State-Space Form:

$$\dot{\vec{x}} = A\vec{x} + B\vec{u} \quad \text{(state eqn.)}$$

$$\vec{y} = C\vec{x} + D\vec{u} \quad \text{(output eqn.)}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\dot{\vec{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

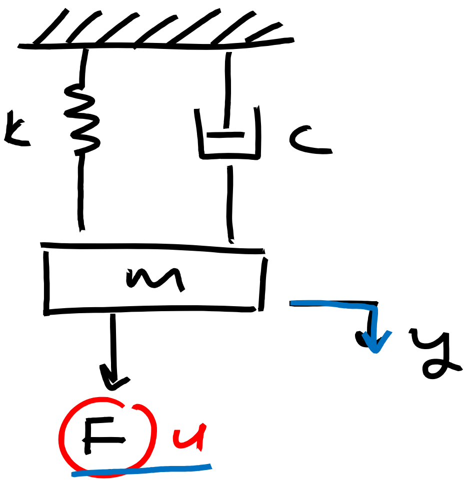
$$x_1 = y \Rightarrow \boxed{\dot{x}_1} = \dot{y} = \boxed{1 x_2}$$

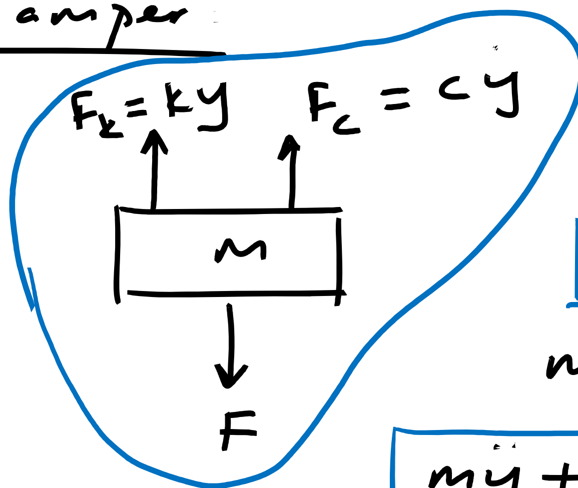
$$x_2 = \dot{y} \Rightarrow \boxed{\dot{x}_2} = \ddot{y} = \boxed{\frac{1}{m} u}$$

$$\underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}}_{\dot{\vec{x}}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}_{B} \underbrace{u}_{u}$$

$$y = x_1 \Rightarrow y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u = C\vec{x} + \overset{0}{D} u$$

Mass-Spring-Damper:





$$\boxed{ma = \sum F}$$

$$m\ddot{y} = F - ky - c\dot{y}$$

$$\boxed{m\ddot{y} + c\dot{y} + ky = F = u}$$

Rearrange:

$$\ddot{y} = \boxed{-\frac{c}{m}\dot{y} - \frac{k}{m}y + \frac{1}{m}u}$$

in state-space form:

$$x_1 = y \Rightarrow \boxed{\dot{x}_1} = \dot{y} = \boxed{1 x_2}$$

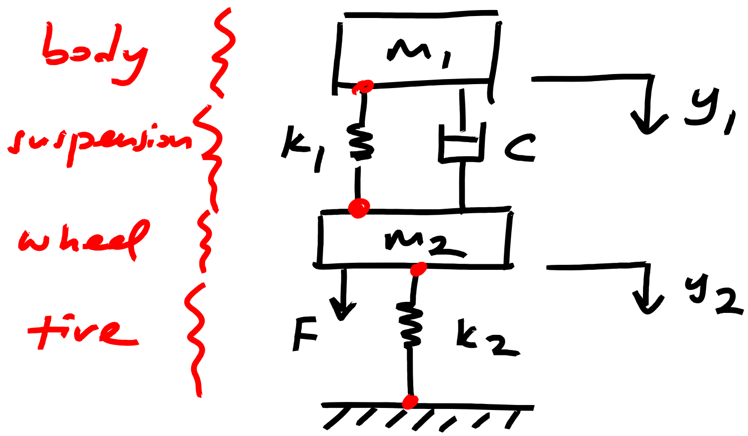
$$x_2 = \dot{y} \Rightarrow \boxed{\dot{x}_2} = \ddot{y} = \boxed{-\frac{c}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}u}$$

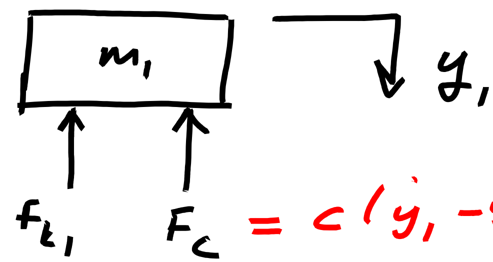
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Automobile suspension:

For n masses, n equations,
n degrees of freedom.

