December 17, 2008

Relativity

P. Koppenburg[1]
Imperial College London

Syllabus

Mechanics and electromagnetism in inertial frames of reference. Invariance of the speed of light. Postulates of special relativity. Light clocks, time dilation, and length contraction. Lorentz transformations. Velocity addition. Relativistic momentum and energy. Mass-energy equivalence. Relativistic mechanics of nuclear and high-energy particle interactions. Four-vectors.

[1] p.koppenburg@imperial.ac.uk

Contents

1 Introduction

This first section places special relativity in its historical context and discusses the experimental facts that have lead to Einstein's postulates.

1.1 Some Books

These notes are a written version of what is being discussed during the lectures. They should be self-contained but may be too dense for a first approach of the subject. Students are advised to use other sources. A different perspective is always very useful.

Hugh D. Young and Roger A. Freedman — *University Physics* [1]
The standard all-in-one textbook. Contains all you need to know except four-vectors. Lacks some enthusiasm.

A.P. French — *Special Relativity* [2]
A good and very clear book at the appropriate level. Some examples are a little out of date.

John B. Kogut — *Introduction to Relativity* [3]
A more modern book, but much shorter. Some difficult topics lack explanation. Many explanations are based on Minkovski diagrams, which take some time to be understood.

Taylor and Wheeler — *Spacetime Physics* [4]
A massive book with a different — very informal — approach. Taylor and Wheeler start from what's invariant and slowly get to the maths. Very detailed and explanation of paradoxes and brainteasers. This is the only place I found a clear and complete explanation of the twin paradox.

Richard Feynman et al. — *Feynman Lectures on Physics* [5]
Feynman's excellent series includes three chapters on relativity. Feynman is always an illuminating background reading. But as usual Feynman only focuses on what's interesting to him.

1.2 What is Relativity?

Definition — Relativity:
Relativity is a theory describing the relation between observations (measurements) of the *same* process by *different* observers in motion *relative* to each other.

Special Relativity refers to the special case of *inertial* observers.

General Relativity refers to the general case of *accelerated* observers and provides a theory of gravity.

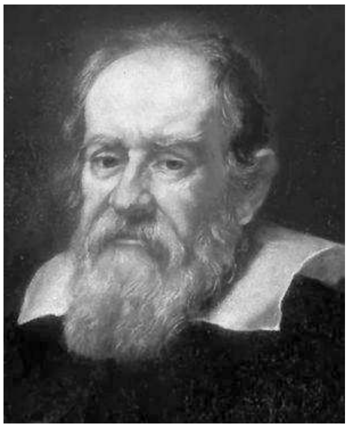
Special relativity starts from very simple postulates and draws conclusions for the results of measurements of lengths and time, as well as mechanics at high speeds. It revises intuitive concepts like simultaneity, addition of velocities or Newton's laws. And $E = mc^2$ also naturally follows from it.

It is a simple theory—which has been confirmed by experiment many times— and more importantly, never been disproved. But it is counter-intuitive to most of us, which is the main reason it needs a detailed study — and a lot of practise.

The speed of light plays a central role in special relativity. So called "*relativistic effects*" are only sizable at high speeds comparable to the speed of light. But the theory is valid at any speeds, even very small ones.

1.3 Galilean Relativity

The concept of relativity dates back much before Einstein. Its first known formulation is from Galileo's *Dialogue Concerning the Two Chief World Systems* and reads as follows [6]:



Galileo Galilei (1564–1642)

> Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction. When you have observed all these things carefully (though doubtless when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. In jumping, you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than toward the prow even though the ship is moving quite rapidly, despite the fact that during the time that you are in the air the floor under you will be going in a direction opposite to your jump. In throwing something to your companion, you will need no more force to get it to him whether he is in the direction of the bow or the stern, with yourself situated opposite. The droplets will fall as before into the vessel beneath without dropping toward the stern, although while the drops are in the air the ship runs many spans. The fish in their water will swim toward the front of their bowl with no more effort than toward the back, and will go with equal ease to bait placed anywhere around the edges of the bowl. Finally the butterflies and flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they will have been separated during long intervals by keeping themselves in the air. And if smoke is made by burning some incense, it will be seen going up in the form of a little cloud, remaining still and moving no more toward one side than the other. The cause of all these correspondences of effects is the fact that the ship's motion is common to all the things contained in it, and to the air also. That is why I said you should be below decks; for if this took place above in the open air, which would not follow the course of the ship, more or less noticeable differences would be seen in some of the effects noted.

This long experimental setup defines what is an inertial frame.

Definition — Inertial frame:
A reference frame in which the first Newton law holds. An isolated body maintains a uniform velocity relative to any inertial frame.

Galileo couldn't use Newton's laws to define inertial frames but states that there is no preferred direction in "*throwing something to your friend, you need throw it no more strongly in one direction than another*" and introduces the concept of frame with "*The cause of all these correspondences of effects is the fact that the ship's motion is common to all the things contained in it, and to the air also.*"

The first sentence "*Shut yourself up [...] below decks*" makes it clear that you are not allowed to look outside. In such conditions "*you [couldn't] tell from any of them whether the ship was moving or standing still.*" which defines the concept or relativity:

Galileo's relativity :
The laws of Mechanics are the same in all inertial frames.

Newton's laws follow this axiom and do not distinguish different inertial frames. For instance when drinking coffee in an aeroplane you don't know if you are at rest on the runway or flying at 1000 km/h.

An inertial observer cannot say he's "at rest" while another "moves", but *accelerations* can be detected. That's why coffee isn't served during takeoff and landing.

Mechanics experiments can distinguish inertial from non-inertial frames, but they cannot distinguish different inertial frames.

1.4 Electromagnetism and Optics

What about non-mechanical physics experiments? Can they distinguish "motion" from "rest"?

1.4.1 Magnet and Conductor

Let's make a simple experiment. We have a coil and a magnet which we can move one through the other.

A moving coil and a magnet at rest (Fig. 1): The free charges in the conducting coil move and experience a magnetic force $\boldsymbol{F} = q\boldsymbol{u} \times \boldsymbol{B}$ as they pass through the magnet. The charges along the segments ab and cd feel the force as they pass through the magnet, inducing a current of opposite sign, respectively.

The coil is at rest and the magnet moves (Fig. 2): This time the magnet moves at speed u and produces a magnetic flux varying with time. Faraday's law says

$$\int_C \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}.$$

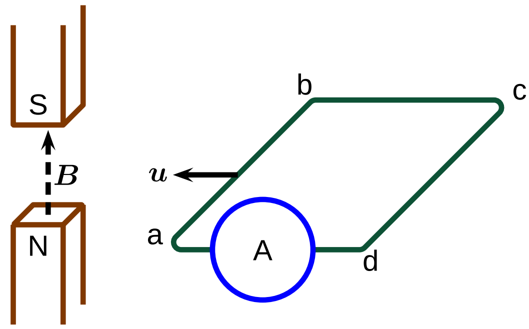


Figure 1: A moving coil and a magnet at rest.

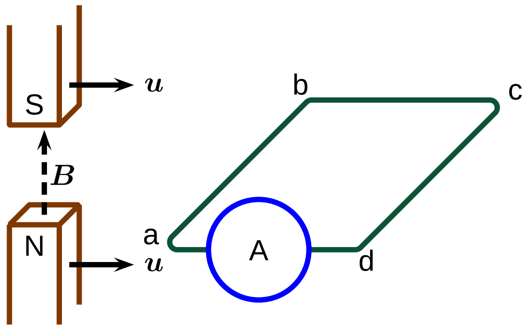


Figure 2: A moving magnet and a coil at rest.

When the magnet reaches segment ab the variation of the magnetic flux through the surface S defined by the coil causes a variation of the electric field $\boldsymbol{E}$ along the path C around the coil. This results in a current measured by the ampere-meter A. Then while the magnet is inside the coil the flux does not vary and there is no current, finally when the magnet crosses cd the flux decreases which induces a current of the opposite sign.

In both cases the measured current pulses are the same (Fig. 3) but the interpretation is different. This experiment does not allow to tell which of the coil or the magnet is at rest, if any.

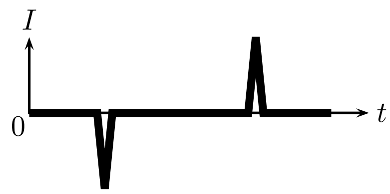


Figure 3: Current shown by the ampere-meter.

1.4.2 Measurement of the Speed of Light

The next experiment deals with the speed of light, which plays a central role in special relativity. One of the biggest debates in the history of physics was about the nature of light. Is it corpuscular or a wave? We now know it's both, but at the end of the XIXth century the problem seemed to have been settled in favour of the wave nature by Huygens and Maxwell.[2] Let's see what predictions we get from these two hypotheses.

If light is corpuscular (emitted like bullets) one expects the speed of light to depend on the speed of the source. Can one infer the speed of the source by measuring the speed of light?

For instance the light emitted in front of a plane travelling at speed u (Fig. 4) would be travelling at speed $c+u$ and the light emitted from the rear at speed $c-u$.

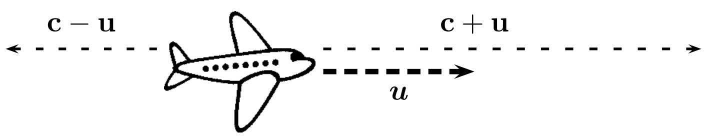


Figure 4: Corpuscular light hypothesis.

If this was the case one could have systems of binary stars in which one star could be seen at two places at the same time.

[2]Einstein's article on the photoelectric effect [7] published the same year as the one proposing special relativity would give a strong argument for the corpuscular nature though.

Light as a wave: Interference and diffraction of light indicates light is a wave. Other waves like sound or water waves require a medium to propagate. The speed of the wave is defined relative to this medium. So what is the light's medium? And can we measure our speed relative to the medium by measuring the speed of light?

At Einstein's time it seemed obvious that there was such a medium—called "*Luminiferous Æther*"—pervading the Universe. But its nature was very controversial. On one hand a wave is a perturbation of the æther and its frequency increases with the force which restores the equilibrium. To accommodate the very large frequencies of visible light the interaction between the medium and the light must be very strong. One the other hand the æther must be completely transparent to matter, allowing the earth to travel trough it without affecting it.[3]

1.4.3 A Thought Experiment

Quite typically, Einstein does not design a real experiment, but a *Gedankenexperiment* (thought experiment), in which he sets the imaginary experimenter at the validity limits of the theory.

In Figure 5 Einstein travels at speed u through the æther. In Einstein's frame there is an "æther wind" at speed $-u$.

The light travel time from Einstein to the mirror at speed $c-u$, and back at speed $c+u$ is

$$t = \frac{L}{c-u} + \frac{L}{c+u} = \frac{2cL}{c^2-u^2} = \frac{2L}{c\left(1-\frac{u^2}{c^2}\right)}.$$

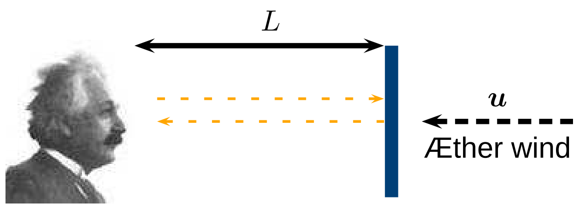


Figure 5: Einstein's Mirror.

By going at speed $u = c$ the time becomes infinite and Einstein would loose his reflection.

1.5 The Michelson-Morley Experiment

The frame in which the æther is at rest defines a special frame of reference in which light propagates at the same speed in all directions. All other frames are moving with respect to it. Many experiments went on trying to to measure the speed of the earth relative to this medium. The most famous and conclusive is discussed below.

The Michelson-Morley experiment was designed to measure the speed of the earth relative to the æther by measuring the speed of light along directions parallel and perpendicular to the earth' movement on its orbit [8, 9].

The setup is shown in Figure 6. Light from a source S falls on an inclined glass plate G_1 with a semitransparent metal coating on its front face. This plate splits the light into two paths. On the path along the x axis the light travels to a mirror M_1 where it is reflected back to plate G_1. It is then reflected by the metal coating and reaches a telescope T.

The light initially reflected by G_1 goes along y to another mirror M_2 where it is reflected back to G_1. A fraction of it will then passes through the semitransparent glass G_1 and reaches the telescope T.

[3] The æther could as well be dragged by the earth, but this would produce optical effects one does not observe. See [2] for the whole story.

