Introduction to Point Processes

Reminder: Point Process Research Questions

- Is the spatial pattern of points random?
- Is there clustering in the events?
- Do events repel each other?

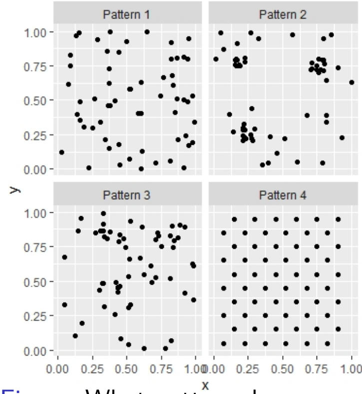

Figure: What pattern do you see?

Steps to Analyzing a Point Pattern

1. Descriptive statistics, such as the average intensity over a region
2. Tests for Complete Spatial Randomness (similar to Moran's I for areal data)
3. Model for the intensity λ (similar to spatial regression for areal data)

Point Processes

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- It is difficult to assess the relationship in point patterns from visualizations, though they can be helpful (just like with areal data)
- We model the **intensity**, λ, of a spatial point pattern to characterize the pattern over a region W
  - How many points do we expect in a given region? (λ)
  - How many points do we expect to be near a specific location? ($\lambda(s)$)
- The intensity is not known and must be estimated

Descriptive Statistics

Intensity Estimate- Point Processes

Basic estimator of the intensity, λ:

$$\hat{\lambda} = \frac{n}{a}$$

where n is the number of points $(s_1, ...s_n)$ and a is the area of the window W.

Descriptive Statistics

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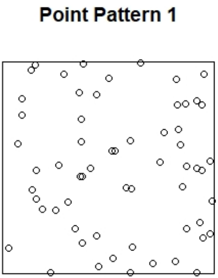

R output:

- Planar point pattern: 60 points
- Window: rectangle = [0, 1] x [0, 1] units
- Window area = 1 square unit
- **Average intensity ($\hat{\lambda}$):**

Descriptive Statistics

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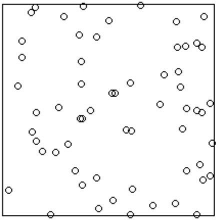
Point Pattern 1



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Descriptive Statistics

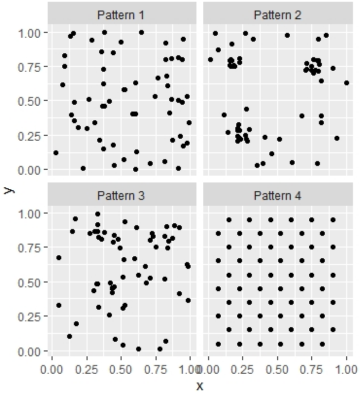

Figure: n = 60 and $W = [0, 1] \times [0, 1]$ for all point patterns

What is the average intensity for each pattern?

Descriptive Statistics

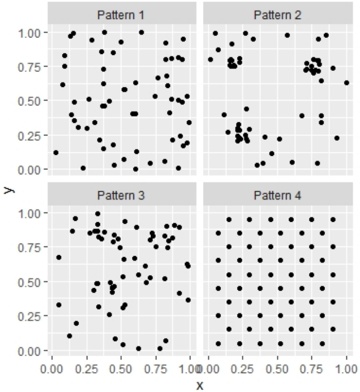

Figure: n = 60 and $W = [0, 1] \times [0, 1]$ for all point patterns

What is the average intensity for each pattern?

$$\hat{\lambda} = \frac{n}{a}$$

Descriptive Statistics- Crime Data

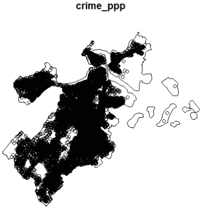

Figure: Crime Data Point Pattern

R output:

- Planar point pattern: 2511520 points
- **Average intensity: 0.009470109 points per square unit**
- Window: polygonal boundary
- Window area = 265205000 square units

Complete Spatial Randomness

Randomness

Is the point process random?

- **Deterministic**: A process is deterministic if the process contains no elements of randomness, can be predicted with certainty
- **Stochastic**: A process is stochastic if the process contains elements of randomness (non-technical definition)

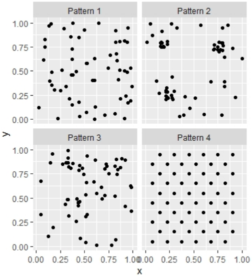

Figure: Which pattern is stochastic? Which is deterministic?

