Complex Analysis

Mathematics 113. Analysis I: Complex Function Theory

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For the students of Math 113

Harvard University, Spring 2013

FOREWORD

Welcome to Mathematics 113: Complex Function Theory! These are the compiled lecture notes for the class, taught by Professor Andrew Cotton-Clay at Harvard University during the spring of 2013. The course covers an introductory undergraduate-level sequence in complex analysis, starting from basics notions and working up to such results as the Riemann mapping theorem or the prime number theorem. We hope that these lecture notes will be useful to you in the future, either as memories of the class or as a handy reference.

These notes may not be accurate, and should not replace lecture attendance.

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Cover image: domain colored plot of the meromorphic function $f(z) = \frac{(z-1)(z+1)^2}{(z+i)(z-i)^2}$.
Source: K. Poelke and K. Polthier. Lifted Domain Coloring. *Eurographics/ IEEE-VGTC Symposium on Visualization*, (2009) 28:3. *Credits: V.Y.*
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CHAPTER 1

INTRODUCTION

Consider $\mathbb{R}$ but throw in $i = \sqrt{-1}$ to get $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$. We will be looking at functions $f : \mathbb{C} \to \mathbb{C}$ which are *complex-differentiable*. Complex-differentiable, or *holomorphic*, functions are quite a bit different from real-differentiable functions.

We can think of the real world as rigid, and the complex world as flexible. Complex analysis has applications in topology and geometry (in particular, consider complex 4-manifolds), and of course, physics and everywhere else.

Anyway, write $\mathbb{C} = \mathbb{R}^2$ with the identification $(a, b) = a + bi$. Addition and multiplication by $a \in \mathbb{R}$ is as for $\mathbb{R}$. For multiplication in general, $(a, b)(c, d) := (ac - bd, ad + bc)$.

Claim. $\mathbb{C}$ is a field: in particular, it's an additively commutative group, and multiplicatively, $\mathbb{C} - \{0\}$ is a commutative group.
Proof. Should be immediate. ■

Claim. Every nonzero complex number $z \in \mathbb{C}$ has a multiplicative inverse.
Proof. In fact, $z^{-1} := \frac{a}{a^2+b^2} - (\frac{b}{a^2+b^2})i$ works. ■

For $z \in \mathbb{C}$, $z = a + bi$, we define the *complex conjugate* $\overline{z} := a - bi = a + b(-i)$.

Claim. $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z}\overline{w}$ for $z, w \in \mathbb{C}$.
Proof. Write it out. ■

We also define the *norm* $|z| := \sqrt{a^2 + b^2} = z\overline{z}$. We can deduce a few properties of the norm:

Claim. (multiplicative property) $|z||w| = |zw|$ for $z, w \in \mathbb{C}$.
Proof. It suffices to work with squares. $|z|^2|w|^2 = z\overline{z}w\overline{w} = zw\overline{zw} = |zw|^2$. ■

Algebraically, can we do better than the complex numbers (e.g. can we throw in $\sqrt{1+i}$, etc.)? The *Fundamental Theorem of Algebra* tells us we're done as soon as we throw in i;

that is, we get all the solutions to polynomials.

In particular, let $p : \mathbb{C} \to \mathbb{C}, p(z) = \sum a_k z^k$ for $a_i \in \mathbb{C}$. If the degree is > 0, then $\exists z \in \mathbb{C} : p(z) = 0$.

Consider the equation $z^2 - (1+i) = 0$. Let's work this out explicitly for square roots. Given $a + bi$, we want $x + iy$ s.t. $(x+iy)^2 = a + bi$, which gives a system of equations:

$$\begin{cases} x^2 - y^2 = a \\ 2xy = b \end{cases} \implies \begin{cases} a^2 = (x^2-y^2)^2 = x^4 - 2x^2y^2 + y^4 \\ (x^2+y^2)^2 = x^4 + 2x^2y^2 + y^4 \end{cases} \implies (x^2-y^2)^2 + (2xy)^2 = a^2 + b^2$$

$$\implies x^2 + y^2 = \sqrt{a^2+b^2} \implies \begin{cases} x^2 = \frac{1}{2}(a + \sqrt{a^2+b^2}) \geq 0 \\ y^2 = \frac{1}{2}(-a + \sqrt{a^2+b^2}) \geq 0 \end{cases}$$

$$\implies x + iy = \pm \left(\frac{\sqrt{a + \sqrt{a^2+b^2}}}{2} + i\operatorname{sgn}(b) \frac{\sqrt{-a + \sqrt{a^2+b^2}}}{2} \right)$$

We can take this big general formula and use it to solve our specific cases.

For $z \in \mathbb{C}, z = a + bi$, we define the *real part* $\operatorname{Re}(z) = a = \frac{z + \overline{z}}{2}$, the *imaginary part* $\operatorname{Im}(z) = b = \frac{z - \overline{z}}{2i}$.

The Geometry of the Complex Plane. We have a multiplicative norm $|z|$. Viewing $z, w \in \mathbb{C}$ as vectors, addition is visualized as a parallelogram, and multiplication is best seen in polar coordinates. Polar coordinates can be obtained by $r = |z|$ and $\theta = \operatorname{Arg}(z)$, the *argument* of z. So we get the pretty important identity

$$\boxed{z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).}$$

For multiplication, try $z_1 z_2$, where $z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

So $|z_1 z_2| = |z_1||z_2|$ and $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) \bmod 2\pi$. Namely, multiplication by $z = r(\cos\theta + i\sin\theta)$ dilates the complex plane by r and rotates it counterclockwise by θ. Another view of the complex number is that $z = r(\cos\theta + i\sin\theta)$ acts on $\mathbb{C} \cong \mathbb{R}^2$ by

$$r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Of course, $\frac{1}{z} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)) = \frac{1}{r}(\cos\theta - i\sin\theta) = \frac{1}{|z|^2}\overline{z}$. So $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.

Problem. Find all cube roots of 1.

Solution. Want $z^3 = 1 = 1(\cos 0 + i\sin 0)$. So $r^3 = 1 \implies r = 1$ and $3\theta = 0 \bmod 2\pi \implies \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$. These give the cube roots of unity, and we can plot them on the unit circle in $\mathbb{C}$. ∎

Notation. $\zeta_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, a primitive n^{th} root of unity. The n^{th} roots of unity are $\zeta_n^k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$, $k = 0, ..., n-1$.

The equation for n^{th} roots of unity is $z^n - 1 = 0 = (z-1)(z^{n-1} + z^{n-2} + ... + z + 1)$. This gives a simple formula for $1 + \cos\theta + \cos 2\theta + ... + \cos(n-1)\theta$. For $z = \cos\theta + i\sin\theta$, $1 + z + ... + z^{n-1} = (\sum_{k=0}^{n-1} \cos k\theta + i \sum \sin k\theta)$. Now multiply by $z - 1 = (\cos\theta - 1) + i(\sin\theta)$ to get

$$\frac{(\cos n\theta - 1) + i\sin n\theta}{(\cos\theta - 1) + i\sin\theta} = \frac{(\cos n\theta - 1)(\cos\theta - 1) - \sin n\theta \sin\theta}{(\cos\theta - 1)^2 + \sin^2\theta}.$$

So

$$\sum_{k=0}^{n-1} \cos k\theta = \frac{(\cos n\theta - 1)(\cos\theta - 1) - i\sin n\theta \sin\theta}{2 - 2\cos\theta}.$$

This isn't the best formula, but we're happy with it.

CHAPTER 2

RIEMANN SPHERE, COMPLEX-DIFFERENTIABILITY, AND CONVERGENCE

Stereographic projection. Consider a sphere $S^2 := \{(u,v,w) : u^2+v^2+w^2=1\}$ in $\mathbb{R}^3$. We have a bijection $\varphi : S^2 - N \to \mathbb{C}$, where N is the north pole (point at ∞), given by stereographic projection. Indeed, we define the *Riemann sphere* $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \cong S^2$.

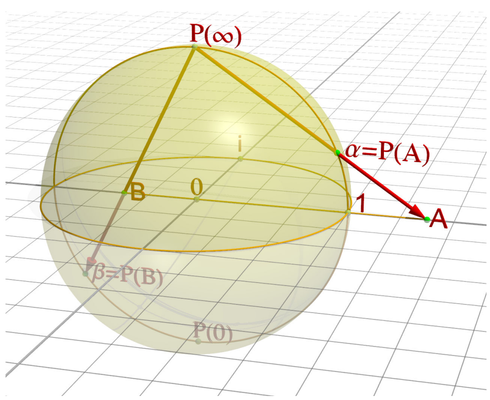


Diagram 1: Stereographic projection. Source: Wikipedia

Given $(u,v,w) \in S^2 - N$ and $(x,y) \in \mathbb{R}^2$, the bijection is given by

$$\varphi : (u,v,w) \mapsto \left(\frac{u}{1-w}, \frac{v}{1-w}\right), \quad \varphi^{-1} : (x,y) \mapsto \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right).$$

Claim. Any circle on S^2 maps to a circle or line in $\mathbb{C}$ under φ and vice-versa.

Proof. I hope you paid attention in lecture. ∎

Now consider the map $(u,v,w) \mapsto (u,-v,-w)$, which swaps $0 \in \mathbb{C}$ with S, the south pole, and $\infty \in \hat{\mathbb{C}}$ with N, the north pole. What does it do to $z = x+iy$? We see that $z \mapsto \frac{u-iv}{1+w}$.

Claim. This is $\frac{1}{z}$.

Proof. Trivial. ■

In fact, $\frac{1}{z}$ is just rotating the sphere around.

Claim. The *inversion* $\frac{1}{z}$ sends (circles and lines) to (circle and lines).

If we have a function $f : \mathbb{C} \to \mathbb{C}$ and we think it's "nice" at ∞, how do we quantify that? We simply use the inversion $\frac{1}{z}$. In particular, we look at $f(\frac{1}{z})$. Suppose $f(\infty) = \infty$; then $\frac{1}{f(\frac{1}{z})}$ sends 0 to 0.

For example, consider a polynomial of degree n, $f(z) = c_0 + ... + c_n z^n$ and $c_n \neq 0$. Take

$$\frac{1}{f(\frac{1}{z})} = \frac{1}{c_0 + ... + c_n z^{-n}} = \frac{z^n}{c_0 z^n + ... + c_n},$$

which is well-defined in a neighborhood of $0 \in \mathbb{C}$ and clearly, sends $0 \mapsto 0$.

Differentiation. Lets talk about something we all like, differentiation. :-)

Definition. Given $f(z) : \mathbb{C} \supset_{\text{open}} U \to \mathbb{C}$, $z \in U$, we say f is *complex-differentiable* (or *holomorphic*) at z if

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists for $h \in \mathbb{C}$.

Proposition. If f, g are holomorphic, then fg and $f + g$ is as well, with $(fg)' = f'g + fg'$ and $(f+g)' = f' + g'$.

Proof. Also trivial. ■

Example. Consider complex polynomials in z, which are holomorphic: $f(z) = \sum c_k z^k$. Then $f'(z)$ exists and $f'(z) = \sum^{n-1} (k+1) c_{k+1} z^k$.

Example. Let $f(z) = \overline{z}$. By showing that the limits in the real and imaginary directions do not agree, show that this is not holomorphic.

Before starting anything else, we would like to review real-differentiable functions. There are "different severities" of being real differentiable. Let $f : \mathbb{R} \supset_{\text{open}} U \to \mathbb{R}$:

1. real differentiable: $f'(x)$ exists $\forall x$ in domain
2. C^1: $f'(x)$ exists and is continuous
3. C^k: $f, f', ..., f^{(k)}$ exists and are continuous
4. C^∞: (smooth) all derivatives exists and are continuous
5. real-analytic: $f(x) = \sum^\infty c_k x^k$ is a convergent power series

The punchline is that homolorphic functions are always C^∞ and analytic! (This is something that should really be appreciated.) Indeed, we oftentimes interchange holomorphic, C^∞, and analytic in the complex world.

To show this, we will go back to convergence. Consider a sequence of functions $\{f_k(z)\}$ defined on $E \subset \mathbb{C}$ that is containable in a compact set. We say that it is pointwise convergence if $\lim_{k\to\infty} f_k(z)$ exists $\forall z$; the pointwise limit is defined as $f(z)$.

Suppose $f_k(z)$ is continuous. It's uniformly convergent if $\forall \epsilon, \exists N : \forall n \geq N : |f_n(z) - f_(z)| \leq \epsilon, \forall z \in E$.

Lemma. If $\{f_k(z)\}$ are continuous and converge uniformly on E, then pointwise limit $f(z)$ is continuous.

Proof. Use triangle inequality and a standard epsilon pushing argument. ∎

Power series. Consider $f(z) = \sum_0^\infty c_k z^k$. We need a condition on $|c_k|$ for this to converge.

Definition. Given a sequence $\{a_k\}_1^\infty$, we define the *limit supremum*

$$\limsup_{k\to\infty} a_k = \lim_{n\to\infty}\left(\sup_{k\geq n} a_k\right) = \overline{\lim_{k\to\infty}} a_k$$

The polynomial condition is then to look at $\limsup |c_k|^{1/k}$.

Theorem. If $\overline{\lim}|c_k|^{1/k} = L$ for $L = 0$, $0 < L < \infty$, or $L = \infty$, then:

1. If $L = 0$, then $f(z) = \sum c_k z^k$ converges for all z.
2. If $0 < L < \infty$, then $\sum c_k z^k$ converges for $|z| < R = \frac{1}{L}$, diverges for $|z| > R$, and R is called the *radius of convergence*. (We do not know for $|z| = R$.)
3. If $L = \infty$, $\sum c_k z^k$ diverges $\forall z \neq 0$.

Proof. $L = 0$: $\forall \epsilon > 0, \exists N : \forall k > N, |c_k|^{1/k} < \epsilon$. Let $\epsilon = \frac{1}{z}|z|$. Then $|c_k|^{1/k} < \frac{1}{2}|z|$, $\mathrm{Re}|c_k| < \frac{1}{2^k}1|z|^k$. We want $|c_k|^{1/k}|z| < \frac{1}{2}$, so $|c_k||z|^k \leq \frac{1}{2}^k$ so $\sum c_k z^k$ converges. We can do this with the M-test, by considering partial sums, etc.

$0 < L < \infty$: $\overline{\lim}|c_k|^{1/k} = L$. Let $|z| = R(1 - 2\delta)$, then $\overline{\lim}|z||c_k|^{1/k} = 1 - 2\delta$ for $R = \frac{1}{L}$. So $\exists N : |z||c_k|^{1/k} < 1 - \delta\ \forall n > N$, and $\sum c_k z^k$ converges because $|c_k|z^k < (1-\delta)^k$.

Likewise, if we suppose $|z| = R(1 + 2\delta)$. Then $\overline{\lim}|c_k|^{1/k}|z| = 1 + 2\delta$. So for infinitely many k, $|c_k|^{1/k}|z| > 1+\delta$ and $|c_k z^k| > 1$. The last condition is proved in the same manner. ∎

Example. $\sum_0^\infty z^k = \frac{1}{1-z}$. Deduce this from the theorem above.

Example. $\sum_0^\infty (k+1)z^k = \frac{1}{(1-z)^2}$. We see this by taking $c_k = 1$ and showing that $\overline{\lim}(k+1)^{1/k} = 1$.

Theorem. Suppose $f(z) = \sum_0^\infty c_k z^k$ converges for $|z| < R$. Then:

1. $\sum_1^\infty k c_k z^{k-1}$ converges for $|z| < R$.
2. $f'(z)$ exists and equals $\sum_1^\infty k c_k z^{k-1}$.

Proof. (1) It suffices to check $\overline{\lim}|c_k|^{1/k} \leq \frac{1}{R} \implies \overline{\lim}(k+1)^{1/l}|c_k|^{1/k} \leq \frac{1}{R}$. Note that $\overline{\lim}(k+1)^{1/k}|c_k|^{1/k} = \lim(k+1)^{1/k}\overline{\lim}|c_k|^{1/k}$.

(2) $R = \infty$: subtract and show that limits $\to 0$.

$$\frac{f(z+h) - f(z)}{h} - \sum k c_k z^{k-1} = \frac{1}{h}\sum c_k[(z+h)^k - z^k] - \sum k c_k z^{k-1}$$

$$= \sum c_k[\frac{1}{h}(z+h)^k - \frac{1}{h}z^k - kz^{k-1}].$$

Recall the binomial theorem,

$$(z+h)^k = \binom{k}{l} h^l z^{k-l},$$

which we can use to rewrite the expression above as

$$= \sum_0^\infty c_k[\sum_{l=1}^k \binom{k}{l} z^{k-l}h^{l-1} - kz^{k-1}] = \sum_0^\infty \sum_2^k \binom{k}{l} z^{k-l}h^{l-1}$$

$$= h\sum_0^\infty \sum_2^k \binom{k}{l} z^{k-l}h^{l-2}.$$

We also have

$$\left|\sum \binom{k}{l} z^{k-l}h^{l-2}\right| \leq \sum \binom{k}{l} |z|^{k-1} = (|z|+1)^k,$$

so $\sum c_k(|z|+1)^k$ converges because $R = \infty$ for $\sum c_k z^k$.

$R < \infty$: Let $|z| = R - 2\delta, |h| < \delta$; then $|z+h| < R - \delta$. Then

$$\frac{f(z+h) - f(z)}{h} - \sum kc_k z^{k-1} = h\sum_0^\infty c_k \sum_2^k \binom{k}{l} z^{k-l}h^{l-2}.$$

Write

$$\binom{k}{l} = \frac{k(k-1)...(k-l+1)}{l!}.$$

For $l \geq 2$ we can bound $\binom{k}{l} \leq k^2\binom{k}{l-2}$. So

$$\left|\sum_2 \binom{k}{l} z^{k-l}h^{l-2}\right| \leq \frac{k^2}{|z|^2}(|z|+|h|)^k$$

and the equation above translates into

$$= \frac{h}{|z|^2}\sum k^2 c_k(|z|+|h|)^k,$$

which converges and $\to 0$ because of the h in the front. ■

CHAPTER 3

POWER SERIES AND CAUCHY-RIEMANN EQUATIONS

Power series. Last time we were dealing with power series, $f(z) = \sum_{k=0}^{\infty} c_k z^k$. We defined the radius of convergence $R = \frac{1}{L}$, where $L = \overline{\lim}|c_k|^{1/k}$ and either $L = 0, 0 < L < \infty$, or $L = \infty$.

Inside the radius of convergence, $f(z)$ is a convergent power series with some R. What are the consequences of this?

Corollary. Inside its radius of convergence, a power series is ∞-differentiable, with expected derivatives as convergent power series.

Corollary. If $R > 0$, then $c_k = \frac{f^{(k)}(0)}{k!}$.

Proof. Take $f^{(k)}(x)$. The corollary follows. ■

There are also several uniqueness properties:

Lemma. If $f(z) = \sum c_k z^k$ is a convergent power series and $f(z_n) = 0$ for a sequence $\{z_n\}_{n=1}^{\infty}$ with $z_n \to 0, z_n \neq 0$, then $c_k = 0 \; \forall k$ and $\mathrm{Re}(f(z)) \equiv 0$.

Proof. $c_0 = f(0) = \lim_{n\to\infty} f(z_n) = 0$. Now form $g_1(z) = \frac{f(z)}{z} = c_1 + c_2 z + c_3 z^2 + \ldots$ (exercise: show that this holds for some radius of convergence). Check $c_1 = g_1(0) = \lim_{n\to\infty} g_1(z_n) = \lim_{n\to\infty} \frac{f(z)}{z} = 0$, and induct on g_i. ■

Proposition. If $f(z) = \sum a_k z^k$, $g(z) = \sum b_k z^k$ and these agree on some set accumulating at 0, then $a_k = b_k \; \forall k$, i.e. $f(z) = g(z)$.

Proof. Consider $\sum (a_k - b_k) z^k$ and apply the lemma. Show that $\overline{\lim}|a_k|^{1/k} \leq L$ and $\overline{\lim}|b_k|^{1/k} \leq L \implies \overline{\lim}|a_k - b_k|^{1/k} \leq L$. ■

Note: To center the power series at $w \in \mathbb{C}$, consider $\sum c_k (z-w)^k$, which shifts the center of the power series from $0 \in \mathbb{C}$ to w and maintains the radius of convergence at R.

Complex-differentiability. We refer to complex-differentiability as either *holomorphic* or *complex-analytic* (some texts, e.g. Ahlfors, simply like to use *analytic*). This is a very nice property of functions that we will be exploring in the upcoming weeks.

Definition. $f : \mathbb{C} \to \mathbb{C}$ is *holomorphic* at $z \in \mathbb{C}$ if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists for $h \in \mathbb{C}$. Equivalently, $\exists$ a number $f'(z)$:

$$0 = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} - f'(z).$$

The Cauchy-Riemann equations. By considering real and imaginary parts of holomorphic functions, we get the celebrated Cauchy-Riemann equations.

Proposition. If $f(z) : \mathbb{C} \to \mathbb{C}$ is holomorphic at z (alternatively, by $\mathbb{R}^2 \cong \mathbb{C}$, we can regard $f(x, y) : \mathbb{R}^2 \to \mathbb{R}^2$), then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and

$$\boxed{i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}.} \tag{3.1}$$

Equivalently, for the decomposition $f(x, y) = u(x, y) + iv(x, y)$ and

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y},$$

what we mean is that, by comparing real and imaginary parts of the equation

$$i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y,}$$

we obtain the equalities

$$\boxed{\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}. \end{cases}} \tag{3.2}$$

Proof. This is a bit tedious to type up, so I hope you paid attention in lecture. The proof follows from a simple computation of the partials; see any textbook. ■

The CR conditions (1) or (2), along with the condition that the partials are continuous, ascertains that f is itself holomorphic.

If the partials are not continuous, consider $f(x, y) = \frac{xy(x+iy)}{x^2+y^2}$, $z \neq 0$, and 0 for $z = 0$. Show that this is not differentiable at 0 but $\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0$.

Proposition. If $f(x, y) = u(x, y) + iv(x, y)$ has continuous partial derivatives and $i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$ (satisfies CR equations), then f is holomorphic.

Proof. Messy as well, but use the mean value theorem and write out the partials. ■

An equivalent (geometric) formulation of the CR equations. Consider the Jacobian of $f : \mathbb{R}^2 \to \mathbb{R}^2$:

$$J_f := \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

and the rotation matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

So multiplication by i is multiplication by the matrix (aside: see the relation to *complex structure*)

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The equivalent form is that $J_f I = I J_f$; e.g. the Jacobian matrix commutes with rotation by $\pi/2$. Check this:

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial y} & -\frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} & -\frac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} -\frac{\partial v}{\partial x} & -\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial x} & -\frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

An algebraic interpretation. Consider complex polynomials, $p(z) = \sum_{k=0}^{n} c_k z^k$. Look at complex-valued polynomials of real variables x, y:

$$p(x,y) = \sum_{m=0}^{n} \sum_{\substack{k+l=m,\\ k,l\geq 0}} c_{k,l} x^k y^l \tag{3.3}$$

with $c_{k,l} = \frac{\partial p}{\partial^k x \partial^l y}(0,0)$, and take a "new basis" as $z = x + iy$ and $\overline{z} = x - iy$. Write the polynomial in $z, \overline{z}$:

$$p(z,\overline{z}) = \sum_{m=0} \sum_{\substack{k+l=m,\\ k,l\geq 0}} \tilde{c}_{k,l} z^k \overline{z}^l. \tag{3.4}$$

We claim that there's a 1-1 correspondence between polynomials given by (3) and (4). Indeed, $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ act on these polynomials:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Claim. We have the equalities

$$\frac{\partial}{\partial \overline{z}}(z^k \overline{z}^l) = l z^k \overline{z}^{l-1},$$

$$\frac{\partial}{\partial z}(z^k \overline{z}^l) = k z^{k-1} \overline{z}^l.$$

Proof. We simply check:

$$\frac{\partial}{\partial \overline{z}}(z^k \overline{z}^l) = \frac{1}{2}[k z^{k-1}\overline{z}^l + l z^k \overline{z}^{l-1}] + \frac{i}{2}[k i z^{k-1}\overline{z}^l + (-i) l z^k \overline{z}^{l-1}] = l z^k \overline{z}^{l-1},$$

and likewise for the second equality. ■

Claim. (equivalent form of CR equations) $f(z)$ satisfies CR equations $\Longleftrightarrow \frac{\partial}{\partial \overline{z}} f = 0$.

Proof. A simple check. ■

The conclusion is that $p(z, \overline{z}) = \sum\sum \tilde{c}_{k,l} z^k \overline{z}^l$ is holomorphic iff $\tilde{c}_{k,l} = 0$ for $l > 0$ (e.g. no $\overline{z}'s$). Alternatively, $p(x, y) = \sum\sum c_{k,l} x^k y^l$ is holomorphic iff it can be written as $p(z) = \sum c_k z^k$.

Let's give some basic hints that holomorphic functions behave a bit like power series:

Lemma. Suppose $f(x, y) = u(x, y) + iv(x, y)$ is holomorphic on a disk of radius R and $u(x, y)$ is constant. Then f is constant.

Proof. $u_x = u_y = 0$, so by CR equations, $u_x = v_y, u_y = -v_x$ and $v_x = v_y = 0$. By the mean value theorem, this shows that v is constant along horizontal and vertical lines in the plane. So v is constant throughout the disk. ■

Lemma. If $f = u + iv$ is holomorphic on a disk in $\mathbb{C}$ and $||f||^2$ is constant, then f is constant.

Proof. We have $u^2 + v^2 = c$, so taking partials by x and y gives $2u_x u + 2v_x v = 0$ and $2u_y u + 2v_y v = 0$. Applying the CR equations, we get $2u_x v - 2v_x u = 0$. Adding equations gives $2u_x(u^2 + v^2) = 0$, and similarly $2v_x(u^2 + v^2) = 0$. So either $u^2 + v^2 = 0$ ($f = 0$), or all partials $= 0$.

In the second case, given any open ball, f is constant in that open ball since its partials $= 0$ in the ball. But the disk is connected, so f is constant everywhere.

CHAPTER 4

DEFINING YOUR FAVORITE FUNCTIONS

Today is about extending your favorite (real analytic) functions from $\mathbb{R}$ to $\mathbb{C}$. When determining their complex analogues, we could (1) ask for the same fundamental properties; (2) use their power series; (2') extend such that the result is holomorphic (actually, this is the same as 2). Happily, these all agree!

1. e^x

   We can consider (1) $e^0 = 1$, $\frac{d}{dx}e^x = e^x, \ldots$ or (2) $e^x = \sum \frac{x^n}{n!}$.

   (1) Let's look for $f(z)$ with $f(x) = e^x$ and $f(z_1 + z_2) = f(z_1)f(z_2)$ and holomorphic. Apparently, $f(x+iy) = e^x f(iy)$. Let $f(iy) = A(y) + iB(y)$, so $f(x+iy) = e^x A(y) + e^x iB(y)$. Then use the Cauchy-Riemann equations to show that $A(y) = \cos y$ and $B(y) = \sin y$ by the Fundamental Theorem of ODEs (namely, any linear ordinary differential equation has an e^z solution). So define $\boxed{e^z = e^x(\cos y + i \sin y),}$ for $z = x + iy$.

   (2) Let
   $$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$
   The radius of convergence is ∞, since $\overline{\lim}_{n\to\infty}(\frac{1}{n!})^{1/n} = 0$ (e.g. it's an *entire* function). Also, $f(z)$ is holomorphic on all of $\mathbb{C}$. We can easily verify the properties that $f'(z) = f(z)$, $f(0) = 1$, $f(z)f(w) = f(z+w)$, etc.

2. $\sin z, \cos z$

   For $f(z) = e^z$, look at $f(iz) = 1 + (iz) + \frac{(iz)^2}{2!} + \ldots = (1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots) + i(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots)$. Then we set $f(iy) =: \cos y + i \sin y$.

   We can verify the usual properties that $\frac{\partial}{\partial z}\cos z = -\sin z$, $\frac{\partial}{\partial z}\sin z = \cos z$, $\cos 0 = 1$, $\sin 0 = 0$, $\cos(-z) = \cos z$, $\sin(-z) = -\sin z$, $e^{iz} = \cos z + i \sin z$, etc.

   Likewise, we can express
   $$\boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.}$$

We can likewise check the addition formulas for $\cos(z_1 + z_2)$ and $\sin(z_1 + z_2)$.

3. $\log z$

   $w = \log z \iff z = e^w$. For $w = a + bi$, $e^w = e^a(\cos b + i \sin b)$. Apparently, $\log z = \log r + i\theta + i(2\pi n)$, so the complex log is multi-valued. Write $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$.

   We define the *principal branch* of $\log z$ as follows. Consider $\mathbb{C} - \mathbb{R}_{\leq 0}$. $\log z$ has imaginary part in $(-\pi, \pi)$. Write $\log z = \log|z| + i\mathrm{Arg}(z)$ for $\mathrm{Arg}(z) \in (-\pi, \pi)$; this is our definition of $\log z$ for the principal branch, which has range $-\pi < \mathrm{Im}(z) < \pi$.

   Claim. $\log z$ is holomorphic given a branch cut, and $\frac{\partial}{\partial z}(\log z) = 1/z$.

   Proof. Consider
   $$\lim_{h\to 0} \frac{\log(z+h) - \log z}{h}.$$
   Set $w_1 = \log(z+h), w = \log z$. Then, noting that log is continuous given a branch cut, we see that
   $$\lim_{h\to 0} \frac{\log(z+h) - \log z}{h} = \lim_{h\to 0} \frac{w_1 - w}{e^{w_1} - e^w} = \frac{1}{\lim_{w_1 \to w} \frac{e^{w_1} - e^w}{w_1 - w}} = \frac{1}{e^w} = \frac{1}{z}.$$
   ■

   Intuitively, $\log z$ looks like a helix above $\mathbb{C} - \{0\}$; we call this the *Riemann surface* for $\log z$.

4. $z^\alpha, \alpha \in \mathbb{R}$

   Define $z^\alpha = e^{\alpha \log z} = e^{\alpha(\log z + 2\pi n i)}$, which is potentially multi-valued. $e^{\alpha \log z}$ is the principal branch; now examine $e^{2\pi n \alpha i}$. If $\alpha \notin \mathbb{Q}$, $e^{2\pi n\alpha i}$ takes infinitely many values. If $\alpha = \frac{k}{l}$ in reduced form, $e^{2\pi nki/l}$ takes l values (e.g. $e^{n\pi i} = \pm 1$ for $\alpha = \frac{1}{2}$).

   For example, when considering $\sqrt{z}$, we still need a branch cut. $\sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta/2}$, $\theta \in (-\pi, \pi)$.