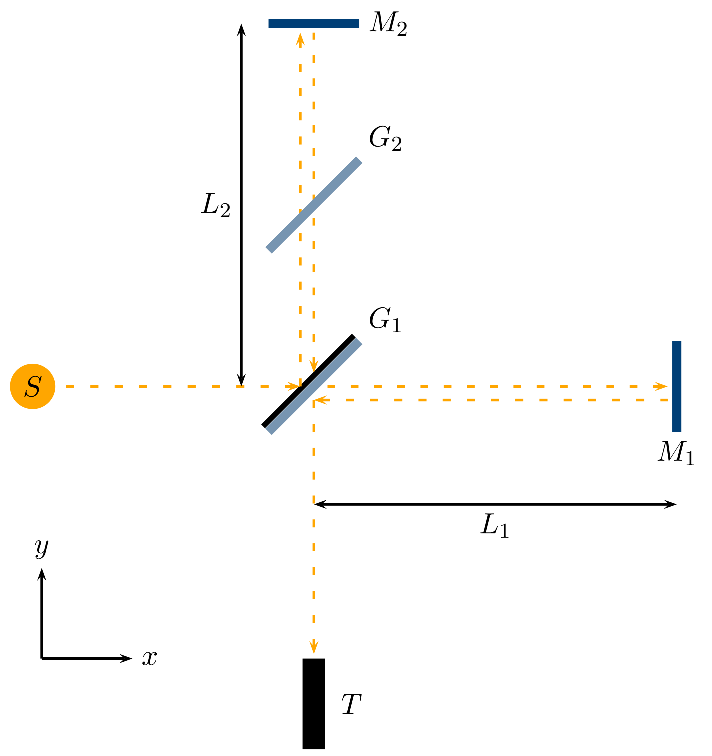


Figure 6: Setup of the Michelson-Morley experiment.

All paths are adjusted to have the same length at a good precision. The light along the x path has to travel through the thickness of the glass plate G_1 three times, while the y path crosses it only once. To compensate for this an additional glass identical to G_1 is placed on the y path in G_2.

If one uses monochromatic light of wavelength λ one should see interferences in the telescope due the different optical path lengths $L_1 = G_1M_1$ and $L_2 = G_1M_2$:

$$2(L_1 - L_2) = n\lambda$$

where n is an integer number. If any of the mirrors is moved by a distance $\lambda/2$ one will see a shift of one interference fringe. Typical interference fringes are shown in Fig 7. Note that an exact positioning of the apparatus is not essential as one is only interested in the difference and not in the actual value of L_1 and L_2. It is practically impossible to adjust these distances so that they are the same within an error of $\lambda \simeq 0.7\ \mu\text{m}$.

Suppose the earth is travelling from left to right, which produces an apparent “æther wind” blowing at speed u from the right, as in the left side of Figure 8. In the laboratory frame, the light on the x path goes slower (at speed $c - u$) on the outbound trip and faster (at $c + u$) on the inbound. The total travel time from G_1 to the mirror M_1 and back is

$$t_1 = \frac{L_1}{c-u} + \frac{L_1}{c+u} = \frac{2L_1}{c\left(1 - \frac{u^2}{c^2}\right)}.$$

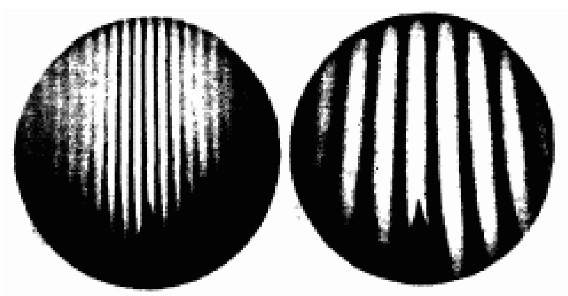


Figure 7: Interference fringes seen through a Michelson-Morley interferometer.

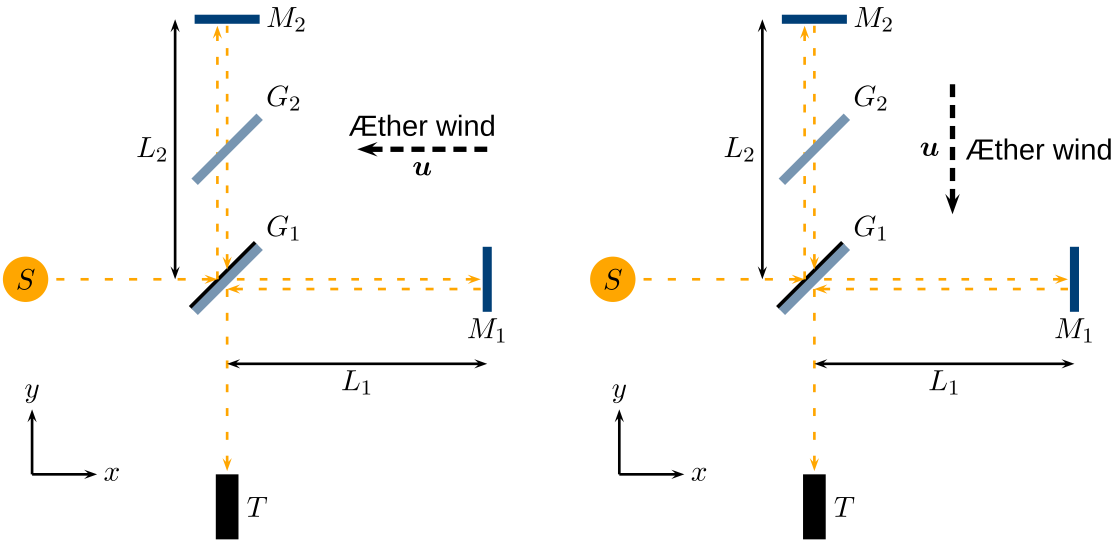


Figure 8: Michelson-Morley experiment in the æther wind.

Let's see the time taken by the light reflected along y. Its total speed is $v = c$ in the æther, but the experimental setup is moving at speed u along the x-axis. The rays which appear along y in the laboratory are actually inclined in the æther frame. Therefore the y-component of the speed must be $v_y = \sqrt{c^2 - u^2}$, both in the æther and in the laboratory frames.

In the lab the time for the $G_1M_2G_1$ round-trip is

$$t_2 = \frac{2L_2}{\sqrt{c^2 - u^2}} = \frac{2L_2}{c\sqrt{1 - \frac{u^2}{c^2}}},$$

and

$$\Delta t = t_1 - t_2 \quad = \quad \frac{2L_1}{c\left(1 - \frac{u^2}{c^2}\right)} - \frac{2L_2}{c\sqrt{1 - \frac{u^2}{c^2}}} \simeq \frac{2(L_1 - L_2)}{c} + \frac{2L_1u^2}{c^3} - \frac{L_2u^2}{c^3},$$

where we used the first order Taylor expansion assuming that $u \ll c$.

If we now turn the apparatus by $90°$ the æther wind blows from the top as in the right hand side of Fig. 8 and we get the time difference

$$\Delta t^{\perp} = t_1^{\perp} - t_2^{\perp} \quad = \quad \frac{2L_1}{c\sqrt{1 - \frac{u^2}{c^2}}} - \frac{2L_2}{c\left(1 - \frac{u^2}{c^2}\right)} \simeq \frac{2(L_1 - L_2)}{c} + \frac{L_1u^2}{c^3} - \frac{2L_2u^2}{c^3}.$$

The relevant quantity is the difference δt of the time differences

$$\delta t = \Delta t - \Delta t^{\perp} \simeq \frac{(L_1 + L_2)u^2}{c^3}$$

which can be converted into a difference of interference fringes

$$\delta n = \frac{c}{\lambda}\delta t = \frac{(L_1 - L_2)u^2}{c^2\lambda} = \frac{2L}{\lambda}\left(\frac{u}{c}\right)^2,$$

if $L_1 = L_2 = L$. In the original experiment [8] Michelson had $L = 1.2\ \mathrm{m}$, $\lambda = 0.6\ \mu\mathrm{m}$ and assumed that u was the speed of the earth on its orbit, which is $u = 30\ \mathrm{km/s}$ (which justifies the assumption $u \ll c$ made before). This would give $\delta n = 0.04$, a quite small difference, but measurable with the setup.

They didn't observe any deviation from 0. They were actually so surprised that they rebuilt the experiment with optical paths 10 time longer than in the previous version. They expected a shift by 0.4 fringes but the observed effect was at most 0.005. They repeated the experiment for any angle with respect to the earth motion and at any time of the year. They set an upper limit at $u < 8\,\mathrm{km/s}$ to the speed of the æther wind at any point on the earth's orbit [9].

Michelson and Morley's experiment is the most famous example of experiments failing to distinguish different inertial frames. No such experiment succeeded so far.

It is not clear whether Einstein was aware of Michelson's result in 1905. In any case he does not refer to it in his first article about special relativity [10].

1.6 Experimental Conclusion

The bottom line is: There is no way of telling which is the frame at rest, if any. All inertial frames are equivalent for all laws of physics.

No experimental test provides any way to distinguish an inertial frame from another.

This negative form of the statement is important, as it is a *prediction* which can be tested experimentally and thus *falsified*. It has never been.

1.7 Postulates of Special Relativity

These observations led Einstein to assert the following postulates:

Postulates of Special Relativity [10]:

1. **The laws of physics are identical in all inertial frames.**
2. **Light is propagated in empty space with a definite velocity c that is independent of the state of motion of the emitting body.**

This value is

$$c = 299,792,458 \text{ (exact) } \simeq 3 \cdot 10^8\ \mathrm{m/s}.$$

It is determined experimentally. Nowadays the exact value is fixed by the definition of the metre.

In principle we are already saying too much. All consequences of relativity could be deduced replacing Postulate 2 by "there is a speed limit". The only difference between Einstein's and Newton's worlds being that there is no speed limit for Newton.[4]

Albert Einstein (1879–1955) in 1905.

The fact that this speed limit is the speed of light is just an experimental observation. But it is very useful to know it as it will allow us to measure times using light as a natural clock.

The invariance of the speed of light for any observer is a direct consequence of the two postulates. Postulate 2 tells that light is always emitted at the same speed and hence is a constant of nature, and Postulate 1 says that the laws of physics, which includes the values of the constants, are independent on the observer. Hence

The speed of light in vacuum has the same value c for all inertial observers.

2 Consequences of the Invariance of c

This fixed value has some important consequences which conflict with "common sense" ideas. This should not be considered as disturbing. Common sense is based on our day-to-day experience which does not involve measurements of speeds close to the speed of light.

In the following we will use three rules

1. $L = vt$ in a given reference frame, with L a distance, v a speed and t the travel time. This is nothing but a definition of velocity.
2. c is invariant.
3. The principle of relativity.

2.1 Simultaneous Events

Figure 9 illustrates one of Einstein's first thought experiments. Imagine that an observer at O in a reference frame $\mathcal{O}$ fixed to the ground sees a train of length L passing by at speed u. This train is hit by a lightning at this moment. He sees two branches of the same lightning hitting each end of the train at the same time. He knows the two branches have hit at the same time because he sees the flashes emitted from A_2 and B_2 at the same time, he knows that the speed of light is constant, and that he's in the middle between A_2 and B_2 (a distance he can easily measure). He computes the time the light took to reach him from A_2 and B_2

$$t_A \overset{\text{Rule 1}}{=} \frac{A_2O}{c} = \frac{L}{2c} = \frac{B_2O}{c} = t_B,$$

[4]Or, equivalently, that the speed limit is infinite in classical mechanics.

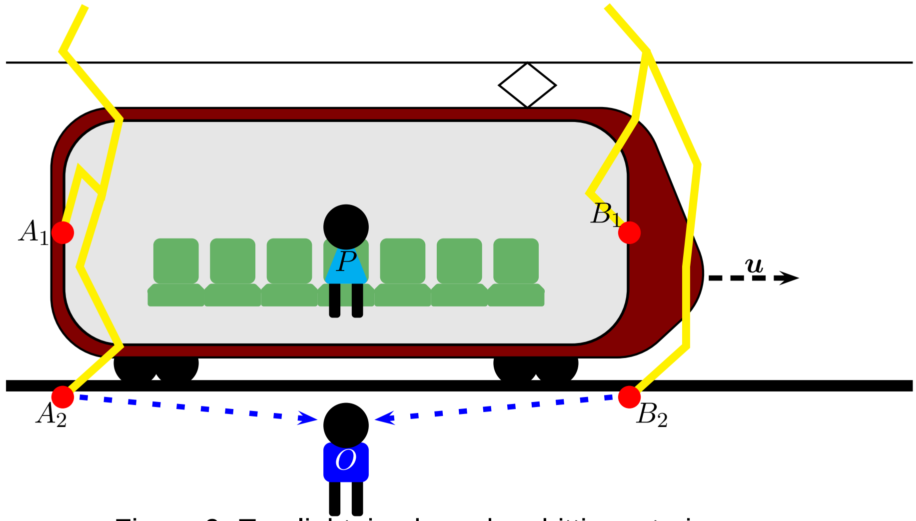


Figure 9: Two lightning branches hitting a train.

and hence the two events are simultaneous.

Another observer P sits in the middle of the train, in reference frame $\mathcal{O}'$. Since she is moving towards flash B_1 emitted from the front of the train she will see this flash first:

$$t'_A > t'_B$$

But she is also in the middle between A_1 and B_1, and the speed of light is also constant for her (Rule 2). So since

$$t'_A \overset{\text{Rule 1}}{=} \frac{A_1P}{c} = \frac{B_1P}{c} > t'_B,$$

she can only conclude that the two lightnings are not simultaneous. We will assume here that B_1 and B_2 are close enough that there is no time shifts due to this distance, as for A_1 and A_2.

From the point of view of O, P is obviously wrong because she's moving. From the point of view of P, he's wrong because he's moving. Who's right?