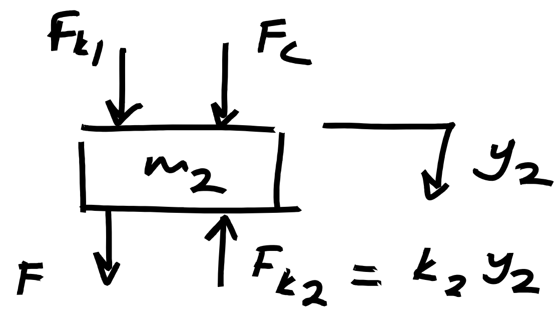




$F_c = c(\dot{y}_1 - \dot{y}_2)$

$F_{k_1} = k y_1 - k y_2$

$F_{k_1} = k(y_1 - y_2)$



$F_{k_2} = k_2 y_2$

Mass 1:

$$m_1 \ddot{y}_1 = -F_{k_1} - F_c$$

$$m_1 \ddot{y}_1 = -k_1(y_1 - y_2) - c(\dot{y}_1 - \dot{y}_2)$$

Mass 2:

$$m_2 \ddot{y}_2 = F + F_{k_1} + F_c - F_{k_2}$$

$$m_2 \ddot{y}_2 = F + k_1(y_1 - y_2) + c(\dot{y}_1 - \dot{y}_2) - k_2 y_2$$

$$m_1\ddot{y}_1 = -k_1(y_1 - y_2) - c(\dot{y}_1 - \dot{y}_2)$$

$$m_2\ddot{y}_2 = F + k_1(y_1 - y_2) + c(\dot{y}_1 - \dot{y}_2) - k_2 y_2$$

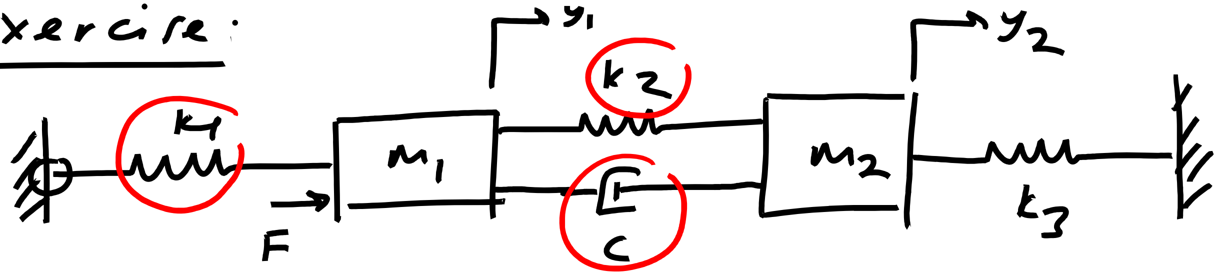
$$m_2\ddot{y}_2 = \underbrace{\boxed{F}}_{u} - k_1(y_2 - y_1) - c(\dot{y}_2 - \dot{y}_1) - k_2(y_2 - 0)$$

RULE: Always $-$, always starting w/ variable of that mass.

mass 1: $m_1\ddot{y}_1 + c\dot{y}_1 + k_1 y_1 = c\dot{y}_2 + k_1 y_2$

mass 2: $m_2\ddot{y}_2 + c\dot{y}_2 + (k_1 + k_2)y_2 = c\dot{y}_1 + k_1 y_1 + u$

Exercise:



Mass 1:

$$m_1\ddot{y}_1 = F - k_1(y_1 - 0) - k_2(y_1 - y_2) - c(\dot{y}_1 - \dot{y}_2)$$

$$m_1\ddot{y}_1 + c\dot{y}_1 + k_1 y_1 + k_2 y_1 = \underbrace{\boxed{F}}_{u} + c\dot{y}_2 + k_1(0) + k_2 y_2$$