Spatial Randomness

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- **Clustering:** there is attraction between points
- **Inhibition:** there is competition among points (leads to regular patterns)

Spatial Randomness

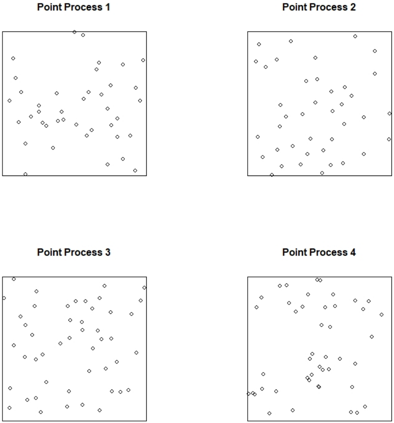


Figure: Which pattern exhibits CSR? Inhibition? Clustering?

Spatial Randomness

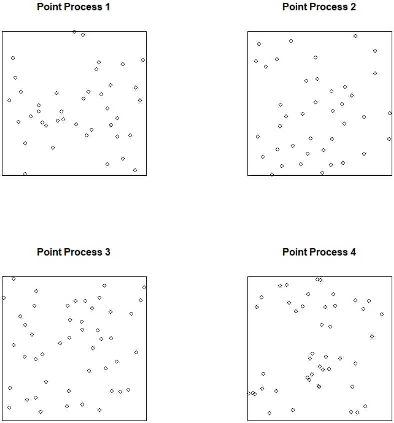


Figure: Which pattern exhibits CSR? Inhibition? Clustering?

1: CSR, 2 & 3: Inhibition, 4: Clustering
It is really difficult to tell from images!!

Nearest Neighbors

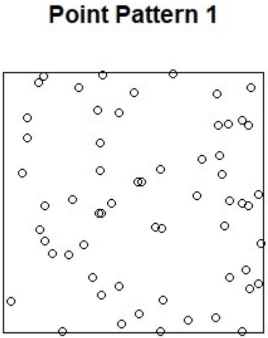
Testing for CSR

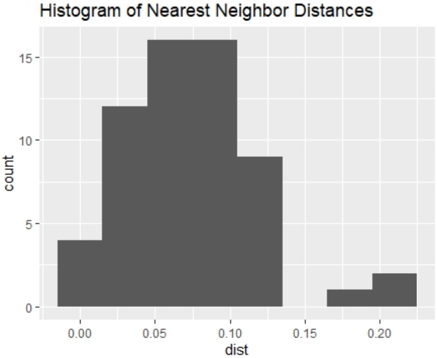
Nearest Neighbors

- Nearest neighbor of a location is the event that is closest (smallest distance) to this location
- Formally, nearest neighbor of event i is event j with shortest distance d from i

$$d_{ij} \leq d_{ik} \forall k$$

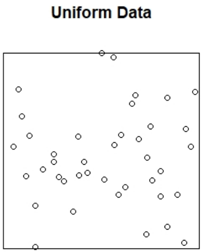
Nearest Neighbor Example

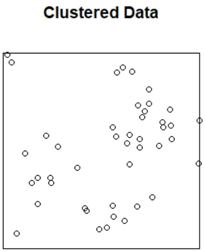




Nearest Neighbor Example

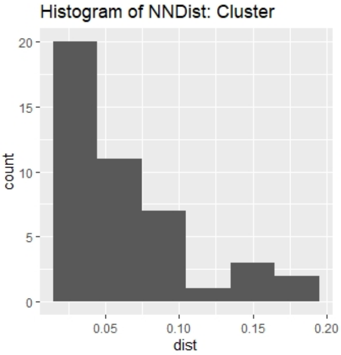
- What would you expect the distribution of nearest neighbor distances to look like for clustering vs CSR?





Nearest Neighbor Example

- Smaller distances for clustering vs uniform data



G Function

Testing for CSR

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- Summary of the distribution of the distances from an arbitrary event to its nearest event

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- When distance r is very large, proportion of events with nearest neighbor at distance less than r is: **large**.

G function

$$\hat{G}(r) = \frac{\#\{d_i : d_i \leq r, \forall i\}}{n}$$

- $d_i = min_j\{d_{ij}, \forall j \neq i\}$ (**distance to nearest event**)
- numerator is the number of elements in the set of n.n. distances that are lower than or equal to r
- n is number of points

G Function

Compare the data to the **G function under CSR**:

$$G(r) = 1 - \exp(-\lambda \pi r^2)$$

where λ is the mean number of events per unit area.