Both are right. Rule 3 ensures that the two reference frames are equivalent. Two events at different places can be simultaneous in one frame of reference and not in another. The postulates of special relativity force us to abandon the concept of absolute time.

Two events simultaneous in one frame need not be simultaneous in another frame.

2.2 A Light Clock

To measure times we need a good clock. The optimal clock makes use of the invariance of the speed of light by measuring time in terms of the travel distance of a beam of light. We build such a clock using two mirrors facing each other and constantly reflecting a ray of light, as shown in Fig. 10. Each round-trip is a "tick".

A clock at rest measures the "proper" time interval between two events:

1. The emission of the pulse from the base
2. The detection of pulse at the base

Both events happen at the *same* position in the frame of the clock.

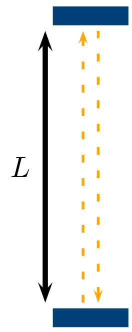


Figure 10: A time clock.

2.3 Time Intervals

Consider a time clock as the one described above on-board of a very fast train. This time let's call $\mathcal{O}$ the frame of the train and $\mathcal{O}'$ the frame of the outside observer.

In the frame $\mathcal{O}$ of the train (Fig. 11) the observer P measures the unit time t' as

$$t = \frac{2L}{c}. \tag{1}$$

From the ground in $\mathcal{O}'$, O sees the clock in the train moving at speed u, and the light has to take a longer path for each trip. According to Pythagoras

$$\begin{aligned} c^2t'^2 &= u^2t'^2 + (2L)^2 \\ t'^2(c^2-u^2) &= 4L^2 \\ \Rightarrow \quad t' &= \frac{2L}{c}\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \gamma t > t \end{aligned} \tag{2}$$

where we use the shortcuts

$$\boldsymbol{\beta = \frac{u}{c}}, \qquad \boldsymbol{\gamma = \frac{1}{\sqrt{1-\beta^2}}} \tag{3}$$

This effect is called "time dilation", because $\gamma \geq 1$. It is often quoted as

Moving clocks run slow.

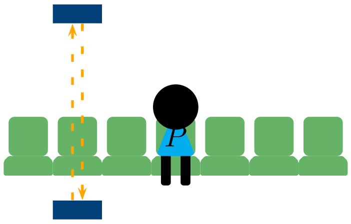


Figure 11: A time clock in a train seen from inside ($\mathcal{O}$).

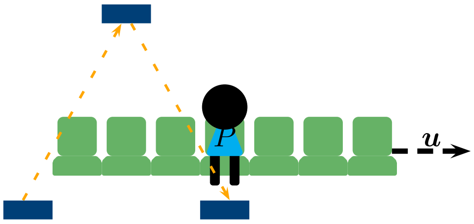


Figure 12: A time clock on a train seen from outside ($\mathcal{O}'$).

At $u = 0$, $\gamma = 1$ and we recover universal time. At $u = c$, $\gamma \to \infty$ and time stands still. Photons don't age.

This fact forces us to abandon the concept of universal time. In Newtonian dynamics time could be used as a parameter independent of the reference frame. Any trajectory could be written as a parametric function depending on time. This is not possible anymore in special relativity.

Although this seems counter-intuitive, it is a very natural consequence of the invariance of c. There is no way of guaranteeing that the speed of light is measured to be the same in any reference frame without affecting the definition of time.

It also gives some sense to the relativity of simultaneity. If there's no universal time, with respect to which clock do we define the time at which events happened? It can only be the frame-dependent clock.

2.4 Relativity of Length

The measurement of the length of an object at rest is easy. One measures the distance between one end and the other using a reference of known length, like a carpenter's rule.

The measurement of the length of a moving object is not trivial, even in Galilean relativity. One wants to measure its length by comparing to something which is in another frame. If one measures the position of one end first and then the position of the other end, one will get a result which depends on the length as well as on the speed and the time between the two measurements. Just any length could be the result of this process including negative lengths!

To achieve a valid measurement one must ensure that the positions of both ends are measured *at the same time*. We thus must not only know the position but also the time.

To make sure we know exactly what the time is, let's start by measuring something simple: our time clock! This time we place it *along* the direction of motion. In the frame $\mathcal{O}$ where the clock is at rest, the length is L. This is the "proper" length. If we place a clock of length L parallel to the object the "proper" time between ticks is $t = 2L/c$.

The same process as seen in frame $\mathcal{O}'$— in which the clock is moving at speed u — is shown in Figure 13.

1. The light travels from A to B in time t'_1, corresponding to a distance ct'_1. But during that time B moved by ut'_1 from B_0 to B_1.

$$\begin{aligned} A_0B_1 &= L' + ut'_1 = ct'_1 \\ \rightarrow \quad t'_1 &= \frac{L'}{c-u} \end{aligned}$$

2. The light travels back from B_1 to A_2 in time t'_2, corresponding to a distance ct'_2.

$$\begin{aligned} B_1A_2 &= L' - ut'_2 = ct'_2 \\ \rightarrow \quad t'_2 &= \frac{L'}{c+u} \end{aligned}$$

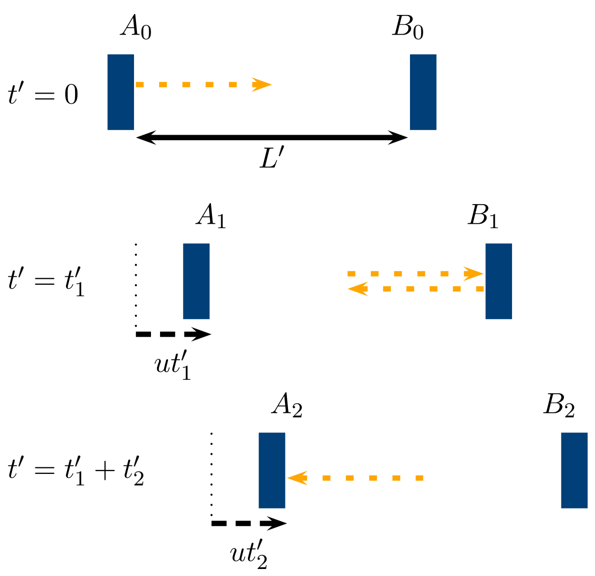


Figure 13: Light clock measured in $\mathcal{O}'$.

Thus the total time between two ticks in the $\mathcal{O}'$ frame is

$$t' = t'_1 + t'_2 = \frac{2L'}{c\left(1 - \frac{u^2}{c^2}\right)}.$$

From the time dilation Eq. (2) we have

$$t' = \gamma t = \frac{2L}{c\sqrt{1 - \frac{u^2}{c^2}}}$$

and thus

$$L' = \frac{ct'}{2}\left(1 - \frac{u^2}{c^2}\right) = \frac{c}{2}\left(1 - \frac{u^2}{c^2}\right)\frac{2L}{c\sqrt{1 - \frac{u^2}{c^2}}} = L\sqrt{1 - \frac{u^2}{c^2}} = \frac{L}{\gamma} \tag{4}$$

and L' is *shorter* by a factor γ:

Moving metre rules appear shorter along their direction of motion.

Here again, it's a counter-intuitive result, but which is a necessary consequence of the previous one. If time runs slow but light speed is constant, how could the light clock give the correct result without being shorter?

Note the direction of motion. There is no contraction along the directions perpendicular to motion, as we'll see in Section 3.

2.5 Summary

Let's summarise Chapter 2:

Length contraction:
The measured length of a body is *greater* in its rest frame than any other frame.

Time dilation:
The measured time difference between the events represented by two readings of a given clock is *less* in the rest frame of the clock than in any other frame.

A body appears to be contracted, and time appears dilated, when seen from *another* frame.

This has to be made very clear: dilation and contraction appear when comparing measurements made in different frames. If you travel at very large speeds you will not *see* your watch running slower or you arm getting shorter if pointed in the direction of motion. This is what observers outside of your frame will observe.

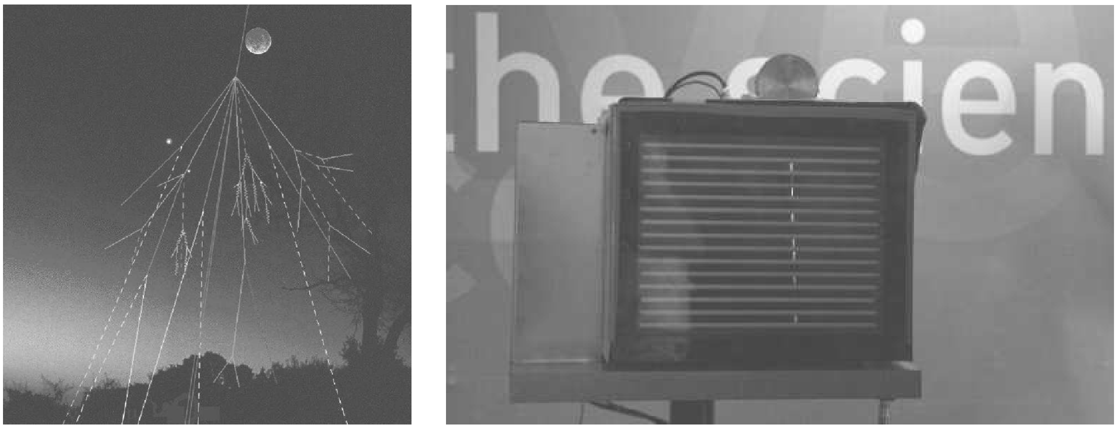


Figure 14: Cosmic muons are produced by high energy cosmic rays (mostly protons) interacting with the higher atmosphere and producing showers of particles (left [11]). Eventually some muons reach the ground and can be seen using a spark chamber (right).

2.6 Example: Cosmic Ray Muons

Muons (μ) are charged leptons like electrons, only 200 times as heavy. They are unstable and decay like radioactive atoms do. Their internal "proper" clock tick is their half-life $\tau_{\frac{1}{2}}$. If we start with N muons, after $\tau_{\frac{1}{2}}$ only $\frac{1}{2}N$ will remain. After $2\tau_{\frac{1}{2}}$, $\frac{1}{4}N$ remain and so on. This half-life is measured to be $\tau_{\frac{1}{2}} = (1.59218 \pm 0.00003) \cdot 10^{-6}\ \mathrm{s}$ [12]. Muons are produced by collisions of cosmic rays in the higher atmosphere ($H \sim 60\mathrm{km}$) and bombard the earth.

What time does it take to reach the ground? Their speed is measured to be $u \simeq 0.9995c$, almost the speed of light.

$$t = \frac{H}{0.9995c} \simeq 200\ \mu\mathrm{s}$$

How many reach the ground? $200\ \mu\mathrm{s}$ is about 125 half-lives. Hence after $200\ \mu\mathrm{s}$ we should see $\left(\frac{1}{2}\right)^{125} \sim 2 \cdot 10^{-38}$ remaining. But we do see a lot of them.

Explanation: We forgot the factor γ. The internal clock of the muon is moving at speed u, so in the earth frame ($\mathcal{O}'$) its time is dilated by a factor

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{1}{\sqrt{1 - 0.9995^2}} \simeq 30.$$

In $\mathcal{O}'$ the half-life is $\gamma\tau_{\frac{1}{2}} \simeq 50\ \mu\mathrm{s}$ and

$$\frac{t}{\gamma\tau_{\frac{1}{2}}} = \frac{200\ \mu\mathrm{s}}{50\ \mu\mathrm{s}} \simeq 4.$$

I.e. $(\frac{1}{2})^4 \simeq \frac{1}{16}$ will reach the ground.

The fact that any muon is detected at sea level is an experimental evidence for time dilation. There is on average one muon traversing your body every second at any time, day and night, and even more when you are on a mountain or in a plane.

But what about the muon's point of view? The muon's clock ticks at $\tau_{\frac{1}{2}}$, so there's no chance to reach the ground in frame $\mathcal{O}$.

But in $\mathcal{O}$ the ground is not at a distance of 60 km. This length is measured in the muon's frame as H/γ. Here again, $(\frac{1}{2})^4 \simeq \frac{1}{16}$ muons will reach the ground.

The result is the same although the interpretation is different.

3 Lorentz Transformations

We have found out that length and time transform when seen from another frame. The Lorentz transformations (LT) are the mathematical form of these transformations. They

- Are a mathematical expression of relativity ;
- Replace light clocks as a tool to solve problems ;
- Relate position and time of the *same* event as measured by *different* observers.

Definition — Event:
An event is a point in space and time. It has a defined position and time.

3.1 Invariance and Covariance

We will use the words "invariant" and "covariant". Here's what it means in this context. There are other possible definitions.

Definition — Invariant:
A physical quantity is invariant if it does not depend on the reference frame.

Examples:

- The speed of light is invariant in special relativity. Galilean relativity says nothing about it.
- Distances and time intervals are invariant in Galilean relativity. They are not in special relativity.
- Mass is invariant in Newtonian physics but we shall see it is only invariant in special relativity if energy is conserved.

Definition — Covariant:
An *equation* is covariant if it holds in any reference frame.

