Mathematical & Computational Modeling of Infectious Diseases **Parker Banks**
Homework 1 **2024**

Instructions:

- Please turn in a single PDF file.
- Please share a link to your code, but do not attach the actual code.
- By the way, if you don't have a github, why not make one? It's easy to do, and a great place to stash your code and link to it. Pop by office hours or phone a friend if you need help. It's never too early (or late) to learn!
- Handwritten math (scanned and included in a PDF) is fine, but please avoid 10MB+ file sizes! Reduce your image quality as needed.
- Don't forget to list anyone you collaborated with.

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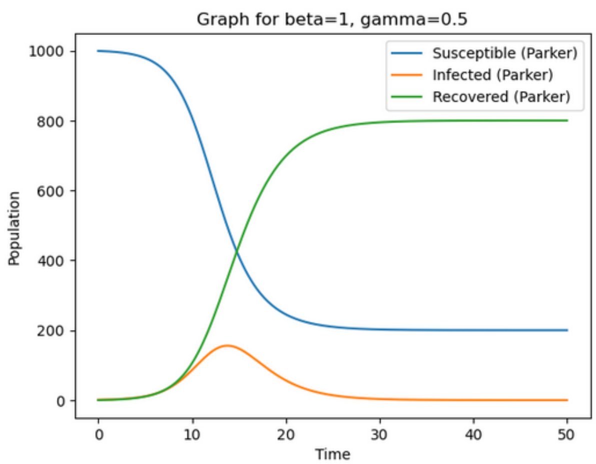
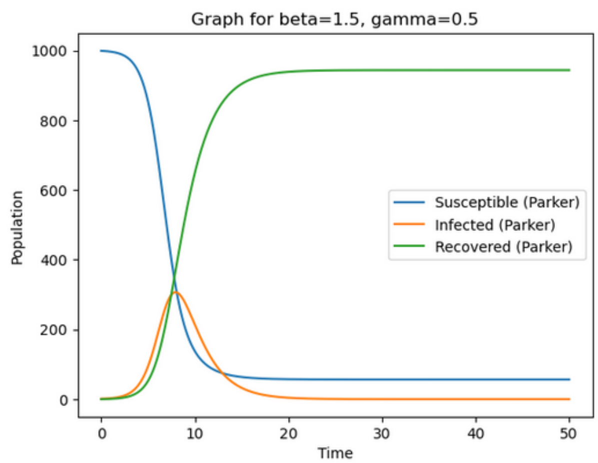
1. The goal of this problem is to get you over any barriers with (i) getting Python set up, (ii), getting the SIR model implemented in a Forward Euler solver, and (iii) getting matplotlib set up.

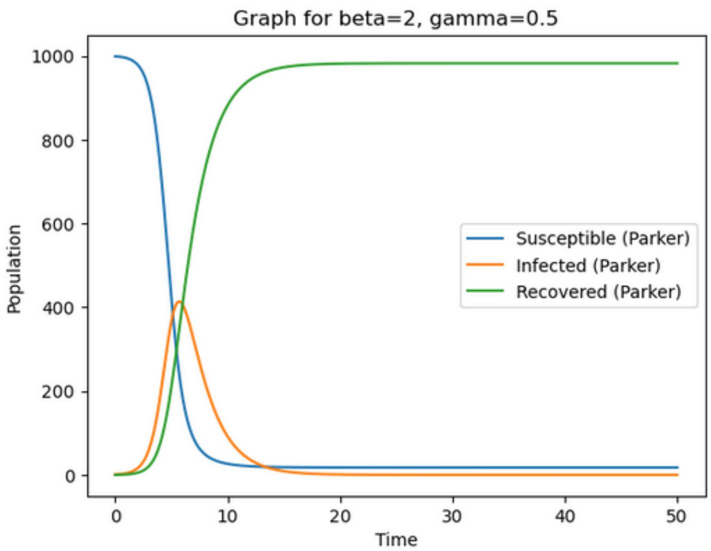
   Write a function in Python that uses the Forward Euler method to simulate the SIR model. Check your work by first reproducing the three plots from Figure 1 of the Week 2 lecture notes. The parameters are: $N = 1000$, $I_0 = 1$, $S_0 = 999$, with

   - $\beta = 1$, $\gamma = 0.5$
   - $\beta = 1.5$, $\gamma = 0.5$
   - $\beta = 2$, $\gamma = 0.5$

   Show that your code works by simply reproducing the plots exactly, but with your first name included in the legend labels, e.g. "S Dan", "I Dan" or something. Link to your code and turn in just the 3 plots.







GitHub: https://github.com/Parker-Banks/CSCI4830-HW1

2. The goal of this problem is to show an important fact about transition rates in compartmental models. It is also a good chance to become refreshed on simple ODE solving and separation of variables. Finally, it makes good on a promise made in Week 2's lecture notes to ask this homework question!

