Imperial College London

Relativity – Lecture 4

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Key concepts of lecture 3

- Events that are simultaneous in one inertial frame and spatially separated, are non-simultaneous in another inertial frame.

- Time dilation: moving clocks run slow.

- Proper time: the time interval measured between 2 events by a stationary clock.

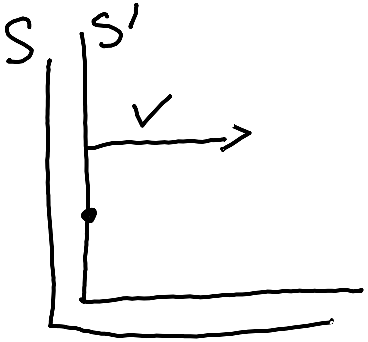
- Length contraction: moving objects are short.

Example: time dilation & length contraction

A negatively-charged pion (π^-) travels at $\beta = 0.998$ in a lab. Its lifetime is measured in the lab frame to be 4.20×10^{-7} s.

1. What distance does it travel in the lab frame?
2. What distance does it travel in the pion's rest frame?
3. What is its rest frame lifetime?

Example: time dilation & length contraction



S: lab frame

S': pion rest frame

$\beta = \frac{v}{c} = 0.998$

$\Delta t = 4.20\times10^{-7}\,s$

① $X = v\Delta t = 0.998c \times 4.20\times10^{-7} = \underline{126\,m}$

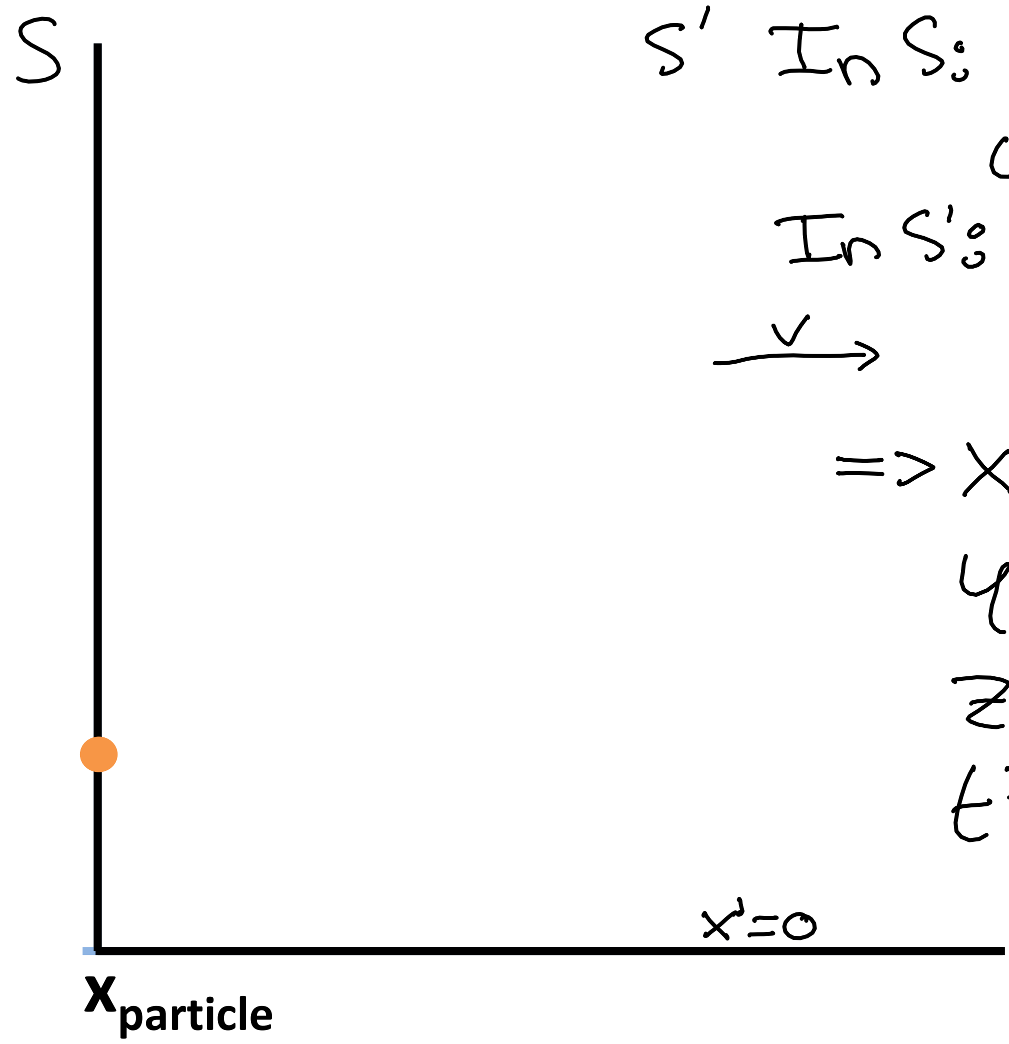
② Length contraction: to the π^-, the lab is moving, so it is shorter: $x' = \frac{X}{\gamma} = \sqrt{1-\beta^2}\times 126 = \underline{8.0\,m}$