Examples:

- Trivially, any equation involving only invariant quantities is covariant (one could say it is invariant then). For instance in Galilean relativity $F = ma$ only involves invariant quantities. It is not the case in special relativity.
- Momentum and energy conservation equations are covariant, although momentum and energy obviously depend on the reference frame. For instance in a collision if

$$\sum_{\text{in}} \boldsymbol{p_i} = \sum_{\text{out}} \boldsymbol{p_o}$$

holds in a given reference frame $\mathcal{O}$ then

$$\sum_{\text{in}} \boldsymbol{p'_i} = \sum_{\text{out}} \boldsymbol{p'_o}$$

will be valid in any other reference frame $\mathcal{O}'$ although the individual values $p_{i,o}$ will be different from $p'_{i,o}$. This guarantees one can play snooker on a ship.

3.2 Galilean transformations

Let's first write the transformation from one frame $\mathcal{O}$ into another frame $\mathcal{O}'$ in Galilean relativity.

An event occurs at point P represented by the vector $r = (x, y, z)$ in $\mathcal{O}$. In a frame $\mathcal{O}'$ moving at velocity u relative to $\mathcal{O}$ the *same* event occurs at P' with $r' = (x', y', z')$.

Let's simplify the problem to avoid some cumbersome and unnecessary algebra (Fig. 15). Choose $\mathcal{O}$ and $\mathcal{O}'$

1. such that the axes are parallel,
2. and the origins coincide at $t = 0$.

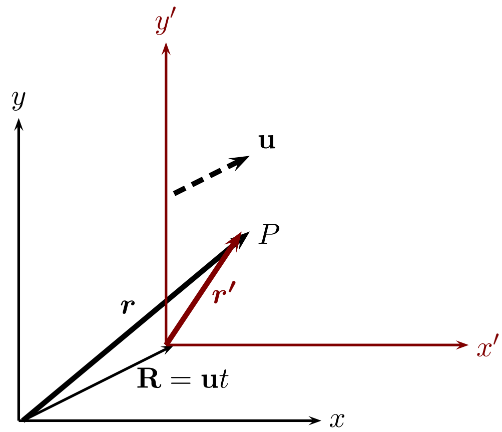


Figure 15: Moving reference frames.

We get the transformations for the position r', speed v' and acceleration a':

$$\boldsymbol{r'} = \boldsymbol{r} - \boldsymbol{R} \quad \rightarrow \quad \boldsymbol{r'} = \boldsymbol{r} - \boldsymbol{u}t, \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \quad \rightarrow \quad \boldsymbol{v'} = \boldsymbol{v} - \boldsymbol{u}, \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \quad \rightarrow \quad \boldsymbol{a'} = \boldsymbol{a}. \tag{7}$$

Equation (5) can be further simplified if we choose the axes such that

3. the velocity u is along the x-axis.

We then have

$$\left.\begin{array}{rrcl} & x' & = & x - ut \\ & y' & = & y \\ & z' & = & z \\ \text{and implicitly} & t' & = & t \end{array}\right\} \tag{8}$$

Eq. (6) is the Galilean transformation of velocities. It expresses the familiar concept of relative velocity.

Eq. (7) says acceleration is invariant. This ensures the covariance of Newton's mechanics.

Eq. (8) are the Galilean transformation (GT) equations. They have not been written as such by Galileo but are based on his formulation of relativity [6]. The last equation about time is added here for completeness. It made no sense to write a transformation equation for time as universal time is one of the foundation axioms of Newtonian mechanics. Newton writes [13]:

> "Absolute, true and mathematical time, to itself, and from its own nature, flows equably without relation to anything external."

Example 1: Momentum conservation

The law of momentum conservation in a two-body collision is:

$$\boldsymbol{p_1} + \boldsymbol{p_2} = m_1\boldsymbol{v_1} + m_2\boldsymbol{v_2} = \text{constant.} \tag{9}$$

Does this satisfy Galileo's relativity principle? Is it covariant under GT?

$$\begin{aligned} \boldsymbol{p'_1} + \boldsymbol{p'_2} &= m_1\boldsymbol{v'_1} + m_2\boldsymbol{v'_2} \\ &\overset{\text{Eq. (6)}}{=} \underbrace{m_1\boldsymbol{v_1} + m_2\boldsymbol{v_2}}_{\text{Const. (Eq. 9)}} - (m_1 + m_2)\boldsymbol{u} \end{aligned} \tag{10}$$

Momentum conservation is covariant if $m_1 + m_2$ is invariant. Momentum conservation requires mass conservation.

Example 2: Speed of light

$\mathcal{O}$ measures the speed of light and gets c. In $\mathcal{O}'$ by Eq. (6) the speed must be $c' = c - u$. GT are incompatible with the invariance of the speed of light.

Similarly, Maxwell equations of electromagnetism do *not* transform under GT. They are not covariant under GT.

3.3 Lorentz transformations

In 1904 (before Einstein!) Lorentz derived transformation equations which were consistent with Maxwell's laws and with relativity principles. They also express the transformations between frames in special relativity. See Young and Freedman [1] Section 37.5 for their derivation.

The Lorentz transformations (LT) are [14]:

$$\left.\begin{aligned} x' &= \gamma(x - ut) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{ux}{c^2}\right) \end{aligned}\right\} \tag{11}$$

assuming $\mathcal{O}'$ moves at speed u along x relative to $\mathcal{O}$.

Some comments on LT:

- The low speed limit for $u \ll c$, i.e. $\gamma \simeq 1$ is

$$\left.\begin{array}{rcl} x' &=& x - ut \\ y' &=& y \\ z' &=& z \\ t' &=& t \end{array}\right\} \equiv \text{GT (Eq. (8))}$$

  This means Galileo's relativity is *not wrong*, but an *approximation* valid at low speeds.

- $t \neq t'$ implies one has to abandon the concept of "universal" time.

- Space and time are "unified" by Lorentz Transformations.

  - For one observer time and space are distinct. There is no "mixing" of space and time for a given observer.
  - For another observer they are also distinct, but with respect to the first observer they are mixed up. If the two observers travel at different speeds they will have to see time and space mixed up for the other observer. We shall come back to this in Section 6.

- The inverse transformations are

$$\left.\begin{array}{rcl} x &=& \gamma(x' + ut') \\ y &=& y' \\ z &=& z \\ t &=& \gamma(t' + \frac{ux'}{c^2}) \end{array}\right\} \text{LT}^{-1} \qquad (12)$$

  The derivation is left as an exercise (Problem 1.1). The inverse transformations (12) are exactly as the direct LT (11) with $u \to -u$, as it must be since $\mathcal{O}$ moves at speed $-u$ relative to $\mathcal{O}'$.

3.4 Observers

What is an observer? One should not be confused by the etymology, "observe" meaning "see". We are not talking about what someone sees at a given moment in time. It is clear that events seen in the distance will be seen with a delay due to the speed of light. This is not what relativity is about. Our observer is a very careful scientist who has paved the space with calibrated clocks.

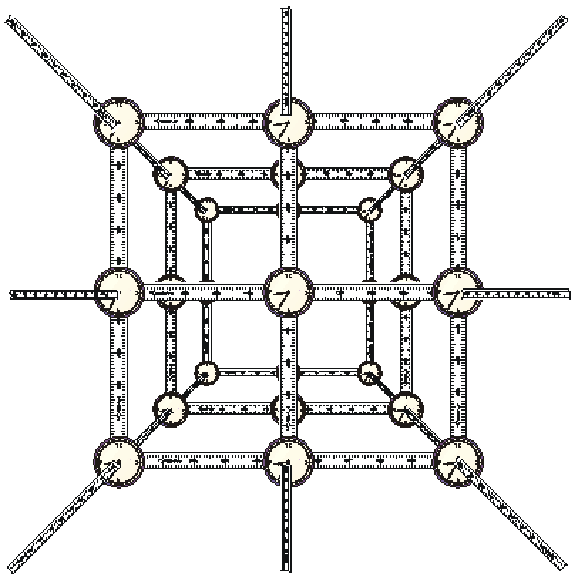


Figure 16: Latticework of metre sticks and clocks. Image from [4].

Figure 16 shows such an imaginary display. All clocks are synchronised using the following procedure: set one clock to an arbitrary time go midway to the next clock, so $1/2\ \mathrm{m}$ if the clocks are one metre away. Emmit two rays of light, one in each direction, and tune the second clock such that the time at which the ray of light arrives at this clock and the reference clock is the same. This is the simplest way of synchronising clocks without making any other assumption than speed of light being constant. Then go on to next clock.

It is not really relevant whether the clocks are $1\ \mathrm{m}$ or $1\ \mu\mathrm{m}$ away. It depends on what you want to measure and the precision you need.

Once this is set up each clock can record the position and the time of moving objects passing in their vicinity. These records are then used by the *observer* to deduce the trajectory of the object.

Definition — Observer:
The observer is a collection of reading clocks associated with a reference frame.

3.5 Lorentz Contraction

A rod has length L_0 measured using a metre rule at rest with respect to it (i.e. L_0 is its proper length).

How do we measure its length when it moves? We have to measure the positions of both ends of the rod *at the same time*. In Figure 17 we have two events at B and F at some time in $\mathcal{O}$.

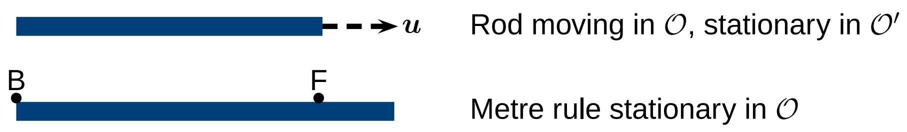


Figure 17: Moving rod.

Let's first look at what we have in frame $\mathcal{O}$:

Event 1: The back end of the rod lines up at B at

$$x_B = 0, \quad t_B = 0.$$

That's our choice of origin for both $\mathcal{O}$ and $\mathcal{O}'$.

Event 2: The front end of the rod lines up at F at

$$x_F = L, \quad t_F = 0,$$

i.e. at the same time! In $\mathcal{O}'$ that's $x'_F = L_0$ and t'_F unknown.

By LT we have

$$\begin{array}{rrcl} & x'_B & = & \gamma(x_B - ut_B) = 0 \\ & t'_B & = & \gamma\left(t_B - u\dfrac{x_B}{c^2}\right) = 0 \\ & x'_F & = & \gamma(x_F - ut_F) \\ \Rightarrow & L_0 & = & \gamma(L - 0). \end{array}$$

The measured length L is shorter than L_0, and we have found the Lorentz contraction again.

We also get

$$t'_F = \gamma\left(t_F - \frac{ux_F}{c^2}\right) = \gamma\left(0 - \frac{uL}{c^2}\right) = -\frac{uL_0}{c^2}$$

and the two measurements are not simultaneous in $\mathcal{O}'$.

If we want to measure the metre rule (at rest in $\mathcal{O}'$) in reference frame $\mathcal{O}$, we need to consider two events which are simultaneous in $\mathcal{O}'$. Hence not B and F! These events will then not be simultaneous in $\mathcal{O}$ and in $\mathcal{O}'$ we also find a contraction by a factor γ.

The non-invariance of simultaneity is the source of this apparent paradox: a metre rule in $\mathcal{O}'$ appears shorter in $\mathcal{O}$, while a similar rule in $\mathcal{O}$ also appears shorter in $\mathcal{O}'$. There's no contradiction: it has to be like this to ensure all frames are equivalent. Else we would know which one is moving.

All apparent paradoxes based on lengths are based on an implied conservation of simultaneity. See for instance the famous pole and barn paradox (classwork).

3.6 Time dilation

Suppose $\mathcal{O}'$ moves at $u = \frac{4}{5}c$ relative to $\mathcal{O}$ and carries a clock which ticks every second. Let's consider the events in $\mathcal{O}'$

$$\begin{array}{lll} 1^{\text{st}} \text{ tick } E_1 & \text{at} & x'_1 = 0; t'_1 = 0 \\ 2^{\text{nd}} \text{ tick } E_2 & \text{at} & x'_2 = 0; t'_2 = 1\,\mathrm{s} \end{array}$$

$1\,\mathrm{s}$ is the "proper" time and E_1 and E_2 are at the same position.

In $\mathcal{O}$ we have

$$\begin{array}{lll} 1^{\text{st}} \text{ tick } E_1 & \text{at} & t_1 \stackrel{(12)}{=} \gamma\left(t'_1 + \dfrac{ux'_1}{c^2}\right) = \gamma(0+0) = 0\,\mathrm{s} \\ 2^{\text{nd}} \text{ tick } E_2 & \text{at} & t_2 \stackrel{(12)}{=} \gamma\left(t'_2 + \dfrac{ux'_2}{c^2}\right) = \gamma(1+0) = \gamma\,\mathrm{s} \end{array}$$

i.e. the clock, which is at rest in $\mathcal{O}'$, takes

$$t_2 = \gamma \times 1\,\mathrm{s} = \frac{1\,\mathrm{s}}{\sqrt{1 - \left(\frac{4}{5}\right)^2}} = \frac{5}{3}\,\mathrm{s}$$

when measured using a clock in $\mathcal{O}$. The clock in $\mathcal{O}'$ "runs slow" as seen in $\mathcal{O}$.

3.7 Measurement of Velocity

So far we have only considered objects at rest either in $\mathcal{O}$ or $\mathcal{O}'$. Let's consider objects moving in $\mathcal{O}$ and $\mathcal{O}'$.

