Shaft Design Problem

An Iterative Shaft Design Project for Alternating Loads | Parker Knopf

Abstract

This report focuses on the procedures of designing a simply supported shaft of two journal bearings subjected to a bending and torsional load. With external forces and design criteria specified as an input from the user, the program will determine the geometrical minimum diameter of the shaft while maintaining eternal structural life due to fatigue. The main criteria to motivate this design relies on this fatigue analysis. Solid mechanics fatigue analysis is inherently an empirically quantified discipline. Numerically determined parameters directly result in an iterative design process, where the desired minimum diameter must be converged to with empirical data. Analysis of design is derived from various curve fitted algorithms, each accounting for different failure criteria. For this analysis a DE-ASME curve fit was used when determining the factor of safety. With all criteria defined, a diameter for the shaft can be determined numerically for this design problem.

Introduction

The forces on a shaft under a dynamic load are characterized by two primary stresses. The amplitude of the varying load from each cycle creates fatigue stress while a mean stress imposes static stresses on the specimen. With a DE-ASME fatigue failure criteria, the minimum diameter of the shaft can be determined with **Equation 1**.

$$d = \left[\frac{16n}{\pi}\left(4\left(\frac{K_f M_a}{S_e}\right)^2 + 3\left(\frac{K_{fs} T_a}{S_e}\right)^2 + 4\left(\frac{K_f M_m}{S_y}\right)^2 + 3\left(\frac{K_{fs} T_m}{S_y}\right)^2\right)^{1/2}\right]^{1/3} \qquad \textbf{Equation 7}$$

External loads can be characterized with **Equations 2-3** to allow for a fatigue analysis of a shaft design. Although by theory it is the stresses that affect a material, for the purposes of calculation, stresses were kept as external moments and torsions, and then accounted for later in calculations in **Equation 1**.

$$M_a = M_{max} - M_m \qquad \textbf{Equation 2.1}$$

$$M_m = \frac{(M_{max} + M_{min})}{2} \qquad \textbf{Equation 2.2}$$

$$T_a = T_{max} - T_m \qquad \textbf{Equation 3.1}$$

$$T_m = \frac{(T_{max} + T_{min})}{2} \qquad \textbf{Equation 3.2}$$

Static loads on a shaft are simple in that they can be analyzed parametrically and with closed form solutions. Fatigue loading on the other hand is completely analyzed with empirically derived data. A result of this is self-dependent geometry and factors making a closed form solution impossible. To go about designing a shaft with these loads, an initial diameter must be known to iteratively converge to a desired shaft diameter. To find this initial diameter or “first guess” a solution for static loading was determined with the closed form solution of **Equation 4**.

$$r = \left(\frac{n^2}{2(\pi S_y)^2}\left(32 * M_{max}^{2} + 24 * T_{max}^{2}\right)\right)^{1/6} \qquad \textbf{Equation 4}$$

With an initial diameter determined, the factors that affect fatigue loading can be determined. The major criteria to analyze fatigue loading is the endurance limit shown in **Equation 5**. When calculating the endurance limit it is standard to take half of the ultimate strength of the material and apply Marine factors, k_i.

$$S_e = k_a k_b k_c k_d k_e\left(\frac{S_{ut}}{2}\right) = (0.5295)d^{-0.157}\left(\frac{S_{ut}}{2}\right) \qquad \textbf{Equation 5}$$

Several criteria and factors affect the material fatigue significantly which can be accounted for in the program with user inputs. A material’s factor of safety for fatigue loading is largely dependent on the geometry, environment, and material conditions. These Marine factors directly contribute to deriving the endurance limit of the specimen. All Marine factor equations can be found in **Appendix A**. The only Marine factor dependent on diameter is k_b, the size modification factor.