Integration. Given $f(z)$ holomorphic, can we find another function $F(z)$ holomorphic with $F'(z) = f(z)$? The answer is, generally, yes. We begin with integration along a contour, $\int_C f(z)dz$ for C a contour.

Definition. The image of $\gamma(t) : [a, b] \to \mathbb{C}$ is a *contour curve* if:

1. γ is continuously differentiable except at finitely many points
2. When differentiable, $\gamma'(t) \neq 0$ except at finitely many points

For C a contour, $\int_C f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$. Call $\gamma_1(t) \sim \gamma_2(t)$ for $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$ if $\exists \lambda(t) : [c, d] \to [a, b]$ s.t. $\gamma_2(t) = \gamma_1(\lambda(t))$.

Claim. This is well defined:
$$\int_a^b f(\gamma_1(t))\gamma_1'(t)dt = \int_c^d f(\gamma_2(t))\gamma_2'(t)dt.$$

Lemma. (complex chain rule) Let $g(t) = f(\gamma(t))$, $f(z)$ holomorphic. Then $g'(t) = \gamma'(t)f'(\gamma(t))$.

Proof. As usual, with limits or whatnot. ■

Example. Compute
$$\int_{S^1} z^k dz$$
for $k \in \mathbb{Z}$, and S^1 the unit circle in $\mathbb{C}$ oriented counterclockwise.

Explanation. Let $\gamma(t) = e^{it} = \cos t + i \sin t$. By chain rule, $\gamma'(t) = ie^{it}$. Then
$$\int_{S^1} z^k dz = \int_0^{2\pi} e^{ikt} ie^{it} dt = i \int_0^{2\pi} e^{i(k+1)t} dt.$$
If $k = -1$, this is
$$\int_{S^1} \frac{1}{z} dz = i \int_0^{2\pi} 1 dt = 2\pi i.$$
If $k \neq 1$, this is
$$\int_0^{2\pi} e^{i(k+1)t} dt = \left. \frac{e^{i(k+1)t}}{i(k+1)} \right|_0^{2\pi} = 0.$$

So z^k for $k \in \mathbb{Z}, k \neq -1$ has an antiderivative in $\mathbb{C} - \{0\}$: $\frac{z^{k+1}}{k+1}$. $1/z$ does not; we'd want $\log z$, but it's multivalued.

Proposition. If $f(z)$ holomorphic with $F(z)$ holomorphic and $F'(z) = f(z)$, then
$$\int_C f(z) dz = F(\text{end}) - F(\text{start}),$$
for C a path from start to end.

Proof. Use the complex chain rule. ■

Next time, we'll consider functions that are defined on all of $\mathbb{C}$ (which we call *entire* functions), we'll show $\exists F(z) : F'(z) = f(z)$. Along the way, we'll prove the *closed curve theorem*.

For motivation, we recall *Green's theorem*:
$$\int_a^b (P, Q) \cdot r'(t) dt = \int \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$
for $r(t) : [a, b] \to \mathbb{R}^2$ a closed curve around a region D. This requires that Q_x, P_y be continuous (or something like that). The theorem we'll prove does not need this requirement.

CHAPTER 5

THE CLOSED CURVE THEOREM AND CAUCHY'S INTEGRAL FORMULA

Today we'll be talking about the **closed curve theorem** and **Cauchy's integral theorem**. Recall from last lecture that, for a parameterization γ of the contour C, we had

$$\int_C f(z)dz = \int f(\gamma(t))\gamma'(t)dt.$$

We also know that the path integral is determined by the antiderivative's value at the curve endpoints:

Proposition. If $F(z)$ is holomorphic, $F'(z) = f(z)$, then

$$\int_C f(z)dz = F(\text{final}) - F(\text{initial}).$$

The closed curve theorem. We will first provide motivation for the closed curve theorem. Given f and γ, consider splitting into real and imaginary parts:

$$f(z) = f(x, y) = u(x, y) + iv(x, y)$$

$$\gamma(t) = a(t) + ib(t)$$

$$\int f(\gamma(t))\gamma'(t)dt = \int (u + iv)(\dot{a} + i\dot{b})dt = \int [(u\dot{a} - v\dot{b}) + i(u\dot{b} + v\dot{a})]dt$$

$$= \int (u, -v) \cdot (\dot{a}, \dot{b})dt + i \int (v, u) \cdot (\dot{a}, \dot{b})df \tag{5.1}$$

If C is the closed, counterclockwise boundary of D, then by Green's theorem (which only applies when all partials are continuous) we have that the above is equal to

$$\int\int_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i \int\int_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy = 0,$$

where the last equality follows from the Cauchy-Riemann equations. We can also note that, in the language of vector fields, (1) is equivalent to

$$\int\int_D \operatorname{curl}(\overline{f})dA + \int\int_D \operatorname{div}(\overline{f})dA,$$

where $\overline{f}$ is thought of as a vector field on $\mathbb{R}^2$ induced by f. The equivalent formulation of the CR equations would then be $\operatorname{curl}(\overline{f}) = \operatorname{div}(\overline{f}) = 0$ as a vector field on $\mathbb{R}^2$.

The conclusion above gives us a glimpse into the closed curve theorem, which, however, does not require the first partials to be continuous. Keep in mind that our objective is the result that f being holomorphic on the interior of a disk implies that f has a convergent power series expansion on that disk: we will do this using the closed curve theorem and friends.

Claim. The closed curve theorem (stated below) is true for linear functions $f(z) = a + bz$.

Proof. Let $F(z) = az + \frac{1}{2}bz^2$, so $F'(z) = f(z)$. So we have $\int_C f(z)dz = F(\text{final}) - F(\text{initial}) = 0$. ∎

Theorem. (closed curve theorem) Suppose $f(z)$ is holomorphic in an open disk D. Let C be any closed contour curve in D. Then

$$\int_C f(z)dz = 0.$$

Remark. Our strategy is in three steps: (1) show this for any polygonal curve; (2) show that $f(z)$ has an antiderivative on D; (3) conclude the result.

Proof for triangle. Let T be a triangle contour. The idea is that f being holomorphic implies that f is well-approximated by linear functions on small scales; see the claim above.

First, we subdivide the triangle. Suppose $|\int_T f(z)dz| = C$. Then $\exists T_i$ with boundary Γ_i such that $|\int_{\Gamma_i} f(z)dz| \geq c/4$. Call this $T^{(1)}$ with boundary $\Gamma^{(1)}$. Keep subdividing to get (dropping the parentheses) $T^1, T^2, T^3, ...$ and $\Gamma^1, \Gamma^2, \Gamma^3, ...$ such that $|\int_{\Gamma^k} f(z)dz| \geq \frac{c}{4^k}$. Note that $\cap_{k=1}^{\infty} T^k = \{\text{point}\}$; call the point z_0. $f(z)$ is holomorphic at z_0, so we have

$$\lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| = 0.$$

Thus

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \epsilon(z)(z - z_0)$$

with $\epsilon(z) \to 0$ as $z \to z_0$. Now consider

$$\int_{\Gamma^k} f(z)dz = \int_{\Gamma^k} [f(z_0) + f'(z_0)(z - z_0) + \epsilon(z)(z - z_0)]dz = \int_{\Gamma^k} \epsilon(z)(z - z_0)dz.$$

Given $\epsilon > 0$, $\exists N$ s.t. for $k \geq N$, $\epsilon(z) < \epsilon$ for $z \in T^k$. Let the perimeter of T be L and the perimeter of T^k be $\frac{L}{2^k}$. The max distance between two points in T^k would then be $\leq \frac{L}{2^k}$. Thus

$$\frac{c}{4^k} \leq \left| \int_{\Gamma^k} \epsilon(z)(z - z_0)dz \right| \leq \frac{L}{2^k} \epsilon \frac{L}{2^k} = \frac{\epsilon L^2}{4^k},$$

where the first term after the equality is length of Γ^k, the second is an upper bound on $\epsilon(z)$, and the third is an upper bound on $|z-z_0|$. We had $c = |\int_\Gamma f(z)dz|$, so $c \le \epsilon L^2 \implies c = 0$. ■

Corollary. (extension to polygons) If P is a polygon and $f(z)$ is holomorphic on an open neighborhood of P with boundary Γ, then $\int_P f(z)dz = 0$.

Proof. Divide P into triangles. ■

Theorem. (antiderivative theorem) If $f(z)$ is holomorphic on an open disk D (or an open polygonally simply connected region $\Omega \subset \mathbb{C}$), then $\exists F(z)$ holomorphic on $\mathbb{D}$ (or Ω) s.t. $F'(z) = f(z)$.

Definition. An open region $\Omega \subseteq \mathbb{C}$ is *polygonally connected* (in topology, connected) if $\forall z, w \in \Omega, \exists$ piecewise linear curve in Ω connecting z, w. A region is *simply polygonally connected* (simply connected) if any closed polygonal curve is the boundary of a union of polygons in Ω.

Proof of antiderivative theorem. Let $p \in D$ (or $\Omega \subseteq \mathbb{C}$). Let $F(z) = \int_p^z f(z)dz$ (the line integral along any polygonal path from p to z, which is well-defined by the result on triangles above and the p.s.c. property of Ω). We check that

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_{z_0}^{z} f(z)dz.$$

Note

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| = \left| \frac{1}{z - z_0} \int_{z_0}^{z} (f(z) - f(z_0))dz \right|$$

because

$$\int_{z_0}^{z} f(z_0)dz = f(z_0) \int_{z_0}^{z} 1dz = f(z_0)(z - z_0).$$

Then we have

$$\left| \frac{1}{z - z_0} \int_{z_0}^{z} (f(z) - f(z_0))dz \right| \le \frac{1}{|z - z_0|} |z - z_0| \epsilon,$$

where the middle term after the inequality is the length of our curve and ϵ is upper bound for $|f(z) - f(z_0)|$ (by continuity). So the LHS $\to 0$ as $\epsilon \to 0$. ■

Corollary. (closed curve theorem for regions) If $f(z)$ is holomorphic on an open disk D (or p.c.r. region Ω), then $\int_C f(z)dz = 0$ for C being any closed contour in D (or Ω).

Proof. Take $F(z)$, the antiderivative of $f(z)$, so that $\int_C f(z)dz = F(\text{final}) - F(\text{initial}) = 0$. ■

Cauchy's integral theorem. Given that $f(z)$ is holomorphic, let's look at

$$g(z) = \begin{cases} \frac{f(z) - f(w)}{z - w} & z \ne w \\ f'(w) & z = w \end{cases}$$

for $w \in \mathbb{C}$.

Claim. $g(z)$ satisfies the closed curve theorem.

Proof. $g(z)$ is continuous because $f'(w)$ exists. The goal is to prove the above result for triangles for $g(z)$; then it will follow $\exists G(z) : G'(z) = g(z)$, along with the closed curve theorem for $g(Z)$.

Consider w. If w is outside T, then $g(z)$ holomorphic in an open neighborhood of T and the old closed curve theorem applies.

If $w \in \Gamma = \partial T$, then we cut off small pieces $\{T_i\}$ so $\int_{\Gamma^i} f(z)dz = 0$ (because they do not contain w). The length of Γ^i can be arbitrarily small at M, the upper bound on $g(z)$ on T, so it goes to 0 in the limit. If w is in the interior, we also use the same trick. ∎

Cauchy integral formula. This is one of the central results of complex analysis. Let C be a circle surrounding $w \in \mathbb{C}$, and $f(z)$ holomorphic on the closed boundary C of the open disk D. Then we have the formula

$$\boxed{f(w) = \frac{1}{2\pi i} \int \frac{f(z)}{z - w} dz}$$

Proof. Suppose C is a circle centered at w. The circle is parameterized by $\gamma(t) = w + Re^{it}$ for $0 \leq t \leq 2\pi$. We know from the closed curve theorem that

$$0 = \int_C \frac{f(z) - f(w)}{z - w} dz,$$

because the integrand is $g(z)$ defined on an interior satisfying the closed curve theorem. Now split this up:

$$\int_C \frac{f(z)}{z - w} dz = f(w) \int_C \frac{1}{z - w} dz, \qquad \int_C \frac{1}{z - w} dz = \int_0^{2\pi} \frac{iRe^{it} dt}{Re^{it}} = 2\pi i$$

$$\implies f(w) = \frac{1}{2\pi i} \int \frac{f(z)}{z - w} dz.$$

∎

Theorem. Suppose $f(z)$ is holomorphic on the open disk $D_k(0)$ of radius R around $0 \in \mathbb{C}$. Then there exists a convergent power series $\sum_{k=0}^{\infty} c_k z^k$ with $\overline{\lim} |c_k|^{1/k} \leq \frac{1}{R}$ such that $f(z) = \sum_{k=0}^{\infty} c_k z^k$.

Proof. From Cauchy's integral formula, we have

$$f(z) = \frac{1}{2\pi i} \int \frac{f(w)}{w - z} dw.$$

Let $|z| < \tilde{R} < R$, and write

$$\frac{1}{w - z} = \frac{1}{w} \frac{1}{1 - \frac{z}{w}} = \frac{1}{w} \left(1 + \frac{z}{w} + \frac{z^2}{w^2} + ... \right)$$

$|\frac{z}{w}| = \frac{|z|}{\tilde{R}} < 1$ so this converges uniformly to $\frac{1}{w-z}$ for $w \in \tilde{C}$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\tilde{C}} f(w) (\frac{1}{w} + \frac{z}{w^2} + \frac{z^2}{w^3} + ...) dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(w)}{w}dw + \left(\frac{1}{2\pi i}\int_C \frac{f(w)}{w^2}dw\right)z + \left(\frac{1}{2\pi i}\int_C \frac{f(w)}{w^3}dw\right)z^2 + ...,$$

i.e.

$$C_k = \frac{1}{2\pi i}\int_{\tilde{C}} \frac{f(w)}{w^k+1}dw.$$

We check this as follows:

$$|C_k| \le \frac{1}{2\pi}2\pi\tilde{R}M\frac{1}{\tilde{R}^{k+1}} = M/R^k,$$

where the terms of the right of the inequality come as follows: $2\pi\tilde{R}$ is the length of R and M is an upper bound on f in $D_{\tilde{R}}(0)$. So $\overline{\lim}|C_k|^{1/k} \le \overline{\lim}\frac{M^{1/k}}{\tilde{R}} = \frac{1}{\tilde{R}}$, and

$$\frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i}\int_C \frac{f(w)}{w^{k+1}}dw$$

for C containing 0. ■

Thus we have the result that f holomorphic $\Longrightarrow$ f analytic $\Longrightarrow$ f infinitely differentiable.

CHAPTER 6

APPLICATIONS OF CAUCHY'S INTEGRAL FORMULA, LIOUVILLE THEOREM, MEAN VALUE THEOREM

Applications of Cauchy's integral formula. Recall that we have the Cauchy integral formula:

$$\boxed{f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw,}$$

for C a curve containing z. For $f(w) = \sum c_k w^k$, note that we can write

$$f(0) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w}dw = \frac{1}{2\pi i}\int_C \sum_{k=1}^{\infty} c_k w^{k-1} dw = \frac{1}{2\pi i}\int_C \frac{c_0}{w}dw = c_0.$$

Let's use this formula to compute some stuff. The general method is to decompose a closed contour (over which the integral is zero) into a sum of directed edges, then evaluating the integral over the other edges to give us results.

Claim.

$$\int_0^{\infty} \frac{\sin x}{x}dx = \frac{\pi}{2}.$$

Proof. Let C_R denote the contour of an upper half-circle of radius R centered at the origin. For $r < R$, we can use Cauchy's integral theorem to write (noting that the integrand is equivalent to the imaginary part of $\frac{e^{ix}}{x}$:

$$0 = \int_r^R \frac{e^{ix}}{x}dx + \int_{C_R} \frac{e^{iz}}{z}dz + \int_{-R}^{-r} \frac{e^{ix}}{x} - \int_{C_r} \frac{e^{iz}}{z}dz.$$

Now parameterize C_r with $z = re^{i\theta}$ for $\theta \in [0,\pi]$ and likewise with $z = Re^{i\theta}$ to see

$$0 = \int_r^R \frac{e^{ix}}{x}dx + \int_0^{\pi} \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} iRe^{i\theta}d\theta + \int_{-R}^{-r} \frac{e^{ix}}{x}dx - \int_0^{\pi} \frac{e^{ire^{i\theta}}}{re^{i\theta}} ire^{i\theta}d\theta.$$

$$= \int_r^R \frac{e^{ix}}{x}dx + \int_0^{\pi} ie^{iRe^{i\theta}}d\theta + \int_{-R}^{-r} \frac{e^{ix}}{x}dx - \int_0^{\pi} ie^{ire^{i\theta}}d\theta.$$

But note that the second integral $\to 0$ as $R \to \infty$ and the fourth integral $\to \pi i$ as $r \to 0$ (show this yourself), so we're done as $\frac{\sin x}{x}$ (the imaginary part of the remaining two integrals) is symmetric over the y-axis and its integral from 0 to ∞ would then be equal to the imaginary part of our answer divided by 2. Namely, we have $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ as desired. ■

Claim (added by Felix, for more practice)

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}.$$

Proof. We first show that $\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$ as follows:

$$\left(\int_{-\infty}^\infty e^{-x^2}dx\right)^2 = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(y^2+z^2)}dydz = \int_0^{2\pi} \int_0^\infty re^{-r^2}drd\theta = \pi,$$

with $y = r\cos\theta$ and $z = r\sin\theta$.

Using what we showed above, we integrate $f(z) = e^{-z^2}$ on the contour defined by $C = \{Re^{it} : 0 < t < \pi/4\}$. Observe that we can bound the integral by

$$0 \le \left|\int_C f(z)dz\right| \le \int_0^{\pi/4} e^{-R^2(\cos^2(t)-\sin^2(t))}Rdt = \int_0^{\pi/4} e^{-R^2\cos(2t)}Rdt,$$

but also (from brute force bounding) we know that $R\int_0^{\pi/4} e^{-R^2\cos(2t)}dt \to 0$ as $R \to \infty$ $(*)$. This is good, because we can invoke Cauchy's theorem to see that $\int_{C'} f(z)$ for $C' = \{re^{it} : 0 < r < R, 0 < t < \pi/4\} = \gamma_1 + \gamma_2 + \gamma_3$ (where γ_1 is the path from 0 to R, γ_2 is the eighths-circle, and γ_3 is the line from $Re^{i\pi/4}$ back to 0), a closed boundary, is zero. Letting $R \to \infty$, we saw from above that $\int_{\gamma_2} f(z) = 0$, so what we have left is (I'll drop the $f(z)$'s to make writing easier)

$$\int_{\gamma_1} + \int_{\gamma_3} = \int_0^\infty e^{-z^2} + \int_{\gamma_3} = \sqrt{\pi}/2 + \int_{\gamma_3} = 0 \Rightarrow \sqrt{\pi}/2 = -\int_{\gamma_3}.$$

So to finish, we evaluate $-\int_{\gamma_3}$:

$$-\int_{\gamma_3} = \int_0^R e^{-(e^{i\pi/4}t)^2}e^{i\pi/4}dt = e^{i\pi/4}\int_0^R e^{-it^2} = e^{i\pi/4}\int_0^R \cos(t^2) - i\sin(t^2)dt.$$

Thus we see that

$$\sqrt{\pi}/2 = e^{i\pi/4}\int_0^\infty \cos(t^2) - i\sin(t^2)dt \Rightarrow \int_0^\infty \cos(t^2) - i\sin(t^2)dt = \frac{\sqrt{\pi}/2}{\sqrt{2}/2 + \sqrt{2}/2i} = \frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}i.$$

Equating real and imaginary parts, we see that $\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}$, as desired.

$(*)$ *Brute force bounding:* For more exercise, we'll explicitly show that

$$R\int_0^{\pi/4} e^{-R^2\cos(2t)}dt \to 0$$

as $R \to \infty$ here. We have

$$R\int_0^{\pi/4} e^{-R^2\cos(2t)}dt = R\int_0^{\pi/4} e^{-R^2\sin(2t)}dt = R\int_0^{\pi/6} e^{-R^2\sin(2t)}dt + R\int_{\pi/6}^{\pi/4} e^{-R^2\sin(2t)}dt.$$

Then, since we can bound $\sin x \geq (1/2)x$ for $0 \leq x \leq 1/3$ ($\frac{d}{dx}\sin x \geq 1/2$ for $x \leq 1/3$) and $\sin x \leq 1/2$ for $\pi/6 \leq \pi/4$, we see that

$$R\int_0^{\pi/6} e^{-R^2\sin(2t)}dt + R\int_{\pi/6}^{\pi/4} e^{-R^2\sin(2t)}dt \leq \int_0^{\pi/6} Re^{-R^2 t}dt + \int_{\pi/6}^{\pi/4} Re^{-R^2/2}dt$$

$$= \frac{\pi}{12}Re^{-R^2/2} + \frac{1}{R}(1 - e^{\frac{-\pi R^2}{6}}) \to 0,$$

as $R \to \infty$. Integrals are fun stuff. ■

Now let's turn to other theorems we can prove using the Cauchy integral formula.

Theorem. (Liouville) A bounded entire function $f(z)$ is constant.

Proof. Suppose $|f(z)| \leq M$ for all z. Consider $f(a), f(b)$ for $|a|, |b| < R$. We would like to show that $f(a) = f(b)$. Let $C_0(R)$ be the circle of radois R about 0. Then

$$|f(a) - f(b)| = \frac{1}{2\pi}\left|\int_{C_0(R)} \frac{f(z)}{z-a}dz - \int_{C_0(R)} \frac{f(z)}{z-b}dz\right|$$

and

$$\frac{f(z)}{z-a} - \frac{f(z)}{z-b} = \frac{(b-a)f(z)}{(z-a)(z-b)}.$$

For $|b-a| \leq 2R$, we can also get

$$\frac{1}{2\pi}\left|\int_{C_0(R)} \frac{f(z)}{z-a}dz - \int_{C_0(R)} \frac{f(z)}{z-b}dz\right| = \frac{1}{2\pi}\left|\int_{C_0(R)} \frac{f(z)(b-a)}{(z-a)(z-b)}dz\right| \leq \frac{1}{2\pi}\frac{2\pi RM|b-a|}{\frac{1}{4}R^2}$$

$$= \frac{4M|b-a|}{R} \to 0$$

as $R \to \infty$. ■

Let's look at polynomials $P(z) = \sum_{k=0}^n c_k z^k, c_n \neq 0$. What can we say about them?

Claim. $\lim_{z\to\infty} P(z) = \infty$ in the sense that $\forall M, \exists r : \forall |z| > R$, $|P(z)| > M$.

Proof. Should be obvious. ■

Theorem. (Fundamental Theorem of Algebra) Given a nonconstant polynomial $P(z) = \sum_{k=0}^n c_k z^k$, $c_n \neq 0$, $n \geq 1$, $\exists w \in \mathbb{C} : p(w) = 0$.

Proof. Suppose not. Then $\frac{1}{P(z)} =: f(z)$ is a bounded entire function. Choose R such that

$$\frac{1}{R} > \sum_{k=0}^{n-1} \frac{|c_k|}{|c_n|}$$

and $R > 1$ so that for $|z| > R$ we have

$$\frac{1}{P(z)} \geq \frac{1}{\frac{1}{2}R|c_n|}$$

Note that, in obtaining this inequality, we noted that the highest power of any polynomial dominates in the limit $|z| \to \infty$. So $f(z)$ is bounded by C for $|z| > R$, and it's similarly bounded on any compact region $D_0(R)$; thus $f(z)$ is constant by Liouville's theorem. ■

Theorem. (Generalization of Liouville's Theorem) If $|f(z)| \leq A + B|z|^N$ and $f(z)$ is entire, then $f(z)$ is a polynomial of degree $\leq N$.

Proof. By induction. The base case is Liouville's theorem. Now define

$$g(z) = \begin{cases} \frac{f(z)-f(\alpha)}{z-\alpha} & z \neq \alpha \\ f'(\alpha) & z = \alpha \end{cases}$$

for any α. Notice that $g(z)$ is holomorphic, since we saw $\exists G(z)$ that is holomorphic with $G'(z) = g(z)$, and we showed that holomorphic implies C^∞ so $g(z)$ is also holomorphic. Notice further that $|g(z)| \leq C + D|z|^{N-1}$.

Addendum by Felix: We can see the last statement more clearly as follows. If to the contrary $|g(z)| > D|z|^{n-1}$, then taking limits as $|z| \to \infty$ we see that the growth rate (growth order $\geq N$) would not match with $\frac{f(z)-f(\alpha)}{z-a}$ or $f'(\alpha)$ (growth order $N-1$).

(Also, as said in lecture, we can see the last statement as follows: look inside a large ball, then outside the large ball. Inside, it's bounded by some constant; outside, because of the induction hypothesis, it has the form $D|z|^{N-1}$.)

So $g(z)$ is a polynomial of degree $\leq N-1$ by induction and $f(z) = g(z)(z-\alpha)$ is a polynomial of degree $\leq N$. This completes the inductive step. ■

Uniqueness theorem. It turns out that holomorphic functions are determined up to their values in a certain region; this is the content of the uniqueness theorem.

Definition. A *region* Ω in $\mathbb{C}$ is a polygonally connected open set.

Proposition. (uniqueness theorem) If f and g are holomorphic on a region $D \subseteq \mathbb{C}$ with $f(z) = g(z)$ on a collection of points accumulating somewhere, then $f(z) = g(z)$ on D.

Proof. Let $A = \{z \in D : \exists$ infinitely many points w in any disk containing z s.t. $f(w) = g(w)\}$. Let $B = D \backslash A$. We claim that both A and B are open.

To show that B is open, note that if $z \in B$, then $\exists$ a disk containing z with only finitely many such w's. Let $\delta < \lim_{such\ w, w \neq z} |z|$. Then every point in the open disk of radius δ is in B.

To show that A is open, suppose $z \in A$. In D, $\exists$ a disk of radius r containing z. On that disk, both f and g being holomorphic implies that both f and g have convergent power series expansions centered at z of radius of convergence $\geq r$. So by uniqueness for power series, $f = g$ on this disk, i.e. every point in this disk in in A.

So we have that A and B are both open, $A \cup B = D$, and $A \cap B = \emptyset$. By elementary topology, one must be the empty set. ■

It turns out that polynomials are the only entire functions that go to infinity with z in the limit. Geometrically, this means that if we consider the Riemannian sphere $\hat{\mathbb{C}}$ with

$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ entire, then $f : \infty \mapsto a$ for a finite implies that $f(z)$ is bounded, i.e. a constant. If $f : \infty \mapsto \infty$, then $f(z)$ is a polynomial.

Proposition. If $f(z)$ is entire and $f(z) \to \infty$ as $z \to \infty$, then $f(z)$ is a polynomial.

Proof. The zeroes of f all lie in a disk of some radius, call it R. There are only finitely many zeroes, else they accumulate somewhere and then $f = 0$. Now consider

$$g(z) = \frac{f(z)}{(z-z_1)^{m_1}(z-z_2)^{m_2}...(z-z_k)^{m_k}}.$$

Note that $g(z)$ has no zeroes and is entire. $\lim f(z) = \infty \implies |g(z)| \geq \frac{1}{|z|^m}$, where $m = \sum m_i$. So $\frac{1}{g(z)}$ is entire and bounded, and by Liouville's theorem it must be a constant. Thus $f(z)$ is a polynomial. ■

We take the time to state one last important result, the *mean value theorem*, without proof:

Theorem. (mean value) If $f(z)$ is holomorphic on $D_\alpha(R)$, then $\forall r : 0 < r < R$, we have

$$f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\theta})d\theta.$$

Corollary. This is true as well for $u = \text{Re}(f)$, $v = \text{Im}(f)$ in place of f.

Proof. Use the integral formula. ■

CHAPTER 7

MEAN VALUE THEOREM AND MAXIMUM MODULUS PRINCIPLE

Today, we'll be talking about the *maximum modulus principle*, the *mean value theorem*, and some topological concerns. Recall from last time that we had the mean value theorem:

Theorem. (mean value) For f holomorphic in $D_R(z)$, and $r < R$,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta.$$

We also have the *maximum modulus principle*, which is one of the most useful theorems in complex analysis:

Theorem. (maximum modulus) Let f be holomorphic in $D_R(z)$. If $|f|$ takes a local maximum value at z in the interior, then f is constant.

Proof. Use the mean value theorem. Alternatively, an equivalent formulation (Ahlfors) is as follows: if f is holomorphic and nonconstant in $D_R(z)$, then its absolute value $|f|$ has no maximum in $D_R(z)$.

To show this, suppose that $w = f(z)$ is any value in $D_R(z)$, so that the neighborhood $D_\epsilon(w)$ is contained in the image of $D_R(z)$. In this neighborhood, there are points of modulus $> |w|$ so $|f(z)|$ is not the maximum of $|f|$. ■

Claim. If $|f|$ is constant and holomorphic, then f is constant on a disk (or region).

Proof. Use the CR equations. ■

Corollary. (minimum modulus) If f is holomorphic and achieves a minimum of $|f(z)|$ in a disk, then $f(z) = 0$ there.

Proof. If $f(z) \neq 0$ at a minumum then $1/f$ is holomorphic in a neighborhood of z. Then use the maximum modulus theorem.

Likewise, we can also use power series to prove this. ■

We went over another proof of the maximum modulus theorem in class using power series; it may be nice to go over this proof as well (which is, again, rather long); see page 87 in the textbook.

Corollary. If f is holomorphic in $D_R(z)$, then on $\overline{D_r(z)}$, $r < R$, the maximum value of $|f(z)|$ on $\overline{D_r(z)}$ is achieved on the boundary.

Proof. Consult the textbook. ■

The following corollary is more general than the above:

Corollary. If f is continuous on a closed bounded set E and holomorphic in its interior $E - \partial E$, then the maximum of $|f|$ is attained on ∂E.

Proof. (Ahlfors) Since E is compact, $|f(z)|$ has a maximum on E. Suppose that it is attained at z_0; if $z_0 \in \partial E$, we're done. Else if z_0 is an interior point, then $|f(z_0)|$ is also the maximum of $|f(z)|$ in a disk $|z - z_0| < \delta$ contained in E. But this is not possible unless $f(z)$ is constant in the component of the interior of E which contains z_0. It follows by continuity that $|f(z)|$ is equal to its maximum on the whole boundary of that component. This boundary is not empty and is contained in the boundary of E, so the maximum is always attained at a boundary point. ■

Proposition ("anti-calculus") If $|f(z)|$ achieves its max value on $\overline{D_r(z)}$ at w, then $f'(w) \neq 0$.

Proof. Sketched in class; see page 88 of the textbook. ■

We will use the maximum modulus principle to prove the following theorem:

Theorem. (open mapping) The map of an open set under a non-constant holomorphic map $f : U \to \mathbb{C}$, $U \subseteq \mathbb{C}$, is open. Note that f continuous at z means that $\forall \epsilon > 0, \exists \delta > 0 : f(B_\delta(z)) \subseteq B_\epsilon(f(z))$.