$\mathcal{O}$ measures a velocity:

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t}, \quad v_y = \frac{\mathrm{d}y}{\mathrm{d}t}, \quad v_z = \frac{\mathrm{d}z}{\mathrm{d}t}.$$

Similarly, $\mathcal{O}'$ measures velocity:

$$v'_x = \frac{\mathrm{d}x'}{\mathrm{d}t'}, \quad v'_y = \frac{\mathrm{d}y'}{\mathrm{d}t'}, \quad v'_z = \frac{\mathrm{d}z'}{\mathrm{d}t'}.$$

By LT we have

$$v'_x = \frac{\mathrm{d}x'}{\mathrm{d}t'} = \gamma\left(v_x - u\right)\frac{\mathrm{d}t}{\mathrm{d}t'} \qquad \text{and} \qquad \left(\frac{\mathrm{d}t}{\mathrm{d}t'}\right)^{-1} = \gamma\left(1 - \frac{uv_x}{c^2}\right) \tag{13}$$

which leads to

$$v'_x = \frac{\gamma\left(v_x - u\right)}{\gamma\left(1 - \frac{uv_x}{c^2}\right)} = \frac{v_x - u}{1 - \frac{uv_x}{c^2}}$$

For the other components one has

$$v'_y = \frac{\mathrm{d}y'}{\mathrm{d}t'} \overset{(11)}{=} \frac{\mathrm{d}y}{\mathrm{d}t'} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}t'} \overset{(13)}{=} \frac{v_y}{\gamma\left(1 - \frac{uv_x}{c^2}\right)},$$

and mutatis mutandis for v'_z.

$$v'_x = \frac{v_x - u}{1 - \frac{uv_x}{c^2}}, \qquad v'_y = \frac{v_y}{\gamma\left(1 - \frac{uv_x}{c^2}\right)}, \qquad v'_z = \frac{v_z}{\gamma\left(1 - \frac{uv_x}{c^2}\right)}. \tag{14}$$

Note that the expressions differ for v_x as opposed to v_y or v_z since v_x is special for being parallel to u. It appears in the denominator of all three expressions. Be careful to always choose u along x. Also note that there is no factor γ in the expression for v'_x.

Warning

Here $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{u}{c}$ refer to the velocity of the *reference frame* $\mathcal{O}'$ in frame $\mathcal{O}$. They are not related to the velocity of the moving object.

The non-relativistic ($u \ll c$) limit of Eq. (14) is

$$v'_x = v_x - u, \qquad v'_y = v_y, \qquad v'_z = v_z.$$

as expected from Galilean relativity. But it does not work like that at high speeds.

3.7.1 Relative Velocity

Suppose two spaceships travel in opposite directions at speed $\frac{1}{2}c$ each. What is the speed of ship A relative to B?

Figure 18: Crossing spaceships.

Let's take B as $\mathcal{O}'$. We have $u = \frac{1}{2}c$ relative to $\mathcal{O}$ and A has speed $v_x = -\frac{1}{2}c$ in $\mathcal{O}$. We want the speed of A in $\mathcal{O}'$:

$$v'_x = \frac{v_x - u}{1 - \frac{uv_x}{c^2}} = \frac{-\frac{1}{2}c - \frac{1}{2}c}{1 - \frac{1}{2}\left(-\frac{1}{2}\right)} = \frac{-c}{1 + \frac{1}{4}} = -\frac{4}{5}c,$$

a bit less than c. More generally, if any object moves at $v \leq c$ in any frame, $v' \leq c$ in any other frame (Problem 2.1).

3.7.2 Invariance of Speed of Light

Suppose a ray of light is emitted by a source at rest in $\mathcal{O}'$ moving at speed u relative to $\mathcal{O}$. What is the speed of the light measured in $\mathcal{O}$?

We use the inverse transformation of Eq. (14), i.e. $u \to -u$ and have $v'_x = c$:

$$v_x = \frac{v'_x + u}{1 + \frac{uv'_x}{c^2}} = \frac{c + u}{1 + \frac{u}{c}} = c.$$

3.7.3 Absolute Speed Limit

We conclude

- **Nothing moves faster than c, according to any observer.**
- **c is the speed limit, a parameter of relativity theory.**
- **It is an *experimental* observation that light travels at speed c.**

3.8 Doppler Effect

The Doppler effect is familiar as a sound phenomenon. The frequency of a sound changes due to the movement of the source. An approaching siren has a frequency f higher by $\Delta f = f - f_0$ and receding siren has a frequency lower by Δf:

$$\frac{\Delta f}{f_0} \simeq \pm\frac{u}{v_s} \quad \Rightarrow \quad \frac{f}{f_0} = \frac{v_s \pm u}{v_s} \tag{15}$$

where u is the speed of the siren and v_s the speed of sound.

In the less familiar case where the siren is still and the observer is moving the ratio of frequencies is

$$\frac{f}{f_0} = \frac{\pm u}{u + v_s}. \tag{16}$$

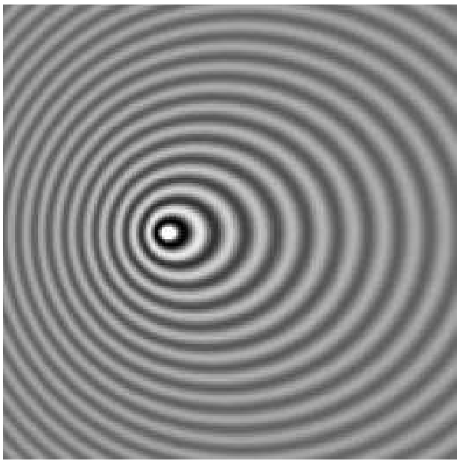


Figure 19: Graphical view of the Doppler effect.

These two cases are not equivalent. By measuring the frequency and the speed of the source one can determine if the source or the observer is moving. This is quite natural as the speed of sound is measured relative to a medium: the air.

3.8.1 Doppler Effect with a Moving Light Source

Something similar occurs with light, with v_s replaced by c, and some additional relativistic effects. A light source at rest in $\mathcal{O}'$ emits light observed by an observer in $\mathcal{O}$. The frequency observed in $\mathcal{O}$ is changed by two effects

1. The source recedes, so the second pulse travels further, increasing the time interval τ in $\mathcal{O}$. This is the same as for the acoustic effect, with v_s replaced by c.
2. The time dilation between $\mathcal{O}$ and $\mathcal{O}'$.

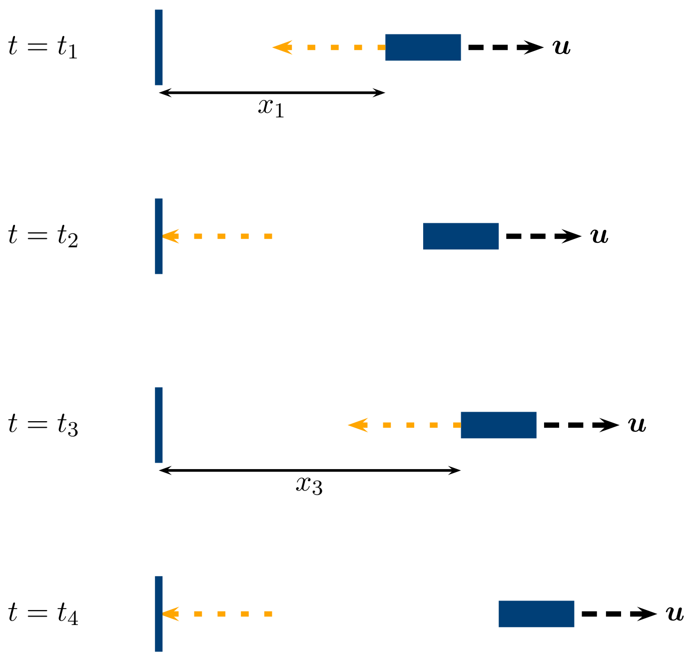


Figure 20: Moving laser as seen in $\mathcal{O}$.

Consider the emission of consecutive pulses by a moving laser at rest in $\mathcal{O}'$ and the reception of the pulses by an observer in $\mathcal{O}$ (Fig. 20). $\mathcal{O}'$ moves at velocity u compared to $\mathcal{O}$.

Event 1: Emission of the first pulse at $x'_1 = 0,\ t'_1 = T$.

Event 2: Reception of the first pulse.

Event 3: Emission of the second pulse at $x'_3 = 0,\ t'_3 = T + \tau_0$.

Event 4: Reception of the second pulse.

Using inverse LT we have

$$\begin{aligned} x_1 &= \gamma\left(x'_1 + ut'_1\right) = \gamma(0 + uT) = \gamma uT \\ t_1 &= \gamma\left(t'_1 + \frac{ux'_1}{c^2}\right) = \gamma T \end{aligned} \tag{17}$$

$$\begin{aligned} x_3 &= \gamma\left(x'_3 + ut'_3\right) = \gamma u(T + \tau_0) \\ t_3 &= \gamma\left(t'_3 + \frac{ux'_3}{c^2}\right) = \gamma(T + \tau_0) \end{aligned} \tag{18}$$

For the reception events we must take into account the time taken by the light to travel to $x_i = 0$:

$$\begin{aligned} t_2 &= t_1 + \frac{x_1}{c} \stackrel{(17)}{=} \gamma T \left(1 + \frac{u}{c}\right) \\ t_4 &= t_3 + \frac{x_3}{c} \stackrel{(18)}{=} \gamma (T + \tau_0) \left(1 + \frac{u}{c}\right) \\ \Rightarrow \quad t_4 - t_2 &= \tau = \gamma \tau_0 \left(1 + \frac{u}{c}\right). \end{aligned}$$

The frequency of the laser is $f_0 = 1/\tau_0$ in $\mathcal{O}'$ and $f = 1/\tau$ in the observer frame $\mathcal{O}$. The ratio of these frequencies is

$$\frac{f}{f_0} = \frac{\tau_0}{\tau} = \frac{1}{\gamma \left(1 + \frac{u}{c}\right)} = \frac{\sqrt{1 - \frac{u^2}{c^2}}}{1 + \frac{u}{c}} = \sqrt{\frac{1 - \frac{u}{c}}{1 + \frac{u}{c}}}. \tag{19}$$

Relativistic Doppler formula:

$$\frac{f}{f_0} = \sqrt{\frac{1 - \beta}{1 + \beta}}, \tag{20}$$

where β takes positive values for a receding and negative values for an approaching source.

Note that in this case only the difference in speeds is relevant. There is no distinction between a moving source and a moving observer, as it should in special relativity.

3.8.2 Example: Doppler Effect Due to the Movement of the Earth

The earth moves at $\sim 30\,\mathrm{km/s}$ with respect to the sun. For a star nearby we have

$$\beta \simeq \frac{3 \cdot 10^4}{3 \cdot 10^8} \simeq 10^{-4}.$$

So

$$\frac{f}{f_0} = \sqrt{\frac{1 - \beta}{1 + \beta}} \simeq \sqrt{(1 - \beta)(1 - \beta)} = 1 - \beta \quad \Rightarrow \quad \frac{\Delta f}{f_0} \simeq -10^{-4}$$

which is a tiny shift to red.

Distant galaxies can have a very large β and the approximation is not valid anymore. The red-shift must be calculated using Eq. (20) directly.

4 Relativistic Mechanics

Having discussed velocities we can now define relativistic momentum and energy. The most important applications of relativistic mechanics is in high-energy particle interactions, for instance collisions or decays. We shall mainly focus on those.

But first let's start with our light clock.

4.1 Energy of the Light Clock

Suppose we have a light clock at rest and free to float. What would happen when we switch it on and the first pulse is emitted?

We know the energy of photons from Planck's law

$$E = h\nu$$

and the momentum from de Broglie

$$p = \frac{h}{\lambda} \quad \text{with} \quad c = \nu\lambda \quad \Rightarrow \quad p = \frac{h\nu}{c} = \frac{E}{c}. \tag{21}$$

A light clock of mass M and length L emits a first pulse of energy E and momentum E/c. By momentum conservation the clock recoils at a speed

$$v = -\frac{p}{M} = -\frac{E}{Mc},$$

where we assume $v \ll c$. After travelling for a time $\Delta t \simeq L/c$ the radiation hits the other end of the clock which brings it back to rest again (Fig. 21). In the process the box has moved by

$$\Delta x = v\Delta t = -\frac{EL}{Mc^2},$$

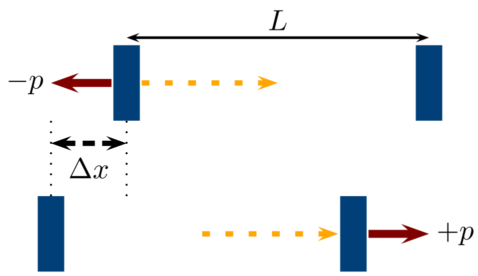


Figure 21: Momentum conservation in a light clock.

where we used $v \ll c$ again. Does this mean the centre of mass of the clock has moved? This cannot be. As no external force acts on the clock the centre of mass cannot have moved. To keep the centre of mass in place we need to *postulate* that the burst of light has transferred some mass m from the left to the right part of the clock such that

$$mL + M\Delta x = 0.$$

This postulated mass can then be calculated as

$$m = -\frac{M}{L}\Delta x = \frac{M}{L}\frac{EL}{Mc^2} = \frac{E}{c^2},$$

or

$$E = mc^2, \tag{22}$$

which you might have seen before.

