Problem Solving Class: Geometric Tests

1 Area formula

Area of Triangle $\triangle abc$: If we want to compute the area of the triangle with vertices $a = (a_x, a_y), b = (b_x, b_y)$ and $c = (c_x, c_y)$, there are two ways to do it: one is to use

$$\text{SignArea}(\triangle abc) = \frac{1}{2}(a_x b_y - a_y b_x + b_x c_y - c_x b_y + c_x a_y - a_x c_y).$$

and the other is just to use the following formula

$$\text{SignArea}(\triangle abc) = \frac{1}{2}\begin{vmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{vmatrix}. \tag{1}$$

Note that
- SignArea($\triangle abc$) > 0 if a, b, c are in *counterclockwise order*,
- SignArea($\triangle abc$) < 0 if a, b, c are in *clockwise order*,
- SignArea($\triangle abc$) $= 0$ if a, b, c are *collinear*.

Area of a Convex Quadrilateral $\square abcd$:

$$\begin{aligned} \text{SignArea}(\square abcd) &= \text{SignArea}(\triangle abc) + \text{SignArea}(\triangle acd) \\ &= \text{SignArea}(\triangle dab) + \text{SignArea}(\triangle dbc) \\ &= a_x b_y - a_y b_x + b_x c_y - c_x b_y + c_x d_y - d_x c_y + d_x a_y - a_x d_y. \end{aligned} \tag{2}$$

Area of a Nonconvex Quadrilateral $\square abcd$: Same as (2). SignArea($\triangle acd$) < 0.

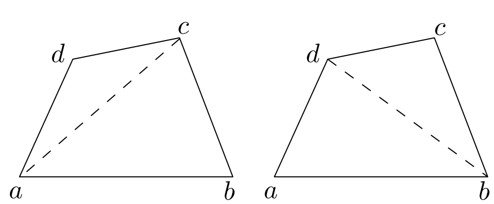
Area from an Arbitrary Center:

Theorem 1.1 *If $\triangle abc$ is a triangle with vertices oriented counterclockwise, and $p \in \mathbb{R}^2$ is any point in the plane, then*

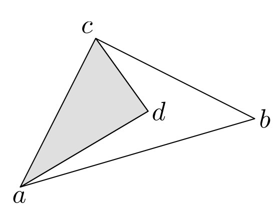
$$\text{SignArea}(\triangle abc) = \text{SignArea}(\triangle pab) + \text{SignArea}(\triangle pbc) + \text{SignArea}(\triangle pca).$$

Theorem 1.2 *Let a polygon (convex or nonconvex) P have vertices $v_1, v_2, \ldots, v_n$ labeled counterclockwise, and let $p \in \mathbb{R}^2$ be any point in the plane. Then*

$$\text{SignArea}(P) = \text{SignArea}(\triangle p v_1 v_2) + \cdots \text{SignArea}(\triangle p v_n v_1) = \sum_{i=1}^{i=n} \text{SignArea}(\triangle p v_i v_{i+1}). \tag{3}$$



Area of a convex quadrilateral



Area of a nonconvex quadrilateral

In the following figure, $\text{SignArea}(\triangle p12), \text{SignArea}(\triangle p67), \text{SignArea}(\triangle p70)$ are negative.

Volume in Three and Higher Dimensions: Formula (1) generalizes to d-dimension as the volume of d-dimensional simplex with $d!$ in the formula

$$\text{SignArea}(\text{Tetrahedron abcd in } \mathbb{R}^3) = \frac{1}{3!} \begin{vmatrix} a_x & a_y & a_z & 1 \\ b_x & b_y & b_z & 1 \\ c_x & c_y & c_z & 1 \\ d_x & d_y & d_z & 1 \end{vmatrix}$$

and thus the volume of any d-dimensional polyhedron is similar as (3).

2 $O(n \log n)$ Convex Hull Algorithm

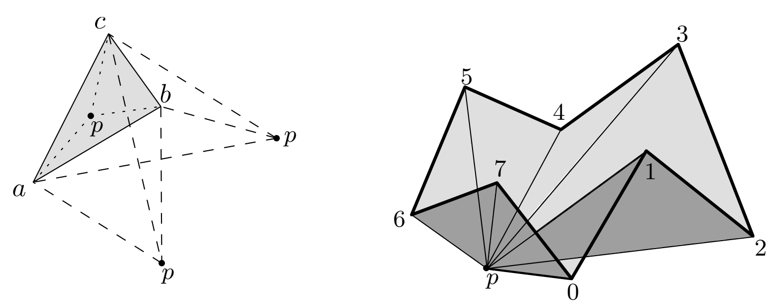
Graham's Scan Algorithm.