Sketch of another proof. If $f'(z) \neq 0$, then

$$J_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

is nonsingular. Then $\det J_f = u_x v_y - u_y v_x = u_x^2 + v_x^2 = |f'(z)|^2 \neq 0$. The inverse function theorem gives the result.

If $f'(z) = 0$, then consider $f(B_\delta(0)) = B_{f(\delta)}(0)$. Write $f(z) = z^k(c_k + c_{k+1}z + ...)$. When f has a zero of order k at 0, i.e. $f(0) = f'(0) = ... = f^{(k-1)}(0) = 0$, it turns out that $f(z) = [g(z)]^k$, $g(z) = Az + ...$ i.e. $g'(0) \neq 0$ (but we cannot prove this as of yet). □

Our proof. See page 93 of the textbook.

Topology toward general closed curve theorem. We would like to take the time to revisit some topological concerns about the complex plane. In particular, recall that we defined the notions of *polygonally connected* and *polygonally simply connected*:

Definition. A set $U \subseteq \mathbb{C}$ is *polygonally connected* if $\forall p, q \in U$, there exists any piece-wise linear path from p to q.

A polygonally connected set is connected, but a connected set may not be polygonally connected (what is an example?). For open sets, they are equivalent:

Proposition. If $U \subseteq \mathbb{C}$ is open, then U is connected iff it's polygonally connected.

Definition. A set $U \subseteq \mathbb{C}$ is *polygonally simply connected* (or, in the topological sense, *simply connected*) if for any closed polygonal curve is the boundary of a union of polygons in U.

Definition. A *region* Ω in $\mathbb{C}$ is an open connected (or polygonally connected) subset of $\mathbb{C}$.

If $E(U)$ denotes the space of directed edges (for purpose of integrating over) in the region U, then by composing elements in $E(U)$ to form a closed contour γ we know from Cauchy's integral theorem that $\int_\gamma f(z)dz = 0$ for f holomorphic.

The notion of connectedness goes into topology, involving such topics as homotopy and homology. For those of you that are familiar with topology, a simply connected set has trivial fundamantal group (e.g. every loop is homotopic to a constant). The Cauchy integral formula also has a more general, homology version involving winding numbers.

CHAPTER 8

GENERALIZED CLOSED CURVE THEOREM AND MORERA'S THEOREM

Today we'll talk about the general closed curve theorem and Morera's theorem.

Theorem. (general closed curve theorem) If D is a region (open, polygonally connected subset) in $\mathbb{C}$ and $\hat{\mathbb{C}} - D$ is connected, then the closed curve theorem holds, i.e.

$$\int_C f(z)dz = 0$$

for f holomorphic on D and C a contour curve in D.

Definition. A set S in $\hat{\mathbb{C}} \cong S^2$ is *connected* if $\nexists A, B$ open in $\hat{\mathbb{C}}$ with $A \cup B \supset S$ and $A \cap B \neq \emptyset$.

The method for proving the general closed curve theorem is:
1. $\hat{\mathbb{C}} - D$ is connected implies D p.s.c.
2. D p.s.c. implies closed curve theorem holds (see textbook for pictures).

To show this, we'll need to go back to last time: first let U be a region in $\mathbb{C}$. Then define

- $T(U) := \{\sum_{i=1}^n a_i T_i\}$, for T_i a triangle in U, $a_i \in \mathbb{Z}$, and any n.
- $E(U) := \{\sum_{i=1}^n a_i E_i\}$, for E_i an edge in U (recall that an *edge* means a pair of points $\{P, Q\}$ with $P \neq Q$).
- $P(U) := \{\sum_{i=1}^n a_i P_i\}$, for P_i a point in U.
- $\partial T := E_1 + E_2 + E_3$ for T a triangle.
- $\partial E = Q - P$; $\partial : E(U) \to P(U)$ and $\partial : T(U) \to E(U)$.
- $e \in E(U)$ is *closed* if $\partial e = 0$, e.g. $e = E_1 + ... + E_n$ is closed iff $\pm E_i$ form loops.

Definition. e_1 is *homologous* to e_2, denoted $e_1 \sim e_2$, if $\exists t : \partial t = e_1 - e_2$. This is an equivalence relation.

Definition. U is p.s.c. if e is closed, e.g. e is null-homologous, which means $e \sim 0$.

Proposition. If U is convex in $\mathbb{C}$, then U is p.s.c.

Proof. Prove this yourself, by dividing the set into triangles. ■

Proposition. If U is p.s.c., then the closed curve theorem holds.

Proof. Note that the antiderivative theorem holds. Let e be a (not necessarily closed) piecewise linear path from P to Q, $\partial e = Q - P$. Then

$$\int_e f(z)dz = \sum a_i \int_{E_i} f(z)dz.$$

Define $F(z) = \int_{z_0}^z f(z)dz$. This makes sense because

$$\int_{e_1} f(z)dz = \int_{e_2} f(z)dz.$$

if $e_1 \sim e_2$, since then

$$\int_{e_1-e_2} f(z)dz = \sum b_j \int_\Gamma f(z)dz$$

where Γ is the boundary of the triangle and the integral is zero. So the antiderivative theorem implies the closed curve theorem as before:

$$\int_C f(z) = f(\text{final}) - f(\text{initial}) = 0,$$

as desired. ■

We'll introduce Morera's theorem since we have some time. First some topology:

Proposition. If U is an open subset in $\mathbb{C}$, U is polygonally connected iff U is connected.

Claim. Let D be a region in $\mathbb{C} \subset \hat{\mathbb{C}}$. If $\hat{\mathbb{C}} - D$ is connected, then D is p.s.c.

Theorem. (Morera) If $f(z)$ is continuous on a region D and $\int_\Gamma f(z)dz = 0$ for Γ a triangle, then $f(z)$ is holomorphic.

Proof. See textbook. ■

Morera's theorem is quite useful. Some of the uses are as follows:

1. Showing $\lim_{n\to\infty} f_n = f$ is holomorphic.
2. Showing that some functions defined by sums or integrals are holomorphic, e.g. the *Riemann zeta function* and the *gamma function*:

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$$

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} dt$$

3. Showing functions that are continuous and holomorphic on all but some set (e.g. some point) are holomorphic.

Proposition. Suppose $\{f_n\}$ is holomorphic in an open set D, and $f_n \to f$ uniformly on compact subsets of D. Then $f = \lim_{n\to\infty} f_n$ is holomorphic.

Proof. It suffices to work in $B_\epsilon(\alpha) \subset D$. $\overline{B_{\epsilon/2}(\alpha)}$ is compact, so by assumption $f_n \to f$ is uniform here. Note that $f_n \to f$ uniformly implies f is continuous. Also

$$\int_\Gamma f(z)dz = \int_\Gamma \lim_{n\to\infty} f_n(z)dz = \lim_{n\to\infty} \int_\Gamma f_n(z)dz = 0$$

(Uniform convergence implies that we can swap $\int$ and lim, as we all know from elementary analysis.) By Morera's theorem, f is holomorphic. ■

The Riemann zeta function. The *zeta function* $\zeta(z)$ is defined as

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z},$$

for $\text{Re}(z) > 1$. We claim that $\zeta(z)$ is convergent for $\text{Re}(z) > 1$, and indeed, is uniformly convergent for $\text{Re}(z) > R > 1$. (Hence by Morera's theorem $\zeta(z)$ is holomorphic.)

Proof of claim. First note that $\frac{1}{n^z} = e^{-(\log n)z}$. We first show uniform convergence. Let

$$f_N(z) := \sum_{n=1}^{N} \frac{1}{n^z} = \sum_{n=1}^{N} e^{-(\log n)z}.$$

We bound $\sum_{n=N+1}^{\infty} |e^{-(\log n)z}|$ for the whole region $\text{Re}(z) \geq R > 1$:

$$|e^{-(\log n)z}| = |e^{-\log n} e^{Re(z)}| \leq R|e^{-\log n}|$$

So we're left with $\sum_{n=N+1}^{\infty} \frac{1}{n^R}$, $R > 1$. The integral test enures convergence, telling us that the tail end is small. ■

CHAPTER 9

MORERA'S THEOREM, SINGULARITIES, AND LAURENT EXPANSIONS

Let's finish the topology stuff, talk about Morera's theorem, and then move on to singularies and Laurent expansions.

First, what does it mean to fix an open set in $\hat{\mathbb{C}} = S^2$?

Definition. A neighborhood of ∞ is given by $\{\infty\} \cup \{z : |z| > R\}$. Then $S \subset \hat{\mathbb{C}}$ is *open* if:
(1) $S \cap \mathbb{C}$ is open in $\mathbb{C}$
(2) If $\infty \in S$, then $\exists R : \{z : |z| > R\} \subset S$.

Lemma. If P is a closed polygonal path which doesn't cross itself (i.e. P is a piecewise linear map from an interval that is injective except at the endpoints), then $\exists A, B$ open in $\hat{\mathbb{C}}$ s.t. $A \cup B = \hat{\mathbb{C}} - P$ and $A \cap B = \emptyset$.

Proof. Note $\exists R : P \cap \{|z| > R\} = \emptyset$ for $z \in \mathbb{C} - P$. Take L, a line segment from z to a point in $S = \{|z| > R\}$, and count the number of times P crosses L to R and the number of times P crosses R to L. This makes sense because $P \cap L$ is a finite union of edges of P and points. Let $z \in A$ if the count is odd; else $z \in B$.

Exercise. Show that $z \in A$ or B is well-defined independent of the choice of L. *Hint: use wedges.* Conclude that nearby points are in the same set. ■

Integration over other regions. We use $\mathbb{C} - \mathbb{R}_{\leq 0}$ to denote the complex plane with a ray on the real axis missing. Note that $\hat{\mathbb{C}} - (\mathbb{C} - \mathbb{R}_{\leq 0}) = \{\infty\} \cup \mathbb{R}_{\leq 0}$ is connected. *Proof: any interval is connected.* Thus we see that $\mathbb{C} - \mathbb{R}_{\leq 0}$ is polygonally simply connected (PSC), and we can integrate along any closed curve in this region by applying the closed curve theorem.

How about $\mathbb{C}^* = \mathbb{C} - \{0\}$? Applying the closed curve theorem for the difference contour

$C_2 - C_1$, we see that

$$\int_{C_2 - C_1} f(z)dz = 0$$

for $f(z)$ holomorphic on $\mathbb{C}^*$. An image of C_2 (the outside counterclockwise contour) and C_1 (the inside counterclockwise contour) is as follows:

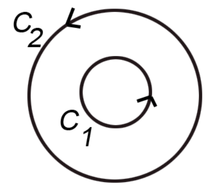


Diagram 1: C_1 and C_2 in $\mathbb{C}^*$.

Now let's pick off with Morera's theorem, which as you recall states

Theorem. (Morera) If f is continuous on an open set $\mathbb{D}$ and $\int_\Gamma f(z)dz = 0$ for Γ being the boundary of a triangle in D, then $f(z)$ is holomorphic.

Corollary. If $f_n(z)$ is holomorphic on D open and $f_n \to f$ uniformly on compact subsets of D, then $f(z)$ is holomorphic on D.

Proof. Proved last time. ■

The Riemann zeta function. The *Riemann zeta function* is a particular form of a *Dirichlet series* or *L-function* given by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

We claim that $\zeta(z)$ is holomorphic on $\mathrm{Re}(z) > 1$, and indeed we showed this last time. □

The gamma function. The *gamma function*, which is useful in probability and number theory, is given by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

We claim that $\Gamma(z)$ is holomorphic for $\mathrm{Re}(z) > 0$.

Remark. $\Gamma(z)$ is a natural extension of the usual factorial function in that $\Gamma(n+1) = n!$. Show this yourself using integration by parts and induction.

Proof of the claim. We show that $\int_0^\infty e^{-t} t^z dt$ is holomorphic in $\mathrm{Re}(z) > -1$. To that extent we work in $-1 < R_1 \leq \mathrm{Re}(z) \leq R_2$; note that any compact subset of $\{z : \mathrm{Re}(z) > -1\}$ is contained in one of these sets. Define

$$f_n(z) := \int_{1/n}^1 e^{-t} t^z dt + \int_1^n e^{-t} t^z dt,$$

which we will show converges uniformly. For $t \geq 1$, we can bound

$$|e^{-t} t^z| \leq e^{-t} t^{\mathrm{Re}(z)} \leq e^{-t} t^{R_2}$$

because

$$t^z = e^{z \log t} = e^{(\mathrm{Re}(z)) \log t} e^{i \mathrm{Im}(z) \log z},$$

the last term of which has norm 1. Then

$$|f_n(z)/2 - f(z)/2| \leq \int_n^\infty e^{-t} t^{R_2} dt,$$

which converges for $n = 1$. So $\forall \epsilon > 0, \exists N : \forall n > N$,

$$\int_n^\infty e^{-t} t^{R_2} dt < \epsilon/2.$$

For $t \leq 1$, $|e^{-t} t^z| = e^{-t} t^{\mathrm{Re}(z)} \leq e^{-t} t^{R_1}$ and

$$\left| \int_{1/n}^1 e^{-t} t^z dt - \int_0^1 e^{-t} t^z dt \right| = \left| \int_0^{1/n} e^{-t} t^z dt \right| \leq \int_0^{1/n} e^{-t} t^{R_1} dt$$

for $R_1 > -1$. Let $s = 1/t$ and $ds = -\frac{1}{t^2} dt$, so that we get $\int_n^\infty e^{-1/s} s^{-2-R_1} ds$. We need $-2 - R_1 < -1$, i.e. $-1 < R_1$. Thus the sequence converge uniformly, so $\forall \epsilon > 0, \exists N : \forall n > N, \int_n^\infty e^{-1/s} s^{-2-R_1} ds < \epsilon/2$. □

Corollary. If f is continuous on D and f is holomorphic on D except at $\alpha \in D$, then f is holomorphic on D.

Proof. Let $\Gamma = \partial T$. Then

$$\int_\Gamma f(z) dz = 0$$

if $0 \notin T$. If $\alpha \in T$, we split the contour so that α is located at the corners of the triangle. If α is at a corner, then

$$0 = \lim_{l \to \infty} \int_{\Gamma_l} f(z) dz = \int_\Gamma f(z) dz$$

by continuity of f. ■

Singularities of functions. (Newman and Bak §9) Singularities of functions will be important when trying to access their behaviors around certain points so that we can define things like Laurent series or perform residue calculus. Needless to say, an understanding of the different types of singularities and how they pertain to classes of functions is important for complex anaysis in general.

Definition. Given $\alpha \in \mathbb{C}$, a *deleted neighborhood* D of α is a neighborhood $-\{\alpha\}$, e.g. $\{z : 0 < |z - \alpha| < \epsilon\}$. f is said to have an *isolated singularity at* α if f is defined and holomorphic on a deleted neighborhood of α.

Types of singularities. For D a deleted neighborhood of α, we have the following types of singularities:

1. *Removable singularity*: $\exists g(z)$ on $D \cup \{\alpha\}$ holomorphic with $g(z) = f(z)$ on D.
2. *Pole of order k*: $\exists A(z), B(z)$ holomorphic on $D \cup \{\alpha\}$ s.t. $A(\alpha) \neq 0, B(\alpha) = 0$, $f(z) = \frac{A(z)}{B(z)}$ and $B(z)$ has a zero of order k at α. Recall that we can expand $B(z)$ into a power series as $B(z) = c_k(z-\alpha)^k + c_{k+1}(z-\alpha)^{k+1}$.
3. *Essential singularity:* Neither of the above.

The following proposition allows us to recognize if something has a removable singularity.

Proposition. If
$$\lim_{z\to\alpha} f(z)(z-\alpha)$$
exists and is equal to 0, then $f(z)$ has a removable singularity at α.

Proof. Define $h(z)$ as
$$h(z) = \begin{cases} f(z)(z-\alpha) & z \neq \alpha \\ 0 & z = \alpha. \end{cases}$$
This is continuous, and holomorphic except at α. So $h(z)$ is holomorphic in a neighborhood of α by the corollary to Morera's theorem. Moreover, we see that $h(\alpha) = 0$, so $g(z) = \frac{h(z)}{z-\alpha}$ is holomorphic and equal to $f(z)$ away from α. (If $h(\alpha) = 0$ and h is holomorphic, then $\frac{h(z)}{z-\alpha}$ is holomorphic; one can prove this easily using power series.) ■

Likewise, the following proposition allows us to recognize a pole of f.

Proposition. Suppose $f(z)$ is holomorphic in a deleted neighborhood of α and $\exists n$:
$$\lim_{z\to\alpha} f(z)(z-\alpha)^n = 0.$$
Then letting $k = 1$ be the least such n, $f(z)$ has a pole of order k at α.

Proof. Again, let
$$h(z) = \begin{cases} f(z)(z-\alpha)^{k+1} & z \neq \alpha \\ 0 & z = \alpha. \end{cases}$$
$h(z)$ is continuous on a neighborhood and holomorphic on a deleted neighborhood, so it's holomorphic on a neighborhood of α again by the corollary to Morera's theorem. $h(\alpha) = 0$ so $g(z) = \frac{h(z)}{z-\alpha}$ is holomorphic in $D \cup \{\alpha\}$ and $g(z) = f(z)(z-\alpha)^k$ on D. Then $f(z) = \frac{g(z)}{(z-\alpha)^k}$. (Note $\lim_{z\to\alpha} g(z) \neq 0$ by assumption; also notice it exists). ■

For essential singularities, we've shown that it must be the case that $\lim_{z\to\alpha} f(z)(z-\alpha)^n$ does not exist for all n. Notice as well the following:

- If f is bounded in a deleted neighborhood, then f has a removable singularity.
- If $f < C_1\frac{1}{|z|^N} + C_2\frac{1}{|z|^{N-1}} + ... + C$ as $z \to 0$, then f has a pole of order $\leq N$.

We have a theorem for essential singularities, which tells us that the image of any deleted neighborhood of an essential singularity under a holomorphic function is necessarily dense in the complex plane:

Theorem. (Casorati-Weierstrass) If f has an essential singularity at α and D is any deleted neighborhood of α, then $f(D) = \{f(z) : z \in D\}$ is dense in $\mathbb{C}$ (i.e for any $\epsilon > 0$, $w \in \mathbb{C}$, $B_\epsilon(w) \cap f(D) \neq \emptyset$).

Proof. Suppose not. Then $\exists B_\epsilon(w)$ with $B_\epsilon(w) \cap f(D) = \emptyset$. This means $|f(z) - w| > \epsilon$, or $\frac{1}{|f(z)-w|} < \frac{1}{\epsilon}$. So $\frac{1}{f(z)-w}$ is defined on D, a deleted neighborhood of α, and bounded there. Thus $g(z) = \frac{1}{f(z)-w}$ is holomorphic on a neighborhood of α.

Now consider $\frac{1}{g(z)} = f(z) - w$, so that $f(z) = \frac{wg(z)+1}{g(z)}$ is a ratio of two holomorphic functions; we see that the singularity must be removable, or $f(z)$ must have a pole.

Exercise left to the reader: prove the converse of the theorem. ■

Example. Consider $f(z) = e^{1/z}$ defined on C^*, which has an essential singularity at $0 \in \mathbb{C}$. We claim that $f(B_\epsilon(0) - \{0\})$ is $\mathbb{C}^*$.

To see this, note that $f(z)$ is the composition of $\frac{1}{z}$ and then e^z. Under $\frac{1}{z}$, $B_\epsilon(0) - \{0\}$ gets inverted to fill $\mathbb{C}$ outside the ball. e^z is $2\pi i$-periodic, so we can draw horizontal lines in the complex plane at $\pi + 2\pi n$, $n \in \mathbb{Z}$. Then the interior of any strip created by these horizontal lines gets mapped under e^z to $\mathbb{C} - \mathbb{R}_{\geq 0}$. Our claim is that there are strips outside $B_0(1/\epsilon)$ so that $f(B_\epsilon(0) - \{0\}) = \mathbb{C}^*$. □

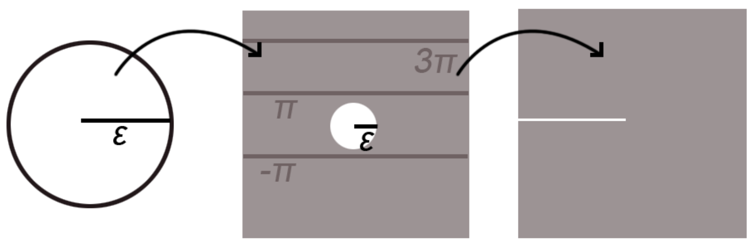


Diagram 2: The mappings $\frac{1}{z} : B_\epsilon(0) - \{0\} \to \mathbb{C} - (B_\epsilon(0) - \{0\})$ *and* $e^z : \{z : \pi \leq Im(z) \leq 3\pi\} \to \mathbb{C} - \mathbb{R}_{\geq 0}$.

Laurent expansions. Simply put, Laurent expansions are Taylor series of the form $f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$.

Theorem. If $f(z)$ is holomorphic on the annulus $A(R_1, R_2) := \{z : R_1 < |z| < R_2\}$, then $f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$ converges on all of $A(R_1, R_2)$. (Taking $R_1 = 0$ and $R_2 = \infty$ is also fine.)

We say $\sum_{k=-\infty}^{\infty} a_k = L$ if $\sum_{k=0}^{\infty} a_k$ exists, $\sum_{-\infty}^{-1} a_k = \sum_{l=1}^{\infty} a_{-l}$ exists, and their sum is L.

Proposition. If

$$\frac{1}{\overline{\lim}_{k\to\infty}|c_k|^{1/k}} \geq R_2 \text{ and } \overline{\lim_{k\to\infty}}|c_{-k}|^{1/k} \leq R_1,$$

then $f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$ converges and is holomorphic on $A(R_1, R_2)$.

Proof. Write

$$f(z) = \sum_{k=0}^{\infty} c_k z^k + \sum_{k=1}^{\infty} c_{-k} \left(\frac{1}{z}\right)^k.$$

The first sum is convergent for $|z| \leq \frac{1}{\overline{\lim}_{k\to\infty}|c_k|^{1/k}}$. The second sum is convergent for $|\frac{1}{z} \leq \frac{1}{\overline{\lim}_{k\to\infty}|c_{-k}|^{1/k}} \implies |z| \geq \overline{\lim}_{k\to\infty}|c_{-k}|^{1/k}$. The first sum is clearly holomorphic. The second sum is holomorphic as a function of $1/z$; e.g. $g(z) = \sum_{k=1}^{\infty} c_{-k} w^k$ is holomorphic for $|w| \leq \frac{1}{R_1}$. The second sum is a composition of $\frac{1}{z}$ and g, so by the chain rule it's also holomorphic. So f is holomorphic. ■

Example. The following Laurent series expansion converges on $A(0, \infty)$:

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + ...$$

$\square$

Example. Near $z = 0$ (pole of order 1), we have

$$\frac{1}{z(1+z)^2} = \frac{1}{z}\frac{1}{(1+z)^2} = \frac{1}{z}(1 - 2z + 3z^2 - 4z^3 + ...).$$

This is because $\frac{\partial}{\partial z}(\frac{1}{1+z}) = -\frac{1}{(1+z)^2}$ and $\frac{1}{1+z} = 1 - z + z^2 - z^3 + ...$ We can do the same near $z = -1$. $\square$

CHAPTER 10

MEROMORPHIC FUNCTIONS AND RESIDUES

Last time, we introduced the Laurent series for $f(z)$ as $f(z) = \sum_{-\infty}^{\infty} c_k(z-\alpha)^k$. This is defined on $A_\alpha(R_1, R_2)$, the annular region centered at α.

Theorem. If $f(z)$ is holomorphic on $A_\alpha(R_1, R_2)$, then $f(z) = \sum_{k=-\infty}^{\infty} c_k(z-\alpha)^k$ and the series converges on $A_\alpha(R_1, R_2)$.

Proof. Do this yourself by considering the function

$$g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & w \neq z \\ f'(w) & w = z \end{cases}$$

and using the closed curve theorem. ■

Remark. Laurent series are unique to every function. We define the *principal part* of $f(z) = \sum$ near α as the sum $\sum_{k=-\infty}^{-1} c_k(z-\alpha)^k$ and observe the following:

(1) α is a removable singularity iff $c_k = 0$ for $k < 0$.
(2) α is a pole of order n iff $c_k = 0$ for $k < -n$, $c_{-n} \neq 0$.
(3) α is an essential singularity iff $c_k \neq 0$ for infinitely many negative k (why?).

Here's an idea: at a pole, we can make the function $f(z)$ still be holomorphic if we expand its range to $\hat{\mathbb{C}}$, so that $f(z) = \infty$ there and holomorphically so.

Proposition. If $f(z)$ has a pole of order k at α, then $\frac{1}{f(z)}$ is holomorphic near α and has a pole of order k.

Proof. Write $f(z) = \frac{1}{(z-\alpha)^k} g(z)$. Then $g(z)$ holomorphic near α, $g(z) \neq 0$, so $\frac{1}{f(z)} = \frac{(z-\alpha)^k}{g(z)}$, $g(\alpha) \neq 0$. This is holomorphic on a neighborhood at α and has zero of order k. ■

Definition. f is *meromorphic* on D if $f(z)$ is holomorphic on D except at isolated singularities at which f has poles.

The proposition above can then be restated as: a function f being meromorphic in D really means that $f : \mathbb{D} \to \hat{\mathbb{C}}$ is holomorphic.

Theorem. Let f be meromorphic in $\mathbb{C}$ and at infinity, and suppose that $\lim_{z\to\infty} f(z) = \infty$ for some $w \in \mathbb{C}$. (The limit means $\forall M, \exists R : |f(z)| > M$ when $|z| > R$.) Then $f(z)$ is a rational function.

Theorem. If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic, then f is a rational function.

Residues of functions. By introducing residues, we seek to deduce the residue theorem as a means for computing integrals more effciently. Given a holomorphic function $f(z)$ in a deleted neighborhood of α, by the Laurent expansion we have $f(z) = \sum_{-\infty}^{\infty} c_k(z-\alpha)^k$. We define the *residue of $f(z)$ at the point α*, $\mathrm{Res}(f;\alpha)$, as

$$\boxed{\mathrm{Res}(f;\alpha) = c_{-1}.}$$

From Cauchy's integral formula, we notice that

$$\frac{1}{2\pi i}\int_{C_R(\alpha)} f(w)dw = c_{-1}.$$

Also recall the expression for c_k:

$$c_k = \frac{1}{2\pi i}\int_{C_R(0)} \frac{f(w)}{w^{k+1}}dw.$$

We see from this that the "limsup stuff" shows that Laurent series are uniformly convergent on $A(r_1, r_2)$ by comparison with geometric series.
On a ball of radius ϵ, we can compute the residue of f at α by using the formula:

$$\mathrm{Res}(f;\alpha) = c_{-1} = \frac{1}{2\pi i}\int_{C_\epsilon(0)} f(w-\alpha)dw = \frac{1}{2\pi i}\int_{C_\epsilon(\alpha)} f(w)dw.$$

There are a few ways to compute residues in general:
(0) Compute the integral $\int_{C_\epsilon(\alpha)} f(w)dw$.
(1) Compute the Laurent series for f centered at α.
(2) If f has a simple pole, then

$$c_{-1} = \lim_{z\to\alpha}(z-\alpha)f(z).$$

Also, if $f(z) = A(z)/B(z)$ for A a nonzero function, and B having a simple zero, then we have:

Lemma. $\mathrm{Res}(f;\alpha) = \frac{A(\alpha)}{B'(\alpha)}$.

We finish by working out an example. Consider

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx.$$

We can compute this by first taking a semicircle contour of radius R, for which there is one pole contained inside ($\alpha = i$). If we compute the residue at this pole, we can use the *residue theorem* to get the value of the integral (this is left to the reader as an exercise).

CHAPTER 11

WINDING NUMBERS AND CAUCHY'S INTEGRAL THEOREM

Today, we'll cover residues, winding numbers, Cauchy's residue theorem, and possibly the argument principle. This is in the textbook, §10.1, §10.2.

Last time, we defined resudies. For f holomorphic in a deleted neighborhood of α, we have

$$\operatorname{Res}(f;\alpha) = \frac{1}{2\pi i}\int_{C_\epsilon(\alpha)} f(z)dz = c_{-1},$$

where c_{-1} appears in the Laurent expansion for f around α, $f = \sum_{-\infty}^{\infty} c_n z^n$. Note that f is holomorphic in $C_\epsilon(\alpha)$ except for the point at α.

Example. $\operatorname{Res}(\frac{1}{1+z^2}; i)$

Method 1: Find the Laurent expansion about i, i.e. in terms of $z-i$.

$$\frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)} = \frac{-i}{2}\frac{1}{z-1} + \frac{1}{4} - \frac{1}{8}(z-i) + \ldots.$$

So $c_{-1} = \frac{-i}{2}$.

Method 2: If $f(z)$ has a simple pole at α, i.e. if $f(z) = \frac{A(z)}{B(z)}$ for $A(\alpha) \neq 0, B(\alpha) = 0, B'(\alpha) \neq 0$, then $\operatorname{Res}(f;\alpha) = \frac{A(\alpha)}{B'(\alpha)}$. Set $\operatorname{Res}(\frac{1}{1+z^2}; i) = \frac{1}{2i} = -i/2$. □

An application of this is as follows. Consider the integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \lim_{R\to\infty} \arctan(R) - \lim_{R\to\infty} \arctan(-R) = \pi/2 - (-\pi/2) = \pi.$$

We can also do this by contour integration over the contours C_1, C_2, C_3, where $C_1 = [-R, R]$, C_2 is the CCW semicircle connecting R to $-R$, and C_3 is the CCW circle omitting the point at i. Then

$$\int_{C_1+C_2-C_3} f(z)dz = 0$$

by the closed curve theorem, and by the estimation lemma we have

$$\left|\int_{C_2} \frac{1}{1+z^2}dz\right| \leq \pi R \cdot \frac{1}{R^2} = \frac{\pi}{R} \to 0$$

as $R \to \infty$. Also,

$$\int_{C_3} \frac{1}{1+z^2}dz = 2\pi i \text{Res}(\frac{1}{1+z^2}; i) = 2\pi i(-i/2) = \pi.$$

So

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^2}dx = \pi$$

after substituting for the values of the integrals over C_1, C_2, C_3.

Winding numbers. These intuitive give the number of times a path loops around a point, and are also dealt with in topology.

Definition. Let $\gamma : [0,1] \to \mathbb{C}$ be a contour curve with $\gamma(t) \neq \alpha \ \forall t$. Then the *winding number of* γ *about* α, denoted $n(\gamma, \alpha)$, is the value

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-\alpha}dz.$$

Theorem. $n(\gamma, \alpha) \in \mathbb{Z}$.