③ EITHER: time dilation: π^- proper time

$\Delta t' = \frac{\Delta t}{\gamma} = 4.20\times10^{-7}\sqrt{1-\beta^2} = \underline{27\,ns}$

OR: $\Delta t' = \frac{x'}{v} = \frac{8.0}{0.998c} = \underline{27\,ns}$

Galilean coordinate transformation



In S: particle moves with $x = vt$

In S': particle remains at $x' = 0$

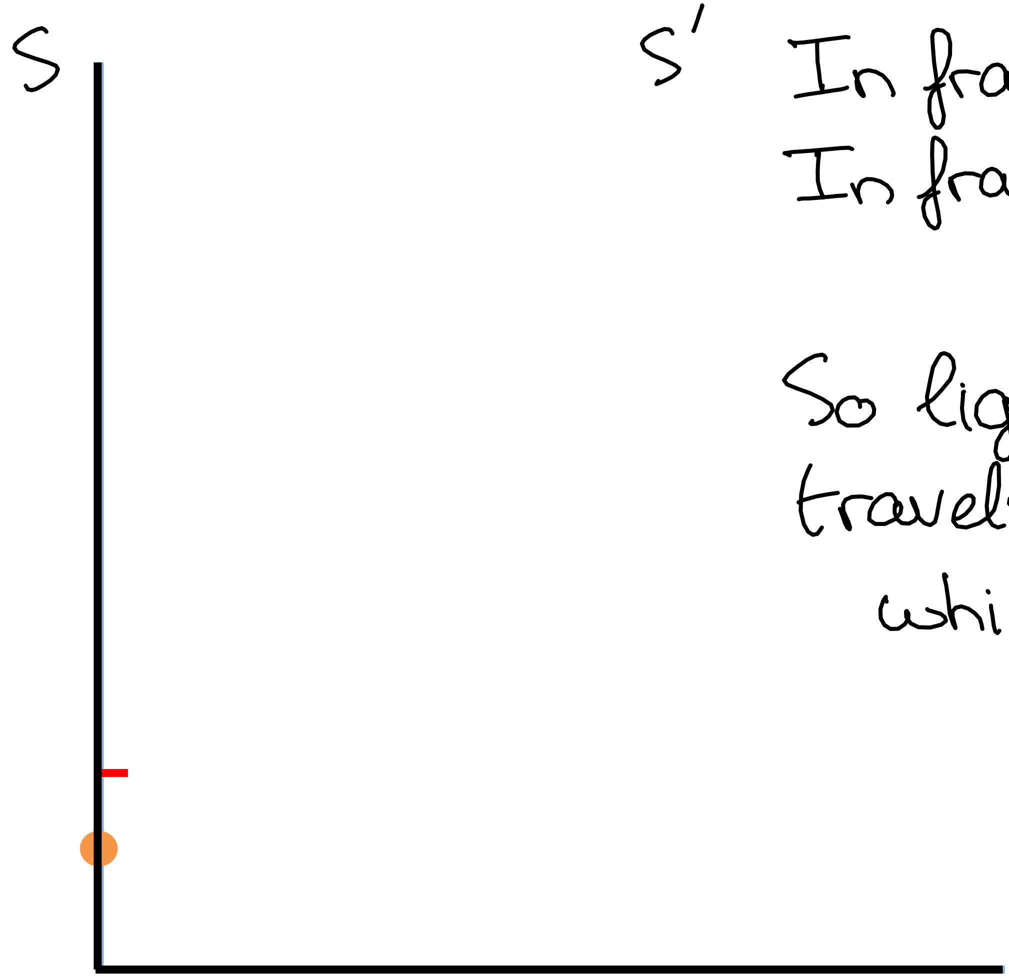
$$\Rightarrow x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t \text{ (one universal time)}$$

Galilean transformations: what about light?