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3. Compare $G(r)$ under CSR and $\hat{G}(r)$ for our dataset
4. **More formal test:** Can also repeatedly simulate datasets under CSR for your window and with the same intensity $\hat{\lambda}$ and compare to the distribution of those simulations
   - Similar to permutation tests in areal data

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Idea of Assessing CSR with the G function

- We have one realization of what the G function (distribution of nearest neighbor distances) looks like for our data
- We can simulate many many data sets that are completely spatially random and calculate the G function for each
- Compare these simulated datasets to our dataset
- Does the G function for the data look different than for the simulated (spatially random) datasets?

Application of G function

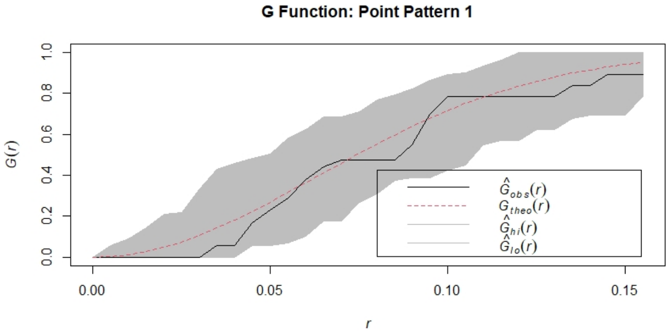

Figure: Is there evidence against CSR?

- Red line: $G(r)$ under Complete Spatial Randomness
- Black line: $\hat{G}(r)$ estimated from the data
- Grey shaded area: bounds from the "simulation envelope" from many many CSR simulations

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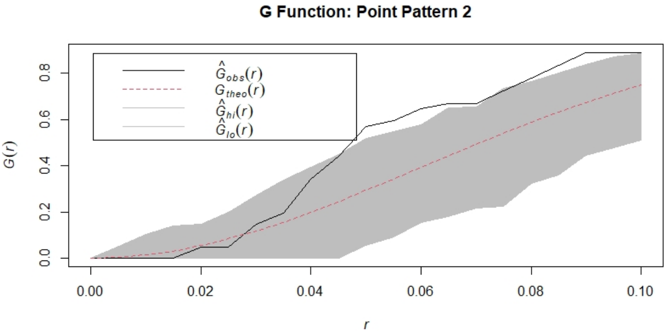

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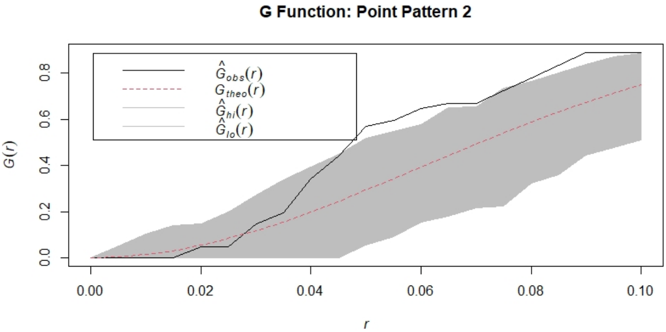

Figure: Is there evidence against CSR?

Intuition: The points are closer together than under CSR (indicates clustering)

Another Application of the G function

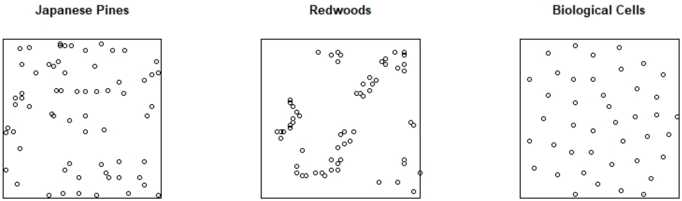

Figure: Which dataset looks like CSR? Inhibition? Clustering?

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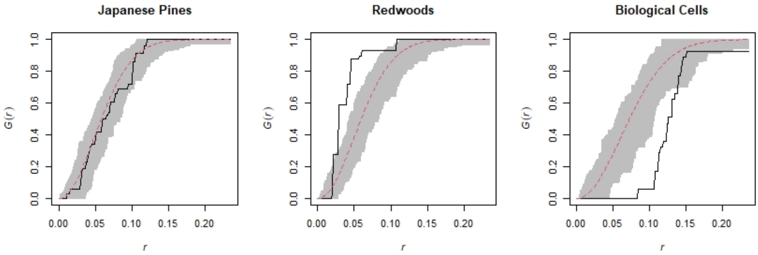

Figure: Which G function gives evidence for CSR? Inhibition? Clustering? (if any)

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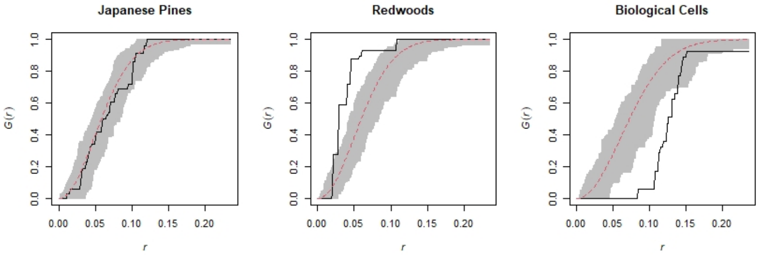

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What is the intuition behind this plot?