Proof. We have

$$\int_\gamma \frac{1}{z-\alpha}dz = \int_0^1 \frac{\gamma'(t)}{\gamma(t)-\alpha}dt.$$

Note that the integrand on the RHS is $\frac{d}{dt}\log(\gamma(t)-\alpha)$, with a caveat that log is a multivalued function. $\log z = \log|z| + i\text{Arg}(z)$ modulo $2\pi i$. We'll show that

$$\int_0^1 \frac{\gamma'(t)}{\gamma(t)-\alpha}dt = 2\pi i k$$

for some k. This $k = n(\gamma, \alpha)$. Let $F(s) = \int_1^s \frac{\gamma'(t)}{\gamma(t)-\alpha}dt$ (secretly, $\log(\gamma(s)-\alpha) - \log(\gamma(1)-\alpha)$. Then $F'(s) = \frac{\gamma'(s)}{\gamma(s)-\alpha}$. We claim that

$$F(s) = \frac{\gamma(s)-\alpha}{\gamma(0)-\alpha}.$$

To see this, let $G(s) = (\gamma(s)-\alpha)e^{-F(s)}$. Then $G'(s) = -\gamma'(s)e^{-F(s)} + \gamma'(s)e^{-F(s)} = 0$, so $G(s)$ is a constant. Check that at $s = 0$, we get $G(0) = (\gamma(0)-\alpha)e^{-F(0)}$, so indeed $e^{F(0)} = \frac{\gamma(0)-\alpha}{\gamma(t)-\alpha}$.

Now $e^{F(1)} = \frac{\gamma(1)-\alpha}{\gamma(t)-\alpha}$. $F(1) = \int_0^1 \frac{\gamma'(t)}{\gamma(t)-\alpha}dt$ and $\gamma(1) = \gamma(0)$, so $e^{F(1)} = 1$. So $F(1) = 2\pi i k$ for $k \in \mathbb{Z}$. ■

Corollary. (version of Jordan curve theorem) Let $\gamma : [0,1] \to \mathbb{C}$ be a closed curve. Then $\mathbb{C} - \text{image}(\gamma)$ is disconnected.

Let's call a simple closed curve one that doesn't intersect itself. For us, if γ has $n(\gamma, \alpha) = 0$ or 1, then we call γ "winding simple." The Jordan curve theorem shows that any closed curve has what we call an interior and exterior. For the interior region B, we would have $n = 1$. For the exterior region A, we would have $n = 0$.

Cauchy's residue theorem. It turns out that we can evaluate any contour integral of a holomorphic function by considering its residues and winding numbers.

Theorem. Let f be holomorphic on a region D with zero first homology (i.e. PSC, i.e. closed curve theorem holds for holomorphic functions on D) except for isolated singularities at $\alpha_1, ..., \alpha_n \in D$. Let γ be any contour curve in D. Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} n(\gamma, \alpha_k) \mathrm{Res}(f; \alpha_k).$$

Proof. The gist is to repeatedly subtract the principal parts of f at each α_k. This proof is a bit long to type up, so it is left as an exercise to the reader. ∎

CHAPTER 12

THE ARGUMENT PRINCIPLE

Recall from last time that we had *Cauchy's residue theorem:*

Theorem. Let D be a region with $\hat{\mathbb{C}} - D$ connected (i.e. closed curve theorem applies), i.e. simply connected (all loops contract to a point in D). Let f be holomorphic on D except at $\alpha_1, ..., \alpha_n$, and let γ be a contour curve in D missing these points. Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f; \alpha_k) n(\gamma, \alpha_k).$$

The argument principle. This is essentially relating the number of zeroes and poles of f to (# of times f winds around 0). Let γ be a *regular* contour curve, e.g. if $\forall \alpha \in \mathbb{C} - \text{im}(\gamma)$, either $n(\gamma, \alpha) = 0$ or $n(\gamma, \alpha) = 1$. Intuitively, this means that the points are either "inside of γ" or outside; the former is given by the set $\{z \in \mathbb{C} - \text{im}(\gamma) : n(\gamma, z) = 1\}$ and the latter is given by the set $\{z \in \mathbb{C} - \text{im}(\gamma) : n(\gamma, z) = 0\}$.

Theorem. If f is holomorphic in a simply connected region D, and γ a regular curve in D, then the following are equal:

(1) (# of zeroes of f with multiplicity) $-$ (# of poles of f with multiplicity inside γ)
(2) The winding number around zero of $f(\gamma(t))$, i.e. $n(f \circ \gamma, 0)$
(3) $\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz$

Remark. This assumes that for $f|_\gamma \neq 0$, there are no poles. The multiplicity is also called the *order*.

Proof. Let's do (2) $\iff$ (3). For $\gamma : [0, 1] \to \mathbb{C}$, we have

$$n(f \circ \gamma; 0) = \frac{1}{2\pi i} \int_{f \circ \gamma} \frac{1}{z} dz = \frac{1}{2\pi i} \int_0^1 \frac{\gamma'(t) f'(\gamma(t))}{f(\gamma(t))} dt = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz.$$

The integrand is actually $\frac{\partial}{\partial z}(\log f)$, which keeps track of how $\text{Arg}(f)$ changes and goes around γ.

(3) $\iff$ (1). To do this we check the residues of f'/f. The only singularities of f'/f occur at the zeroes or poles of f. Near a zero or pole α of f, we have $f(z) = (z - \alpha)^n g(z)$

for $g(z)$ holomorphic near α, $g(\alpha) \neq 0$. Also,

$$f'(z) = n(z-\alpha)^{n-1}g(z) + (z-\alpha)^n g'(z),$$

so

$$\frac{f'(z)}{f(z)} = \frac{n}{z-\alpha} + \frac{g'(z)}{g(z)},$$

where we note that the residue at α of $\frac{n}{z-\alpha}$ is n, and $g(z) \neq 0$ implies that the term on the right is holomorphic and is the residue at α. Hence

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}dz = \sum_{\substack{\alpha \text{ pole of } f \\ \text{outside } \gamma}} n(\gamma,\alpha)\cdot \text{order}(f,\alpha) - \sum_{\substack{\alpha \text{ pole of } f \\ \text{inside } \gamma}} n(\gamma,\alpha)\cdot \text{order+poles}(f,\alpha)$$

And $n(\gamma, \alpha) \to 1$ in all these sums. ■

Moreover, we can say that the "zeroes of order n look like z^n"; they wrap around 0 n times CCW, and "the poles of order n look like $\frac{1}{z^n}$"; they wrap around 0 n times CW.

The idea of computing the number of zeroes in a curve by computing an integral is quite nice. Most of the time, however, we just use the following corollary (also dropping the idea of poles for now, since they aren't really used in it):

Corollary. (Rouché Theorem) (Assume f has no zeroes on γ.) Let f, g be holomorphic in the unit disk (in γ), and $||g(z)|| < ||f(z)||$ on the unit circle (on γ). Then # of zeroes of f in the unit disk = # of zeroes of $f+g$ in the unit disk, for γ regular.
Examples. The # of zeroes of $3z^8(+iz^6+1)$ in the unit disk is the same as $3z^8$, which has 8 zeroes. iz^8+3z^6+1 has 6 zeroes in the unit circle. iz^8+z^6+3 has no zeroes. □

Proof of Rouché's Theorem. First note that for any functions A and B,

$$\frac{(AB)'}{AB} = \frac{A'}{A} + \frac{B'}{B}$$

Now write

$$(f+g) = f\left(1+\frac{g}{f}\right),$$

and note $|\frac{g}{f}| < 1$ on γ. The # of zeroes of $f+g$ in γ is given by

$$\frac{1}{2\pi i}\int_\gamma \frac{(f+g)'}{f+g}dz = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}dz + \frac{1}{2\pi i}\int_\gamma \frac{(1+\frac{g}{f})'}{(1+\frac{g}{f})}dz,$$

but this is just

$$= (\text{\# of zeroes with multiplicity of } f \text{ in } \gamma) + n\left(1+\frac{g\circ\gamma}{f\circ\gamma};0\right).$$

Note that $1+\frac{g\circ\gamma}{f\circ\gamma}$ is contained inside $B_1(1) \subseteq \{\text{Re}(z) > 0\}$, the upper half-plane. So the winding number is zero, and the above is

$$= \frac{1}{2\pi i}\int_\rho \frac{1}{z}dz$$

where ρ is the curve traced out by the fact that $1 + \frac{g\circ\gamma}{f\circ\gamma} < 1$, which has an antiderivative given by the principal branch of $\log z$ in $\mathrm{Re}(z) > 0$. ■

. . . .

Generalized Cauchy's integral formula. Let γ be regular, and z be inside γ. Then

$$f^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{k+1}}dw.$$

Proof. The residue of $\frac{f(w)}{(w-z)^{k+1}}$ at zero is

$$\frac{1}{w-z}\left(\frac{f(w)}{(w-z)^{k+1}}\right) = (w-z)^k f(w) = \frac{f^{(k)}(z)}{k!}.$$

■

Corollary. Suppose f_n is holomorphic and $f_n \to f$ uniformly on compact sets (hence holomorphic by Morera's theorem). Then $f_n' \to f'$ uniformly as well.

Proof. We have

$$f_n'(z) - f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f_n(w) - f(w)}{(w-z)^2}dw$$

for $\gamma = C_r(\alpha)$, $z \in D_r(\alpha)$. On $D_{r/2}(\alpha)$, we have

$$|f_k'(z) - f'(z)| = \frac{1}{2\pi}\left|\int_\gamma \frac{f_n(w)-f(w)}{(w-z)^2}dw\right| \leq \frac{1}{2\pi}\cdot 2\pi r \cdot \frac{4}{r^2}\cdot \max_{C_r(\alpha)}|f_n(w) - f(w)|,$$

where the $\frac{4}{r^2}$ comes as an upper bound on $\frac{1}{|w-z|^2}$ because $|w-z| \geq r/2 \implies \frac{1}{|w-z|^2} \leq \frac{4}{r^2}$. Now choose n large enough so that

$$|f_n(w) - f(w)| < \frac{\epsilon r}{4}$$

on $C_r(\alpha)$. Then $|f_n(z) - f(z)| < \epsilon$ for $z \in D_{r/2}(\alpha)$, $n > N$.

Now cover the compact set in euqation by disks of half the radius. Because the set is compact, finitely many suffiice. Take the largest N_α among these. ■

Theorem. (Hurwitz) Suppose $f_n \to f$ holomorphic in D, and uniformly so on compact sets. Suppose that none of the f_n's are zero in D. Then either $f = 0$ on D or f has no zeroes.

Proof. Suppose $f \neq 0$, and let $f(\alpha) = 0$ for some $\alpha \in D$. α is not the limit of zeroes of f (other than 0), because then $f = 0$ by the uniqueness theorem. Hence there is some closed disk $\overline{D_\epsilon(\alpha)}$, $\epsilon > 0$, with no zeroes of f in $\overline{D_\epsilon(\alpha)}$ other than α. We have $f_n' \to f'$ uniformly, and there are no zeroes on $C_\epsilon(\alpha)$ of f; none of f_n are zero, so $\frac{1}{f_n} \to \frac{1}{f}$ uniformly on $C_\epsilon(\alpha)$ (show this yourself). Thus

$$\frac{f_n'}{f_n} \to \frac{f'}{f}$$

uniformly on $C_\epsilon(\alpha)$. It follows that

$$\lim_{n\to\infty}\frac{1}{2\pi i}\int_{C_\epsilon(\alpha)}\frac{f_n'}{f_n} = \frac{1}{2\pi i}\int_{C_\epsilon(\alpha)}\frac{f'}{f}.$$

So
$$\lim_{n\to\infty} \# \text{ of zeroes of } f_n \text{ in } D_\epsilon(\alpha) = \# \text{ of zeroes of } f \text{ in } D_\epsilon(\alpha) = 1,$$
a contradiction, so we can't have this isolated zero of f. ■

CHAPTER 13

INTEGRALS AND SOME GEOMETRY

Recall the residue theorem: let α_k be the singularities of f. Then

$$\int_\gamma f(z)dz = 2\pi i \sum \text{Res}(f;\alpha_k)n(\gamma,\alpha_k).$$

There are many applications of this. If P, Q are polynomials, $Q(x) \neq 0$ for real x, and $\deg Q \geq \deg P + 2$, then

$$\int_{-\infty}^{\infty} \frac{P(x)dx}{Q(x)dx}dx = 2\pi i \sum_{k=1}^{n} \text{Res}\left(\frac{P(z)}{Q(z)}, \alpha_k\right).$$

You can show this with the usual residue calculation (yes, the ones that we beat to death in our review session...).

Example.

$$\int_{-\infty}^{\infty} \frac{1}{x^6+1}dx$$

The poles are where $z^6 = -1$, e.g. at $e^{i\theta}$ for $\theta = \frac{\pi}{6}, \frac{3\pi}{6}, ...$

Recall that, for $f(z) = A(z)/B(z)$ with $B(\alpha) = 0, A(\alpha) \neq 0$, if $B'(\alpha) \neq 0$ then this is a simple pole, and the residue is given by $\text{Res}(f;\alpha) = A(\alpha)/B'(\alpha)$.

So using this we get that the residues are

$$\text{Res}(f(z); e^{i\pi/6}) = \frac{1}{6e^{5\pi i/6}}$$

$$\text{Res}(f(z); e^{i\pi/2}) = \frac{1}{6e^{\pi i/2}}$$

$$\text{Res}(f(z); e^{5i\pi/6}) = \frac{1}{6e^{\pi i/6}}$$

Then

$$\int_{-\infty}^{\infty} \frac{1}{x^6+1}dx = \frac{2\pi i}{6}(e^{-5\pi i/6} + e^{-i\pi/2} + e^{-i\pi/6}) = \frac{2\pi}{3}.$$

Geometry of complex functions. Recall the open mapping theorem: if f is holomorphic and nonconstant, and if U is open in $\mathbb{C}$, then $f(U)$ is open, i.e. given $\epsilon, \exists\delta : B_\delta(z) \subseteq f^{-1}(B_\epsilon(w)) = \{x \in \mathbb{C} : f(x) \in B_\epsilon(w)\}$, i.e. $|z-\alpha| < \delta \Longrightarrow |f(z) - f(w)| < \delta$. Open means that $\forall\delta, \exists\epsilon : f(B_\delta(z)) \supseteq B_\epsilon(w)$, so $|f(z) - \beta| < \epsilon \Longrightarrow \exists\alpha : |z-\alpha| < \delta, f(\alpha) = \beta$.

Proof. Let $\alpha \in \mathbb{C}$. Wlog $f(\alpha) = 0$; take C inside $B_\delta(\alpha)$. Take C inside $B_\delta(\alpha)$; $\min f(C)$ exists and it's not zero, so call it z_ϵ. Let $w \in B_\epsilon(0)$. Then for $z \in C$, $|f(z) - w| \geq |f(z)| - |w| \geq 2\epsilon - \epsilon = \epsilon$. For $z = \alpha$, $|f(\alpha) - w| = |-w| < \epsilon$. Hence $\min |f(z) - w|$ for $z \in B_{\delta/2}(\alpha)$ is not achieved on the boundary. Thus, by the minimum modulus theorem, $\exists$ a zero, e.g. $f(z) - w = 0$ in B.

Theorem. (Schwarz Lemma) Let $\mathbb{D}$ be the unit disk. If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic (extending continuously to the boundary) and $f(0) = 0$, then

$$\begin{cases} |f(z)| \leq |z| \\ |f'(0)| \leq 1 \end{cases}$$

with equality in either iff $f(z) = e^{i\theta}z$.

Proof. Define

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0, \end{cases}$$

which is holomorphic on the circle of radius δ; $|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{\delta}$. By the maximum modulus principle, we know in fact $|g(z)| \leq 1$ for all $z \in \mathbb{D}$. Let $\delta \to 1$.

So if a max is achieved on the interior, then $f(z)$ is constant, i.e. $g(z)$ is constant if $|f(z)| = |z|$ for any $z \in D_1(0)$, $z \neq 0$, or $|f'(0)| = 1$. This means $g(z) = e^{i\theta}$; then $g(z) = f(z)/z$ $(f'(0))$, so $f(z) = e^{i\theta}z$. ∎

Corollary. If a map from $\mathbb{D} \to \mathbb{D}$, $f(0) = \alpha$, has a maximum value of $|f'(0)|$ among maps $\mathbb{D} \to \mathbb{D}$, then f is surjective.

Proof. Consider $h(z) = B_\alpha(f(z))$. $h(0) = f'(0)B'_\alpha(0), h(z)\mathbb{D} \to \mathbb{D}, h(z) : 0 \mapsto 0, |h'(0)| \leq 1$.

The conclusion is that

$$|f'(1)| \leq \frac{1}{B'_\alpha(\alpha) = 1 - |\alpha|^2}.$$

Notice that this is achieved for inverse of B_α, which is B_α.

$$B_\alpha(B_\alpha(z)) = z.$$

Also note that

$$B_\alpha(z) = \frac{z - \alpha}{1 - \overline{\alpha}z},$$

and

$$B'_\alpha(z) = \frac{1}{1 - |\alpha|^2},$$

so $B'_\alpha(0) = 1 - |\alpha|^2$, $B_\alpha(0) = 0$, $|B_\alpha(z)| \leq 1$ for $|z| \leq 1$. ∎

We used $h'(0) \leq 1$. We also know $|h(z)| \leq |z|$ and $h(z) = B_\alpha(f(z))$; $|B_\alpha(f(z))| \leq |z|$.

We can therefore conclude that

$$|f(z)| \leq |B_\alpha(z)|$$

if $f(0) = \alpha, f : \mathbb{D} \to \mathbb{D}$.

Riemann mapping theorem. (casually stated) Given a simply connected region $U \subseteq \mathbb{C}$, $U \neq \mathbb{C}$, $\exists f : \mathbb{D} \to U$ where $\mathbb{D}$ denotes the open unit disk, which is a holomorphic bijection with holomorphic inverse (and essentially unique).

CHAPTER 14
FOURIER TRANSFORM AND SCHWARZ REFLECTION PRINCIPLE

We will go over contour integration via Fourier transforms and infinite sums, §11, the Schwarz reflection principle §7.2, and the Mobius transformations §13.2.

The Fourier Transform. Let $f : \mathbb{R} \to \mathbb{R}$ or $\mathbb{R} \to \mathbb{C}$. The *Fourier transform* of f is given by

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x y} dx.$$

The *inversion formula* is

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(y) e^{2\pi i y x} dy.$$

This is writing f in terms of $e^{2\pi i y x}$, and has applications in physics, probability theory (proving CLT via generating functions), etc. Look at Wikipedia for a summary of how useful it is.

Example. $f(x) = \frac{1}{1+x^2}$.

$$\hat{f}(y) = \int_{-\infty}^{\infty} \frac{1}{1+x^2} e^{-2\pi i y x} dx.$$

Assume $y \le 0$. Let $C(R) = C_1(R) + C_2(R)$ denote the semicircle contour with $C_1(R)$ denoting the part on the real axis and $C_2(R)$ the semicircle part. Then consider

$$\int_{C_1(R)+C_2(R)} \frac{1}{1+z^2} e^{-2\pi i y z} dz = 2\pi i \cdot \operatorname{Res}(g_y(z); i)$$

where $g_y(z)$ is the integrand. With a simple bounding argument, we note that

$$\int_{C_1(R)} g_y(z) = \int_{-R}^{R} \frac{1}{1+x^2} e^{-2\pi i y x} dx.$$

Also, $\text{Res}(g_y(z); i) = \frac{e^{2\pi y}}{2i}$. Thus $\hat{f}(y) = \pi e^{2\pi y}$ for $y \leq 0$. Now for $y \geq 0$ we use the same contour reflected over the real axis. Then $\text{Res}(g_y(z); -i) = \frac{e^{-2\pi y}}{-2i} \Longrightarrow$

$$\int_{-R}^{R} \frac{1}{1+x^2} e^{-2\pi i y x} dx = 2\pi i \cdot \mathfrak{n}(C, -i) \cdot \text{Res}(g_y(z); i) = \pi e^{-2\pi y}.$$

So

$$\boxed{\hat{f}(y) = \pi e^{-2\pi |y|}.}$$

□

The applications of Fourier transformations abound. Frankly Prof. Cotton-Clay hasn't used it much, but natural scientists do. A very nice source for getting to know the complex-analytic details of Fourier transforms is Stein and Shakarchi.

Example. Verify that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

We can do this by showing $\sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3}$. This is left as an exercise to the reader (or see the textbook). Note that this is $\zeta(2)$, and the method that you use for computing this generalizes to $\zeta(2n), n \in \mathbb{Z}^+$; this was first computed by Euler. □

Conformal mappings. These are functions that preserve angles. The *Schwarz reflection principle* is useful as a tool here.

Theorem. (Schwarz reflection) Let $\mathbb{H}$ denote the upper half-plane, $\{z \in \mathbb{C} : \text{Im}(z) > 0\}$. Suppose that f is holomorphic in a region $D \subseteq \mathbb{H}$, that f is continuous on $D \cup L$ where $L = \partial D \cup \mathbb{R}$, and that f is real on L. Then

$$g(z) := \begin{cases} f(z) & z \in D \cup L \\ \overline{f(\overline{z})} & z \in \overline{D} \end{cases}$$

is holomorphic on $D \cup L \cup \overline{D}$.

Proof. First, g is continuous because both parts agree on L. Second, g is holomorphic on D, and on $\overline{D}$:

$$\lim_{h\to 0} \frac{g(z+h) - g(z)}{h} = \lim_{h\to 0} \frac{\overline{f(\overline{z} + \overline{h})} - \overline{f(\overline{z})}}{h} = \overline{f'(\overline{z})}$$

exists, so g is holomorphic on $\overline{D}$. Then use Morera's theorem to show that $g(z)$ is holomorphic. ■

Mobius transformations. This answers the question: what are the injective, holomorphic maps $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$?

Recall that if $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic (meromorphic on $\mathbb{C}$), then f is a rational function. Solving $\alpha = \frac{P(z)}{Q(z)}$, we have $P(z) - \alpha Q(z) = 0$, and the LHS is a polynomial of degree $\max(\deg P, \deg Q)$. For $f(z)$ to be injective, we need $\max \deg = 1$. So

$$f(z) = \frac{az+b}{cz+d},$$

where (a, b) and (c, d) are linearly independent; equivalently, $ad - bc \neq 0$. Maps of the form above are called *Mobius transformations.* It is obvious that the inverse f^{-1} is given by

$$f^{-1}(w) = \frac{-dw + b}{cw - a}$$

with $ad - bc \neq 0$.

Proposition. If

$$f(z) = \frac{az + b}{cz + d}, \qquad g(z) = \frac{\alpha z + \beta}{\gamma z + \delta},$$

then letting

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

we have

$$f(g(z)) = \frac{Az + B}{Cz + D}.$$

Proof. This is just a simple check. ■

CHAPTER 15

MÖBIUS TRANSFORMATIONS

Today we'll talk about Mobius transformations §13.2, cross ratios, and automorphisms of $\mathbb{D}$ and $\mathbb{H}$. Recall that we defined a Mobius transformation to be a function $f(z)$ of the following form:

$$f(z) = \frac{az+b}{cz+d}.$$

These are the *automorphisms* of $\hat{\mathbb{C}}$, i.e. the holomorphic maps $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with holomorphic inverse. We showed that composition translates into matrix multiplication:

$$f_{M_1}(f_{M_2}(z)) = F_{M_1 M_2}(z),$$

where M's are the matrix forms of the transformation.

These 2×2 invertible matrices with complex coefficients are known as the *general linear group of order* 2 *over* $\mathbb{C}$:

$$GL_2\mathbb{C} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C}, ad - bc \neq 0 \right\}.$$

$GL_2\mathbb{C}$ has Lie group structure (it is an algebraic group endowed with manifold structure, e.g. the group operations are C^∞). Let $\mathcal{M} = \{\text{mobius transformations}\}$. Then, modding out by complex scalars $\neq 0$, we have in fact that

$$\mathcal{M} = \frac{GL_2\mathbb{C}}{\mathbb{C}^\times} = PGL_2\mathbb{C},$$

where $PGL_2\mathbb{C}$ is the *projective linear group of order* 2 *over* $\mathbb{C}$, which is also a Lie group. We mod out by scalar multiples of $\mathbb{C}$ because

$$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = f\begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix},$$

has no other coincidences; e.g. multiplication by a complex scalar gives the same transformation. We now claim that the generators for $\mathcal{M}$ are as follows:

Generators for $\mathcal{M}$:

(1) Dilations/rotations: $z \mapsto az$.
(2) Translations: $z \mapsto z + b$.
(3) Inversion: $z \mapsto \frac{1}{z}$.

Claim. These generate $\mathcal{M}$.

Proof. We get the map $z \mapsto az + b$ from dilation and translation. Suppose $c \neq 0$; we want a map of the form $z \mapsto \frac{az+b}{cz+d}$. Well,

$$z \mapsto_{(1,2)} cz + d \mapsto_{(3)} \frac{1}{cz+d} \mapsto_{(1,2)} \frac{az+b}{cz+d}.$$

Let's show that we can achieve the latter map. We can write

$$\frac{az+b}{cz+d} = A\left(\frac{1}{cz+d}\right) + B = \frac{A + Bcz + Bd}{cz+d} = -\left(\frac{ad-bc}{c}\right)\left(\frac{1}{cz+d}\right) + \frac{a}{c},$$

for which we want $B = \frac{a}{c}$ and $A = b - \frac{a}{c}d = -\frac{ad-bc}{c}$. Note that setting $c = 0$ we have $f(z) = \frac{a}{d}z + \frac{b}{d}$ by (1,2). ∎

Corollary. {Mobius transformations} : {circles and lines} $\to$ {circles and lines}.

Proof. All of the generators do. ∎

Let's examine the dimensions of the groups we have above. Note that $GL_2\mathbb{C}$ has 4 complex dimensions, or 8 real dimensions (by the obvious identification $\mathbb{C} \cong \mathbb{R}^2$). This is because there are 4 degrees of freedom in choosing elements in the matrix; another way would be to show this is to note that $GL_2\mathbb{C}$ is open in $\mathrm{Mat}_2\mathbb{C}$ under the continuous determinant map, the latter of which has complex dimension 4.

Since we are eliminating one degree of freedom by imposing the condition $\det(A) \neq 0$ for $A \in \mathcal{M}$, we see that $\dim_{\mathbb{C}} \mathcal{M} = 3$, or $\dim_{\mathbb{R}} \mathcal{M} = 6$. Given this, we can say something about the action on the 3 points $\{\infty, 0, 1\}$:

Lemma. Given $f \in \mathcal{M}$ and $f \neq \mathrm{id}$, f has at most 2 fixed points (points $w \in \mathbb{C} : f(w) = w$) in $\{\infty, 0, 1\}$.

Proof. Let $c \neq 0$, so that $f(z) = \frac{az+b}{cz+d}$ and setting $f(z) = z$ we have $az + b = cz^2 + dz$, which maps $\infty \mapsto \frac{a}{c} \neq \infty$ because $c \neq 0$. For $c = 0$, we have $\infty \mapsto \infty$, and $az + b = z$ has ≤ 1 solution. ∎

Proposition. There exists a unique Mobius transformation $f(z)$ sending $\alpha_1, \alpha_2, \alpha_3$ to $\infty, 0, 1$ respectively, and $f(z)$ is given by

$$f(z) = \frac{(z-\alpha_2)(\alpha_3-\alpha_1)}{(z-\alpha_1)(\alpha_3-\alpha_2)}.$$

Note that if any $\alpha_j = \infty$, we cross out both terms in which it appears.

Proof. The existence of $f(z)$ is trivial. Uniqueness follows thus: if we have two transformations f, g satisfying the above properties, we consder $g \circ f^{-1}$, which sends $\infty \mapsto \infty, 0 \mapsto 0, 1 \mapsto 1$. So three points are fixed, which implies $g \circ f^{-1}(z) = z \implies f(z) = g(z)$. ∎

This suggests that $\dim_{\mathbb{C}} \mathcal{M} = 3$ is no mistake. Indeed, we can define the *cross-ratio*

$[\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4]$ as the number $w \in \mathbb{C}$ where α_4 is sent if $\alpha_1 \mapsto \infty, \alpha_2 \mapsto 0$, and $\alpha_3 \mapsto 1$ by a Mobius transformation. Note that this number is equal to

$$\frac{(\alpha_4 - \alpha_2)(\alpha_3 - \alpha_1)}{(\alpha_4 - \alpha_1)(\alpha_3 - \alpha_2)}.$$

Proposiiton. The cross-ratio is preserved by Mobius transformations, i.e. if $f(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$, then $[\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4] = [f(\alpha_1); f(\alpha_2); f(\alpha_3); f(\alpha_4)]$.

Proof. Let g be a Mobius transformation sending $\alpha_1 \mapsto \infty, \alpha_2 \mapsto 0, \alpha_3 \mapsto 1$. We have $[\alpha_1; \alpha_2; \alpha_3; \alpha_4] = g(\alpha_4)$. Then $g \circ f^{-1}$ sends $f(\alpha_1) \mapsto \infty, f(\alpha_2) \mapsto 0, f(\alpha_3) \mapsto 1$, and $g \circ f^{-1}(f(\alpha_4)) = g(\alpha_4)$. ■

Proposition. There exists a non-unique Mobius transformation sending $z_1 \mapsto w_1, z_2 \mapsto w_2$ and $z_3 \mapsto w_3$, and it satisfies, for $w = f(z)$,

$$\frac{(w - w_2)(w_3 - w_1)}{(w - w_1)(w_3 - w_2)} = \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}.$$

Proof. Existence comes from the formula and the cross-ratios. Uniqueness follows as before. ■

Let's use the above propositions above to find a Mobius transformation f sending $\mathbb{D} \to \mathbb{H}$ bijectively. If we visualize how f sends $\mathbb{D} \subset \mathbb{C} \to \mathbb{H} \subset \mathbb{C}$, we can also visualize how it maps $\hat{\mathbb{C}}$. In particular, it should send $-1 \mapsto \infty, 0 \mapsto i, 1 \mapsto 0, i \mapsto 1, -1 \mapsto -1$. Let $w = f(z)$. Then using the equation above we have

$$\frac{(w - i)(0 - \infty)}{(w - \infty)(0 - i)} = \frac{(z - 0)(1 + 1)}{(z + 1)(1 - 0)} \Longrightarrow iw + 1 = \frac{2z}{z + 1} \Longrightarrow w = i\frac{-z + 1}{z + 1}.$$

Claim. This sends $\mathbb{D}$ to $\mathbb{H}$.

Proof. Let's go with a more geometric proof. Note that $f(i) = 1$. We have a subclaim: given 3 points in $\hat{\mathbb{C}}$, there exists a unique circle or line through the 3 points.