What does that mean? It is *not* the mass of the light burst. Light has no mass. It is the amount of mass which has been lost by the left side of the clock and absorbed by the right side in the process.

Einstein postulated in his second relativity article [15] that any change of energy in the reference frame of a body is associated to a change of mass. The left side of the clock becomes lighter by a mass E/c^2 and the right side heavier. Mass is not conserved in a process involving a change of energy.

Do not quote this as the proof of $E = mc^2$. We have cheated a bit using de Broglie's laws (1924) which were not known by Einstein in 1905. He reached the same conclusion in a similar way without using them. Have a look at the original paper if you'd like to find out how [15].

It looks as if we haven't used any of the postulates of special relativity here and that Newton's laws could have predicted the same result. This is not really true as we have implicitly used that light is emitted as speed c and that it propagates at speed c.[5]

This result postulates the equivalence of mass and energy. Mass is just a form of energy at rest. Note that it is a very dense form of energy: c^2 is a huge factor. The energy produced by a $1\,\mathrm{GW}$ power plant in a year is equivalent to only $350\,\mathrm{g}$ of mass.

Let's now define more precisely the relativistic momentum and energy.

4.2 Energy and Momentum Conservation

In special relativity energy and momentum conservation is actually more useful than the relativistic form of Newton's second law.

The classical momentum conservation in an elastic collision says:

$$m_1 \boldsymbol{v}_{\mathbf{1}}^{\text{in}} + m_2 \boldsymbol{v}_{\mathbf{2}}^{\text{in}} = m_1 \boldsymbol{v}_{\mathbf{1}}^{\text{out}} + m_2 \boldsymbol{v}_{\mathbf{2}}^{\text{out}}.$$

But this is not covariant under LT. (Problem 2.3). It would be a very bad idea to write this when solving problems.

We need to redefine the momentum to preserve the law. We need

1. A definition of the momentum $\boldsymbol{p}$ such that it is conserved.
2. The low-speed limit must be $\boldsymbol{p} = m\boldsymbol{v}$.
3. The conservation laws must be covariant under LT.

We postulate

Definition — Momentum:

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma_v m\boldsymbol{v} \tag{23}$$

where

$$\gamma_v = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

is the Lorentz factor associated to a speed v. This will be discussed more in detail in Section 4.3.

Does this definition address the constraints listed above?

[5] Although Newton himself wrote *Are not gross bodies and light convertible into one another [...]?* in 1730.

1. The conservation has to be tested experimentally.
2. The low-speed limit is $p = mv$ if $v \ll c$.
3. If $\boldsymbol{p_1} + \boldsymbol{p_2} =$ constant, is $\boldsymbol{p_1'} + \boldsymbol{p_2'} =$ constant$'$ when $\boldsymbol{p_i}$ and $\boldsymbol{p_i'}$ are related by a LT?

Let's simplify by having a momentum along x, i.e. $\boldsymbol{p} = (p, 0, 0)$. Using Eq. (23) we have

$$\begin{aligned} p' &= \frac{mv'}{\sqrt{1-\frac{v'^2}{c^2}}} \quad \text{with} \quad v' \overset{(14)}{=} \frac{v-u}{1-\frac{uv}{c^2}} \\ &= \gamma p - \gamma \frac{u}{c^2} \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}, \end{aligned} \tag{24}$$

after some algebra. Note that as usual

$$\gamma = \sqrt{\frac{1}{1-\frac{u^2}{c^2}}}$$

refers to velocity u of $\mathcal{O}'$ in $\mathcal{O}$. It has nothing to do with velocities v or v'.

Thus with $p = p_1 + p_2$ we get

$$p_1' + p_2' = \underbrace{\gamma\,(p_1 + p_2)}_{\text{constant}} - \underbrace{\gamma \frac{u}{c^2}}_{\text{constant}} \left(\frac{m_1 c^2}{\sqrt{1-\frac{v_1^2}{c^2}}} + \frac{m_2 c^2}{\sqrt{1-\frac{v_2^2}{c^2}}} \right).$$

The momentum is conserved provided the quantity

$$\frac{m_1 c^2}{\sqrt{1-\frac{v_1^2}{c^2}}} + \frac{m_2 c^2}{\sqrt{1-\frac{v_2^2}{c^2}}}$$

is also conserved.

This is not different from Galilean relativity (Eq. (10)) where momentum is conserved provided that mass is conserved. Here we have no requirement on mass (we know it's related to energy) but on the quantity above. We shall define as the energy:

Definition — Energy:

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma_v mc^2. \tag{25}$$

So momentum conservation applies in all inertial frames, *if* energy is conserved. Here again there is no mathematical proof that this quantity is indeed conserved. It's an experimental fact.

4.3 Useful Shortcuts

Dangerous, but sometimes useful shortcuts are

$$\beta_v = \frac{v}{c} \qquad\qquad \gamma_v = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{26}$$

As β and γ are nothing but functions of velocity, we can also write them as a function of v. This is very useful when you want to Lorentz-transform into the frame in which the moving body is at rest, like we did with the cosmic muon in Section 2.6. Then $u = -v$ and

$$\gamma_v = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \gamma_u.$$

In that case one can write

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma_v mc^2.$$

It becomes really dangerous when people start writing

$$E = \gamma mc^2.$$

implying that γ refers to the speed of the moving body. Particle physicists tend to do this.

4.4 Energy-Momentum Relations

In Newtonian mechanics we relate the kinetic energy K to the momentum by

$$K = \frac{p^2}{2m}.$$

How is this relation in relativistic mechanics? Using Eq. (23) and (25) we get

$$\frac{p}{E} = \frac{mv}{mc^2} = \frac{v}{c^2} \quad \Rightarrow \quad v = \frac{p}{E}c^2, \tag{27}$$

and

$$E^2 = \frac{m^2c^4}{1-\frac{v^2}{c^2}} = \frac{m^2c^4}{1-\frac{p^2}{E^2}c^2} = \frac{E^2m^2c^4}{E^2-p^2c^2} \quad \Rightarrow \quad E^2 - p^2c^2 = m^2c^4$$

or

$$\boxed{E^2 = p^2c^2 + m^2c^4,} \tag{28}$$

which is the relativistic energy-momentum relation.

4.4.1 Rest Energy

An object at rest has $p = 0$ and therefore

$$E_0 = mc^2. \tag{29}$$

A particle at rest has an energy equal to its mass (times some constant). The mass at rest ("rest mass" or "proper mass") is a property of the particle.

4.4.2 Kinetic Energy

If $E_0 = mc^2$ is the rest energy one can define the kinetic energy as

$$K = E - E_0 = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - mc^2 = mc^2\left(\gamma_v - 1\right) \tag{30}$$

4.4.3 Non-relativistic approximation

At low speeds $v \ll c$ we get the momentum

$$p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \Rightarrow \qquad p \simeq mv$$

and the kinetic energy

$$K = mc^2\left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1\right) = mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots - 1\right) \simeq \frac{1}{2}mv^2,$$

and recover Newton's quantities.

4.4.4 Ultra-relativistic approximation

At very high speeds $v \simeq c$ (but smaller!) the rest energy of the particle is negligible compared to the total energy. We then have

$$E^2 = p^2c^2 + m^2c^4 \simeq p^2c^2 \qquad \Rightarrow \qquad E \simeq pc.$$

This is a very good approximation for the muon in Section 2.6.

4.4.5 About Photons

Where does the photon fit into all this? We know from Planck and de Broglie that $E = pc$ (Eq. (21)). But we also know $E^2 = p^2c^2 + m^2c^4$, so the photon must have no mass. Using Eq. (27):

$$v = \frac{p}{E}c^2 = c$$

Photons travel at speed of light at any energy. What differs is their frequency, i.e. their colour.

4.4.6 High-Energy Electrons

In an X-ray gun electrons are accelerated by an electric potential of $U \sim 10^6\,\mathrm{V}$. What's the electron's speed?

Newton would have said

$$K = eU = \frac{1}{2}mv^2$$

with $m_e \sim 10^{-30}\,\mathrm{kg}$ and $e = 1.6 \cdot 10^{-19}\,\mathrm{C}$,

$$v^2 = \frac{2eU}{m_e} \simeq \frac{2 \cdot 1.6 \cdot 10^{-19} \cdot 10^6}{10^{-30}} = 32 \cdot 10^{16}\,\frac{\mathrm{m}^2}{\mathrm{s}^2} \qquad \Rightarrow \qquad v \simeq 6 \cdot 10^8\,\frac{\mathrm{m}}{\mathrm{s}} \simeq 2c,$$

clearly an invalid use of Newtonian mechanics. We have comparable kinetic energy $K = 16 \cdot 10^{-14}\,\mathrm{J}$ and rest mass energy $m_e c^2 = 9 \cdot 10^{-14}\,\mathrm{J}$.

The correct way is to use Eq. (27)

$$v = \frac{p}{E}c^2 \quad \text{with} \quad E = K + m_e c^2 = 25 \cdot 10^{14}\,\mathrm{J}$$

and the momentum is obtained from Eq. (28)

$$p^2c^2 = E^2 - m_e^2 c^4 \qquad \Rightarrow \qquad \frac{pc}{E} = \sqrt{1 - \left(\frac{m_e c^2}{E}\right)^2} = \frac{v}{c}$$

and therefore

$$v = c\sqrt{1 - \left(\frac{9}{25}\right)^2} \simeq 0.9c.$$

It takes a million volts to accelerate an electron to 90% of the speed of light.

4.5 Some Useful Relations

We now have some useful relations between E, p, β_v and γ_v:

For a particle of mass m, velocity v, momentum p and total energy E

$$\beta_v = \frac{v}{c} = \frac{pc}{E}, \qquad \gamma_v = \frac{E}{mc^2}. \tag{31}$$

5 Applications

5.1 Units

Charged particles attain high energies via electric fields, as in the example above. For fundamental particles of charge $\pm e$, the best is to measure the energy in units of fundamental charge times electric potential difference.

The kinetic energy of a particle of charge e accelerated by $1\mathrm{V}$ is 1 eV $= 1.6 \cdot 10^{-19}\mathrm{J}$. The use of multiples of eV is more practical for atomic or sub-atomic physics than Joules (see Problem 2.4).

Energies are measured in eV (typical energies of atoms in gas at room temperature), keV (kinetic energy of electrons in an old-style TV set), MeV (as the electron above), GeV (at big accelerators), and even TeV ($= 10^{12}$ eV $\simeq 10^{-7}$ J, at the biggest accelerators like the Large Hadron Collider at CERN).

Masses are then measured in units of eV$/c^2$. For instance

$$1\ \text{GeV}/c^2 = 10^9 \frac{1\text{V} \cdot e}{c^2} = 10^9\ \frac{1.6 \cdot 10^{-19}}{9 \cdot 10^{16}} \simeq 2 \cdot 10^{-27}\,\text{kg}$$

is about the mass of the proton. The electron mass is 511 keV$/c^2$.

Similarly one measures momenta in eV$/c$.

Since the factors c and e are absorbed into the units, one can write Eq. (28) as

$$E^2 = p^2 + m^2$$

when using these units. This avoids clutter of c's and e's in equations, like in our electron speed calculation above.

5.1.1 High-Energy Electrons Revisited

Using these relations and units we solve the problem in Section 4.4.6 in one line

$$\beta_v = \frac{p}{E} = \frac{\sqrt{E^2 - m^2}}{E} = \frac{\sqrt{1 - \left(\frac{1}{2}\right)^2}}{1} \simeq 0.9.$$

5.2 Nuclear Physics

A nucleus is a *bound* state of *nucleons* (p,n). Where bound means that

$$E_{\text{Nucleus}} < E_{\text{free constituents}}$$

else it would be advantageous to be free and the nucleus would fall apart. Since rest energy is related to mass, we have

$$m_{\text{Nucleus}} < \sum m_{\text{constituents}}.$$

Example: The simplest nucleus is the deuterium $\mathrm{D} = {}^2_1\mathrm{H} = (p, n)$. The masses are [16]:

$$\begin{aligned} m_p &= 938.272\ \text{MeV}/c^2 \\ m_n &= 939.565\ \text{MeV}/c^2 \\ m_\mathrm{D} &= 1875.613\ \text{MeV}/c^2 \end{aligned}$$

The difference in energies $(m_\mathrm{D} - m_p - m_n)c^2 = 2.22$ MeV is the *binding energy* of the deuteron nucleus. This means the fusion of a proton and a neutron would free about 2 MeV of energy in form of high-energy photons. However, free neutrons are not available in nature as they decay.

But fusion of deuterium works. In the sun the fusion

$$\mathrm{D} + \mathrm{D} \Rightarrow {}^3_2\mathrm{He} + n + 3\ \text{MeV}$$

happens all the time and is providing the fuel for life on earth.