In frame S: $x = ct$

In frame S': $x' = ct - vt$

$= (c-v)t$

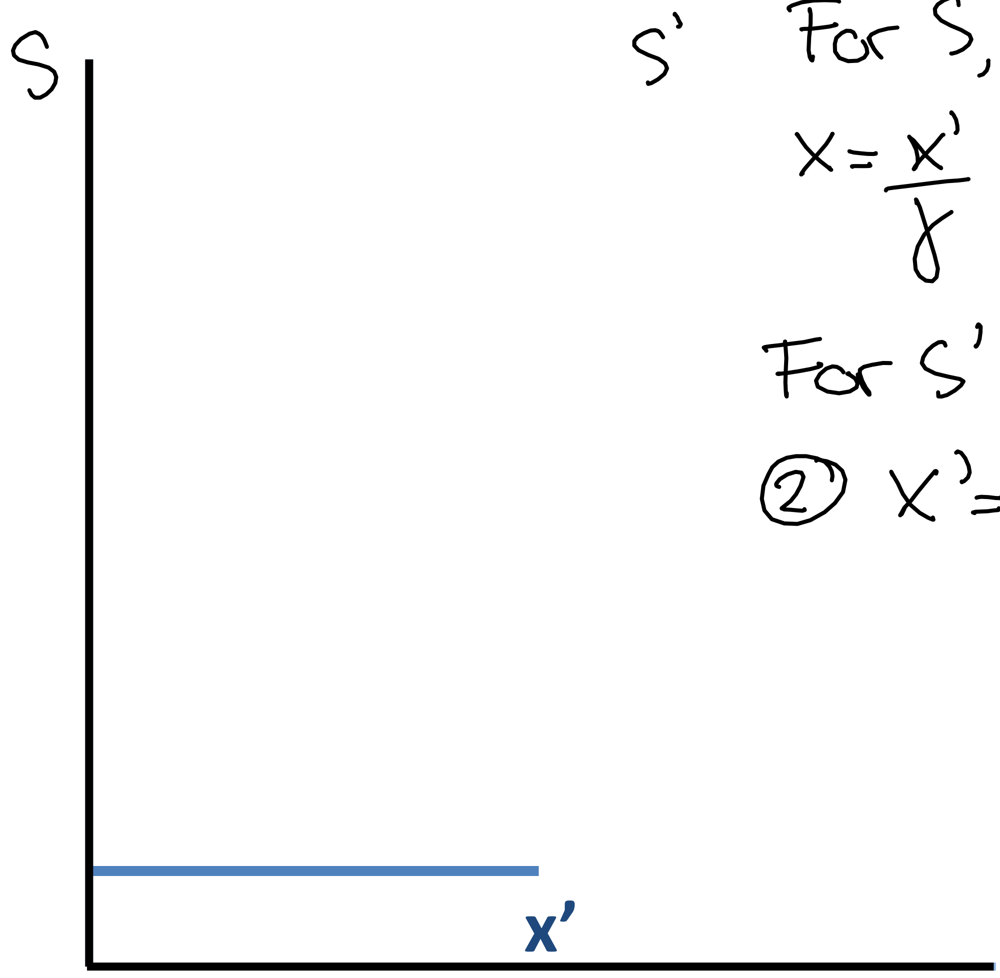
So light in frame S' travels at speed $c-v$, which is $< c$!

$x' = (c-v)t$

$x = ct$

Galilean transformations, like Newtonian mechanics, do not treat light correctly.

Lorentz transformations



For S, S' is moving at v:

$$x = \frac{x'}{\gamma} + vt$$

$$\Rightarrow (1)\quad x' = \gamma(x - vt)$$

For S', S is moving at -v:

$$(2)\quad x' = \frac{x}{\gamma} - vt'$$

$$x = \frac{x'}{\gamma} \text{ at } t = t' = 0$$

Lorentz transformations

① and ② Solve for t': $\gamma(x - vt) = \frac{x}{\gamma} - vt'$

$$vt' = \frac{x}{\gamma} - \gamma x + \gamma vt = x\left(\frac{1}{\gamma} - \gamma\right) + \gamma vt$$

$$= x\left(\frac{1-\gamma^2}{\gamma}\right) + \gamma vt \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