Proof of subclaim. Send the 3 points to $\infty, 0, 1$ by a Mobius transformation. There's a unique circle/line through these, namely $\mathbb{R}$. We showed that Mobius transformations sends circles and lines to circles and lines. □

Now f sends the unit circle to the real line by the formula $f(e^{it}) = \tan(t/2)$, where $0 \leq t < 2\pi$. We make another subclaim, namely that f sends points inside $\mathbb{D}$ to points inside $\mathbb{H}$.

Proof of subclaim. Use the following proposition on curves that are circles and omits -1 (a "Pac-Man" curve). □

Proposition. Let $f(z)$ be a Mobius transformation, $\alpha \in \mathbb{C}$, γ a contour curve, and $\alpha \notin$ the interior of γ. Suppose $\mathfrak{n}(f(\gamma); f(\infty)) = 0$. Then $\mathfrak{n}(\gamma; \alpha) = \mathfrak{n}(f(\gamma); f(\alpha))$.

Proof. Under dilations,

$$\mathfrak{n}(\gamma; \alpha) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - \alpha} = \frac{1}{2\pi i} \int_0^1 \frac{\gamma'(t)dt}{\gamma(t) - \alpha} dt.$$

The equalities for the other generators follow similarly. In particular, note that for inversion we want to show the formula $\mathfrak{n}(\frac{1}{\gamma}; \frac{1}{\alpha}) = \mathfrak{n}(\gamma; \alpha) + \mathfrak{n}(\frac{1}{\gamma}; 0)$, from which we can conclude the proposition. This is a simple check using the definition of winding number and the chain rule.

Maps from regions into other regions. We know that log maps to a horizontal strip. In particular, $\log(z) : \mathbb{H} \to \{z : 0 < \mathrm{Im}(z) < \pi\}$, so

$$\frac{1}{\pi}\log(z) : \mathbb{H} \to \mathbb{R} \times [0, 1].$$

We then see that

$$\mathbb{D} \to_f \mathbb{H} \to_g \mathbb{R} \times [0, 1]$$

for $f(z) = i\frac{1-z}{1+z}, g(z) = \frac{1}{\pi}\log(z)$. Thus we have an automorphism from the unit disc to a strip given by

$$\frac{1}{\pi}\log(i\frac{1-z}{1+z}) : \mathbb{D} \to \mathbb{R} \times [0, 1].$$

CHAPTER 16

AUTOMORPHISMS OF D AND H

We were looking at $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Claim. $f(z) = -i\frac{z-1}{z+1}$ maps $\mathbb{D}$ to $\mathbb{H}$, and $g(z) = \frac{-z+i}{z+i}$ maps $\mathbb{H}$ to $\mathbb{D}$. Each are one-to-one, holomorphic, and inverses of each other.

Proof. Check the last part; we will show that f maps $\mathbb{D}$ to $\mathbb{H}$. Note that, under f, $1 \mapsto 0, -1 \mapsto \infty, i \mapsto -i(\frac{i-1}{i+1}) = 1$. Hence, like last time, we see that the unit circle is mapped to the real line, since the 3 points determine a circle or a line. (Also note $0 \mapsto i$, $-i \mapsto -1$.) Apparently, the real axis is mapped to the imaginary axis. Now fill up $\mathbb{D}$ by circular arcs through -1 and 1. These map to straight lines "through 0 and ∞", all $\subset \mathbb{H}$ because they intersect the upper half of the unit circle, which is the image of the imaginary axis $\subseteq \mathbb{D}$.

Another proof is the use the argument principle. Notice that the winding of $f(\gamma(t))$ around $\alpha \in \mathbb{C}$ is the number of zeroes of $f(z) = \alpha$... but this argument is sketchier, so we leave it to you to fill in the gaps. ∎

Here's an unnecessary but amusing claim:

Claim. Let $\gamma(t)$ be a contour curve and $0, \alpha \notin \mathrm{im}(\gamma)$. Then

$$n\left(\frac{1}{\gamma}, \frac{1}{\alpha}\right) = n(\gamma, \alpha) + n\left(\frac{1}{\gamma}, 0\right) = n(\gamma, \alpha) - n(\gamma, 0).$$

Proof. We have

$$n(1/\gamma, 0) = \int_{1/\gamma} \frac{dz}{z-0} = \int_0^1 \frac{d(\frac{1}{\gamma t})}{\frac{1}{\gamma(t)} - 0} = \int_0^1 \frac{-\gamma'(t)}{\gamma(t)} dt = -n(\gamma, 0).$$

On the other hand, we have

$$n(1/\gamma, 1/\alpha) = \int_0^1 \frac{d(\frac{1}{\gamma(t)})}{\frac{1}{\gamma(t)} - \frac{1}{\alpha}} = \int_0^1 \frac{-\gamma'(t)/\gamma(t)^2}{1/\gamma(t) - 1/\alpha} = \int_0^1 \frac{-\alpha\gamma'(t)}{\alpha\gamma(t) - \gamma(t)^2} dt = \int_0^1 \frac{\alpha\gamma'(t)}{\gamma(t)[\gamma(t) - \alpha]} dt$$

$$= \int_0^1 \frac{-\gamma'(t)}{\gamma(t)} dt + \int_0^1 \frac{\gamma'(t)}{\gamma(t) - \alpha} dt = n(1/\gamma, 0) + n(\gamma, \alpha),$$

which gives the result. ■

Automorphisms of $\mathbb{D}$ and $\mathbb{H}$. What are the injective, surjective, holomorphic maps $\mathbb{D} \to \mathbb{D}$ with holomorphic inverses? Any such map is in the *automorphism group of* $\mathbb{D}$, denoted $\text{Aut}(\mathbb{D})$. Recall that the Schwarz lemma tells us that if $f : \mathbb{D} \to \mathbb{D}$ is such that $f(0) = 0$, we have $|f(0)| \leq 1$ and $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$, with equality in either where $f(z) = e^{i\theta} z$.

Corollary. If $f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0$, and f is an automorphism of the disk (there exists a holomorphic inverse $f^{-1} : \mathbb{D} \to \mathbb{D}$), then $f(z) = e^{i\theta} z$.

Proof. $|f(z)| \leq |z| \implies |f^{-1}(z)| \leq |z|$ so $|z| \leq |f(z)|$, e.g. equality is obtained and $f(z) = e^{i\theta} z$. ■

Recall that $B_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}$ for $|\alpha| < 1$. This maps $\mathbb{D} \to \mathbb{D}$ and $0 \mapsto \alpha, \alpha \mapsto 0$. Hence $B_\alpha(B_\alpha(z))$ maps $\mathbb{D} \to \mathbb{D}$, $0 \mapsto 0, \alpha \mapsto \alpha$, and their composition is the identity map.

Theorem. The automorphisms of $\mathbb{D}$ are precisely the maps

$$f(z) = e^{i\theta} \left(\frac{\alpha - z}{1 - \overline{\alpha} z} \right).$$

Proof. Suppose f is an automorphism of $\mathbb{D}$ with $f(\alpha) = 0$. Then $f(B_\alpha(w))$ takes 0 to 0 and $f(B_\alpha(w)) = e^{i\theta} w$. Take $w = B_\alpha(z)$ and $B_\alpha(w) = z$. Then $f(z) = e^{i\theta} B_\alpha(z) = e^{i\theta} (\frac{\alpha - z}{1 - \overline{\alpha} z})$. ■

Corollary. The automorphisms of $\mathbb{D}$ are the Mobius transformations taking $\mathbb{D}$ to itself.

Corollary. The automorphisms of $\mathbb{H}$ are the Mobius transformations taking $\mathbb{H}$ to itself.

Theorem. The automorphisms of $\mathbb{H}$ are precisely the maps $f(z) = \frac{az+b}{cz+d}$ with $ad - bc > 0$ and $a, b, c, d \in \mathbb{R}$.

Proof. Take $x_1 \mapsto \infty, x_2 \mapsto 0, x_3 \mapsto 1$ in $\mathbb{R} \cup \{\infty\}$ via a Mobius transformation for x_1, x_2, x_3 all distinct (note that there is a unique such Mobius transformation). Write it down:

$$f(z) = \frac{(z - x_2)(x_3 - x_1)}{(z - x_1)(x_3 - x_2)}.$$

The collection of these is the collection of maps $f(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$. Now check when $f(i) \in \mathbb{H}$, $f(i) = \frac{(bd+ac)+i(ad-bc)}{c^2+d^2}$. We see that f maps into $\mathbb{H}$ iff $ad - bc > 0$. ■

Here is a classification of the automorphisms of $\mathbb{D}$ (and $\mathbb{H}$), up to conjugation (e.g. for $f, g : \mathbb{D} \to \mathbb{D}$, $g^{-1} f g : \mathbb{D} \to \mathbb{D}$ is allowed):

- Identity: $f(z) = z$
- Elliptic automorphisms: $f(z) = e^{i\theta} z$ (rotation on $\mathbb{D}$)
- Parabolic: $f(z) = z + b$ on $\mathbb{H}$
- Hyperbolic: $f(z) = az$ on $\mathbb{H}$, for $a \in \mathbb{R}$

Claim. Every automorphism of $\mathbb{D}$ (or $\mathbb{H}$) is conjugate to precisely one of these.

Proof. Assume f is not the identity. We claim that every automorphism of $\mathbb{H}$ (or $\mathbb{D}$) fixes either one point inside $\mathbb{H}$, one point on $\mathbb{R}$, two points on $\mathbb{R}$, and no others. To see this, note that every Mobius transformation fixes one or two points. If one is in $\mathbb{H}$, then the full Mobius transformation is given by the Schwarz reflection

$$g(z) = \begin{cases} f(z) & z \in \mathbb{H} \\ \text{something in } \mathbb{R} \cup \{\infty\} & z \in \mathbb{R} \cup \{\infty\} \\ \overline{f(\overline{z})} & z \in \overline{\mathbb{H}}. \end{cases}$$

So if we have one in $\mathbb{H}$, then there can be no more. Either this happens, or one/two points are on $\mathbb{R}$.

If one fixed point is in $\mathbb{H}$, we conjugate to $\mathbb{D}$ and see it's at $\alpha \in \mathbb{D}$; conjugate by B_α to get it at 0, then get a rotation. In particular, let $f : \mathbb{H} \to \mathbb{H}, f(\beta) = f(\beta)$. Take our $F : \mathbb{D} \to \mathbb{H}$, $F^{-1} : \mathbb{H} \to \mathbb{D}$. $F^{-1} \circ f \circ F$ then fixes $F^{-1}(\beta) = \alpha$. Then taking B_α (which is its own inverse), we have $B_\alpha \circ F^{-1} \circ f \circ F \circ B_\alpha$ sends $0 \mapsto 0, \mathbb{D} \to \mathbb{D}$, so it equals $e^{i\theta}z$.

If we fix one point on $\mathbb{R}$, we conjugate to the disk, then conjugate by rotation so that the fixed point on the unit circle is at -1. Then we conjugate back to $\mathbb{H}$ so that the fixed point is at ∞ (of Mobius transformation); i.e. $f(z) = az + b$. We need $b \neq 0$, else $0 \mapsto 0$. We also have $a > 0$ by $ad - bc > 0$. If $a \neq 0$, we fix another point so that $a = 1$ and $f(z) = z + b$. If we fix two points in $\mathbb{R}$, we again place one at ∞ and conjugate the other by $A(z+1)$ for appropriate A; then $f(z) = az, a > 0$. ■

In fact, these are the symmetries of the hyperbolic plane (taking the line $y = 0$ to be a sort of "line at infinity"). As an aside, the *hyperbolic metric* on $\mathbb{H}$ is given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

for $y > 0$. The arc length of $\gamma(t)$ in the hyperbolic metric is $\int_\gamma ds = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt$ for $\gamma(t) = (x(t), y(t)) : [a, b] \to \mathbb{H}$.

CHAPTER 17

SCHWARZ-CHRISTOFFEL AND INFINITE PRODUCTS

Today we'll be talking about Schwarz-Christoffel functions §13.3 and infinite products and Euler's formula for $\sin(\pi\theta)$, §17.3.

Schwarz-Christoffel maps. These are holomorphic maps from $\mathbb{H}$ to a single polygon. We'll see next time how it follows from the Riemann mapping theorem. But first off we can note that there are only a limited number of ways to do this.

Let's draw a polygon with vertices v_i, interior angles $\pi\beta_i$, and exterior angles $\pi\alpha_i$ ($\pi\alpha_i + \pi\beta_i = \pi$). Note that $-1 < \alpha_i < 1$, and $\sum \pi\alpha_i = 2\pi \implies \sum \alpha_i = 2$.

Proposition. Given angles $\pi\alpha_1, ..., \pi\alpha_n$ and that $a_1, ..., a_{n-1}$ are points on $\mathbb{R}$, then

$$f(z) = \int_0^z \frac{dz}{(z-a_1)^{\alpha_1} ... (z-a_{n-1})^{\alpha_n - 1}}$$

gives a holomorphic map from $\mathbb{H}$ to a polygon with angles α_i, and has a continuous extension to $\mathbb{R}$ with $f(a_i) = v_i, f(\infty) = a_n$.

Remarks on the function. This is a contour integral taken on $\mathbb{H} \cup \mathbb{R}$. We can take a branch of $\frac{1}{(z-a_j)^{\alpha_j}} = (z-a_j)^{-\alpha_j}$ which is positive for z real, $> a_j$ and extends downward, e.g. is defined on $\mathbb{C} - \{a_j - ir\}$, $r > 0$. We can require $0 \neq a_j$, but in fact this still makes sense if 0 is one of those: $\int_0^1 \frac{1}{x^\lambda} dx$ converges for $\lambda < 1$. Let's examine the argument of $f'(z) = \prod_{j=1}^{n-1} (z-a_j)^{-\alpha_j}$ on $\mathbb{R}$:

Claim. $\text{Arg}(f'(z))$ is constant on the connected segments of $\mathbb{R} - \cup\{\alpha_j\}$. Also, $\text{Arg}(f'(z))$ increases by $\pi\alpha_j$ when preserving a_j.

Proof. This follows from:

Claim. Let $z^{-\alpha}$ be defined on $\mathbb{C} - \{\text{negative imaginary axis}\}$, and be and positive real for z real and positive. Then $\text{Arg}(x^{-\alpha}) = 0$ for $x > 0$. If x is real, $\text{Arg}(x^{-\alpha}) = -\pi\alpha, x < 0$.

Proof. Write $z = re^{i\theta}$. Then $z^{-\alpha} = r^{-\alpha}e^{-i\alpha\theta}$ gives the desired branch, which is in particular our negative real axis. So if x is negative, i.e. $\lambda = re^{i\pi}$, then we get $x^{-\alpha} = r^{-\alpha}e^{-i\alpha\pi}$, i.e. $\text{Arg}(x^{-\alpha}) = -\alpha\pi$. We can conclude that on the boundary ($\mathbb{R}$) of $\mathbb{H}$, $f(z)$ maps to straight lines, turning by angles $\pi\alpha_k$ to L. $\blacksquare$

Claim. This closes up, e.g. $\int_{-\infty}^{\infty} \prod (x - a_j)^{\alpha_j}\, dx = 0$.

Proof. Use a semicircle contour, but skip over the poles on the real line. Then use a combination of the closed curve theorem and bounding on the arc. $\blacksquare$

Example. $\sin z$ maps $(-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, \infty)$ to $\mathbb{H}$. We can view the domain as a triangle with exterior angles $\pi/2, \pi/2, \pi$. So, in some sense, $\sin^{-1} z : \mathbb{H} \to$ this strip. We can then write

$$\sin^{-1}(z) = \int_0^z \frac{dz}{\sqrt{1-z^2}} = \int_0^z \frac{dz}{(1+z)^{1/2}(1-z)^{1/2}},$$

for some branch of the root function. (This is because $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.) $\square$

Infinite products. We want to segue into the question, when does the infinite product $\prod_{k=1}^{\infty} a_k$ converge?

Lemma. $\prod a_k$ converges iff $\sum \log(a_k)$ converges, for $\mathrm{Re}(a_k) > 0$.

Proof. Let $p_n := \prod_{k=1}^{n} a_k$, $s_n = \sum_{k=1}^{n} \log(a_k)$. Then $\log p_n = s_n$ and $p_n = e^{s_n}$. e and log are continuous, so convergence for one gives convergence for another. $\blacksquare$

Lemma. $\prod(1 + a_k)$ converges if $\sum |a_k|$ converges.

Proof. If $\sum |a_k|$ converges, then past some point N, $|a_k| < 1/z$. So we check: $\sum_{k=N}^{\infty} \log(a_k)$ converges.

$$|\log(1+a_k)| = \left| a_k - \frac{a_k^2}{2} + \frac{a_k^3}{3} - \frac{a_k^4}{4} + ... \right| \le |a_k| \left(1 + \frac{1}{2} + \frac{1}{4} + ...\right) \le 2|a_k|.$$

So $\sum_N^{\infty} |\log(1 + a_k)| \le \sum 2|a_k|$ converges, hence $\sum_N^{\infty} \log(1 + a_k)$ converges; finally, this implies that $\prod_N^{\infty}(1 + a_k)$ converges. $\blacksquare$

Proposition. Suppose $f_k(z)$ is holomorphic on a region D, with $\sum |f_k(z)|$ uniformly convergent on compact subsets of D. Then $\prod_{k=1}(1 + f_k(z))$ is uniformly convergence on compact subsets and hence convergent.

Proof. Use the proof of the lemma in uniform convergence along with Morera's theorem. $\blacksquare$

Example. (Euler, 1734) We have the infinite product formula for $\sin(\pi z)$:

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Theorem. (Weierstrass product) There exists an entire, holomorphic function with zeroes only at $\{\alpha_k\}_{k=1}$ if $a_k \to \infty$. Furthermore, the order of zero is the number of times repeated in the list.

Proof. First attempt to go about it: $f(z) = \prod(z - a_k) \Longrightarrow f(z) = \prod(1 - \frac{z_k}{a_k})$ converges if $\sum \frac{1}{|a_k|}$ converges.

Claim. If $\sum \frac{1}{|a_k|^2}$ converges, then

$$f(z) = \prod \left[\left(1 - \frac{z}{a_k}\right) e^{z/a_k} \right].$$

Proof. This converges if $\sum |(\log(1 - \frac{z}{a_k}) + \frac{z}{a_k})|$ does. Suppose $\frac{|a_k|}{2} > |z|$ for $|z| \leq R$; this is possible for $k > N$ since $a_k \to \infty$. Then we can bound

$$\leq \frac{|z|^2}{a_k^2} \left(1 + \frac{1}{2} + \frac{1}{4} + ...\right) \leq 2\frac{|z|^2}{|a_k|^2},$$

i.e. suppose $\sum \frac{1}{|a_k|^2}$ converges.

Back to the full proof of Weierstrass's theorem: Let

$$f(z) = \prod \left[\left(1 - \frac{z}{a_k}\right) E_k(z) \right],$$

for

$$E_k(z) = e^{\frac{z}{a_k} + \frac{z^2}{2a_k} + ... + \frac{z^k}{ka_k^k}}.$$

Check via a diagonal argument that this works; in particular, consult your textbook. ■

Proposition. (*Euler's formula*)

$$\sin \pi z = \pi z \prod \left(1 - \frac{z^2}{k^2}\right)$$

Proof. Take the quotient

$$Q(z) = \frac{\pi z \prod \left(1 - \frac{z^2}{k^2}\right)}{\sin \pi z}.$$

This is an entire function with no zeroes. We'll argue that this has growth at most Ae^z; let's claim that it follows that $Q(z)$ is constant.

We note that $Q(z)$ is even. So, taking the log of $Q(z)$, we have $\int_0^z \frac{Q'(z)}{Q(z)} dz$. This has growth at most constant $= |z|^{3/2}$ for large values. So $\log Q(z)$ is linear, i.e. $Q(z) = Ae^{Bz}$ with $B = 0$ since Q is even, i.e. $Q(z)$ is constant. $Q(z)$ is constant because $\lim_{z\to 0} \frac{\sin(\pi z)}{\pi z} = 1$ and $\lim_{z\to 0} \prod_{k=1}^{\infty}(1 - \frac{z^2}{k^2}) = 1$.

Now let's talk about the growth. $\frac{1}{\sin \pi z}$ is bounded on a square with sides from $-N - 1/2$ to $N + 1/2$ (exercise), and likewise on the edges. To bound the rest...actually, we'll save that for next time because we're out of time.

CHAPTER 18

RIEMANN MAPPING THEOREM

Guest lecture: Joe Rabinoff
Note: These notes are particularly messy since the scribe wasn't paying too much attention, so use in discretion.

You're lucky that we cover the Riemann mapping theorem, which is one of the most interesting results you'll learn in this class. The context is the following:

Let regions $R_1, R_2 \subsetneq \mathbb{C}$ be open and simply connected, with R_1 connected and $\hat{\mathbb{C}} - R$ connected. Then there exists $\varphi : R_1 \to R_2$ that is onto, one-to-one, and holomorphic. Thus $\varphi^{-1} : R_2 \to R_1$ is also holomorphic. As far as complex analysis is concerned, these regions are "identically the same." Now let's introduce a bit of reductionism here. Assume $\psi : R_2 \to U = \{x \in \mathbb{C} : |x| < 1\}$, so that we have the following diagram:

$$\begin{tikzcd}
R_1 \arrow[r, "\varphi"] \arrow[d, "\varphi_i"'] & R_2 \arrow[dl, "\psi_i"] \\
U &
\end{tikzcd}$$

We need $R_1 \neq \mathbb{C}$: if $\varphi : \mathbb{C} \to U$, then φ is bounded, then constant by Liouville's theorem.

Theorem. Let $R \subsetneq \mathbb{C}$ be open, $R \neq \emptyset$, and $R \neq \mathbb{C}$. Choose a $z_0 \in R$. Then there exists a unique, one-to-one, holomorphic map $\varphi : R \to U$ such that $\varphi(z_0) = 0$ with $\varphi'(z_0) \in \mathbb{R}_{\geq 0}$.

Proof. Uniqueness: suppose $\varphi_1, \varphi_2 : R \to U$ both satisfy the theorem. Then $\varphi : \varphi_2 \circ \varphi_1^{-1} : U \to U$ is one-to-one, onto, and holomorphic, and $\varphi(0) = \varphi_2(\varphi^{-1}(0)) = \varphi_2(z_0) = 0$.

By the Schwarz lemma, we know that $|\varphi(z)| \leq |z|$ for all $z \in U$, and if φ is onto, $\varphi(z) = e^{i\theta} z$. Then $\varphi^{-1}(0) = e^{i\theta} \in \mathbb{R}_{\geq 0} \implies e^{i\theta} = 1 \implies \varphi(z) = z$. This implies $\varphi_2 = \varphi_1^{-1} = Id \implies \varphi_2 = \varphi_1$, as $\varphi'(0) = \frac{\varphi_2'(0)}{\varphi_1'(0)} > 0$. So uniqueness is easy. Let's work out an example for the other parts:

Example. Let $R = U$, $\varphi : R = U \to U$. If $\varphi(0) = 0$, then by Schwarz's lemma, we know that φ is onto $\iff |\varphi'(0)| = 1 \implies |\varphi'(0)|$ is the largest possible among all maps $U \to U$ if $0 \mapsto 0$.

If $\varphi : U \to U, \varphi(0) = \alpha \neq 0$, then $B_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}$ is such that $B_\alpha : U \to U$, $B_\alpha(0) = \alpha$, $B_\alpha(\alpha) = 0$; e.g. B_α swaps 0 and α. Then $|(\varphi \circ B_\alpha)'(0)| = |\varphi'(\alpha)||B'_\alpha(0)| \iff |\varphi'(\alpha)|$ is constant. It turns out that φ is one-to-one and onto $\implies |\varphi'(\alpha)|$ is maximized.

In general, we want to find $\varphi : R \to U$ that's one-to-one and holomorphic, with $\varphi(z_0) = 0, |\varphi'(z_0)|$ maximized. Even more precisely, we will take

$$\mathcal{F} = \{f : R \to U : f \text{ is one-to-one, holomorphic, and } f'(z_0) = 0\}.$$

We'll show

(a) $\mathcal{F} \neq \emptyset$

(b) $\sup_{f \in \mathcal{F}} f'(z_0) = M < \infty$

(b') There exists $f \in \mathcal{F} : f'(z_0) = M$.

(c) If $\varphi = f$ from above, then $\varphi : U \to U$ is one-to-one and onto, $\varphi(z_0) = 0$, and $\varphi'(z_0) > 0$.

In this case, the function we will cook up looks like B_α.

First we want $f : R \to U$ that is one-to-one and holomorphic. Suppose there exists $D(p_0, \delta) := \{z : |z - p_0| < \delta\}$ such that $D(p_0, \delta) \cap R - \emptyset$. Let $f(z) = \frac{\delta}{z - p_0}$. Then this is one-to-one since it's a Mobius transform, if $z \in R \implies |f(z)| = \frac{\delta}{|z - p_0|} < \frac{\delta}{\delta} = 1$.

What is no such disc exists? Choose any $p_0 \notin R$, and define $g(z) = \sqrt{\frac{z - p_0}{z_0 - p_0}}$ with $g(Z_0) = 1$, one-to-one since $g(z)^2$ is. Do this by choosing the appropriate square root. How do we choose a branch of $\sqrt{\ }$? Well,

$$\sqrt{\frac{z - p_0}{z_0 - p_0}} = \exp\left(\frac{1}{2} \log\left(\frac{z - p_0}{z_0 - p_0}\right)\right)$$

and

$$\log\left(\frac{z - p_0}{z_0 - p_0}\right) = \log(z - p_0) - \log(z_0 - p_0) = \int_{z_0}^{z} \frac{d\zeta}{\zeta - p_0}.$$

This makes sense by the closed curve theorem, as $\frac{1}{\zeta - p_0}$ is holomorphic on R.

Claim. There exists $\delta > 0$ such that $|g(z) + 1| > \delta$ for all $z \in R$.

Proof. If not, then $\exists z_0, z_1, ... \in R$ such that $g(z_i) \to -1$. Then $\sqrt{\frac{z_i - p_0}{z_0 - p_0}} \to -1$, and squaring $\frac{z_i - p_0}{z_0 - p_0} = 1 \implies \lim(z_i - p_0) = \lim(z_0 - p_0) \implies \lim z_i = z_0$. But by continuity of g, $\lim g(z_i) = g(z_0) = 1$, a contradiction. This proves (a).

Proof of (b). Since R is open, $D(z_0, 2\delta) \subseteq R$ for some $\delta > 0$. If $f \in \mathcal{F}$, then

$$|f'(z_0) = \left| \frac{1}{2\pi i} \int_{C(z_0, \delta)} \frac{f'(z)}{(z - z_0)^2} dz \right| \leq \frac{1}{2\pi} \int \frac{|f'(z)|}{|z - z_0|^2} dz \leq \frac{1}{2\pi} \int \frac{1}{\delta^2} dz = \frac{1}{\delta}$$

$$\implies \sup_{f \in \mathcal{F}} |f'(z)| \leq \frac{1}{\delta} < \infty$$

For (b'), choose $f_1, f_2, ... \in \mathcal{F}$ such that $\lim_{n\to\infty} |f_i(z_0)| = M$. Suppose that we knew that $f_i \to \varphi$ uniformly, where φ is holomorphic on R.

Claim. $\varphi \in \mathcal{F}$.

Proof. $f_i'(z) \to \varphi'(z_0) = M$. Also, for all $z \in R$, $|\varphi(z)| = \lim |f_n(z)| \leq 1$ so $\varphi : R \to \overline{U} =$ closed ball.

By the open mapping theorem, $\varphi(R)$ is open, so $\varphi(R) \subseteq U$.

Why is φ one-to-one? We need the fact that a limit of one-to-one functions is again one-to-one. Suppose not. Then $\exists z_1, z_2 \in R$ such that $\varphi(z_1) = \varphi(z_2) = a \in U$, and let D_1, D_2 be discs around z_1, z_2 such that $D_1 \cap D_2 = \emptyset$. Now $\varphi - a$ is not constant, since $\varphi - a$ has a zero on D_i. Thus $f_i - a$ has a zero on D_1 for infinitely many i, which implies that $f_i - a$ has no zeroes on D_2 for all such i. Thus $f_i(z_0) \to a$. Apply Hurwitz to the subsequence of such f_i on D_2, so $\varphi - a$ has no zeroes on D_2, which implies $\varphi(z_1) = a$, a contradiction. So we've checked that φ is one-to-one, holomorphic, and onto.

We need to show that the f_i converge uniformly on compact subsets $K \subseteq R$. We'll only use that $|f_i| \leq 1$ for all i. This is *Montel's theorem.*

Theorem. If $f_1, f_2, ... : R \to U$ is any sequence of functions, then there is some subsequence $f_{i_1}, f_{i_2}, ...$ that converges uniformly on compact subsets $K \subseteq R$.

Proof. Let $\{\xi_1, \xi_2, ...\} \subseteq R$ be a countable dense subset, e.g. $\{x + iy \in R : x, y \in \mathbb{Q}\}$. Since $\{f_n(\xi_i)\}_{n=i}^{\infty} \subseteq U$ is bounded, there is a subsequence $\{f_{1,n}\} \subseteq \{f_n\}$ such that $f_{1,n}(\xi_i) \to$ some number $\varphi(\xi_i)$. Likewise, there is $\{f_{2,n}\} \subseteq \{f_n\}$ such that $f_{2,n}(\xi_i) \to \varphi(\xi_i)$...

For every m we get $\{f_{m,n}\}$ such that $f_{m,n}(\xi_i) \to \varphi(\xi_i)$ for all $i \leq m$. Let $\varphi_n = f_{n,n}$, the "diagonal" elements, so that $\{\varphi_n\}_{n \neq m} \subseteq \{f_{m,n}\}_{n \geq m}$. Thus $\varphi_n(\xi_m) \to \varphi(\xi_n)$ for all $m \geq 1$.

Claim. $\{\varphi_n\}_{n=1}^{\infty}$ converges on all of R.

Proof. Let $D = D(x', 3d) \subseteq R$ be some disk, and let $K = \overline{D(z', d)} \subset D(z', 3d) \subseteq \mathbb{R}$ be a compact subset of R. Moreover, for all $z \in K$, $C(z, d) \subseteq D$. Since $|\varphi_n| < 1, \forall z \in K$ we have

$$|\varphi_n'(z)| = \left| \frac{1}{2\pi i} \int_{C(z,d)} \frac{\varphi_n(\xi)}{(\xi - z)^2} d\xi \right| \leq \frac{1}{d}.$$

$\implies \forall z_1, z_2 \in K$,

$$|\varphi_n(z_1) - \varphi_n(z_2)| = \left| \int_{z_1}^{z_2} \varphi_n(z) dz \right| \leq \left| \int_{z_1}^{z_2} |\varphi_n(z)| dz \right| \leq \left| \int_{z_1}^{z_2} \frac{dz}{d} \right| = \frac{1}{d} |z_1 - z_2|.$$

Hence $\{\varphi_n\}$ is equicontinuous on K in that $\forall n, z_1, z_2 \in K$, if $|z_1 - z_2| < d\epsilon$, then $|\varphi_n(z_1) - \varphi_n(z_2)| < \epsilon$.

