a^{(-)}(x)$$

$$\psi_a^{(+)}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm} u_a(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}$$

$$\psi_a^{(-)}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm} v_a(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}^{\dagger}\mathrm{e}^{\mathrm{i}p\cdot x}$$

$$\bar{\psi}_a^{(+)}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm} \bar{v}_a(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}$$

$$\bar{\psi}_a^{(-)}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\lambda=\pm} \bar{u}_a(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}^{\dagger}\mathrm{e}^{\mathrm{i}p\cdot x}$$

$\langle 0|\,\psi_a(x)\bar{\psi}_b(y)\,|0\rangle$

利用产生湮灭算符的反对易关系和自旋求和关系，推出

$$
\begin{aligned}
&\langle 0|\,\psi_a(x)\bar{\psi}_b(y)\,|0\rangle \\
=\ &\langle 0|\,\psi_a^{(+)}(x)\bar{\psi}_b^{(-)}(y)\,|0\rangle \\
=\ &\int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\sum_{\lambda\lambda'}\langle 0|\,u_a(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}\bar{u}_b(\mathbf{q},\lambda')a^\dagger_{\mathbf{q},\lambda'}\mathrm{e}^{\mathrm{i}q\cdot y}\,|0\rangle \\
=\ &\int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot x-q\cdot y)}}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\sum_{\lambda\lambda'}u_a(\mathbf{p},\lambda)\bar{u}_b(\mathbf{q},\lambda')\,\langle 0|\,\{a_{\mathbf{p},\lambda},a^\dagger_{\mathbf{q},\lambda'}\}\,|0\rangle \\
=\ &\int \frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot x-q\cdot y)}}{(2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\sum_{\lambda\lambda'}u_a(\mathbf{p},\lambda)\bar{u}_b(\mathbf{q},\lambda')\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
=\ &\int \frac{\mathrm{d}^3p\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{(2\pi)^3 2E_{\mathbf{p}}}\sum_{\lambda}u_a(\mathbf{p},\lambda)\bar{u}_b(\mathbf{p},\lambda) \\
=\ &\int \frac{\mathrm{d}^3p}{(2\pi)^3}\,(\not{p}+m)_{ab}\,\frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{2E_{\mathbf{p}}}
\end{aligned}
$$

$\langle 0|\,\bar{\psi}_b(y)\psi_a(x)\,|0\rangle$

再推出

$$
\begin{aligned}
&\langle 0|\,\bar{\psi}_b(y)\psi_a(x)\,|0\rangle \\
=\;&\langle 0|\,\bar{\psi}_b^{(+)}(y)\psi_a^{(-)}(x)\,|0\rangle \\
=\;&\int\frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\sum_{\lambda\lambda'}\langle 0|\,\bar{v}_b(\mathbf{p},\lambda)b_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot y}v_a(\mathbf{q},\lambda')b^\dagger_{\mathbf{q},\lambda'}\mathrm{e}^{\mathrm{i}q\cdot x}\,|0\rangle \\
=\;&\int\frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot y-q\cdot x)}}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\sum_{\lambda\lambda'}v_a(\mathbf{q},\lambda')\bar{v}_b(\mathbf{p},\lambda)\,\langle 0|\,\{b_{\mathbf{p},\lambda},b^\dagger_{\mathbf{q},\lambda'}\}\,|0\rangle \\
=\;&\int\frac{\mathrm{d}^3p\,\mathrm{d}^3q\,\mathrm{e}^{-\mathrm{i}(p\cdot y-q\cdot x)}}{(2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\sum_{\lambda\lambda'}v_a(\mathbf{q},\lambda')\bar{v}_b(\mathbf{p},\lambda)\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
=\;&\int\frac{\mathrm{d}^3p\,\mathrm{e}^{-\mathrm{i}p\cdot(y-x)}}{(2\pi)^3 2E_{\mathbf{p}}}\sum_{\lambda}v_a(\mathbf{p},\lambda)\bar{v}_b(\mathbf{p},\lambda) \\
=\;&\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,(\not{p}-m)_{ab}\,\frac{\mathrm{e}^{\mathrm{i}p\cdot(x-y)}}{2E_{\mathbf{p}}}
\end{aligned}
$$