1. Find the rightmost lowest point; label it p_0.
2. Sort all other points angularly about p_0. In case of tie, delete the point closer to p_0. If there are multiple copies, delete all but one.
3. Stack $S = (p_1, p_0) = (p_t, p_{t-1})$; t indexes top.
4. $i = 2$;
   while $i < n$ do
       if p_i is strictly left of $p_{t-1}p_t$
       then `Push(`p_i, S`)` and set $i \rightarrow i + 1$.
       else `Pop(S)`

3 Triangulation of a Simple Polygon

Let $\mathcal{P}$ be a simple polygon with n vertices $v_1, v_2, \cdots, v_n$ in counterclockwise order.

- A *triangulation* of $\mathcal{P}$ is a decomposition of the polygon into a set of triangles.
- A *diagonal* of P is a line segment between two of its vertices a and b that are clearly visible to one another. That is, $\overline{ab}$ is inside $\mathcal{P}$.



Area from arbitrary center p

- A vertex v of given polygon $\mathcal{P}$ is called an *ear* of $\mathcal{P}$ if its two neighbors (predecessor, successor) generate a diagonal, i.e., the segment `prev(v)next(v)` is inside $\mathcal{P}$. Note that if v is an ear then the vertices v, `prev(v)` and `next(v)` form a triangle.

3.1 Existence of Triangulation and Properties

Lemma 3.1 *Every polygon must have at least one strictly convex vertex.*

Proof. The lowest part of the polygon must contain a strictly convex vertex. ◻

Lemma 3.2 (Meisters) *Every polygon of $n \geq 4$ vertices has a diagonal.*

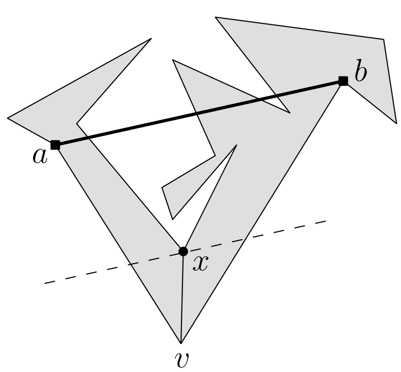
Proof. By the above lemma, there is a strictly convex vertex v. Let a and b be the vertices adjacent to v. If $\overline{ab}$ is a diagonal, we are done. Otherwise, the triangle $\triangle vab$ contains at least one vertex of $\mathcal{P}$ other than a, v, b. Let x be the first point in $\triangle vab$ hit by a line parallel to $\overline{ab}$ moving from v to $\overline{ab}$. Then x is a vertex of $\mathcal{P}$ and $\overline{vx}$ must be a diagonal. ◻

Theorem 3.3 (Triangulation) *Every polygon $\mathcal{P}$ can be partitioned into triangles by the addition of diagonals.*

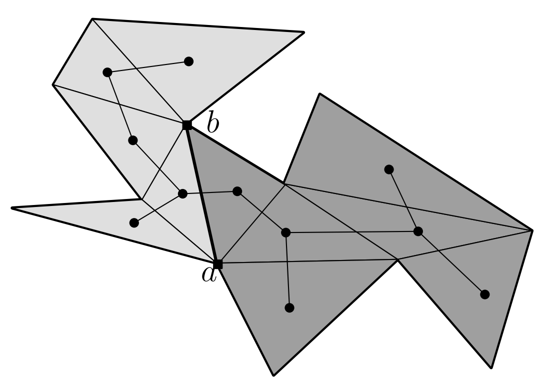
Proof. The proof is by induction. If $n = 3$, trivial. Assume that every polygon with at most k vertices can be triangulated and consider a polygon $\mathcal{P}$ with $k+1$ vertices. By Meisters' Lemma, there is a diagonal $\overline{ab}$ in $\mathcal{P}$. By the diagonal, $\mathcal{P}$ is partitioned into $\mathcal{P}'$ and $\mathcal{P}''$ sharing ab as edges. $\mathcal{P}'$ and $\mathcal{P}''$ have at most k vertices each, so we can triangulated them. This gives us a triangulation of $\mathcal{P}$. ◻

Theorem 3.4 *Given a simple polygon $\mathcal{P}$ with $n \geq 4$ vertices,*

- *Every triangulation of $\mathcal{P}$ has exactly $n - 3$ diagonals and $n - 2$ triangles.*
- *The dual graph of any triangulation of $\mathcal{P}$ is a tree.*



vx is a diagonal



Triangulation of P and its dual

- *There exist at least two nonoverlapping ears.*

Proof. - The proof is by induction. For $n = 3$, trivially true. Consider a polygon $\mathcal{P}$ with k vertices. A diagonal $\overline{ab}$ partitions $\mathcal{P}$ into $\mathcal{P}'$ and $\mathcal{P}''$ sharing ab as edges. $\mathcal{P}'$ has $k_1 \leq k-1$ vertices and $\mathcal{P}''$ has $k_2 \leq k-1$ vertices, where $k_1 + k_2 = k + 2$. By the assumption, the triangulation of $\mathcal{P}'$ consists of $k_1 - 3$ diagonals and $k_1 - 2$ triangles, and the triangulation of $\mathcal{P}''$ consists of $k_2 - 3$ diagonals and $k_2 - 2$ triangles. So $\mathcal{P}$ has $(k_1 - 3) + (k_2 - 3) + 1 = k - 3$ diagonals and $(k_1 - 2) + (k_2 - 2) = k - 2$ triangles.
- Dual graph of the triangulation is the graph, one node per triangle and one edge per diagonal.
- A leaf node in a triangulation dual corresponds to an ear. A tree having at least two nodes has at least two leaves. □