For $z \in K, n, m \geq 0, k \geq 0$,

$$|\varphi_n(z) - \varphi_m(z)| \leq |\varphi_n(z) - \varphi_n(\xi_k)| + |\varphi_n(\xi_k) - \varphi_m(\xi_k)| + |\varphi_m(\xi_k) - \varphi_m(z)|.$$

If we choose $\xi_k \in K$ such that $|z - \xi_k| < \frac{1}{3}\epsilon d$, and choose N s.t. $\forall n, m \geq N$, $|\varphi_n(\xi_k) - \varphi_m(\xi_k)| < \epsilon/3$, then

$$|\varphi_n(z) - \varphi_m(z)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Thus $\{\varphi_n(z)\}$ is Cauchy, and $\varphi_n(z) \to \varphi(z)$. We've proven $\varphi_n \to \varphi$ pointwise.

Claim. φ is uniformly continuous on K if $|z_1 - z_2| < d\epsilon$.

$$|\varphi(z_1) - \varphi(z_2)| = \lim |\varphi_n(z_1) - \varphi_n(z_2)| \leq \epsilon.$$

Uniform convergence on K? Suppose $\varphi_n \to \varphi$. Let $\epsilon > 0$, and $S_j := \{z \in K : |\varphi_n(z) - \varphi(z)| < \epsilon \text{ for all } n > j\}$. S_j is open: indeed, if $z_1 \in S_j$ and $|z_1 - z_2| < d\delta$ for $\delta > 0$, then

$$|\varphi_n(z_1) - \varphi(z_2)| \leq |\varphi_n(z_1) - \varphi_n(z_1)| + ... \leq |\varphi_n(z_1) - \varphi(z)| + 2\delta < \epsilon/2$$

for all $n > j$. Taking δ small enough, $z_2 \in S_j$. This implies $D(z, \delta) \subseteq S_j$.

$$\bigcup_{j=1}^{\infty} S_j = K$$

Compactness implies $K + S_j$ for some j implies uniform convergence. All you do now is fiddle around with B_α's.

CHAPTER 19

RIEMANN MAPPING THEOREM AND INFINITE PRODUCTS

We will continue with the Riemann mapping theorem, which we got through most of last time:

Theorem. (Riemann mapping) Let $R \subset \mathbb{C}$ be open and simply connected (for us, use $\hat{\mathbb{C}} - R$ is connected, i.e. can apply closed curve theorem). Also suppose $R \neq \mathbb{C}$. Then $\exists!$ map f that is holomorphic, with holomorphic inverse, s.t. $f : R \to \mathbb{D}$, with $f(\alpha) = 0, \mathbb{R}_{\geq 0} \ni f'(\alpha) > 0$.

Improperly Stated Generalization. (Uniformization theorem) Every simply connected Riemann surface is conformal to, and has a holomorphic map with holomorphic inverse to, precisely one of
(1) $\hat{\mathbb{C}} = S^2$
(2) $\mathbb{C}$
(3) $\mathbb{D}$
The nice thing is that these correspond to different geometries. $\mathbb{D}$ is the hyperbolic geometry, S^2 you've done on homework, and $\mathbb{C}$ is Euclidean. We don't have the tools to do this properly yet.

Proof of the Riemann mapping theorem so far. Let

$$\mathcal{F} = \{f : R \to \mathbb{D} : f \text{ is holomorphic, 1-1, and } f'(\alpha) > 0\}.$$

We show that
(A) $\mathcal{F} \neq \emptyset$.
(B) $\sup_{f \in \mathcal{F}} |f'(\alpha)| < \infty$ and $\exists g \in \mathcal{F}$ achieving this sup, e.g. $|g'(\alpha)| = \sup_{f \in \mathcal{F}} |f'(\alpha)|$.
(C) Given this g achieving the sup, g is surjective and $g(w) \neq 0$ for $w \in \mathbb{R}$ (so that we get a holomorphic inverse).
We can also draw pictures for A and B, which we showed last time with the help of the following:

Lemma. (Montel's theorem) A uniformly bounded sequence of holomorphic functions has a subsequence which converges on compact subsets of the domain, and the limit is holomorphic (using Morera's theorem).

For more info here, see also the Arzela-Ascoli theorem.

Let's finish up with C.

Claim. This limit g has the property that $g(\alpha) = 0$ and is surjective (any 1-1 function f can have $f'(w) = 0$ for any w.).

Proof. g sends $0 \mapsto 0$. If not, $g(\alpha) = w \neq 0$, and $B_h \circ g : \alpha \mapsto 0$. Then $|(B_w \circ g)'(\alpha)| = |B_w'(g(\alpha))g'(\alpha)| = \frac{1}{1-|w|^2}|g'(\alpha)| > 1$, and there is a larger such g.

g is surjective. Suppose not, that that it misses w. Wlog $w = 0$, and consider $(B_w \circ g)$, which sends $\alpha \mapsto w$ and misses 0. Then $|(B_w \circ g)'(\alpha)| = |B_w'(g(\alpha))||g'(\alpha)| = (1-|w|^2)|g'(\alpha)|$. Now consider the following claim:

Claim. $\exists$ a holomorphic branch of $\sqrt{B_w \circ g}$ (which maps $R \to \mathbb{D}$)

Proof. We can write

$$\log(B_w \circ g)(z) - \log(B_w \circ g)(\alpha) = \int_\alpha^z \frac{(B_w \circ g)'(z)}{(B_w \circ g)(z)} dz = *.$$

Then take $e^{\frac{1}{2}(*)}$.

We have $(\sqrt{\ } \circ B_w \circ g) : w \mapsto \sqrt{w}$ by construction. Then $(\sqrt{\ } \circ B_w \circ g)'(\alpha)| = |\sqrt{\ }'(w)|(1 - |w|^2)|g'(\alpha)| = \frac{1}{2|w|^{1/2}}(1 - |w|^2)|g'(\alpha)|$. Also, $(B_{\sqrt{w}} \circ \sqrt{\ } \circ B_w \circ g) : \alpha \mapsto 0$, and $(B_{\sqrt{w}} \circ \sqrt{\ } \circ B_w \circ g)'(\alpha) = |B_{\sqrt{w}}'(\sqrt{w})| \ldots = \frac{1}{1-|w|}\frac{1}{2|w|^{1/2}}(1 - |w|^2)|g'(\alpha)|$.

Set $|w| = r^2$, for $r \in \mathbb{R}_{\geq 0}$ (with w not 0 b/c $g'(\alpha) > 0$). We claim that

$$\left(\frac{1}{1-r^2}\right)\left(\frac{1}{2r}\right)(1 - r^4) > 1$$

for $0 < r < 1$. To see this, note that

$$1 + r^2 - 2r = (r-1)^2 > 0$$

for $0 < r < 1$, so $\frac{1+r^2}{2r} > 1$.

So g is surjective. Hence g is injective. We claim that $g'(w) \neq 0$ for any w. To see this, suppose $g'(w) = 0$. Wlog suppose $w = 0$. Then $g'(0) = 0$ and $g(z) = z^k h(z)$ for $k \geq 2$, with h holomorphic and $h'(0) \neq 0$. Hence

$$\int_{C_\epsilon(0)} \frac{g'(z)}{g(z)} dz = \mathbb{Z}(g)$$

inside $D_\epsilon(0) = k$. Take β with $|\beta|$ small; then

$$k = \int_{C_\epsilon(0)} \frac{g'(z)}{g(z) - \beta} dz = \mathbb{Z}(g - \beta)$$

in $D_\epsilon(0)$. If so, $g(z) - \beta$ is 0 at inverse images of β, but all of the same small ball around 0 is an inverse image of something. So $g'(0) = 0$ and g is constant. ∎

Here are some consequences of the Riemann mapping theorem:

(1) Every connected and simply connected region in $\mathbb{R}^2$ is topologically equivalent (in the sense of homeomorphic) to a disk. For example, consider taking out a Cantor set in $\mathbb{R}^2$.

(2) See moduli spaces of curves and their relations to string theory.

Back to infinite products and the gamma function. Let's finish the proof of the following theorem:

Theorem.

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Corollary. Set $z = 1/2$, and

$$1 = \frac{\pi}{2} \prod_{k=1}^{\infty} \left(10 \frac{1}{(2k)^2}\right) = \frac{\pi}{2} \prod_{k=1}^{\infty} \left(\frac{(2k-1)(2k+1)}{(2k)(2k)}\right).$$

Then

$$\frac{\pi}{2} = \left(\frac{2 \cdot 2}{1 \cdot 3}\right) \left(\frac{4 \cdot 4}{3 \cdot 5}\right) \left(\frac{6 \cdot 6}{5 \cdot 7}\right) \cdots.$$

This has a taste of number theory.

Proof of the theorem. The gist of the proof is to write

$$Q(z) = \frac{\pi z \prod_{k=1}^{\infty}(1 - \frac{z^2}{k^2})}{\sin(\pi z)},$$

which has growth order of at most something like $Ae^{|z|^{3/2}}$. We conclude that $Q(z)$ is a constant, which taking limits we see to be $= 1$. See the textbook for a more thorough proof. ■

This is presumably the flavor of analytic number theory, where we can't show things definitely but can use complex analysis to give a (poor) bound that translates into something nice. We'll continue with the gamma function $\Gamma(z)$ next time.

CHAPTER 20

ANALYTIC CONTINUATION OF GAMMA AND ZETA

Analytic continuation of gamma and zeta. So we have these two functions:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad \zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z}$$

The first is holomorphic for $\mathrm{Re}(z) > 0$ and the second for $\mathrm{Re}(z) > 1$; this is just an application of Morera's theorem. We'll talk about the gamma function for a while; what we'd like is for these functions to be holomorphic in all of the complex plane. It turns out that the properties of the gamma function will be important for understanding prime numbers. Let's start with gamma—recall what happened last time. We have

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt = \int_0^1 e^{-t} t^{z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt.$$

The right integral uniformly converges on $B_R(0)$, i.e. it converges on compact subsets of $\mathbb{C}$. Now look at the left integral. We saw that it converges uniformly for compact subsets of $\mathrm{Re}(z-1) > -1$, i.e. $\mathrm{Re}(z) > 0$.

How do we extend this to all of $\mathbb{C}$? The extension will have various poles, and is meromorphic in $\mathbb{C}$.

Claim. $\Gamma(z+1) = z\Gamma(z)$.

Proof. We integrate by parts. Consider

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Take $u = t^{z-1}, dv = e^{-t}dt \Longrightarrow du = (z-1)t^{z-2}dt, v = -e^{-t}$. Then

$$\int_0^\infty e^{-t} t^{z-1} dt = -e^{-t} t^{z-1}\big|_0^\infty + \int_0^\infty (z-1)e^{-t} t^{z-2} dt = (z-1)\Gamma(z-1),$$

as desired. ∎

Corollary. $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}_{\geq 1}$.

Proof. Check $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$ from the claim. ■

From the above, we can see that
$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$
is meromorphic (holomorphic) for $\mathrm{Re}(z+1) > 0$, i.e. $\mathrm{Re}(z) > -1$ with a pole only at $z = 0$. Repeating this k times, we see that
$$\Gamma(z) = \frac{\Gamma(z+k+1)}{z(z+1)...(z+k)}$$
is meromorphic for $\mathrm{Re}(z+k+1) > 0$, i.e. $\mathrm{Re}(z) > -k+1$, with poles only at nonnegative integers. This gives a continuation of Γ into the left side of $\mathbb{C}$.

The residues at the poles is given by
$$\mathrm{Res}(\Gamma(z); 0) = \frac{\Gamma(1)}{1} = 1,$$
$$\mathrm{Res}(\Gamma(z); -k) = \frac{\Gamma(1)}{(-k)(-k+1)...(-1)} = \frac{(-1)^k}{k!}.$$
These poles are indeed simple by our calculations above. We will show another nice claim:

Claim. $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$.

Corollary. Γ has no zeroes.

Proof. This property follows directly from the claim. $\sin(\pi z)$ is nonzero for z not an integer; for z an integer, we already know that Γ has a pole if $z \in \mathbb{Z}_{\leq 0}$ or $\Gamma(z) = (z-1)! \neq 0$ if $z \in \mathbb{Z}_{\geq 1}$. ■

To prove the initial claim, we'll need to show another fact, that for
$$e^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n$$
we have
$$\Gamma(z) = \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}dt,$$
e.g. we can switch the lim and $\int$. Then
$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}dt = \lim_{n\to\infty} \frac{1}{n^n} \int_0^n (n-t)^n t^{z-1}dt,$$
and from integration by parts we have
$$\int_0^n (n-t)^n t^{z-1}dt = \frac{t^z}{z}(n-t)^n \Big|_0^n + \int_0^n \frac{t^z}{z} n(n-t)^{n-1}dt.$$
So
$$\lim_{n\to\infty} \frac{1}{n^n} \int_0^n (n-t)^n t^{z-1}dt = \lim_{n\to\infty} \frac{1}{n^n} \frac{n}{z} \int_0^n t^z (n-t)^{n-1}$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \frac{n(n-1)...1}{z(z+1)...(z+n-1)} \int_0^n t^{z+n-1} dt$$

$$= \lim_{n\to\infty} \left(\frac{n^z}{z} \left(\frac{1}{z+1} \right) ... \left(\frac{n}{z+n} \right) \right) \qquad \text{since} \qquad \left. \frac{t^{z+n}}{z+n} \right|_0^n = \frac{n^n n^z}{z+n}.$$

This gives

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} \frac{z}{n^z} \left(\frac{z+1}{1} \right) ... \left(\frac{z+n}{n} \right) = \lim_{n\to\infty} \frac{z}{n^z} \prod_{k=1}^{n} \left(1 + \frac{z}{k} \right),$$

which looks something like the expression for $\sin(\pi z)$ we saw before. Let's rewrite the last expression as

$$\lim_{n\to\infty} \frac{z}{n^z} \prod_{k=1}^{n} \left(1 + \frac{z}{k} \right) = \lim_{n\to\infty} zn^{-z} \exp\left(z + \frac{z}{2} + \frac{z}{3} + ... + \frac{z}{n} \right) \prod_{k=1}^{\infty} \left[\left(1 + \frac{z}{k} \right) e^{-z/k} \right],$$

and the product by itself converges as shown before. Rewriting once again with $n^{-z} = e^{-(\log n)z}$,

$$\lim_{n\to\infty} zn^{-z} \exp\left(z + \frac{z}{2} + \frac{z}{3} + ... + \frac{z}{n} \right) \prod_{k=1}^{\infty} \left[\left(1 + \frac{z}{k} \right) e^{-z/k} \right]$$

$$= \lim_{n\to\infty} \exp\left[z \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) \right] z \prod_{k=1}^{\infty} \left[\left(1 + \frac{z}{k} \right) e^{-z/k} \right].$$

Also,

$$\lim_{n\to\infty} \left[\sum_{k=1}^{n} \left(\frac{1}{k} \right) - \log n \right] = \gamma \approx 0.577,$$

the *Euler-Mascheroni constant.* Then

$$\frac{1}{\Gamma(z)} = e^{-\gamma z} z \prod_{k=1}^{\infty} \left[\left(1 + \frac{z}{k} \right) e^{-z/k} \right].$$

From this we see that

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{1}{-z\Gamma(z)\Gamma(-z)} = \frac{1}{-z} e^{-\gamma z} e^{\gamma z} z(-z) \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2} \right) = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2} \right) = \frac{\sin(\pi z)}{\pi},$$

which is a cute equation. It also gives the values

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \qquad \text{“}(1/2)!\text{”} = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

The zeta function. We turn to $\zeta(z)$, defined as

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \qquad \mathrm{Re}(z) > 1.$$

We claim the following:

Claim. $\zeta(z)$ has a meromorphic extension to all of $\mathbb{C}$ with a simple pole only at 1.

We'll show next time that $\zeta(z)$ has no zeroes on $\mathrm{Re}(z) = 1$, and from this deduce the prime number theorem, which relates to the growth rate of the number of prime numbers. You may have also heard of the *Riemann hypothesis*, which says that the nontrivial zeros of ζ are all on the line $\mathrm{Re}(z) = 1/2$; this gives us a stronger estimate of the growth rate of the primes.

Claim. For $\mathrm{Re}(z) > 1$,

$$\zeta(z) = \prod_{p \text{ prime}} \left(\frac{1}{1 - p^{-z}} \right).$$

Proof. We first write

$$\frac{1}{1 - 1/p^z} = 1 + \frac{1}{p^z} + \frac{1}{p^{2z}} + \frac{1}{p^{3z}} + ...$$

This gives us something of the form $1 + 1/2^z + 1/3^z + ...$ If we write $n = p_1^{\alpha_1} ... p_k^{\alpha_k}$, then

$$\frac{1}{n^z} = \frac{1}{p_1^{\alpha_1} z} ... \frac{1}{p_k^{\alpha_k} z}$$

in the product. Then

$$\left| \sum_{n=1}^{N} \frac{1}{n^z} \right| \leq \left| \prod_{p<N} \left(1 + \frac{1}{p^z} + ... + \frac{1}{p^{Mz}} \right) \right|$$

for some M. Moreover,

$$\left| \prod_{p<N} \left(1 + \frac{1}{p^z} + ... + \frac{1}{p^{Mz}} \right) \right| \leq \left| \prod_{p<N} \left(\frac{1}{1 - p^{-z}} \right) \right| \leq \left| \prod_{p \text{ prime}} \left(\frac{1}{1 - p^{-z}} \right) \right|.$$

In the limit, we thus have

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^z} \right| \leq \left| \prod_{p \text{ prime}} \left(\frac{1}{1 - p^{-z}} \right) \right|$$

For the reverse direction, we have

$$\prod_{p<N} \left(1 + \frac{1}{p^z} + ... + \frac{1}{p^{Mz}} \right) \leq \sum_{n=1}^{N} \frac{1}{n^z},$$

etc. Then

$$\left(1 - \frac{1}{2^z} \right) \zeta(z) = 1 + \frac{1}{3^z} + \frac{1}{5^z} + ...$$

and

$$\prod_{p<N} \left(1 - \frac{1}{p^z} \right) \zeta(z) = \sum_{n \text{ has no prime factors} < N} \frac{1}{n^z} \to 1 \text{ as } N \to \infty,$$

giving the desired equality. ■

Now let's extend $\zeta(z)$ to all of $\mathbb{C}$ (with a pole). With the change of variables $s = nt$, we have

$$\int_0^\infty e^{-nt} t^{z-1} dt = \int_0^\infty e^{-s} \frac{s^{z-1}}{n^{z-1}} \frac{ds}{n} = n^{-z}\Gamma(z).$$

Then

$$\zeta(z) = \sum_{n=1}^\infty \frac{1}{\Gamma(z)} \int_0^\infty e^{-nt} t^{z-1} dt = \frac{1}{\Gamma(z)} \int_0^\infty \left(\sum_{n=1}^\infty e^{-nt} \right) t^{z-1} dt = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt$$

for $\text{Re}(z) > 1$. This is defined on the entire complex plane:

$$\zeta(z) = \frac{1}{\Gamma(z)} \left[\int_0^1 \frac{t^{z-1}}{e^t - 1} dt + \int_1^\infty \frac{t^{z-1}}{e^t - 1} dt \right].$$

The right integral is holomorphic by Morera's theorem and the fact that it converges uniformly on compact subsets of $\mathbb{C}$. For the first integral, we break the integrand up. Note that the function $f(z) = \frac{1}{e^z - 1}$ has a simple pole at $z = 0$ near the real line, and is otherwise holomorphic. Its Laurent expansion around $z = 0$ converges on $A(0, 2\pi)$, and uniformly and absolutely on $A(\varepsilon, 2\pi - \varepsilon)$:

$$\frac{1}{e^t - 1} = \frac{1}{t} + c_0 + c_1 t + c_2 t^2 + \ldots$$

Note that $c_{-1} = 1$. Rewriting the integral with the Laurent expansion, we have

$$\int_0^1 \frac{t^{z-1}}{e^t - 1} dt = \sum_{n=-1}^\infty \int_0^1 t^{z-1+n} c_n dt,$$

and we can commute the $\sum$ and $\int$ by absolute convergence of the integrand. Then we can simplify:

$$\sum_{n=-1}^\infty \left[\frac{c_n}{z+n} \right] = \frac{1}{z-1} + \frac{c_0}{z} + \frac{c_1}{z+1} + \ldots.$$

[Alternatively, the $\int_0^1$ is holomorphic and converges since

$$\int_0^1 \frac{x^{s-1}}{e^x - 1} dx = \sum_{m=0}^\infty \frac{B_m}{m!(s+m-1)},$$

where B_m denotes the m-th Bernoulli number defined by

$$\frac{x}{e^x - 1} = \sum_{m=0}^\infty \frac{B_m}{m!} x^m.$$

Then $B_0 = 1$, and since $\frac{z}{e^z - 1}$ is holomorphic for $|z| < 2\pi$, we must have $\limsup_{m\to\infty} |B_m/m!|^{1/m} = 1/2\pi$.] Anyway, we can claim

Claim. This gives a holomorphic function on all of $\mathbb{C}$.

Claim. $\sum_{n=1}^{\infty} \frac{c_n}{z+n}$ converges uniformly for z in $B_R(0) - \cup_{n=-1}^{\lfloor R \rfloor} B_\epsilon(-n)$. The same argument also shows it's meromorphic by excluding the term $\frac{1}{z+k}$ and $B_\epsilon(-k)$.

The punchline here is that *there exists a meromorphic function on* $\mathbb{C}$ *which equals* $\sum_{n=1}^{\infty} \frac{1}{n^z}$ *for* $Re(z) > 1$ *and has a pole only at* $z = 1$*.*

CHAPTER 21

ZETA FUNCTION AND PRIME NUMBER THEOREM

Functional equations towards the prime number theorem. Last time we had

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt, \qquad \mathrm{Re}(z) > 1,$$

where

$$\frac{1}{e^t - 1} = \frac{1}{t} + c_0 + c_1 t + \ldots$$

We used this to see that $\zeta(z)$ extends to a meromorphic function in all of $\mathbb{C}$ with a simple pole at $z = 1$ (and this is the only pole). We can see this by breaking the integral down into

$$\frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt = \frac{1}{\Gamma(z)} \left(\int_0^1 \frac{t^{z-1}}{e^t - 1} dt + \int_1^\infty \frac{t^{z-1}}{e^t - 1} dt \right),$$

and noting that the right hand integral is holomorphic by Morera's theorem and uniform convergence on compact subsets of $\mathbb{C}$, while the left hand integral is also holomorphic and converges.

Functional equation. Define a *theta function* $\vartheta(t)$ as

$$\vartheta(t) := \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}.$$

This has the property that

$$\vartheta(t) = t^{-1/2} \vartheta(1/t)$$

for $t \in \mathbb{R}_{>0}$, which is on the problem set and uses the Fourier transform.

Theorem.

$$\pi^{-z/2} \Gamma(z/2) \zeta(z) = \frac{1}{2} \int_0^\infty u^{z/2-1} (\vartheta(u) - 1) du = \int_0^\infty u^{z/2-1} \psi(u) du,$$

where
$$\psi(u) := \frac{\vartheta(u)-1}{2} = \sum_{n=1}^{\infty} e^{-\pi n^2 u}.$$

Proof. First we rewrite
$$\frac{1}{2}\int_0^\infty u^{z/2-1}(\vartheta(u)-1)du = \int_0^\infty \left(\sum_{n=1}^{\infty} u^{z/2-1}e^{-\pi n^2 u}\right) du.$$
Note that
$$\int_1^\infty u^{z/2-1}(\vartheta(u)-1)du$$
converges nicely because $\vartheta(u)-1$ is bounded by the decreasing exponential:
$$\left|\sum_{n=1}^{\infty} e^{-\pi n^2 u}\right| \le \left|\sum_{m=1}^{\infty} e^{-\pi m u}\right| = \frac{e^{-\pi u}}{1-e^{-\pi u}}.$$
So for $u = 1$, this is bounded and the integral converges. Near 0, we consider
$$\frac{1}{2}\int_0^1 u^{z/2-1}(\vartheta(u)-1)du.$$
and as $u \to 0$, $\vartheta(u)$ is well-behaved and the integral converges. Take the original integral and swap the sum and integral (justifying the interchange of summation and integral yourself) to get
$$\int_0^\infty \left(\sum_{n=1}^{\infty} u^{z/2-1}e^{-\pi n^2 u}\right) du = \sum_{n=1}^{\infty}\int_0^\infty u^{z/2-1}e^{-\pi n^2 u}du$$
Let $t = \pi n^2 u$, $dt = \pi n^2 du$. Then
$$\sum_{n=1}^{\infty}\int_0^\infty u^{z/2-1}e^{-\pi n^2 u}du = \sum_{n=1}^{\infty}\int_0^\infty t^{z/2-1}e^{-t}dt \cdot (\pi n^2)^{-z/2} = \sum_{n=1}^{\infty}\Gamma(z/2)\pi^{-z/2}n^{-z}$$
$$= \Gamma(z/2)\pi^{-z/2}\zeta(z),$$
giving us the desired result. ■

What do we do with this theorem? Let's define the *xi function*
$$\xi(z) = \Gamma(z/2)\pi^{-z/2}\zeta(z).$$

Theorem. $\xi(z)$ is holomorphic for $\mathrm{Re}(z) > 1$, extends to a meromorphic function on all of $\mathbb{C}$ with poles at 0 and 1, and
$$\xi(z) = \xi(1-z).$$

Proof. First we have $\vartheta(u) = u^{-1/2}\vartheta(1/u)$. Then
$$\psi(u) = \frac{\vartheta(u)-1}{2} = \frac{u^{-1/2}\vartheta(1/u)-1}{2} = u^{-1/2}\frac{\vartheta(1/u)-1}{2} + \frac{1}{2u^{1/2}} - 1/2$$

$$= u^{-1/2}\psi(1/u) + \frac{1}{2u^{1/2}} - 1/2.$$

We break the integral up into two pieces:

$$\xi(z) = \int_0^1 u^{z/2-1}\psi(u)du + \int_1^\infty u^{z/2-1}\psi(u)du.$$

Let $t = -1/u, du = -1/u^2 du, du = -1/t^2 dt$. Then

$$\int_0^1 u^{z/2-1}\psi(u)du = \int_1^\infty \frac{t^{-z/2+1}}{t^2}\left(t^{1/2}\psi(t) + \frac{t^{1/2}}{2} - \frac{1}{2}\right)dt$$

$$= \int_1^\infty t^{-z/2-1/2}\psi(t)dt + \int_1^\infty \left(t^{-z/2-1/2} - \frac{1}{2}t^{-z/2-1}\right)dt$$

$$= \int_1^\infty t^{-z/2-1/2}\psi(t)dt + \frac{1}{2}\left.\frac{t^{-(z-1)/2}}{-\frac{z-1}{2}}\right|_1^\infty + \frac{1}{2}\left.\frac{t^{-z/2}}{-z/2}\right|_1^\infty$$

$$= \frac{1}{z-1} - \frac{1}{z} + \int_1^\infty t^{-z/2-1/2}\psi(t)dt$$

$$\implies \xi(z) = \frac{1}{z-1} - \frac{1}{z} + \int_1^\infty \psi(u)\left(u^{z/2-1} + u^{-z/2-1/2}\right)du,$$

and the whole thing is symmetric through $z \mapsto 1-z$. ■

The conclusion is that ξ is symmetric about $x = 1/2$. There are no zeroes with $x > 1$, and by symmetry there are no zeroes with $x < 1$. Note that $\Gamma(z)$ has simple poles at nonnegative integers, while $\Gamma(z/2)$ has simple poles at $0, -2, -4, \ldots$. Then

$$\zeta(z) = \frac{\pi^{z/2}\xi(z)}{\Gamma(z/2)}$$

has trivial zeroes at $-2, -4, \ldots$, and we'll show that there are no zeroes right of $x = 1$ (also, there is neither a pole nor zero at 0).

Claim. $\zeta(z)$ has no zeroes in $\{\mathrm{Re}(z) > 1\}$.
Proof. $\zeta(z) = \prod_{p \text{ prime}}(1 - 1/p^z) \geq 1$ for $\mathrm{Re}(z) > 1$. ■

Theorem. $\zeta(z) \neq 0$ for $\mathrm{Re}(z) = 1$.
Proof. Deferred to below.

Theorem. (Prime Number) Let $\pi(N) = \#$ of primes $\leq N$. Then

$$\pi(N) \sim \frac{N}{\log N},$$

i.e.

$$\frac{\pi(N)}{N/\log N} \to 1 \quad \text{as} \quad N \to \infty.$$

We won't show this theorem today, but we will show that $\zeta(z)$ has no zeroes for $\mathrm{Re}(z) = 1$.

To do this we need a few other results:

Lemma.
$$\log \zeta(z) = \sum_{m \in \mathbb{Z}, p \text{ prime}} \frac{1}{mp^{mz}}.$$

Proof.
$$\zeta(z) = \prod_{p \text{ prime}} \left(\frac{1}{1 - 1/p^z} \right)$$
$$\log \zeta(z) = -\sum_{p \text{ prime}} \log\left(1 - \frac{1}{p^z}\right) = \sum_{p \text{ prime}} \sum_{m=1}^{\infty} \frac{(p^{-z})^m}{m} = \sum_{m \in \mathbb{Z}, p \text{ prime}} \frac{1}{mp^{mz}}.$$
Note that this is much like the zeta function:
$$\sum_{p \text{ prime}} \sum_{m=1}^{\infty} \frac{(p^{-z})^m}{m} = \sum_{m \in \mathbb{Z}, p \text{ prime}} \frac{1}{mp^{mz}} = \sum_{n=1}^{\infty} \frac{c_n}{n^z},$$
where
$$c_n = \begin{cases} 1/m & n = p^m \\ 0 & \text{otherwise} \end{cases}$$
and $c_n \geq 0$. This is an instance of a Dirichlet series, which we won't talk about in detail. (Take Math 229x for this.) ■

Lemma. $3 + 4\cos\theta + \cos 2\theta \geq 0$.