A shaft, in addition to conditions, is also subjected to stress concentrations due to geometrical features. In the case of this shaft, a change in diameter with a filet is the stress concentration that is analyzed for this shaft. The stress concentration factors are also empirically determined and reliant on the diameter. The curve-fit equations for both bending and torsional stress concentration factors are shown in **Equation 6.1 and 6.2**.

$$K_t = C_1 + C_2\left(\frac{2t}{D}\right) + C_3\left(\frac{2t}{D}\right)^2 + C_4\left(\frac{2t}{D}\right)^3$$

	$0.1 \le t/r \le 2.0$	$2.0 \le t/r \le 20.0$
C_1	$0.947 + 1.206\sqrt{t/r} - 0.131t/r$	$1.232 + 0.832\sqrt{t/r} - 0.008t/r$
C_2	$0.022 - 3.405\sqrt{t/r} + 0.915t/r$	$-3.813 + 0.968\sqrt{t/r} - 0.260t/r$
C_3	$0.869 + 1.777\sqrt{t/r} - 0.555t/r$	$7.423 - 4.868\sqrt{t/r} + 0.869t/r$
C_4	$-0.810 + 0.422\sqrt{t/r} - 0.260t/r$	$-3.839 + 3.070\sqrt{t/r} - 0.600t/r$

$$K_{tn} = C_1 + C_2\left(\frac{2t}{D}\right) + C_3\left(\frac{2t}{D}\right)^2 + C_4\left(\frac{2t}{D}\right.$$

	$0.25 \le t/r \le 4.0$
C_1	$0.905 + 0.783\sqrt{t/r} - 0.075t/r$
C_2	$-0.437 - 1.969\sqrt{t/r} + 0.553t/r$
C_3	$1.557 + 1.073\sqrt{t/r} - 0.578t/r$
C_4	$-1.061 + 0.171\sqrt{t/r} + 0.086t/r$

Equation 6.1 and 6.2

To apply the geometric stress concentration factor to the analysis of fatigue loading, **Equation 7** is used to perform the conversion.

$$K_f = 1 + \frac{K_t - 1}{1 + \frac{\sqrt{a}}{\sqrt{r}}} \qquad \textbf{Equation 7}$$

where *a* is empirically determined with curve-fitted data. With all geometrical, environmental, and material conditions accounted for, with an initial guess using static closed solution theory, **Equation 1** can be used to solve this shaft design problem.

Results

With this shaft design problem inherently an iterative process, the following steps and procedures were followed to develop a minimum allowable diameter for fatigue loadings.

Procedure:

1. Find the minimum diameter allowable for the shaft with static loading criteria. (Initial diameter guess)
2. Find the endurance limit of the material, considering all geometrical, environmental, and material factors. Use the current known diameter.
3. Develop the stress concentration factors for the fillet that exists at a change in diameter. Use the current known diameter.
4. Compute the new minimum diameter for fatigue loading.
5. Repeat steps 2 - 4 until a certain error metric is reached.

Program

To perform this analysis a program was implemented to iterate and converge the shaft design solution to a minimum allowable diameter. The repository for this program can be found in **Appendix B**.

Material Selection

To determine the minimal diameter of the shaft, a code base of materials was necessary for the project. Having a bank of materials allows for the user to determine which material would perform more optimally for this design solution scenario. The importance of having a bank of materials is to understand and characterize material properties to be used in the design process. These properties consisted of yield and ultimate stresses. **Figure 1** is an example of one material with its properties in the database.

```
%% Materials

steel1010CD.sy = 44 * 1000; % psi
steel1010CD.sut = 53 * 1000; % psi
steel1010CD.name = 'Steel_1010_CD';
```

Figure 1: Database material properties for ASCII Steel 1010 Cold Drawn

Geometry Stress Concentrations

Stress Concentrations are crucial to the calculations as the shaft will fail first at these locations. To calculate stress concentrations, the change in diameters, the diameter, and the size of the radius must be known. With the solution to be solved being the diameter, this is one factor where it will need to be iterated on in the program. Updating this value will allow a more accurate minimum diameter to be found. The following **Figure 2** illustrates for example the stress concentration factor for bending.

```
if (m=='b' || m=='a')
    a = 0.246 - 3.08*(10^-3)*sut + 1.51*(10^-5)*sut^2 - 2.67*(10^-8)*sut^3;
    k = 1 + (stressCon_Bend(d, D, r) - 1) / (1 + a / sqrt(d/2));
```

Figure 2: Database material properties for ASCII Steel 1010 Cold Drawn

Endurance Limit

The endurance limit is the most important factor for shaft design for fatigue stresses as it is directly compared to the induced stresses. The endurance limit in this program is also iteratively determined as k_b, the size factor depends on the diameter of the shaft.

```
se = ka * kb * kc *kd * ke * (sut/2);
```
Equation 8