3.2 $O(n^2)$ algorithm for a triangulation of $\mathcal{P}$

1. For each vertex v, check if v is an ear and store the information;
   How to check if v is an ear :
   - Check if v is convex (Left(`prev(v)`,v, `next(v)`) = TRUE) and then
   - Check if the segment `prev(v)next(v)` does not intersect any edge of $\mathcal{P}$, i.e., check if Diagonal(`prev(v)`,`next(v)`) = TRUE.
2. While there's any ear left in the polygon, find an ear and do
   - Cut the ear off and list it to a triangle in our triangulation.
   - Update the information of *earness* for `prev(v)` and `next(v)`.

Analysis:
Step 1: $O(n^2)$
Step 2: Whenever we cut off an ear (in total $n-2$ ears)

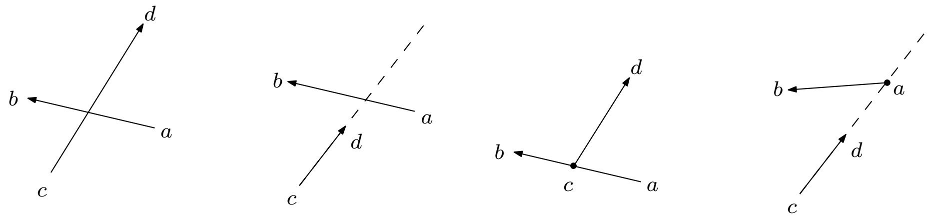
- Finding an ear: $O(n)$
- Updating the information for `prev(v)` and `next(v)`: 2 $O(n)$

So Step 2 has time complexity $O(n^2)$.

3.3 Implementation of Triangulation

Let SignArea$(a, b, c) = \frac{1}{2}(a_x b_y - a_y b_x + b_x c_y - c_x b_y + c_x a_y - a_x c_y)$. Then

- Left(a,b,c): TRUE if c is left of ab;
  c is left of ab if SignArea$(a, b, c) > 0$, right if SignArea$(a, b, c) < 0$ and c is on ab if SignArea$(a, b, c) = 0$.
- Diagonal(a,b): TURE if ab is a diagonal of $\mathcal{P}$.
  - return (InCone(a,b) && InCone(b,a) && Diagonalie(a,b));
- InCone(a,b): TRUE if ab is inside $\mathcal{P}$ in the vicinity of vertex a.



Proper Intersection   Improper Intersection   Collinearity is not sufficient

- Let `next(a)` and `prev(a)` be the next and the preceding vertices of a.
- if (LeftOn(a, `next(a)`, `prev(a)`))
  return (Left(a, b, `prev(a)`) && Left(b, a, `next(a)`));
- else
  return !(LeftOn(a, b, `next(a)`) && LeftOn(b, a, `prev(a)`));

• Diagonalie(a,b): TRUE if ab does not intersect any edge of $\mathcal{P}$.

  - For all i, do the following:
    if $((v_i, v_{i+1} \neq a, b)$&& Intersect$(a, b, v_i, v_{i+1}))$ return FALSE;

  - return TRUE;

• Intersect(a,b,c,d): TRUE if ab and cd intersect.

  - if (IntersectProp(a,b,c,d)) return TRUE;
  - else if (Between(a,b,c)||Between(a,b,d)||Between(c,d,a)||Between(c,d,b)) return TRUE;
  - else return FALSE;

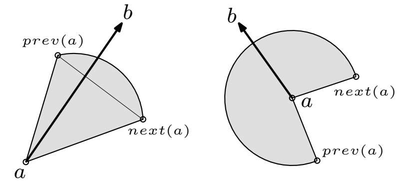
• IntersectProp(a,b,c,d): TRUE if ab and cd intersect properly. Instead of using

$$(\text{SignArea}(a, b, c) * \text{SignArea}(a, b, d) < 0)\&\&(\text{SignArea}(c, d, a) * \text{SignArea}(c, d, b) < 0),$$

use the following:

  - if (collinear(a,b,c) || collinear(a,b,d) || collinear(c,d,a) || collinear(c,d,b)) return FALSE;
  - return Xor(Left(a,b,c), Left(a,b,d)) && Xor(Left(c,d,a), Left(c,d,b))

• Between(a,b,c): TRUE if collinear(a,b,c) is TRUE and c is between a and b.(how can check this?)



InCone(a,b) when a is convex/reflex