Proof. Trivial. ■

Lemma. For $z = x + iy, x > 1$,
$$\log |\zeta(x)^3 \zeta(x+iy)^4 \zeta(x+2iy)| \geq 0.$$

Proof. We have
$$\begin{aligned}\log |\zeta(x)^3 \zeta(x+iy)^4 \zeta(x+2iy)| &= 3\log|\zeta(x)| + 4\log|\zeta(x+iy)| + \log|\zeta(x+2iy)| \\ &= 3\mathrm{Re}(\log\zeta(x)) + 4\mathrm{Re}(\log\zeta(x+iy)) + \mathrm{Re}(\zeta(x+2iy)) \\ &= \sum_n n^{-x}(3 + 4\cos(y\log n) + \cos(2y\log n)) \geq 0,\end{aligned}$$
as desired. ■

Proof of the theorem. We wish to show that $\zeta(z) \neq 0$ for $\mathrm{Re}(z) = 1$. Suppose $\zeta(x+2iy) = 0$. Then consider $\zeta(x)^3\zeta(x+iy)^4\zeta(x+2iy) = A$. If $\zeta(1+iy) = 0$, then as $x \to 1^+$, $A \to 0$, a contradiction. This is because $\zeta(z)$ has a simple pole at $z = 1$, $\zeta(z)$ has zero at $z = 1 + iy$, and $\zeta(z)$ is holomorphic (no pole) at $z = 1 + 2iy$. ■

Chebyshev/Tchebychev's $\psi(x)$ function. (unrelated to $\psi(u)$ earlier) Define
$$\psi(x) := \sum_{p^m < x} \log p = \sum_{n < x} \Lambda(n) = \sum_{p < x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p,$$

for the *von Mangoldt* Λ *function* defined as

$$\Lambda(n) = \begin{cases} \log p & n = p^m \\ 0 & n \neq p^m, \end{cases}$$

and p a prime.

Proposition. $\psi(x) \sim x \iff \pi(x) \sim \frac{x}{\log x}$.

Proof. We will only show the forward direction. Note that

$$\psi(x) = \sum_{p<x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p \leq \sum_{p<x} \log x = \log x \cdot \pi(x).$$

So

$$\frac{\psi(x)}{x} \leq \frac{\pi(x) \log x}{x},$$

and if $\psi(x)/x \to 1$, then

$$1 = \liminf \frac{\pi(x) \log x}{x}.$$

For the opposite inequality, note that

$$\psi(x) \geq \sum_{p<x} \log p \geq \sum_{x^\alpha < p < x} \log p \geq (\pi(x) - \pi(x^\alpha)) \log(x^\alpha)$$

for $\alpha \in (0, 1)$. Hence

$$\psi(x)/x + \frac{\alpha \pi(x^\alpha) \log x}{x} \geq \alpha \frac{\pi(x) \log x}{x},$$

i.e.

$$\frac{\psi(x)}{x} + \frac{\alpha \log x}{x^{-\alpha}} \geq \frac{\alpha \pi(x) \log x}{x}.$$

So if $\frac{\psi(x)}{x} \to 1$,

$$1 \geq \alpha \limsup \frac{\pi(x) \log(x)}{x}$$

for all $\alpha \in (0, 1)$, so

$$1 \geq \limsup \frac{\pi(x) \log x}{x}$$

and $\phi(x)/x \to 1 \implies \frac{\pi(x) \log x}{x} \to 1$. ■

CHAPTER 22

PRIME NUMBER THEOREM

Prime number theorem. Recall from last time that if $\pi(x)$ is the number of primes $\leq x$, then the prime number theorem states that $\pi(x) \sim \frac{x}{\log x}$. We'll go about proving this today.

A bit of history: while the prime number theorem was proved in the 1800s, an elementary proof of it was still in the air in the 1940s. Selberg and Erdös actually had a bitter fight over it: Selberg came up with a formula that implied the Prime Number Theorem but didn't work out the details until Erdös had already did, and they disagreed on who got the credit for it. Eventually Selberg won out, much to Erdös's chagrin. There is an even simpler proof using basic complex analysis, given by Newman, one of the authors of our textbook.

Anyways, recall that we had

$$\log \zeta(z) = \sum_{p \text{ prime}, n \geq 1} \frac{1}{np^{nz}}$$

for $\text{Re}(z) > 1$. Since

$$\sum_{p \text{ prime}, n \geq 2} \frac{1}{np^{nz}} = \sum_{p \text{ prime}} \left(\frac{1}{2p^{2z}} + \frac{1}{3p^{3z}} + ... \right)$$

is holomorphic where $\text{Re}(z) > 1/2$,

$$\sum_{p \text{ prime}} \frac{1}{p^z} = \log \zeta(z) - \sum_{p \text{ prime}, n \geq 2} \frac{1}{np^{nz}}$$

is holomorphic where $\text{Re}(z) > 1/2$. Note that $\log(z-1)\zeta(z)$ is entire, holomorphic on a subset of $\text{Re}(z) \geq 1$, so that $\sum_{p \text{ prime}} \frac{1}{p^z} + \log(z-1)$ is as well, along with $\sum_{p \text{ prime}} \frac{-\log p}{p^z} + \frac{1}{z-1}$.

Let ϕ be defined as

$$\phi(z) = \sum_{p \text{ prime}} \frac{\log p}{p^z}.$$

Lemma. $\phi(z) - \frac{1}{z-1}$ is holomorphic where $\text{Re}(z) \geq 1$.

Proof. See above.

We also consider the function $\theta(x) = \sum_{p \le x} \log p$ for real x.

Theorem. $\theta(x) \sim x \Longrightarrow \pi(x) \sim x$.

Proof. We have $\theta(x) = \sum_{p \le x} \log p \le \sum_{p \le x} \log x = \pi(x) \log x$. ■

Chebyshev showed that $\theta(x) \le C \cdot x$ in limit, i.e. $\limsup \frac{\theta(x)}{x} < c$, which we will show later. For the moment, we turn to proving another useful theorem first.

Theorem. Suppose

$$g(z) = \int_0^\infty e^{-zt} f(t) dt$$

for f bounded and integrable, and is holomorphic for $\mathrm{Re}(z) \ge 0$. Then

$$g(0) = \int_0^\infty f(t) dt$$

is well-defined.

Proof. Suppose $T > 0$ and let $g_T(z) := \int_0^T e^{-zt} f(t) dt$. Let C be the boundary of the region $\{z : |z| \le R, \mathrm{Re}(z) \ge \delta$ for small $\delta = \delta(R) > 0$ and large R so that $g(z)$ is holomorphic on the region and its boundary. Applying Cauchy's integral formula, we get

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_C (g(z) - g_T(z)) e^{Tz} \left(1 + \frac{z^2}{R^2}\right) (1/z) dz.$$

Now bound this: on the part of C on the upper half-plane (denote this as C_1), we can write

$$|g(z) - g_T(z)| = \left| \int_T^\infty e^{-tz} f(t) dt \right| \le B \int_T^\infty |e^{-tz}| dt = \frac{Be^{-T\mathrm{Re}(z)}}{\mathrm{Re}(z)},$$

where $B = \max_{t \ge 0} |f(t)|$ and since $|e^{Tz}| = e^{T\mathrm{Re}(z)}$ and $|1 + \frac{z^2}{R^2}| = \frac{2\mathrm{Re}(z)}{R}$, we see that $|\int_{C_1}| \le \frac{B}{R}$. For the other piece of the contour C_2, we can replace the integral for g_T (since g_T is entire) by the contour of a semicircle in the lower half plane C_2' (which, similar to the above, has $|\int_{C_2'}| \le \frac{B}{R}$), while the other integral $\to 0$ as $T \to \infty$ by a simple bounding argument. Taking $R \to \infty$ concludes the proof. ■

Now we can get a bound for $\theta(x)$ in the flavor of Chebyshev:

Lemma. $\theta(x) \le (4 \log 2) x$.

Proof. $\binom{2n}{n}$ by definition contains every prime number from $n+1, ..., 2n$ in its factorization; thus $\prod_{n < p \le 2n} p \le \binom{2n}{n}$. But we can also expand

$$(1+1)^{2n} = \binom{2n}{0} + \binom{2n}{1} + ... + \binom{2n}{2n} \Longrightarrow \binom{2n}{n} \le 4^n.$$

Thus

$$\prod_{n<p\le 2n} p \le 4^n \Longrightarrow \sum_{n<p\le 2n} \log p \le n \log 4.$$

If n takes on values from $1, 2, 4, ..., 2^M$, where M is the highest such that $2^{M+1} \geq x$, we get

$$\theta(x) \leq \sum_{1<p\leq 2^{M+1}} < 2^{M+1}\log 4 < 2x\log 4 = (4\log 2)x,$$

as desired. ■

We're ready to finish off the proof of the prime number theorem. If $\mathrm{Re}(z) > 1$, then

$$\phi(z) = \sum_p \frac{\log p}{p^z} = z\int_0^\infty e^{-zt}\cdot\theta(e^t)dt,$$

and from the above, we have $\lim_{x\to\infty}\frac{\theta(x)}{x^2} = 0$. This implies

$$\frac{\phi(z+1)}{z+1} = \int_0^\infty e^{-(z+1)t}\theta(e^t)dt \Longrightarrow \frac{\phi(z+1)}{z+1} - \frac{1}{z} = \int_0^\infty e^{-(z+1)t}(\theta(e^t)e^{-t} - 1)dt.$$

We can then take $g(z) = \int_0^\infty f(t)e^{-zt}dt$ and $f(t) = \theta(e^t)e^{-t} - 1$, so that the lemma we proved above tells us

$$\int_0^\infty f(t)dt = \int_1^\infty \frac{\theta(x)-x}{x^2}dx < \infty.$$

This furthermore implies that $\theta(x) \sim x$:

Lemma. If $g(x)$ is nondecreasing and

$$\int_1^\infty \frac{g(x)-x}{x^2}dx < \infty,$$

then $g(x) \sim x$.

Proof. Almost trivial. ■

So we've proved the prime number theorem.

CHAPTER 23

ELLIPTIC FUNCTIONS

Elliptic Functions. These are doubly periodic, meromorphic functions on $\mathbb{C}$. We won't talk too much about the history of the word "elliptic." If w_1, w_2 are two nonparallel vectors in $\mathbb{C}$ ($\frac{w_1}{w_2} \notin \mathbb{R}$), then an elliptic function f are periodic in both directions:

$$f(z) = f(z + w_1) \quad \text{and} \quad f(z) = f(z + w_2).$$

These had origins in determining elliptic arclengths. By precomposing with multiplication by a nonzero complex number α, we may assume $w_1 = 1, w_2 = \tau = \frac{w_2}{w_1}$: that is, take $g(z) = f \circ m_\alpha(z)$, $g(z) = f(\alpha z)$. Taking $\alpha = w_1$ gives $g(z+1) = f(w_1 z + w_1) = g(z)$ and $g(z+\tau) = f(w_1 z + w_2) = f(w_1 z) = g(z)$.

Wlog, we have $\tau \in \mathbb{H}$, i.e. $\mathrm{Im}(\tau) > 0$. Then $\mathrm{Im}(\tau), \mathrm{Im}(1/\tau)$ have opposite signs because $\mathrm{Im}(\tau) \neq 0$ or else it's collinear with 1.

The objective now is to study these meromorphic functions f on $\mathbb{C}$ such that $f(z+1) = f(z)$ and $f(z+\tau) = f(z)$. The fundamental domain for translation by the lattice

$$\Lambda = \{a + b\tau : a, b \in \mathbb{Z}\}$$

is then given by the parallelogram

$$P = \{a + b\tau : (a, b) \in [0, 1)^2\}.$$

Now we have some facts about f:

1. **Fact.** If f is holomorphic, then f is constant.

   Proof. f is bounded by its maximum on $\overline{P}$.

2. **Fact.** $\sum_{\alpha \in P} \mathrm{Res}(f; \alpha) = 0$.

   Proof. First, if there are no poles on ∂P, then

$$\int_{\partial P} f(z)dz - 2\pi i \sum_{\alpha \in P} \mathrm{Res}(f; \alpha).$$

   By taking the integral along P, we put

$$\int_{\partial P} f(z)dz = \int_0^1 f(t)dt + \int_0^1 f(1 + \tau t)\tau dt - \int_0^1 f(1 + \tau - t)dt - \int_0^1 f(\tau - t)dt.$$

The integrals cancel in pairs by f being elliptic, and we are left with 0. If there is a pole along ∂P, then the claim is that $\exists \alpha, \epsilon : P + \alpha\epsilon$ has no pole where $\alpha = (1+\tau) \in \mathbb{C}$ is not collinear with 1 and τ. To see this, suppose not; then we get an accumulation of poles at distinct points. Now note that $P + \alpha\epsilon$ is also a fundamental domain, and we can reason similarly to the above.

Remark on terminology: The *order* of an elliptic function is the total order of its poles in the fundamental domain P.

Corollary. There are no elliptic functions with orders 0 or 1.

Proof. If f has order 0, then it is holomorphic, giving a contradiction. If f has order 1, then there is one simple pole, and the residue is $\neq 0$, giving another contradiction. ■

Proposition. If f is an elliptic function of order k, then TFAE:

(a) f has k total order of zeroes in P.

(b) $f(z) - c$ has total order of zeroes equal to k.

Proof. $(a) \implies (b)$. Use the argument principle. $(b) \implies (a)$ follows similarly. ■

Weierstrass $\wp$ functions. These are the order 2 elliptic functions with double poles at the lattice points (e.g. 0). Here's a first attempt to define such a function:

$$\sum_{w\in\Lambda} \frac{1}{(z+w)^2} = \sum_{m,n\in\mathbb{Z}} \frac{1}{(z+m+n\tau)^2}.$$

The problem here is that this is not absolutely convergent.

Lemma. $\sum_{(m,n)\neq(0,0)} \frac{1}{(|m|+|n|)^r} < \infty$ for $r > 2$.

Proof.

$$\sum_{m\in\mathbb{Z}} \frac{1}{(|m|+|n|)^r} = \frac{1}{|n|^r} + 2\sum_{m=1}^{\infty} \frac{1}{(|m|+|n|)^r} = \frac{1}{|n|^r} + 2\sum_{l=|n|+1}^{\infty} \frac{1}{l^r} \leq \frac{1}{|n|^r} + 2\int_{|n|}^{\infty} \frac{1}{l^r} dl$$

$$= \frac{1}{|n|^r} + 2\frac{|n|^{-r+1}}{-r+1}.$$

Sum this over n:

$$\sum_n \left(\frac{1}{|n|^r} + C\frac{1}{|n|^{r-1}} \right)$$

summable for $r - 1 > 1$, i.e. $r > 2$.

Corollary. $\sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^r}$ converges absolutely for any $r > 2$, $\tau \in \mathbb{H}$.

Proof. Left as an exercise. ■

The right way to define such a function is

$$\wp(z) = \frac{1}{z^2} + \sum_{w\in\Lambda-\{0\}} \left(\frac{1}{(z+w)^2} - \frac{1}{w^2} \right).$$

The claim is that this converges uniformly on compact subsets of $\mathbb{C} - \Lambda$ to a meromorphic function with double poles at Λ.

Claim. $\wp$ is the function we're looking for.

Proof. Let $|z| < R$. Then

$$\wp(z) = \frac{1}{z^2} + \sum_{|w|<2R} \left(\frac{1}{(z+w)^2} - \frac{1}{w^2} \right),$$

which is a finite sum. We add on another term:

$$\frac{1}{z^2} + \sum_{|w|<2R} \left(\frac{1}{(z+w)^2} - \frac{1}{w^2} \right) + \sum_{|w|>2R} \left(\frac{1}{(z+w)^2} - \frac{1}{w^2} \right).$$

The claim is that $\sum_{|w|>2R} \left(\frac{1}{(z+w)^2} - \frac{1}{w^2} \right)$ with $w \in \Lambda$ converges uniformly to a holomorphic function on $|z| < R$. to see this, note that $\frac{1}{(z+w)^2} - \frac{1}{w^2} = \frac{(z+w)^2 - w^2}{(z+w)^2 w^2} = \frac{z^2+2zw}{w^2(z+w)^2}$, where the denominator is like w^4 and the numerator is like w where $|z|$ is bounded. So the sum converges uniformly on $|z| < R$. Thus the whole sum converges uniformly. ■

Proposition. $\wp$ is elliptic: $\wp(z+1) = \wp(z)$ and $\wp(z+\tau) = \wp(z)$.

Proof. Consider $\wp'(z)$ on the swiss cheese disk. We have the limit fo holomorphic functions, so the derivative converges uniformly. Note that

$$\wp'(z) = -2 \sum_{w \in \Lambda} \frac{1}{(z+w)^2}$$

is doubly periodic. Then we have that $\wp(z+1) = \wp(z) + a$ where a is a fixed constant because the derivative is 1-periodic, while $\wp(z+\tau) = \wp(z) + b$ where b is fixed because the derivative is τ-periodic. The fact that $a = b = 0$ follows from the fact that $\wp$ is even: in particular, $\wp(1/2) = \wp(-1/2)$ and $\wp(-\tau/2) = \wp(\tau/2)$. ■

Remark. We note that $\wp$ is an order 2 elliptic function, as it has a double pole at only the origin in P. Likewise, $\wp'$ is an order 3 elliptic function, since it has a triple pole at only the origin in P.

Theorem. Every elliptic function is a rational function in $\wp$ and $\wp'$.

Proof. First let's understand the relationship between $\wp$ and $\wp'$. We note that $\wp'$ is odd, and $\wp'(1/2) = \wp'(-1/2) = -\wp'(1/2)$, so that $\wp'(!/2) = 0$ and similarly for $\tau/2, (1+\tau)/2$. These are all the zeroes of $\wp'$ since $\wp'$ is of order 3, while $\wp'$ has a pole of order 3 at the origin.

For $\wp$, this means that $\wp(z) - \wp(\alpha) = 0$ for $\alpha = 1/2, \tau/2, (1+\tau)/2$, and has a double root (and nothing else since $\wp$ is of order 2). $\wp(z) - \wp(\alpha) = 0$ has a simple root at $\alpha_j - \alpha$ distinct, for $\alpha \neq 0, 1/2, \tau/2, (1+\tau)/2 \in P$.

Theorem. We have

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

for $e_1 = \wp(1/2), e_2 = \wp(\tau/2), e_3 = \wp((1+\tau)/2)$.

Proof. Look at the zeroes and the poles of each side. $\wp'(z)^2$ has a pole of order 6 at 0,z eroes of order 2 at $1/2, \tau/2, (1+\tau)/2$, and no others. The right hand side has poles of order

6 at 0, zeroes of order 2 at the same points, and no others. The function

$$\frac{\wp'(z)^2}{4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)}$$

is holomorphic, doubly periodic, and hence constant. What's the constant? If we Laurent expand at 0, we get

$$\wp(z)=\frac{1}{z^2}+\ldots \text{ and } \wp'(z)=-\frac{-2}{z^2}+\ldots,$$

so

$$4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)=4\left(\frac{1}{z^6}+\ldots\right)$$

and

$$\wp'(z)^2=\frac{4}{z^6}+\ldots$$

so the constant is 1. ■

Geometric interpretation. $\wp$ is surjective from P to $\hat{\mathbb{C}}$ with "order 2." Note that $P/\Lambda \cong \mathbb{T}^2$.

A (sketch of a) geometric picture is as follows: consider skewering your donut on a stick through four points on the lattice; then taking the quotient by flipping over, we get half-torus that is one-to-one except for two points that are two-to-one.

Theorem. Every elliptic function that is doubly periodic under 1 and τ is a rational function in $\wp_\tau, \wp'_\tau$.

Proof. First, we do this for even elliptic functions. Let F denote one such, which has even order (possibly 0) of zeroes and poles at 0. Hence $F(z)\wp(z)^m, m \in \mathbb{Z}$ has no zero or pole at 0. Let $\alpha_1,\ldots,\alpha_k$ denote the zeroes of F with multiplicity; these come in pairs $\pm\alpha_i$ if $\alpha_i \neq 1/2, \tau/2, (1+\tau)/2$. Likewise, list the poles $\beta_1,\ldots,\beta_k$ of F; the same comment holds. The claim is that

$$G(z)=\frac{\prod_i(\wp(z)-\wp(\alpha_i))}{\prod_j(\wp(z)-\wp(\beta_j))}$$

has the same zeroes and poles with multiplicity as F, so $F=cG$.

Now for general elliptic F, let $F=F_{even}+F_{odd}$. Then $F_{even}=\frac{F(z)+F(-z)}{z}$, $F_{odd}=\frac{F(z)-F(-z)}{2}$, and they are both elliptic. F_{even} is a rational function in $\wp$, while $F_{odd}/\wp'$ is even and elliptic, and thus a rational function in $\wp$. The result follows. ■

Note that, for different τ, P/Λ are not generally the same torus.

Anyway, this story of elliptic curves has applications to number theory and cryptography, and algebraic geometry and other areas of math. Next time, we'll start with the formula $(\wp')^2=4\wp'-g_2\wp-g_3$ for some g_2, g_3, and get into other uses and generalizations for elliptic functions.

CHAPTER 24

WEIERSTRASS'S ELLIPTIC FUNCTION AND AN OVERVIEW OF ELLIPTIC INVARIANTS AND MODULI SPACES

Elliptic functions. There is a relationship between elliptic functions and elliptic integrals. Consider integrals of the form

$$\int \frac{dx}{\sqrt{\text{cubic}}} \text{ or } \int \frac{dx}{\sqrt{\text{quartic}}}.$$

These integrals show up when we want to compute the arc length of ellipses. The second is a Schwarz-Christoffel transformation from $\mathbb{H}$ to a quadrilateral. As before, starting from their double-periodicity and some transformation properties, we want to study the fundamental function $\wp(z)$.

Today, we'll start with $\wp(z)$, and consider the equation

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

which arises when we consider the integral

$$\int_z^\infty \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}} = \int_{\wp(z)}^{\wp'(z)} \frac{\wp'(w)dw}{\sqrt{\wp'(w)^2}}$$

and substitute $\wp(w) = z$; this gives $\wp$ as the inverse of the integral. Note that, under the elliptic integral given above, the three roots of the cubic in the denominator and ∞ go to the vertices of the quadrilateral in the image, since the elliptic integral is basically a Schwarz-Christoffel map. One can show here that an elliptic function is the inverse to the Schwarz-Christoffel map.

Recap on what we know about $\wp$. First we recall that Weierstrass's elliptic function $\wp$ is defined for any fixed τ as

$$\wp_\tau(z) = \frac{1}{z^2} + \sum_{w\in\Lambda^*} \left(\frac{1}{(z+w)^2} - \frac{1}{w^2} \right),$$

where $\Lambda^* = \Lambda - \{0\}$ and the lattice Λ for a fixed τ is $\Lambda = \{n + m\tau : m, n \in \mathbb{Z}\} \subseteq \mathbb{C}$. $\wp$ is meromorphic with double poles at lattice points. We also had

$$\wp'(z) = -2\sum_{w\in\Lambda} \frac{1}{(z+w)^3} = -2\sum_{n,m\in\mathbb{Z}} \frac{1}{(z+n+m\tau)^3}$$

We showed that if $\wp(1/2) = e_1, \wp(\tau/2) = e_2$, and $\wp(\frac{1+\tau}{2}) = e_3$, then

$$\wp'(z)^2 = 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3).$$

Today, we'll compute $\wp'(z) = 4\wp(z)^3 - g_2\wp(z) - g_3$. We do this by introducing *Eisenstein series*:

Eisenstein series. These are functions given by

$$E_k(\tau) = \sum_{w\in\Lambda^*} \frac{1}{w^2} = \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^k}.$$

Here are a few facts, which we'll leave to the reader to prove:

- *Fact 1:* If $k \geq 3$, then $E_k(\tau)$ is a holomorphic function of τ in $\mathbb{H}$: the sum converges uniformly in τ for $\mathrm{Im}(\tau) > \delta > 0$.
- *Fact 2:* If k is odd, then $E_k(\tau) = 0$.
- *Fact 3:* We have the following transformation relations:

  $$E_k(\tau+1) = E_k(\tau) \quad \text{and} \quad E_k(-1/\tau) = \tau^k E_k(\tau)$$

  E.g. for the last relation we have $(m + n(-1/\tau))^k = \tau^{-k}(m\tau - n)^k$.

These properties tell us that E_{2k} is a *weakly modular function* of weight $2k$. Note that, since the Eisenstein series converge depending on τ, we can just write them as constants.

Proposition. If z is near 0, then

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)E_{2k+2}z^{2k} = \frac{1}{z^2} + 3E_4z^2 + 5E_6z^4 + ...$$

Proof. We can write

$$\wp(z) = \frac{1}{z^2} + \sum_{w\in\Lambda^*}\left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right),$$

where we reindex the lattice under the transformation $w \mapsto -w$. Then

$$\frac{1}{(z-w)^2} = \frac{1}{w^2}\frac{1}{(1-z/w)^2}$$

Noting that $\frac{1}{1-x} = 1 + x + x^2 + ...$ for $|x| < 1$ and $\frac{1}{(1-x)^2} = \frac{\partial}{\partial x}(\frac{1}{1-x}) = 1 + 2x + 3x^2 + ...$, we have for $|z/w| < 1$ that

$$\frac{1}{(z-w)^2} = \frac{1}{w^2}\left(1 + 2\frac{z}{w} + 3\left(\frac{z}{w}\right)^2 + ...\right).$$

So

$$\wp(z) = \frac{1}{z^2} + \sum_{w\in\Lambda^*} \frac{1}{w^2} \sum_{n=1}^{\infty}(n+1)\frac{z^n}{w^n} = \frac{1}{z^2} + \sum_{n=1}^{\infty}\left(\sum_{w\in\Lambda^*} \frac{1}{w^{n+2}}\right)(n+1)z^n$$

Since $E_{n+2} = \sum_{w\in\Lambda^*} \frac{1}{w^{n+2}}$, we have

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)E_{2k+2}z^{2k},$$

as $E_{n+2} = 0$ when n is odd. ■

Theorem. If $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$, then $g_2 = 60E_4$ and $g_3 = 140E_g$.

Proof. We can write out some terms of the identity we derived above:

$$\wp(z) = \frac{1}{z^2} + 3E_4z^2 + 5E_6z^4 + ...$$

$$\wp'(z) = -\frac{2}{z^3} + 6E_4z + 20E_6z^3 + ...$$

$$\wp'(z)^2 = 4\wp(z)^3 + ... = \frac{4}{z^6} - 24E_4\frac{1}{z^2} - 80E_6 + ...$$

$$\wp(z)^3 = \frac{1}{z^6} + 9E_4\frac{1}{z^2} + 15E_6 + ...$$

Now take the difference

$$\wp'(z)^2 - 4\wp(z)^3 + 60E_4\wp(z) + 140E_6.$$

This is holomorphic near 0; each term is holomorphic away from Λ and is doubly-periodic, so the difference is holomorphic, doubly-periodic, and takes 0 to 0. Then, from earlier problems, we know that it's 0, and the equality

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

gives $g_2 = 60E_4$ and $g_3 = 140E_g$ as desired. ■

What is this explicit formula telling us? On the one hand, we have $\mathbb{C}/\Lambda$, the "domain of $\wp$ and $\wp'$" which we identify with a torus. Consider a map to $\mathbb{C}^2$ given by

$$z \mapsto (\wp(z), \wp'(z));$$

we can draw $\mathbb{C}^2$ (which we don't really discuss in class) by drawing $\mathbb{R}\times\mathbb{R}$ on the axes in 2-space, and the curve it traces out is the set of solutions to the polynomial function $y^2 = 4x^3 - g_2x - g_3$, a cubic curve. What you're seeing is that this complex torus is a cubic curve in $\mathbb{C}^2$. We can check injectivity and surjectivity and other nice things, but let's get out of this story portion and move into....

The space of elliptic curves. Now we want to consider all possible τ's, and we note that the lattice Λ doesn't change under the action of $PSL_2\mathbb{Z} \subseteq \mathrm{Aut}(\mathbb{H}) = PSL_2\mathbb{R}$. This is just the space of transformations of the form

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

for $a, b, c, d \in \mathbb{Z}$, generated by $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$; it's what you can do to τ to not change your lattice.

(The elliptic functions compose with dilations; e.g. they're maps from coplex tori from one direction to the other direction, just by dilation. You can't just stretch it in one direction, since then it won't be holomorphic.)

So let me show you a fundamental domain D in $\mathbb{H}$. This is the part of $\mathbb{H}$ above the circle $|z| = 1$ and bounded by the lines $\mathrm{Re}(z) = \{-1/2, 1/2\}$; e.g. $D = \{z \in \mathbb{H} : |z| > 1, -\frac{1}{2} < \mathrm{Re}(z) < \frac{1}{2}\}$. This is the *moduli space of complex 1-tori (elliptic curves).* In other words, D is the fundamental domain of the action of $PSL_2\mathbb{Z}$. $\mathbb{H}/PSL_2\mathbb{Z}$ is the moduli space of elliptic curves.

Let's do some justice to why these things are nice. $E_{2k}(\tau)$ is sort of a function on $\mathbb{H}/PSL_2\mathbb{Z}$, defined on D, and transforms under the involution $\tau \mapsto -1/\tau$:

$$E_{2k}(-1/\tau) = \tau^{2k} E_{2k}(\tau).$$

When we say these things are called modular forms, we mean that they "don't change the correct way," and the reason here is the power coefficient. We can, however, write

$$\lim_{\mathrm{Im}(\tau)\to\infty} E_{2k}(\tau) = \lim_{\mathrm{Im}(\tau)\to\infty} \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^{2k}} = 2\sum_{m=1}^{\infty} \frac{1}{m^{2k}} = 2\zeta(2k),$$

since for $n \neq 0$ the center experssion has the $n\tau$ term $\to 0$ for $\mathrm{Im}(\tau) \to \infty$. We also have

$$E_4(\infty) = 2\zeta(2k).$$

Then

$$g_2(\infty) = 60E_4(\infty) = 60(2)(\frac{\pi^2}{90}) = \frac{4}{3}\pi^4.$$

$$g_3(\infty) = 140E_6(\infty) = 140(2)(\zeta(6)) = \frac{8}{27}\pi^6.$$

Setting the *modular discriminant* $\Delta = g_2^3 - 27g_3^2$, we note that $\Delta(\infty) = 0$; the right hand side expression, $g_2^3 - 27g_3^2$, is the discriminant of $4x^3 - g_2x - g_3$. This is a weight 12 *modular form*, by which we mean that E_{2k} is a transformation of weight $2k$, and it's *modular* if $\lim_{\mathrm{Im}(\tau)\to\infty} E_{2k}(\tau)$ exists and holomorphic at ∞, $E_{2k}(\tau)$ is holomorphic on $\mathbb{H}$, and $E_{2k}(\tau + 1) = E_{2k}(\tau)$. We have another modular form of weight 12, which is simply g_2^3. Define

$$j = \frac{1728g_2^3}{\Delta}.$$

This has weight 0, so it's really defined as a function on $\mathbb{H}/PSL_2\mathbb{Z}$ (e.g. a meromorphic function from $\mathbb{H} \to \mathbb{C}$ invariant under the action of $SL_2\mathbb{Z}$), which is the moduli space of all elliptic curves. It's called a *j-invariant of elliptic curves.* There's a theorem that states:

Theorem. This is holomorphic on $\mathbb{H}$, invertible under $PSL_2\mathbb{Z}$, has $j(\infty) = \infty$, with residue 1 at ∞, and gives a surjection $\mathbb{H} \to \mathbb{C}$ that injects from the quotient $\mathbb{H}/PSL_2\mathbb{Z}$.