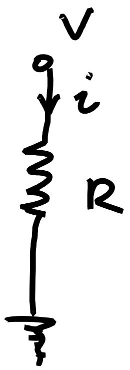
Mass 2:

$$m_2\ddot{y}_2 = -k_3(y_2 - 0) - k_2(y_2 - y_1) - c(\dot{y}_2 - \dot{y}_1)$$

$$m_2\ddot{y}_2 + c\dot{y}_2 + (k_2 + k_3)y_2 = k_2 y_1 + c\dot{y}_1$$

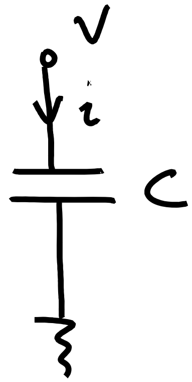
Modeling Electrical Systems:

3 passive components:



Resistance

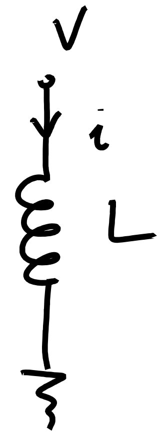
$$V = i \cdot R$$



Capacitance

$$i = C \frac{dV}{dt}$$

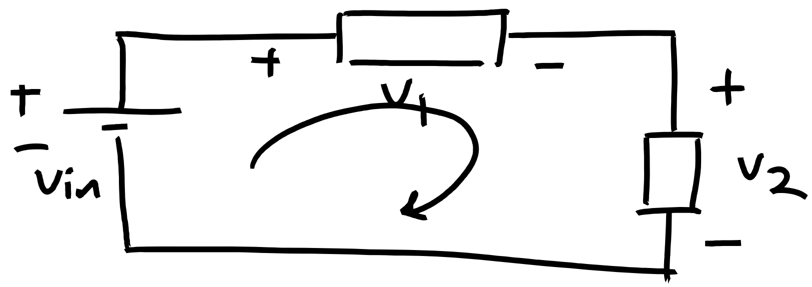
$$V = \frac{1}{C} \int i \, dt$$



Inductance

$$V = L \frac{di}{dt}$$

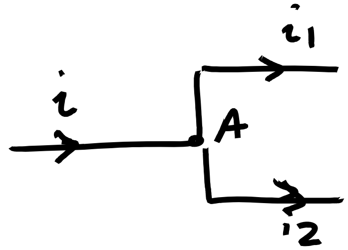
Kirchoff's voltage Law (KVL)



$$-V_{in} + V_1 + V_2 = 0$$

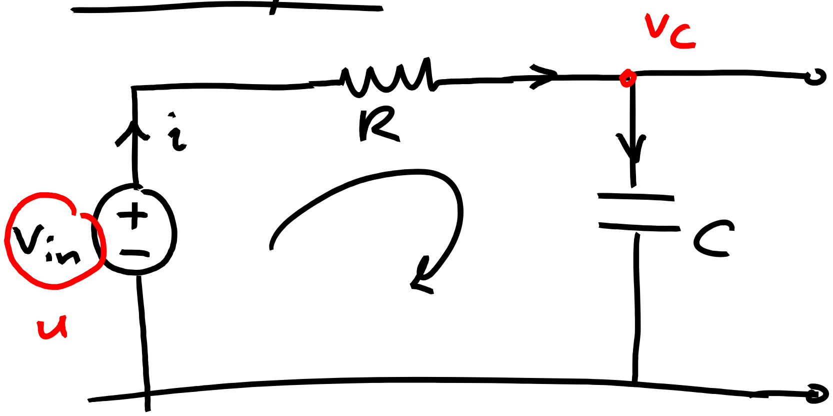
$$\boxed{V_{in} = V_1 + V_2}$$

Kirchoff's Current Law (KCL)



$$\boxed{i = i_1 + i_2}$$

Example:



$V_{out} = y = V_c$

KVL: $-V_{in} + V_R + V_c = 0$

Resistor: $V_R = i \cdot R$

Capacitance: $i = C \dfrac{dV_c}{dt}$

$$V_{in} = i \cdot R + V_c$$

$$V_{in} = RC \frac{dV_c}{dt} + V_c$$

$$u = RC\,\dot{y} + y \quad \Rightarrow \quad \boxed{\dot{y} + \frac{1}{RC} y = \frac{1}{RC} u}$$