   Imagine that we are interested in SIR dynamics, but everyone starts out either infected or recovered, and no one starts out susceptible.

   a. Use this information to simplify the typical equation for $\dot{I}$.

$$\dot{I} = \frac{\beta S I}{N} - \gamma I \tag{1}$$
$$\dot{I} = \frac{\beta 0 I}{N} - \gamma I \tag{2}$$
$$\dot{I} = -\gamma I \tag{3}$$
$$\frac{dI}{dt} = -\gamma I \tag{4}$$
$$\frac{1}{I} dI = -\gamma dt \tag{5}$$
$$\int \frac{1}{I} dI = -\int \gamma dt \tag{6}$$
$$\int \frac{1}{I} dI = -\int \gamma dt \tag{7}$$
$$\ln(I) = -\gamma t + C \tag{8}$$
$$e^{\ln(I)} = e^{-\gamma t + C} \tag{9}$$
$$I = e^{C} e^{-\gamma t} \tag{10}$$

   b. Solve your simplified differential equation with the initial condition $I(0) = I_0$.

$$I(0) = I_0 \tag{11}$$
$$I_0 = e^{C} e^{-\gamma(0)} \tag{12}$$
$$I_0 = e^{C} \tag{13}$$
$$I(t) = I_0 e^{-\gamma t} \tag{14}$$

   c. Manipulate your solution to derive the fraction of the initially infected people who are still infected at time t.

$$I(t) = I_0 e^{-\gamma t} \tag{15}$$

$$\frac{I(t)}{I(0)} = e^{-\gamma t} \quad \text{Fraction of initially infected that are still infected} \tag{16}$$

$$1 - \frac{I(t)}{I(0)} = 1 - e^{-\gamma t} \quad \text{Fraction of initially infected that are recovered} \tag{17}$$

d. Discuss this equation. What does it do over time? How is it related to the fraction of infected people who have *left* the infected class?

Over time, this equation decreases, at a decreasing rate. It asymptotically approaches 0. This means that the fraction of infected people who are still infected decreases, eventually approaching zero. On the other hand, the fraction of infected people who have left the infected class is the mirror of this equation. This number increases at a decreasing rate, asymptotically approaching 1 (100%).

e. This formula produces values between 0 and 1, and it tells us the probability that a randomly chosen infected person is still infected at time t. How does this relate to the cumulative distribution function (CDF) that describes the probability that someone is infected for less than or equal to t units of time? Take a derivative of the CDF to get a PDF for the duration of infection lengths is. Then, find out what this famous probability distribution is called, and write down its expected value.

The CDF can be found from the previous equation by taking its complement:

$$CDF = 1 - e^{-\gamma t}$$

Now, we find the PDF by taking the derivative:

$$CDF = 1 - e^{-\gamma t} \tag{18}$$

$$PDF = \gamma e^{-\gamma t} \quad \text{by chain rule} \tag{19}$$

Now, finding the expected value:

$$E[X] = \int_0^\infty t \cdot \gamma e^{-\gamma t}\, dt \tag{20}$$

f. Use your results to explain how the recovery rate γ is related to the typical amount of time a person remains infectious.

3. The goal of this problem is to (i) figure out how to solve the final epidemic size equation, and (ii) test the equation's predictions.

   a. First, explain how an epidemic's total size, also called its cumulative incidence, is related to s_∞ and r_∞.

      s_∞ is essentially the total number of people who never became infected. Similarly, r_∞ is the total number of people who recovered. This tells us the cumulative incidence, because everyone who recovered had to have been infected. Therefore, the cumulative incidence is equal to r_∞.

   b. Recall that $r_\infty = 1 - e^{-R_0 r_\infty}$. Though we can't solve this equation, we can use a valuable graphical technique: if we set $f(r_\infty) = r_\infty$ and $g(r_\infty) = 1 - e^{-R_0 r_\infty}$, we can plot both $f(r_\infty)$ vs r_∞ and $g(r_\infty)$ vs r_∞, and see where $f = g$. Create four plots for $R_0 \in \{0.9, 1.0, 1.1, 1.2\}$ with f in black and g in red. Use the **fsolve** function to find the intersection point, and use matplotlib's **scatter** function to plot a blue circle at the intersection.[1]

      Ran out of time - sorry!

   c. Comment on what you see in the plots in the context of what we have learned about R_0. What do you see in your figures? What happens when $R_0 < 1$?

   d. Finally, test the predictions made by this final-size equation by using your SIR code and $\beta = 1$, $\gamma = 0.5$ by creating a new version of that epidemic with a green dotted line at the height of r_∞. Does this final size prediction work?[2]

[1] Hint, use the fsolve docs, and note that, because fsolve wants to find roots (points where a function is zero), you can create an auxiliary function $h(r_\infty) = f(r_\infty) - g(r_\infty)$ which will be equal to zero at the point where f and g intersect!

[2] Take care, as the units of our SIR plots and the units of our final size prediction are not the same! You may have to do a conversion...

4. (Grad / EC): In class, we showed that the SIR model's disease-free equilibrium is stable when $s < \frac{1}{R_0}$ and unstable otherwise. Using $N = 10^6$, and $\varepsilon = \frac{1}{N}$ as your perturbation, produce a single figure *using your simulation code and its output* that illustrates this point. Write a caption that explains the principle of stability, and explain how your figure illustrates it.[3]

[3] There are many possible ways to make such a figure, and write its caption! One could use all of the outputs of the simulation, for instance, but one could also use just those that illustrate the intended point.