Dirac 旋量场的 Feynman 传播子

✂️ 于是，**Dirac 旋量场的 Feynman 传播子**为

$$
\begin{aligned}
& S_{\mathrm{F},ab}(x-y) = \langle 0| \mathsf{T}[\psi_a(x)\bar{\psi}_b(y)] |0\rangle \\
= & \theta(x^0-y^0) \langle 0| \psi_a(x)\bar{\psi}_b(y) |0\rangle - \theta(y^0-x^0) \langle 0| \bar{\psi}_b(y)\psi_a(x) |0\rangle \\
= & \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left[\theta(x^0-y^0)(\gamma_\mu p^\mu + m)_{ab} \frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{2E_{\mathbf{p}}} \right. \\
& \qquad \left. - \theta(y^0-x^0)(\gamma_\mu p^\mu - m)_{ab} \frac{\mathrm{e}^{\mathrm{i}p\cdot(x-y)}}{2E_{\mathbf{p}}} \right] \\
= & (\gamma_0)_{ab} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} [E_{\mathbf{p}}\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - E_{\mathbf{p}}\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}] \\
& + (\gamma_i)_{ab} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} [p^i\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} - p^i\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}] \\
& + m\delta_{ab} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)} + \theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)} \right]
\end{aligned}
$$

Dirac 旋量场的 Feynman 传播子

类似于前面得到的等式

$$\theta(x^0-y^0)\frac{\mathrm{e}^{-\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}+\theta(y^0-x^0)\frac{\mathrm{e}^{\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}=\int\frac{\mathrm{d}p^0}{2\pi}\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{p^2-m^2+\mathrm{i}\epsilon}$$

$$\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2-m^2+\mathrm{i}\epsilon}$$

对于**任意解析函数** $f(p^0)$，可以推出

$$\theta(x^0-y^0)\frac{f(E_{\mathbf{p}})\mathrm{e}^{-\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}+\theta(y^0-x^0)\frac{f(-E_{\mathbf{p}})\mathrm{e}^{\mathrm{i}E_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}}$$

$$=\int\frac{\mathrm{d}p^0}{2\pi}\frac{\mathrm{i}f(p^0)\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{p^2-m^2+\mathrm{i}\epsilon}$$

$$\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[f(E_{\mathbf{p}})\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+f(-E_{\mathbf{p}})\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]$$

$$=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}f(p^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2-m^2+\mathrm{i}\epsilon}$$

左边**第一项**对应于 $p^0=E_{\mathbf{p}}$ 处的**留数**，**第二项**对应于 $p^0=-E_{\mathbf{p}}$ 处的**留数**

Dirac 旋量场的 Feynman 传播子

- 取 $f(p^0)=p^0$，把 $S_{\mathrm{F},ab}(x-y)$ 的第一项变成

$$(\gamma_0)_{ab}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[E_{\mathbf{p}}\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}-E_{\mathbf{p}}\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]$$
$$=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}(\gamma_0p^0)_{ab}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2-m^2+\mathrm{i}\epsilon}$$

- 再将 $S_{\mathrm{F},ab}(x-y)$ 式的后两项化成

$$(\gamma_i)_{ab}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}[p^i\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}-p^i\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}]$$
$$+m\delta_{ab}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}\right]$$
$$=[(\gamma_i)_{ab}\mathrm{i}\partial_x^i+m\delta_{ab}]\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}+\theta(y^0-x^0)\mathrm{e}^{\mathrm{i}p\cdot(x-y)}\right]$$
$$=[(\gamma_i)_{ab}\mathrm{i}\partial_x^i+m\delta_{ab}]\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2-m^2+\mathrm{i}\epsilon}=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}(\gamma_ip^i+m)_{ab}}{p^2-m^2+\mathrm{i}\epsilon}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}.$$

- 把这三项加起来，得到 **Dirac 旋量场的 Feynman 传播子**

$$S_{\mathrm{F},ab}(x-y)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}(\not{p}+m)_{ab}}{p^2-m^2+\mathrm{i}\epsilon}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

等价形式

将 **Dirac 旋量场的 Feynman 传播子**写成**旋量空间矩阵**的形式，得

$$S_{\mathrm{F}}(x-y)=\langle 0|\,\mathsf{T}[\psi(x)\bar{\psi}(y)]\,|0\rangle=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}(\not{p}+m)}{p^2-m^2+\mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

由 $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$ 有

$$\not{p}\not{p}=p_\mu p_\nu\gamma^\mu\gamma^\nu=\frac{1}{2}p_\mu p_\nu(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)=p_\mu p_\nu g^{\mu\nu}=p^2$$

从而得到 $(\not{p}+m)(\not{p}-m)=\not{p}\not{p}-m^2=p^2-m^2$，则

$$(\not{p}+m)(\not{p}-m+\mathrm{i}\epsilon)=p^2-m^2+\mathrm{i}\epsilon(\not{p}+m)$$

$\mathrm{i}\epsilon(\not{p}+m)$ 是**无穷小量**，因而上式右边**与** $p^2-m^2+\mathrm{i}\epsilon$ **等价**，故

$$S_{\mathrm{F}}(x-y)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}(\not{p}+m)\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{(\not{p}+m)(\not{p}-m+\mathrm{i}\epsilon)}=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}}{\not{p}-m+\mathrm{i}\epsilon}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

最后的表达式更为**简洁**，但在**矩阵**的意义上不好理解，应将它转化回**原式**来理解