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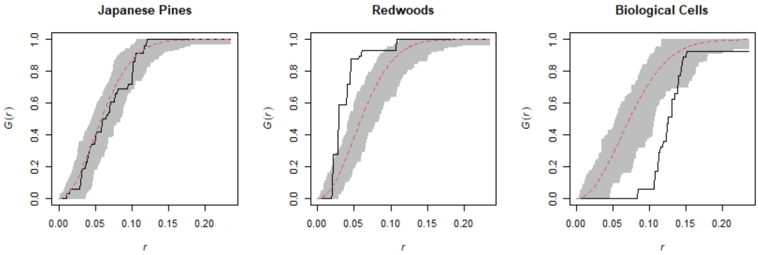

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What is the intuition behind this plot?

With the redwood data (clustered), there are more points than expected under CSR at many radii. **Events are closer than expected.**
With the cell data (inhibition), there are fewer points than expected under CSR at most radii. **Events are further than expected.**

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  - Measure of average space left between events
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- Also called the **Empty Space Function**
  - Measure of average space left between events
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- Estimated from the data

F Function from the data

$$\hat{F}(r) = \frac{\#\{d_{ik} : d_{ik} \leq r, \forall i\}}{n}$$

- d_{ik} distance from point i to nearest neighbor event k
- Compare the data to the F function under CSR:

$$F(r) = 1 - \exp(-\lambda \pi r^2)$$

Application of the F function

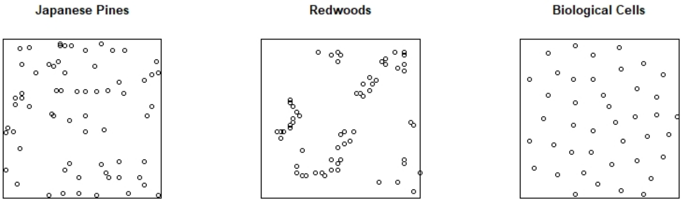


Figure: Reminder of the datasets

Empty Space Illustration

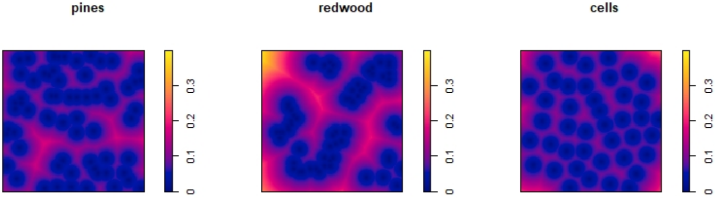
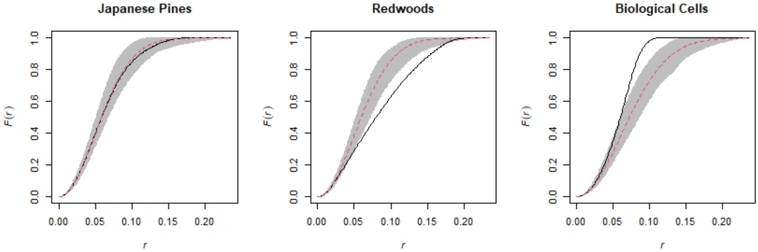


Figure: Comparison of empty space between datasets

- Blue: where points are located
- Pink/yellow: empty space

Application of the F function



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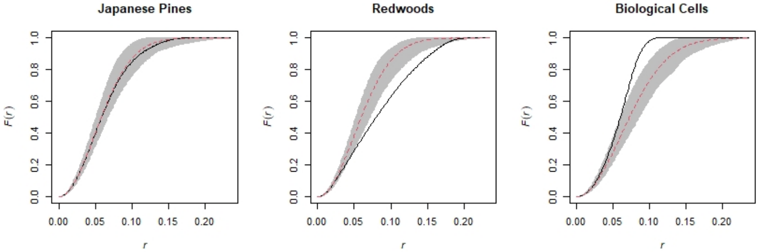

Figure: Which F function gives evidence for CSR? Inhibition? Clustering? (if any)

- Interpretation is opposite of the G function
- **Inhibition:** Empty spaces are closer to events than expected under CSR
- **Clustering:** Empty spaces are further than events than expected under CSR