Trail and correction Procedure

This program is built on the foundation of iteratively determining the diameter. To do so a while loop was implemented to determine the factors and the minimum diameter until an error metric was reached.

```
error = 0.01;
e = Inf;
while e > error
    r = d * dtor;
    kf = stressCon(mat.sut, d, D, r, 'b');
    kfs = stressCon(mat.sut, d, D, r, 't');
    d_new = (16*n/pi * (4*(kf*Ma/Se)^2 + 3*(
    e = abs(d - d_new) / d;
    Se = enduranceLim(mat.sut, d);
    d = d_new;
    D = d*dtoD;
end
```

Figure 3: Iterative shaft design code

Fatigue Failure Design Criteria

The design of a structure based on static and dynamic loadings is complex in nature and only determined through empirical data. There exists several curve-fitted failure theories to characterize Factors of Safety (FOS) for different materials under varying external loads. For this design problem the DE-ASME fatigue failure criteria was used. A form of the following Equation 9 was used to determine the minimum diameter.

$$n_f = \left[\left(\frac{\sigma_a}{S_e}\right)^2 + \left(\frac{\sigma_m}{S_y}\right)^2\right]^{-1/2}$$

Equation 9

Checking for Yielding

In this program, material yielding is checked by the first guess of the minimum diameter. The initial guess is created not randomly, but by solving for yield stress the shaft will be subjected to. This diameter when iterated on for fatigue stresses will grow to compensate for the more destructive load at smaller stresses.

Example Code Outputs

Table 1: Minimum Diameter for various Materials

Material	Steel 1020 CD	Steel 1045 CD	Steel 1050 CD
Minimum Diameter (n = 1.5)	2.44 in	2.25 in	2.19 in
Minimum Diameter (n = 1)	2.12 in	1.95 in	1.9 in

Discussion

The program developed for this design solution has many advantages for the user. With user inputs for external loads, factors of safety, diameter differences, and fillet radii, various solutions can be achieved with different material properties and geometric constraints. It can be seen in the results that for an increasing strength of material, under the same conditions, decreases minimum shaft diameter. It can also be seen that for smaller factors of safety, the minimum shaft diameter decreases which aligns with intuition, theory, and empirical data.

This data presented in **Table 1** will always be the case since the user does not specify an initial guess. The initial guess not being defined by the user in this program means it can not be varied for the same configuration. This method of guessing is more efficient for the user and makes the program easier to use.

Converging to a minimum diameter is also automated by the program by using an error metric between calculated diameter values. With an error metric of 1%, the program will terminate the loop and output the final diameter found.

Conclusion

The program developed for this design solution is reliable, however limited to general design. This program only calculates for simple shaft designs with bending and torsional external loads. With the external loads known at exact locations the program is also limited to the user knowing physical internal forces.

Although this program solves a small portion of shaft design problems it is developed with fundamental theories that could be expanded upon to solve more complicated shaft design problems.

Appendix A

[1] Endurance Limit: Marin Factors

$$k_a = a \cdot (S_{ut})^b$$

$$k_b = 0.91d^{-0.157}$$

$$k_c = k_d = 1$$

$$k_e = 0.753$$

[2] Stress Concentration Numerical Parameters

Bending or axial:

$$\sqrt{a} = 0.246 - 3.08(10^{-3})S_{ut} + 1.51(10^{-5})S_{ut}^2 - 2.67(10^{-8})S_{ut}^3 \qquad 50 \le S_{ut} \le 250 \text{ kpsi}$$

$$\sqrt{a} = 1.24 - 2.25(10^{-3})S_{ut} + 1.60(10^{-6})S_{ut}^2 - 4.11(10^{-10})S_{ut}^3 \qquad 340 \le S_{ut} \le 1700 \text{ MPa} \tag{6–35}$$

Torsion:

$$\sqrt{a} = 0.190 - 2.51(10^{-3})S_{ut} + 1.35(10^{-5})S_{ut}^2 - 2.67(10^{-8})S_{ut}^3 \qquad 50 \le S_{ut} \le 220 \text{ kpsi}$$

$$\sqrt{a} = 0.958 - 1.83(10^{-3})S_{ut} + 1.43(10^{-6})S_{ut}^2 - 4.11(10^{-10})S_{ut}^3 \qquad 340 \le S_{ut} \le 1500 \text{ MPa} \tag{6–36}$$

Appendix B

[1] Program Repository

[GitHub]