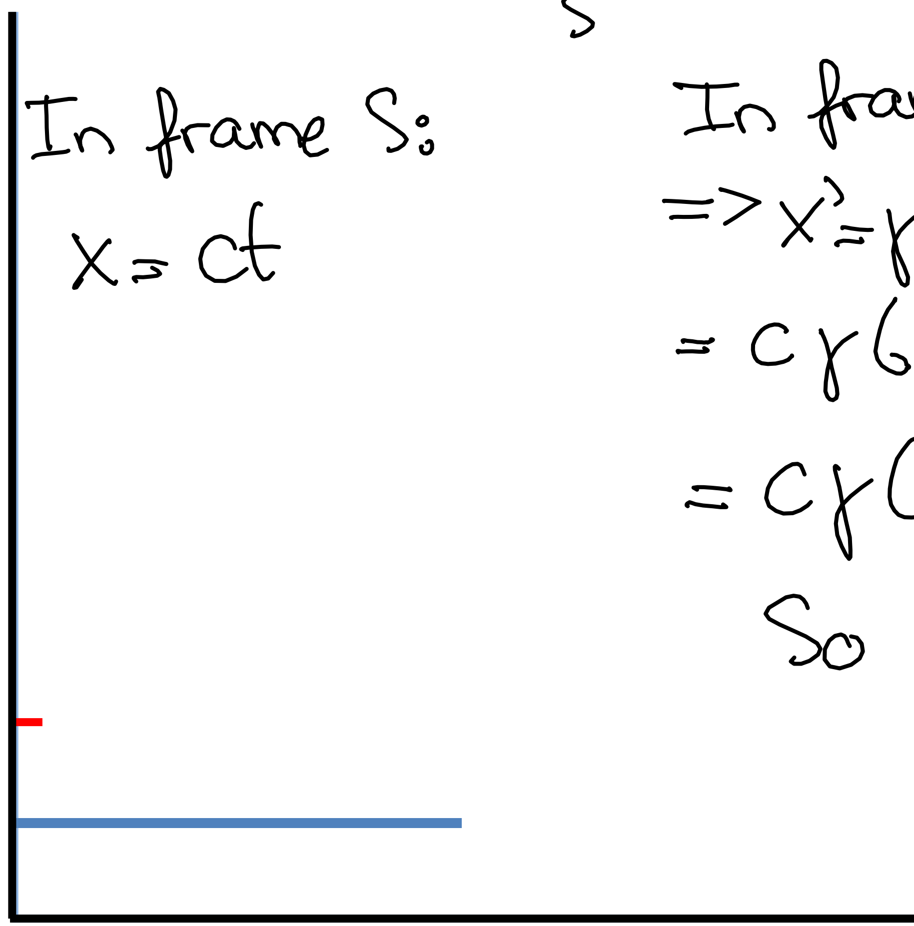
$$= x\left(\sqrt{1-\beta^2}\left(1 - \frac{1}{1-\beta^2}\right)\right) + \gamma vt =$$

$$= x\left(\sqrt{1-\beta^2}\left(\frac{1-\beta^2-1}{1-\beta^2}\right)\right) + \gamma vt = -\gamma\beta^2 x + \gamma vt$$

$$= \gamma(vt - \beta^2 x) \quad \underline{\text{OR}} \quad t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

Lorentz transformations: what about light?

S   S'



In frame S:

$x = ct$

In frame S': $x' = \gamma(x - vt)$

$\Rightarrow x' = \gamma(ct - vt)$

$= c\gamma\left(t - \frac{v}{c^2}ct\right)$

$= c\gamma\left(t - \frac{v}{c^2}x\right) = ct'$

So $x' = ct'$

$x' = ct'$

$x = ct$

$\Rightarrow$ Light still travels at c in frame S'!

Example: Lorentz transformations

Another look at the pion in the lab. $\beta = 0.998$

① $x = 125.748\,\text{m}$, $t = 4.20 \times 10^{-7}\,\text{s}$
(126)

② $x' = \gamma(x - vt) = \dfrac{1}{\sqrt{1 - 0.998^2}}\left(125.748 - 0.998c \times 4.2 \times 10^{-7}\right)$

$= \underline{\underline{0}}$

③ $t' = \gamma\left(t - \dfrac{vx}{c^2}\right) = \dfrac{1}{\sqrt{1 - 0.998^2}}\left(4.2 \times 10^{-7} - \dfrac{0.998}{c} \times 125.748\right)$

$= \underline{\underline{27\,\text{ns}}}$

Velocity Addition

S

In lab:

$x = ut$

S'

In S': $x' = u't'$

$x' = \gamma(x - vt)$

$t' = \gamma\left(t - \frac{v}{c^2}x\right)$

$\xrightarrow{v}$

$$u' = \frac{x'}{t'} = \frac{\cancel{\gamma}(x - vt)}{\cancel{\gamma}\left(t - \frac{v}{c^2}x\right)}$$

$x = ut$:

$$u' = \frac{ut - vt}{t - \frac{v}{c^2}ut} =$$

$\xrightarrow{u'}$

$$\boxed{\frac{u - v}{1 - \frac{uv}{c^2}} = u'}$$

Or $$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

Example: velocity addition

① A spaceship moving at $\frac{1}{2}c$ launches a rocket at $\frac{3}{4}c$ (in its own frame). What is the rocket's velocity as seen from earth?

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{\left(\frac{3}{4} + \frac{1}{2}\right)c}{1 + \frac{3}{4} \times \frac{1}{2}} = \frac{\frac{5}{4}c}{\frac{11}{8}} = \frac{10}{11}c$$

$u' = \frac{3}{4}c$

$v = \frac{1}{2}c$

What if u and $v \ll c$? What if $u' \rightarrow c$?

Summary of formulae

Lorentz transformations (1D):

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - \frac{vx}{c^2})$$

Velocity addition:

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$