Fusion is possible when the binding energy per nucleon increases with the number of nucleons in the nucleus. Fission is possible when this energy decreases. The binding energy is shown for all nuclei in Figure 22. At the top of the curve is iron ${}^{56}\mathrm{Fe}$, which is the most bound nucleus. Nuclear fusion in stars stops at iron, that's why there is so much iron in the Universe, as compared to similar atoms. The earth core is mainly made of iron.

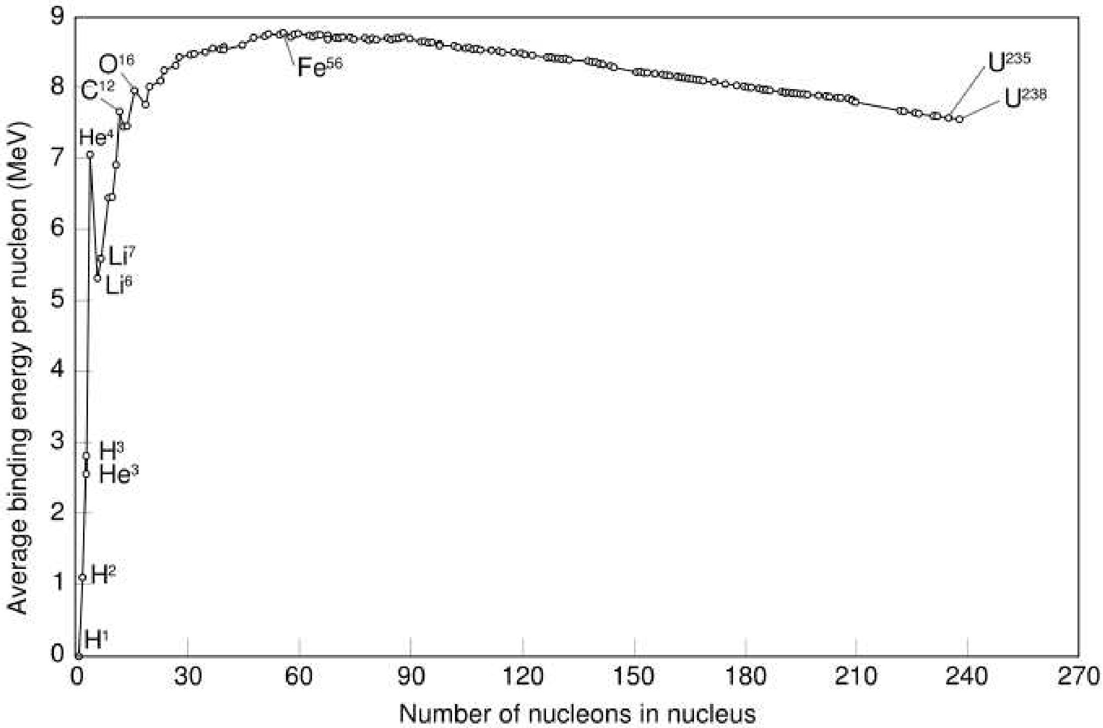


Figure 22: Binding energy of various nuclei.

5.3 Particle Collisions

Momentum and energy conservation in collisions and decays is written

$$\left.\begin{aligned} \sum_i \boldsymbol{p}_i &= \sum_o \boldsymbol{p}_o \\ \sum_i E_i &= \sum_o E_o \\ \text{with } E^2_{(i,o)} &= p^2_{(i,o)}c^2 + m^2_{(i,o)}c^4 \end{aligned}\right\} \tag{32}$$

where i stands for *incoming* and o for *outgoing*. In an elastic collision the same particles appear on both sides of the equations. In a decay there is only one incoming particle. The general case is the inelastic collision where there can be any number of particles on each side. The values of $\boldsymbol{p}$ and E differ for observers in different inertial frames, but the relations above remain valid in *any* inertial frame. This is the *covariance of energy and momentum relations*.

A few examples are given below.

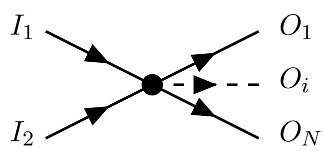


Figure 23: Collision, decay...

5.3.1 Particle Decay at Rest

Take the decay of the neutral kaon to two pions $K^0 \to \pi^+\pi^-$ in its rest frame.

Momentum conservation: $0 = \boldsymbol{p}_1 + \boldsymbol{p}_2$

$p_1 = p_2 \quad (= p_\pi$, say)

Energy conservation: $E_{K^0} = m_{K^0}c^2 = E_1 + E_2$

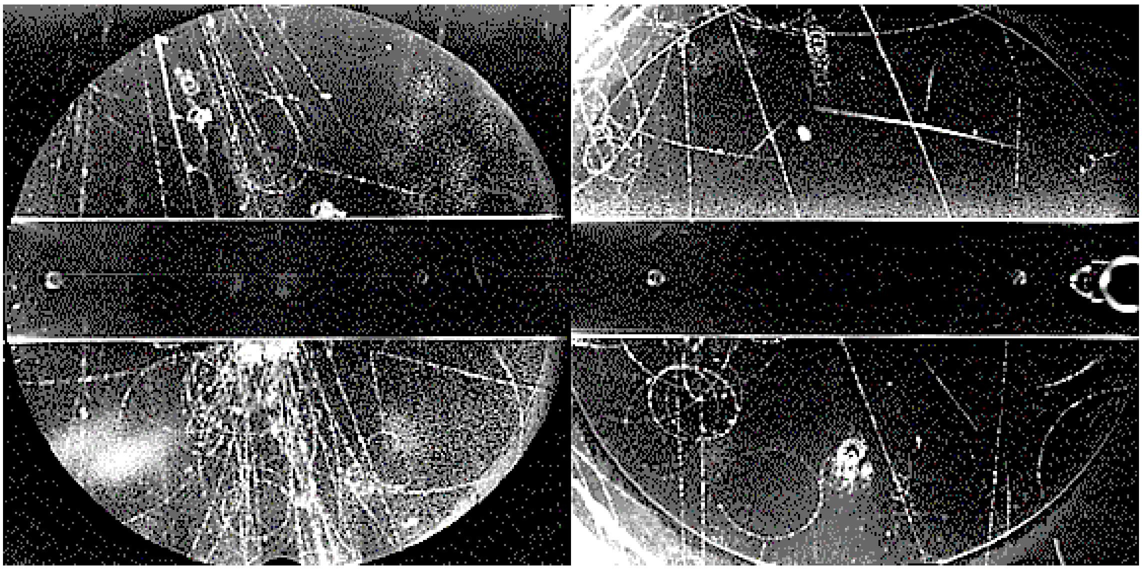


Figure 24: Decays of a K^0 (left) and a K^+ (right), Manchester, December 20, 1947 [17].

where

$$E^2_{(1,2)} = p^2_\pi c^2 + m^2_\pi c^4 \quad \Rightarrow \quad E_1 = E_2 \quad (= E_\pi, \text{ say}).$$

Experiment gives

$$\begin{aligned} m_{K^0} &= 498\,\text{MeV}/c^2 \quad \Rightarrow \quad E_\pi = \frac{1}{2} m_{K^0} c^2 = 249\,\text{MeV} \\ m_\pi &= 139\,\text{MeV}/c^2 \end{aligned}$$

and

$$p^2_\pi c^2 = E^2_\pi - m^2_\pi c^4 = 249^2 - 139^2 \quad \Rightarrow \quad p_\pi = 206\,\text{MeV}/c.$$

The two pions fly away with momenta of 206 MeV$/c$ and speeds of $\beta = \frac{206}{249} \simeq 0.7$.

This process is purely relativistic. Newton's laws do not allow mass to be converted into kinetic energy.

5.3.2 Particle Decay in Flight

In the real world a K^0 is almost never at rest. It is flying with a momentum $\boldsymbol{p}_{K^0}$ in the laboratory frame $\mathcal{O}$, as in Figure 24. But we know the analysis is simple in the K^0 frame $\mathcal{O}'$: $\boldsymbol{p}'_1 = \boldsymbol{p}'_2 = 206$ MeV$/c2$. So we simply need to Lorentz-transform this result into the laboratory frame.

To do this we need Lorentz transformations for momenta.

5.3.3 Lorentz Transformations for Momentum and Energy

The momentum is defined as

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

and we already know how to transform it using Eq. (24):

$$p'_x = \gamma p_x - \gamma \frac{u}{c^2} \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \left(p_x - \frac{uE}{c^2} \right),$$

where we also used the definition of the Energy, Eq. (25).

A similar analysis leads to the transformations:

$$\left.\begin{array}{rcl} p'_x &=& \gamma\left(p_x - \frac{uE}{c^2}\right) \\ p'_y &=& p_y \\ p'_z &=& p_z \\ E' &=& \gamma\left(E - up_x\right) \end{array}\right\} \tag{33}$$

where $\gamma = \gamma_u$ as usual.

We then get the Lorentz transformations:

Lorentz Transformations ($\boldsymbol{x}, ct$)

$$\left.\begin{array}{rcl} x' &=& \gamma\left(x - \beta ct\right) \\ y' &=& y \\ z' &=& z \\ ct' &=& \gamma\left(ct - \beta x\right) \end{array}\right\} \tag{34}$$

Lorentz Transformations ($\boldsymbol{p}, E$)

$$\left.\begin{array}{rcl} p'_x &=& \gamma\left(p_x - \beta\frac{E}{c}\right) \\ p'_y &=& p_y \\ p'_z &=& p_z \\ \frac{E'}{c} &=& \gamma\left(\frac{E}{c} - \beta p_x\right) \end{array}\right\} \tag{35}$$

$\boldsymbol{p}$ transforms like $\boldsymbol{x}$ and E/c like ct.

Inverse transformations also take the usual form.

Warning

In equation (35) γ and β refer to the velocity of the frame $\mathcal{O}'$ in frame $\mathcal{O}$. They have nothing to do with the velocity of the particle of momentum $\boldsymbol{p}$ (except if you transform into the particle frame). Be very careful not to mix up γ and β obtained from Eq. (35) and $\gamma_{\boldsymbol{v}}$ and $\beta_{\boldsymbol{v}}$ given in Eq. (31).

5.3.4 Particle Decay in Flight

Back to our K^0. Consider a decay along the x-axis, the axis of motion of the K^0. From Section 5.3.1 we have

$$\begin{array}{rclcrcl} (p'_1)_x &=& 206\,\text{MeV}/c & \qquad & (p'_2)_x &=& -206\,\text{MeV}/c \\ E'_1 &=& 249\,\text{MeV}/c & & E'_2 &=& 249\,\text{MeV}/c \end{array}$$

Let's transform this to the laboratory frame $\mathcal{O}$ using inverse LT:

$$(p_1)_x = \gamma\left[(p'_1)_x + \beta\frac{E'_1}{c}\right].$$

with $\beta = u/c$ and u is the speed of $\mathcal{O}'$ relative to $\mathcal{O}$.

Here we are in the special case where the velocity v of our object of momentum p is equal to u. We can thus use the useful relations of Eq. (31):

$$u = \beta c = \frac{p_{K^0}}{E_{K^0}}c^2.$$

For example a K^0 with $p_{K^0} = 500$ MeV/c in the laboratory has

$$\begin{aligned}
E_{K^0} &= \sqrt{p_{K^0}^2 c^2 + m_{K^0}^2 c^4} \simeq 700 \text{ MeV} \\
\beta &= \frac{p_{K^0} c}{E_{K^0}} \simeq \frac{5}{7} \qquad \gamma = \frac{E_{K^0}}{m_{K^0} c^2} \simeq \frac{7}{5} \\
(p_1)_x &= \frac{7}{5}\left[+206 + \frac{5}{7}249\right] \simeq 539 \text{ MeV}/c \\
(p_2)_x &= \frac{7}{5}\left[-206 + \frac{5}{7}249\right] \simeq -39 \text{ MeV}/c,
\end{aligned}$$

and we have one pion going forward at a large speed ($\beta = 0.98$) and the other one backward with a low speed ($\beta = 0.27$).

6 Four-Vectors

Lorentz transformations teach us that there is no absolute space and no absolute time. Time and space mix depending on the speed of the observer. Each observer knows what his space and his time is and doesn't feel like they mix, but when talking to another observer in another frame of reference, one has to apply Lorentz transforms, in particular

$$x' = \gamma\,(x - \beta ct) \qquad ct' = \gamma\,(ct - \beta x)\,. \tag{36}$$

x' is an admixture of x and t and t' an admixture of t and x.

That's similar to rotations in two dimensions. If you speak about the position of a point P with an observer whose reference frame is rotated by an angle α with respect to yours you will have to apply the transformation

$$x' = x\cos\alpha + y\sin\alpha \qquad y' = y\cos\alpha - x\sin\alpha, \tag{37}$$

as in Figure 25. x' is an admixture of x and y.

Is a Lorentz transform just a rotation is space-time? Not really, the key problem being the presence of the two minus signs in the Lorentz transform while there is a minus and a plus in the rotation. See Figure 26 for a graphical representation.