Proof. See Serre's *Course in Arithmetic.* ■

I won't say too much more about this subject; we've got j as an invariant in the moduli space, but we won't talk about why number theorists and cryptographers are interested in it. For this, consult any reference on elliptic curves. We'll use the remainder of the time for questions.

APPENDIX A

SOME THINGS TO REMEMBER FOR THE MIDTERM

Complex expressions:

1. Remember polar form: $z = re^{i\theta}$.
2. Stereographic projection is given by $\varphi : S^2 - \{p\} \to \mathbb{R}^2, \varphi : (u, v, w) \mapsto (\frac{u}{1-w}, \frac{v}{1-w}), \varphi^{-1} : (x, y) \mapsto (\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1})$.
3. $\cos z = \frac{e^{iz}+e^{-iz}}{2}, \sin z = \frac{e^{iz}-e^{-iz}}{2i}$.
4. $\log z = \log r + i\theta + i(2\pi n)$.

Holomorphic functions:

1. To show that a function f is holomorphic, use the Cauchy-Riemann equations and show that the partials are continuous. You can also any rule from real analysis (the chain rule, the product rule, the quotient rule, etc.), show that f has a power series development, or use Morera's theorem.
2. Recall that the **Cauchy-Riemann** equations are

$$\begin{cases} \frac{\partial u}{\partial x} & = & \frac{\partial v}{\partial y} \\ -\frac{\partial v}{\partial x} & = & \frac{\partial u}{\partial y}. \end{cases}$$

3. **Morera's theorem**: If $f(z)$ is continuous on a region D and $\int_\Gamma f(z)dz = 0$ for Γ a triangle (or closed curve), then $f(z)$ is holomorphic. This is useful for:

   1. Showing $\lim_{n\to\infty} f_n = f$ is holomorphic.

   2. Showing some functions defined by sums or integrals are holomorphic.

   3. Showing functions that are continuous and holomorphic on all but some set (e.g. some point) are holomorphic.

4. f is holomorphic $\iff$ f complex analytic $\iff$ f smooth; this means you can substitute any holomorphic function with its power series, and can differentiate it endlessly.
5. The **closed curve theorem** tells us that the integral of a function that is holomorphic in the open disk D over a contour $C \subset D$ is 0.

6. Per the **antiderivative theorem**, if $f(z)$ is holomorphic on an open disk D, then it has an antiderivative $F(z)$ such that $F'(z) = f(z)$.
7. **Cauchy's integral formula** allows us to determine the value of a holomorphic function f at a point. In particular,

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-w} dz,$$

where f is holomorphic in an open disk D and C contains w.

8. The **uniqueness theorem** tells us that holomorphic functions are determined uniquely up to their values in a certain region.
9. The **mean value theorem** allows us to evaluate a holomorphic function at point. If $f(z)$ is holomorphic on $D_\alpha(R)$, then $\forall r : 0 < r < R$, we have

$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + re^{i\theta}) d\theta.$$

10. The **maximum modulus principle**: if f is holomorphic in any region, then its maximum is achieved at the boundary.
11. **Harmonic functions** are characterized by the fact that their Laplacian is zero, and possess nice properties as you discovered in the problem sets.

Entire and meromorphic functions:

1. Entire functions are functions that are holomorphic in all of $\mathbb{C}$.
2. **Liouville's theorem** tells us that any bounded entire function is constant. A generalization is that if $|f(z)| \leq A + B|z|^N$ and $f(z)$ is entire, then $f(z)$ is a polynomial of degree $\leq N$.
3. Likewise, if $f(z)$ is entire and $f(z) \to \infty$ as $z \to \infty$, then $f(z)$ is a polynomial.
4. A nonconstant polynomial always has a root, per the fundamental theorem of algebra.
5. Remember the different types of singularities and why they're useful, namely poles of order k and essential singularities.
6. Meromorphic functions are holomorphic functions on $\hat{\mathbb{C}}$. In other words, they have a finite set of removable singularities, and can always be expressed as as a ratio of two holomorphic functions.
7. A weak formulation of **Picard's little theorem** tells us that if f is entire, then its image is dense.

Evaluating integrals:

1. There are generally two ways we're learned to do nontrivial integrals:

   1. Use the closed curve theorem to give a contour integral as zero, then decompose the contour and bound the individual parts to find a specific integral.

   2. Use the residue theorem by first calculating residues:

(0) Compute the integral $\int_{C_\epsilon(\alpha)} f(w)dw$

(1) Compute the Laurent series for f centered at α, and take c_{-1}

(2) If f has a simple pole, then $c_{-1} = \lim_{z\to\alpha}(z-\alpha)f(z)$

(3) If $f(z) = A(z)/B(z)$ for B having a simple pole and $A \neq 0$, then $c_{-1} = \operatorname{Res}(f;\alpha) = A(\alpha)/B'(\alpha)$.

The value of the integral over a curve containing these poles is $2\pi i \times$ (sum of residues).

Series:

1. Know the definitions of **radius of convergence**, $\overline{\lim}|a_n|^{1/n} = L$, and convergence tests/arguments for power series. These may be things that range from proving that a power series of a holomorphic function is uniformly convergent to finding the radii of convergence of some functions.
2. **Laurent series** are power series that extend in both $-\infty$ and ∞ directions, and are typically defined on annuli (why?). To compute Laurent series, you can generally

    1. Start with known power series and make the appropriate substitutions/adjustments.

    2. Use known geometric series. Being able to decompose a function into its partial fraction expansion is a good trick to know here.

Others:

1. Know the **topological aspects**: polygonally connected (path connected), polygonally simply connected (simply connected), what is meant by a region, etc. Note any open path connected set is simply connected.
2. Know when to interchange $\sum\sum$, $\sum\int$, $\partial\int$, $\lim\int$, etc. To deal with interchange of sums, show absolute convergence. To deal with interchange of limits, show uniform convergence. (This is covered in real analysis, and also in baby Rudin.)
3. A helpful tool is that you can bound the absolute value of any integral over C_R by $2\pi R$ times the integrand; this is known as the ML theorem or the estimation lemma.

APPENDIX B

ON THE FOURIER TRANSFORM

This is an addendum to the class notes on Fourier transforms, and is based on Chapter 4 of Stein and Shakarchi's *Complex Analysis*. It is important to note that a function $f : \mathbb{R} \to \mathbb{C}$ must satisfy appropriate regularity and decay conditions to possess a Fourier transform. If it does (we'll make this notion precise below), then the *Fourier transform* is defined by

$$\boxed{\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx, \qquad \xi \in \mathbb{R}.} \tag{B.1}$$

The *Fourier inversion formula* is then

$$\boxed{f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi, \qquad x \in \mathbb{R}.} \tag{B.2}$$

There is a deep and natural connection between complex analysis and the Fourier transform. For example, given a function f initially defined on the real line, the possibility of extending it to a holomorphic function is closely related to the rapid decay at infinity of its Fourier transform $\hat{f}$.

Proposition. The functions $f(x) = e^{-\pi x^2}$ and $f(x) = \frac{1}{\cosh \pi x}$ are their own Fourier transforms.

Proof. Show this yourself. ■

Definition. For $\alpha > 0$, denote by $\mathfrak{F}_\alpha$ the class of all functions that satisfy the following two conditions:

(i) f is holomorphic in the horizontal strip $S_\alpha = \{z \in \mathbb{C} : |\mathrm{Im}(z) < a\}$.

(ii) There is a constant $A > 0$:

$$|f(x + iy)| \leq \frac{A}{1 + |x|^{1+\epsilon}}$$

for any $\epsilon > 0$ and all $x \in \mathbb{R}$ and $|y| < a$. This condition is known as *moderate decay.* As we will soon see, $\mathfrak{F}_\alpha$ can be regarded as the *class of functions with well-defined Fourier* *transforms.*

Intuitively, one can think of $\mathfrak{F}_\alpha$ as those holomorphic functions of S_α that are of "moderate decay" on each horizontal line $\mathrm{Im}(z) = y$, uniformly in $-a < y < a$. Some functions that belong to $\mathfrak{F}_\alpha$ include

$$f(z) = e^{-\pi z^2}$$

for all α, and

$$f(z) = \frac{1}{\pi}\frac{c}{c^2 + z^2}$$

for $0 < a < c$.

Proposition. If $f \in \mathfrak{F}_\alpha$, then $f^{(n)} \in \mathfrak{F}_\beta$ for $0 < \beta < \alpha$.
Proof. Use the Cauchy integral formula. ■

Proposition. If $f \in \mathfrak{F}_\alpha$ for some $\alpha > 0$, then

$$|\hat{f}(\xi)| \leq Be^{-2\pi\beta|\xi|}$$

for some B and any $0 \leq \beta < \alpha$.
Proof. Use contour integration. ■

Let's drop the α in $\mathfrak{F}_\alpha$ to make notation simpler. The previous proposition tells us that if $f \in \mathfrak{F}$, then $\hat{f}$ has *rapid decay at infinity.* The Fourier inversion formula then works in the following sense:

Proposition. If $f \in \mathfrak{F}$, then the Fourier inversion formula holds, and we have

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi} d\xi$$

for all $x \in \mathbb{R}$.

Properties of the Fourier transform.

Now that we have some basis for knowing when a Fourier transform or inversion is well-defined, let's talk about some of the things we do with them.

Theorem. (Basic properties) Let $f, g \in \mathfrak{F}$. Then the following are true:
(1) If $h(x) = af(x) + bg(x)$ for $a, b \in \mathbb{C}$, then $\hat{h} = a\hat{f} + b\hat{g}$.
(2) If $h(x) = f(x - x')$ for $x' \in \mathbb{R}$, then $\hat{f}(\xi) = e^{-2\pi i x'\xi}\hat{f}(\xi)$.
(3) If $h(x) = e^{2\pi i x\xi'}f(x)$ for $\xi' \in \mathbb{R}$, then $\hat{h}(\xi) = \hat{f}(\xi - \xi')$.
(4) If $h(x) = f(ax)$ for $a \in \mathbb{R} - \{0\}$, then $\hat{h}(\xi) = \frac{1}{|a|}\hat{f}\left(\frac{\xi}{a}\right)$.
(5) If $h(x) = \overline{f(x)}$, then $\hat{h}(\xi) = \overline{\hat{f}(-\xi)}$.
(6) Let $\mathcal{F}$ denote the Fourier transform. $\mathcal{F}^2(f)(x) = f(-x)$ and $\mathcal{F}^4(f) = f$.
Proof. These are all easy to verify.

Theorem. (Relation to differentiation)

$$\boxed{\mathcal{F}\left[\left(\frac{d}{dx}\right)^m ((-ix)^n f)\right] = (i\xi)^m \left(\frac{d}{d\xi}\right)^n \hat{f},}$$

where $\mathcal{F}$ denotes the Fourier transform. In particular, the Fourier transform of $(-ix)f(x)$ is $\frac{d}{d\xi}\hat{f}(\xi)$, and the Fourier transform of f' is $(i\xi)\hat{f}(\xi)$.

Proof. Use differentiation under the integral sign and integration by parts.

Theorem. (Relation to convolution) If $f * g$ denotes the *convolution* of f and g defined by

$$(f * g)(\xi) := \int_{-\infty}^{\infty} f(\xi)g(\xi - t)dt,$$

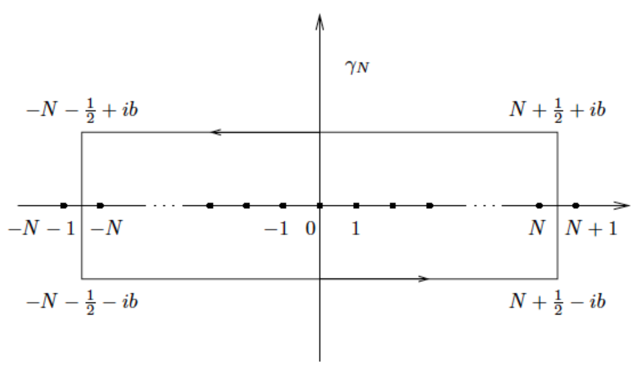
then $\widehat{f * g}(\xi) = \hat{f} \cdot \hat{g}$.

This latter theorem is important for functional analysis, but we won't discuss it to make this note self-contained. We can also prove an important tool known as the *Poisson summation formula:*

Theorem. (Poisson summation formula) If $f \in \mathfrak{F}$, then

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n). \tag{B.3}$$

Proof. Let $f \in \mathfrak{F}_\alpha$ and choose a $\beta : 0 < \beta < \alpha$. The function $\frac{1}{e^{2\pi iz}-1}$ has simple poles with residue $\frac{1}{2\pi i}$ at the integers. So $\frac{f(z)}{e^{2\pi iz}-1}$ has simple poles at the integers n, with residues $f(n)/2\pi i$. Applying the residue formula to the contour γ_N below,



we get

$$\sum_{|n|\leq N} f(n) = \int_{\gamma_N} \frac{f(z)}{e^{2\pi iz} - 1} dz.$$

Letting $N \to \infty$, and recalling the moderate decay of f, we see that the sum converges to $\sum_{n\in\mathbb{Z}} f(n)$, and that the integral over the vertical segments goes to 0. So in the limit we get

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{L_1} \frac{f(z)}{e^{2\pi iz} - 1} dz - \int_{L_2} \frac{f(z)}{e^{2\pi iz} - 1} dz, \tag{B.4}$$

where L_1 and L_2 are the real line shifted down and up by b, respectively. Now we use the fact that if $|w| > 1$, then

$$\frac{1}{w-1} = w^{-1} \sum_{n=0}^{\infty} w^{-n}$$

to see that on L_1, where $|e^{2\pi iz}| > 1$, we have

$$\frac{1}{e^{2\pi iz} - 1} = e^{-2\pi iz} \sum_{n=0}^{\infty} e^{-2\pi inz}.$$

If $|w| < 1$, then $\frac{1}{w-1} = -\sum_{n=0}^{\infty} w^n$, so on L_2 we have

$$\frac{1}{e^{2\pi iz} - 1} = -\sum_{n=0}^{\infty} e^{2\pi inz}.$$

Substituting these equalities into equation (4) above, we see that

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{L_1} f(z) \left(e^{-2\pi iz} \sum_{n=0}^{\infty} e^{-2\pi inz} \right) dz + \int_{L_2} f(z) \left(\sum_{n=0}^{\infty} e^{2\pi inz} \right) dz$$

$$= \sum_{n=0}^{\infty} \int_{L_1} f(z) e^{-2\pi i(n+1)z} dz + \sum_{n=0}^{\infty} \int_{L_2} f(z) e^{2\pi inz} dz = \sum_{n=0}^{\infty} \hat{f}(n+1) + \sum_{n=0}^{\infty} \hat{f}(-n) = \sum_{n\in\mathbb{Z}} \hat{f}(n),$$

where we shift L_1 and L_2 back to the real line by replacing $x \mapsto x - ib$. ■

The Poisson summation formula gives many useful identities in complex analysis and beyond (namely, number theory). For example, the Fourier transform of the function $f(x) = e^{-\pi t(x+a)^2}$ is $\hat{f}(\xi) = t^{-1/2} e^{-\pi\xi^2/t} e^{2\pi ia\xi}$. Applying the Poisson summation formula to this pair gives the following identity:

$$\sum_{n=-\infty}^{\infty} e^{-\pi t(n+a)^2} = \sum_{n=-\infty}^{\infty} t^{-1/2} e^{-\pi n^2/t} e^{2\pi ina}.$$

This identity can be used to prove the following transformation law for the *theta function* $\vartheta(t) := \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$:

$$\vartheta(t) = t^{-1/2} \vartheta(1/t),$$

for $t > 0$. It can also be used to prove the key functional equation of the *Riemann zeta function*, which gives its analytic continuation:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

Finally, Fourier transforms can tell us much about a function. For example, the following theorem describes the nature of those functions whose Fourier transforms are supported; in particular, it relates the decay properties of a function with the holomorphicity of its Fourier transform:

Theorem. (Paley-Wiener) Suppose f is continuous and of moderate decrease on $\mathbb{R}$. Then f has an extension to the complex plane that is entire with $|f(z)| \leq Ae^{2\pi M|z|}$ for some $A > 0$ if and only if $\hat{f}$ is supported in the interval $[-M, M]$.

Applications of the Fourier transform.

One can think of the Fourier transform as a mapping that converts a function $f(t)$ in the time domain (t has units of seconds) to a function $\hat{f}(\omega)$ in the frequency domain (ω has units of hertz or something).

Fourier transforms can be used to solve differential equations (c.f. below), and is clearly widely used in physics, probability, engineering, and many other areas. For example, one usually proves the central limit theorem in probability theory using Fourier transforms, and can prove a mathematical formulation of the Heisenberg uncertainty principle. For all the Fourier analysis you can get, Stein and Shakarchi's *Fourier Analysis* is a good standard textbook (and one you can find online...).

Examples involving the Fourier transform.

Here is a flavor of some of the problems one can ask about Fourier transforms.

Example 1. This example generalizes some of the properties of $e^{-\pi x^2}$ related to the fact that it is its own Fourier transform. Suppose $f(z)$ is an entire function that satisfies $|f(x+iy)| \leq ce^{-ax^2+by^2}$ for some $a, b, c > 0$. Let $\hat{f}(\zeta) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\zeta}dx$. Then $\hat{f}$ is an entire function of ζ that satisfies $|\hat{f}(\zeta+i\eta)| \leq c'e^{-a'\zeta^2+b'\eta^2}$ for some $a', b', c' > 0$.

Proof. Note that $\hat{f}$ is clearly an entire function since we can just commute $\frac{d}{dz}$ with the integral, and $f(x)e^{-2\pi i x\zeta}$ is entire. Thus $\hat{f}$ does not have any poles, and in particular we can change the contour from the real axis to the line $x - iy$ for some $y > 0$ fixed and $-\infty < x < \infty$ without affecting the value of the integral. Assume $\zeta > 0$, and observe that

$$\hat{f}(\zeta) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\zeta}dx = \int_{-\infty}^{\infty} f(x-iy)e^{-2\pi i(x-iy)\zeta}dx = \int_{-\infty}^{\infty} f(x-iy)e^{-2\pi(xi+y)\zeta}dx.$$

Then note

$$\left|\int_{-\infty}^{\infty} f(x-iy)e^{-2\pi(xi+y)\zeta}dx\right| \leq \left|\int_{-\infty}^{\infty} ce^{-ax^2+by^2}e^{-2\pi(xi+y)\zeta}dx\right| = \left|\int_{-\infty}^{\infty} ce^{-ax^2-2\pi xi\zeta+by^2-2\pi\zeta y}dx\right|.$$

Since $|e^{-ax^2-2\pi xi\zeta}| \to 0$ as $|x| \to \infty$, we see that $\hat{f}(\zeta) = O(e^{by^2-2\pi\zeta y})$. Setting $y = d\zeta$ where d is a small constant, we have that $\hat{f}(\zeta) = O(e^{b(d\zeta)^2-2\pi\zeta(d\zeta)}) = O(e^{(bd^2-2\pi d)\zeta^2})$, as desired. Also observe that

$$|\hat{f}(\zeta+i\eta)| = \left|\int_{-\infty}^{\infty} f(x)e^{-2\pi ix(\zeta+i\eta)}dx\right| = \left|\int_{-\infty}^{\infty} f(x-iy)e^{-2\pi(xi+y)(\zeta+i\eta)}dx\right|$$

$$= \left|\int_{-\infty}^{\infty} f(x-iy)e^{-2\pi(x\zeta i-x\eta+\zeta y+y\eta i)}dx\right| \leq \left|\int_{-\infty}^{\infty} ce^{-ax^2+by^2-2\pi(x\zeta i-x\eta+\zeta y+y\eta i)}dx\right|$$

Repeating the above argument, we see that $\hat{f}(\zeta) = O(e^{(by^2-2\pi\zeta y)+(-ax^2+2\pi x\eta)})$. Now let $y = d\zeta$ and $x = d_0\eta$ for small d, d_0 in the bound to see that $|\hat{f}(\zeta+i\eta)| = O(e^{(bd^2-2\pi d)\zeta^2+(-ad_0^2+2\pi d_0)\eta^2})$. So $|\hat{f}(\zeta+i\eta)| \leq c'e^{-a'\zeta^2+b'\eta^2}$ for $a' = 2\pi d - bd^2$, $b' = 2\pi d_0 - ad_0^2$ (which are > 0 for small d, d_0), and some suitable scale factor $c' > 0$. ■

Example 2. The problem is to solve the differential equation

$$a_n \frac{d^n}{dt^n} u(t) + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} u(t) + ... + a_0 u(t) = f(t),$$

where $a_0, a_1, ..., a_n$ are complex constants, and f is a given function. Here we suppose that f has bounded support and is smooth.

Proof. We do this in steps:

(a) Let $\hat{f}(z) = \int_{-\infty}^{\infty} f(t)e^{-2\pi izt}dt$. Observe that $\hat{f}$ is an entire function, and using integration by parts show that $|\hat{f}(x+iy)| \leq \frac{A}{1+x^2}$ if $|y| \leq a$ for any fixed $a \geq 0$.

(b) Write $P(z) = a_n(2\pi iz)^n + a_{n-1}(2\pi iz)^{n-1} + ... + a_0$. Find a real number c so that $P(z)$ does not vanish on the line $L = \{z : z = x + ic, x \in \mathbb{R}\}$.

(c) Set

$$\boxed{u(t) = \int_L \frac{e^{2\pi izt}}{P(z)} \hat{f}(z)dz.}$$

Check that

$$\sum_{j=0}^{n} a_j (\frac{d}{dt})^j u(t) = \int_L e^{2\pi izt} \hat{f}(z)dz$$

and

$$\int_L e^{2\pi izt} \hat{f}(z)dz = \int_{-\infty}^{\infty} e^{2\pi ixt} \hat{f}(x)dx.$$

We then conclude by the Fourier inversion theorem that $\sum_{j=0}^{n} a_j(\frac{d}{dt})^j u(t) = f(t)$. Note that the solution u depends on the choice of c.

(a) $\hat{f}$ is an entire function because we can commute $\frac{d}{dz}$ with the integral and the integrand is also an entire function. We know that $f \in C^2$, so now we use integration by parts: let $u = f(t)$, $du = f'(t)dt$, $dv = e^{-2\pi izt}dt$, and $v = -\frac{1}{2\pi iz}e^{-2\pi izt}$ so that

$$|\hat{f}(z)| = \left| \int_{-\infty}^{\infty} f(t)e^{-2\pi izt}dt \right| = \left| -\frac{f(t)}{2\pi iz} e^{-2\pi izt}]_{t=-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{f'(t)}{2\pi iz} e^{-2\pi izt}dt \right|$$

Now let $u = f'(t)$, $du = f''(t)dt$, $dv = \frac{e^{-2\pi izt}}{2\pi iz}dt$, and $v = -\frac{1}{(2\pi iz)^2}e^{-2\pi izt}$ so that

$$|\hat{f}(z)| = \left| -\frac{f(t)}{2\pi iz} e^{-2\pi izt}]_{t=-\infty}^{\infty} + \left(-\frac{f'(t)}{(2\pi iz)^2} e^{-2\pi izt}]_{t=-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{f''(t)}{(2\pi iz)^2} e^{-2\pi izt}dt \right) \right|.$$

Since $\hat{f}$ is entire, we can (as in the previous exercise) change the contour of integration since $\hat{f}$ does not have any poles. So we change the contour from t to $t - is$, as in the argument on page 115 of the text. Rewriting, we have

$$|\hat{f}(x+iy)| \leq \left| -\frac{f(t-is)}{2\pi i(x+iy)} e^{-2\pi i(x+iy)(t-is)}]_{t=-\infty}^{\infty} \right| + \left| \frac{f'(t-is)}{(2\pi i(x+iy))^2} e^{-2\pi i(x+iy)(t-is)}]_{t=-\infty}^{\infty} \right|$$
$$+ \left| \int_{-\infty}^{\infty} \frac{f''(t+is)}{(2\pi i(x+iy))^2} e^{-2\pi i(x+iy)(t-is)}dt \right|.$$

If f' and f'' also decay, we can bound this again:

$$|\hat{f}(x+iy)| \leq \left|-\frac{M_0}{2\pi i(x+iy)}e^{-2\pi s(x+iy)}\right| + \left|\frac{M_1}{(2\pi i(x+iy))^2}e^{-2\pi s(x+iy)}\right|$$
$$+\left|\int_{-\infty}^{\infty}\frac{M_2}{(2\pi i(x+iy))^2}e^{-2\pi s(x+iy)}dt\right|.$$

Note that the first two terms are bounded, and examining the integral we see that we can extract the $\frac{1}{z^2}$ term. Since y is also bounded, we have that $|\hat{f}(x+iy)| = O(\frac{1}{x^2})$, and can therefore conclude that $|\hat{f}(x+iy)| \leq \frac{A}{1+x^2}$ for a suitable scale factor A.

(b) P is a polynomial of degree n, so it only has finitely many roots and we can choose such a c so that the line L does not pass through a root. So we can choose $c_1 = 1$, and let $c_n = c_{n-1}+1$ if P vanishes with the choice of c_{n-1}. Then c will be the c_n for which P does not vanish, and it'll surely be found after n iterations.

We can find some constraints on c if we notice that we want $|P(x+ic)| > 0$ and can bound $|P(x+ic)|$ as follows:

$$|P(x+ic)| \geq |a_n||x+ic|^n - \sum_{j=1}^{n-1}|a_j||x+ic|^j \geq |a_n||x+ic|^n - \left(\sum_{j=1}^{n-1}|a_j|\right)|x+ic|^{n-1}$$

for $|x+ic| > 1$ (we can repeat the same procedure with a different bound of $|x+ic|$ instead of $|x+ic|^{n-1}$ in the rightmost term if $|x+ic| < 1$, and again solve accordingly). Since we want $|a_n||x+ic|^n - (\sum_{j=1}^{n-1}|a_j|)|x+ic|^{n-1} > 0$, we see that $|x+ic| > \sum_{j=1}^{n-1}\frac{|a_j|}{|a_n|}$. We also know that $|x+ic| \geq |c|$, so if $|x+ic| > 1$ we can choose $c > \max(1, \sum_{j=1}^{n-1}\frac{|a_j|}{|a_n|})$ to satisfy the inequality and we're done. In particular, one c for which the polynomial does not vanish is $c = \max(1, \sum_{j=1}^{n-1}\frac{|a_j|}{|a_n|}) + 1$.

(c) We differentiate under the integral sign to see that

$$\sum_{j=0}^{n}a_j(\frac{d}{dt})^j u(t) = \sum_{j=0}^{n}a_j(\frac{d}{dt})^j\int_L\frac{e^{2\pi izt}}{P(z)}\hat{f}(z)dz = \sum_{j=0}^{n}a_j\int_L(\frac{d}{dt})^j\frac{e^{2\pi izt}}{P(z)}\hat{f}(z)dz$$

$$= a_n\int_L(2\pi iz)^n\frac{e^{2\pi izt}}{P(z)}\hat{f}(z)dz + a_{n-1}\int_L(2\pi iz)^{n-1}\frac{e^{2\pi izt}}{P(z)}\hat{f}(z)dz + ... + a_0\int_L(2\pi iz)^0\frac{e^{2\pi izt}}{P(z)}\hat{f}(z)dz$$
$$= \int_L\frac{P(z)e^{2\pi izt}}{P(z)}\hat{f}(z)dz = \int_L e^{2\pi izt}\hat{f}(z)dz,$$

as desired.

Furthermore, we claim that the equality

$$\int_L e^{2\pi izt}\hat{f}(z)dz = \int_{-\infty}^{\infty}e^{2\pi ixt}\hat{f}(x)dx$$

follows by observing that, as in the previous problem, $e^{2\pi izt}\hat{f}(z)$ is clearly an entire function and thus its integral is also entire by commuting $\frac{d}{dz}$. So the integral does not have any poles, and in particular we can change the contour of integration from the line $L = \{z : z = x+ic, x \in \mathbb{R}\}$ (where c is chosen so that $P(z)$ does not vanish) to the real axis without affecting the value of the integral.

Finally, because we're applying the Fourier inversion to the Fourier transform $\hat{f}$ of f and both functions satisfy the decay conditions in the hypothesis, we get from the Fourier inversion theorem that $\sum_{j=0}^{n} a_j(\frac{d}{dt})^j u(t) = \int_{-\infty}^{\infty} e^{2\pi ixt}\hat{f}(x)dx = f(t)$, and we are done. Thus the solution to the differential equation

$$a_n \frac{d^n}{dt^n}u(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}u(t) + ... + a_0 u(t) = f(t),$$

is

$$\boxed{u(t) = \int_L \frac{e^{2\pi izt}}{P(z)}\hat{f}(z)dz,}$$

for $P(z) = a_n(2\pi iz)^n + a_{n-1}(2\pi iz)^{n-1} + ... + a_0$ and $c \in \mathbb{R}$ so that $P(z)$ does not vanish on the line $L = \{z : z = x + ic, x \in \mathbb{R}\}$.

■

APPENDIX C

REFERENCES

We cite many sources in these notes. The main course textbook is:

J. Bak and D. J. Newman, *Undergraduate Texts in Mathematics: Complex Analysis.* Springer, 2010.

One that is oftentimes referenced (and is a gem of a textbook), is

E. M. Stein and R. Shakarchi, *Princeton Lectures in Analysis 2: Complex Analysis.* Princeton UP, 2003.

And of course, the classic textbook is none other than Ahlfors:

L. V. Ahlfors, *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable.* McGraw-Hill, 1979.

AFTERWORD

You survived! If you enjoyed what you saw in Math 113, you should consider Math 213a (graduate complex analysis) or Math 229x (analytic number theory). Math 113 provides adequate (dare I say ample) background for both in most years. 213a deals with complex analysis, but on a graduate level: you'll revisit some of the topological basics, and learn some nice and beautiful mathematics like infinite product expansions and more on elliptic and meromorphic functions. Math 229x is number theory using complex analysis: you'll begin with things like the prime number theorem and move on quickly. These courses can vary a bit, so do take our recommendation with caution.

If you aren't pursuing further work in mathematics, we hope that you enjoyed the course and remember how nice complex analysis is.

Thanks for being a great class, in terms of problem sets and everything else. Keep in touch with Andy or any of us CA's if you'd like, and have a great summer!

Felix and Anirudha