But there are similarities. In two dimensions there is one obvious quantity which is conserved by rotations: the distance r:

$$r'^2 = x'^2 + y'^2 = x^2 + y^2 = r^2,$$

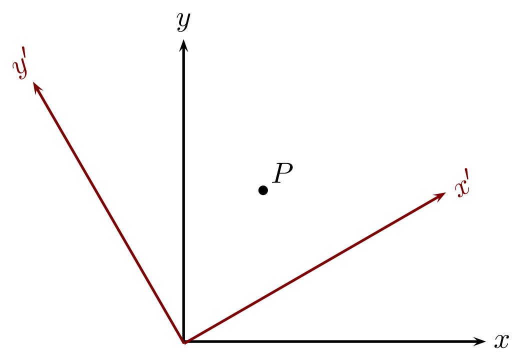


Figure 25: Rotation in 2D.

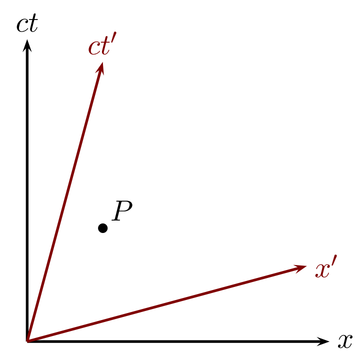


Figure 26: Lorentz-transform.

which you can easily prove using Eq. (37). r is an invariant of 2D-rotations and can easily be generalised to 3D-rotations

$$r'^2 = x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2 = r^2. \tag{38}$$

Is there an equivalent for Lorentz Transforms? Yes, it is

$$c^2t'^2 - x'^2 - y'^2 - z'^2 = c^2t^2 - x^2 - y^2 - z^2 \tag{39}$$

Note the minus signs! The proof is in Problem 3.1.

Given this invariant it is quite natural to extend three-dimensional vectors to four-vectors including time. They are usually written as

Definition — Space-time four-vector:

$$a \equiv (ct, x, y, z) = (ct, \boldsymbol{x})$$

where $\boldsymbol{x}$ is the spacial three-vector.

Some authors write them with the space component first and the time last. They can also be written in columns. Finally some write the first term as $-ct$. It's just a matter of conventions and does not matter as long as one gets Eq. (40) and (42) below right. There is also no standard notation for four-vectors unlike three-vectors which are shown in bold-type. In these notes we shall always call a space-time four-vector a or b.

Unlike for three-vectors we shall define the squared modulus as follows:

Definition — Squared Modulus of a four-vector:

$$a^2 \equiv (ct)^2 - x^2 - y^2 - z^2 \tag{40}$$

a^2 is *invariant* under Lorentz transformations as shown in Eq. (39).

So for any four-vector a and any Lorentz transformation LT:

$$a^2 = [\mathsf{LT}(a)]^2\,. \tag{41}$$

Students who like linear algebra might try to write LT as a 4×4 symmetric square matrix (Problem 3.4). But we won't need this.

6.1 Example

For example go back to Figures 11 and 12 on p. 11. Consider the two events

- Emission of a ray of light from the base of the clock, (which we set to be the origin O)
- Reception of this ray back on the base.

In the first case (Fig. 11) we have the light clock at rest in the train. If we suppose the clock is $h = 2$ metres high, the light will take a time $ct' = 4\,\mathrm{m}$ for the round trip. The four-vector is then

$$s' = (4, 0, 0, 0) \quad [\mathrm{m}].$$

In the frame of the earth the clock is moving with the train. Suppose the base has moved by $3\,\mathrm{m}$ during the time the light travels in the clock. At the end of the round trip the base will be at $x = 3\,\mathrm{m}$. And the time taken is given by Pythagoras: $(ct)^2 = x^2 + (2h)^2 = 5^2$.

The four-vector in the earth frame is

$$s = (5, 3, 0, 0) \quad [\mathrm{m}].$$

And we have

$$s^2 = 5^2 - 3^2 = 4^2 = s'^2 \quad [\mathrm{m}^2].$$

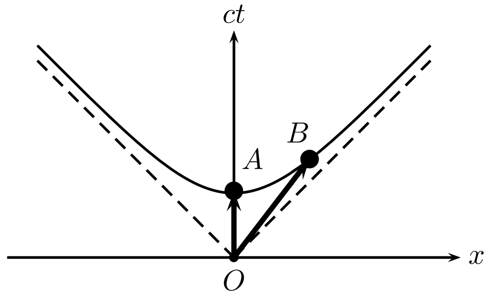


Figure 27: Graphical representation of the invariance.

Figure 27 shows a 2D projection of four dimensional space-time where the two space-time vectors are shown. The round-trip trajectory of the ray of light in the frame of the train is $OA \equiv s'$, with $x = 0$ staying constant. In the frame of the earth the light ray goes forward in x and takes longer in t: this is the $OB \equiv s$ vector. Both are on a hyperbola at constant a^2. The dashed diagonals show trajectories at speed of light (so their slope is $\mathrm{d}x/\mathrm{d}ct = 1$). They are the limit for train speeds close to the speed of light.

The fact that the curve is a hyperbola is a consequence of the minus sign in the modulus of four-vectors. In the case of the 3D distance invariance r^2 we would draw a circle.

6.2 Space-Time Geometry

Unlike $\boldsymbol{x}^2$ which is always positive, a^2 can take negative values. We can classify four-vectors according to the sign of a^2:

If $a^2 > 0$, a **is called *timelike*,**

If $a^2 < 0$, a **is called *spacelike*,**

If $a^2 = 0$, a **is called *lightlike*.**

Note that $\sqrt{a^2}$ has only a physical meaning for timelike vectors. It's the time measured by an observer moving along the four-vector.

6.2.1 Timelike

Lets extend Fig. 27 to the past. Suppose an event occurs at the origin O at $\boldsymbol{x} = 0, t = 0$. Another event occurs at point P at $x = 0, t < 0$. It is in the past at the same place as O and hence can have some effect on O. Event Q also occurred in the past, but at a different place. An object from Q could still reach O while travelling at speed below c. The event Q could be the Piccadilly line starting from Knightsbridge, P be you arriving on the South Kensington platform and O you boarding the Piccadilly line.

Everything in zone ① below the dashed lines is in O's past domain of influence. These are events that can affect O. Everything in zone ② above the dashed lines in the future is in O's future domain of influence. These are events that could be influenced by O. R could be you leaving the Piccadilly line at Hammersmith. All points in these two domains have $ct^2 > \boldsymbol{x}^2$ and are thus *timelike*. It's the part of the universe that affects O now or that O is affecting.

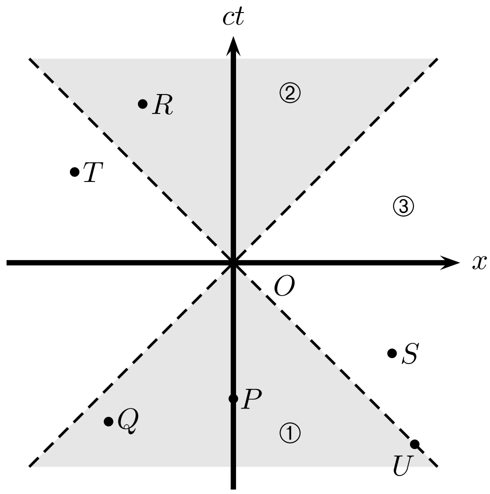


Figure 28: Space-time geometry.

6.2.2 Spacelike

Event S is also in the past but too far away to affect O. Information about event S cannot not have reached O. It might reach the line $x = 0$ in the future but not at time $t = 0$. For instance if Sirius exploded yesterday we wouldn't know it before 8 years. Events in these regions (zone ③) have $ct^2 < \boldsymbol{x}^2$ and are *spacelike*. The same applies to T although it's in the future. S and T are not really different as one can always find a reference frame for which S is *after* O and T, for instance.

This is not possible for Q and O. Q will always be seen as before O by any observer. Else causality would be meaningless.

6.2.3 Lightlike

The dashed lines are O's light-cone. It's a three-dimensional surface of all events which you can see while boarding the Piccadilly line (if in the past), or which can see O in the future. Four-vectors on these lines have $ct^2 = \boldsymbol{x}^2$ and are *lightlike*.

Point U for instance could be the emission of a photon from the sun arriving at South Kensington at O.

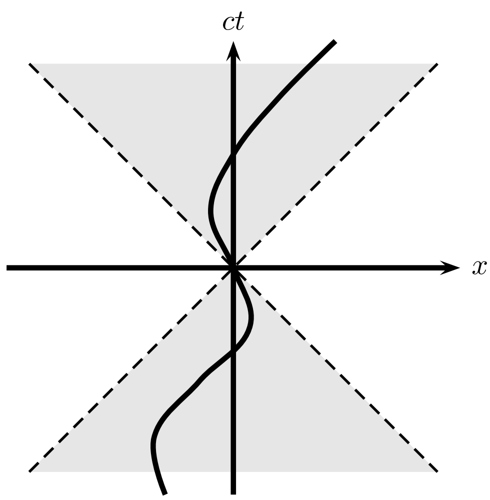


Figure 29: A "worldline": a trajectory of a particle through space-time.

6.2.4 Time Travel

While four-vectors can be spacelike, the four-vector between two positions of the same particle cannot be. Along the trajectory of a particle in four-dimensional space-time (called "worldline") starting from the origin, the squared squared modulus a^2 of its position four-vector is always positive and always increasing along the worldline at speeds $u < c$.

In the own proper frame time is increasing. There's no turning back. The particle in Fig 29 crosses $x = 0$ three times, but never goes back in time and never exceeds speed of light. Although there is a minus sign in Eq. (40) the speed limit c will prevent us reducing a^2. The distance r (Eq. (38)) can become 0 and one can come back to the initial point. But not from a trip in space-time. Time travel is excluded by special relativity.

6.3 Scalar Product

For three-vectors the squared modulus is nothing but a special case of the scalar ("dot") product.

$$\boldsymbol{x_1} \cdot \boldsymbol{x_2} = x_1 x_2 + y_1 y_2 + z_1 z_2 = |\boldsymbol{x_1}|\,|\boldsymbol{x_2}| \cos\theta$$

for any two vectors $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$, where θ is the angle of the two vectors. This quantity is obviously conserved under rotations as the latter does not affect moduli nor angles.

Similarly for four-vectors we have:

Definition — Scalar product of two four-vectors:

$$a \cdot b \equiv c^2 t_a t_b - x_a x_b - y_a y_b - z_a z_b = c^2 t_a t_b - \boldsymbol{x_1} \cdot \boldsymbol{x_2}. \tag{42}$$

The scalar product is also *invariant* under Lorentz transformations.

The proof is in Problem 3.1.

6.4 Four-Momentum

Equations (34) and (35) show that $\boldsymbol{p}$ transforms like $\boldsymbol{x}$ and E/c like t. All we have said for $a = (ct, \boldsymbol{x})$ is also valid for $P = (E/c, \boldsymbol{p})$.

Definition — Energy-momentum four-vector:

$$P \equiv \left(\frac{E}{c}, p_x, p_y, p_z\right) = \left(\frac{E}{c}, \boldsymbol{p}\right). \tag{43}$$

The full name *energy-momentum four-vector* is often shortened *four-momentum* in physicists' jargon. That's what it is called on `wikipedia`. Taylor and Wheeler [4] invent the neologism *momenergy*, which it is hardly ever used elsewhere.

We now have a new Lorentz-invariant, the scalar product of two four-momentum vectors

$$P_1 \cdot P_2 \equiv \frac{E_1 E_2}{c^2} - \boldsymbol{p_1} \cdot \boldsymbol{p_2}. \tag{44}$$

which will prove very useful for problem-solving. If we take $P = P_1 = P_2$ we get the squared modulus of a four-momentum

$$P^2 = \frac{E^2}{c^2} - \boldsymbol{p}^2 = m^2 c^2, \tag{45}$$

which is as invariant as it gets.

6.4.1 Creation of anti-protons

Anti-protons $\overline{p}$ are created in collisions of accelerated protons with a target in the reaction

$$p + p \rightarrow p + p + p + \overline{p}$$

where the first proton is the incoming one and the second a proton of the target. They both also come out, but transfer some energy for the creation of a $p + \overline{p}$ pair. What is the minimal energy for this reaction to happen?

The first two protons have four-momenta in the laboratory frame $\mathcal{O}$

$$P_1 = \left(\frac{E}{c}, |p|, 0, 0\right) \quad \text{and} \quad P_2 = \left(\frac{mc^2}{c}, 0, 0, 0\right).$$

The incoming 4-momentum is the sum

$$P_{\text{in}} = P_1 + P_2 = \left(\frac{E + mc^2}{c}, |\boldsymbol{p}|, 0, 0\right).$$

This is equal to the outgoing 4-momentum P_{out}, but the latter is difficult to relate to anything useful about the four outgoing particles.

It is much easier in the centre-of-mass frame $\mathcal{O}'$. At the threshold, i.e. when the energy is just enough to produce the additional pair, the pack of three protons and the anti-proton is at rest in $\mathcal{O}'$. The total momentum in this frame is 0 and the energy $4mc^2$

$$P'_{\text{out}} = \left(\frac{4mc^2}{c}, 0, 0, 0\right).$$

The prime reminds us that we are in a different frame. Using Eq. (45) we can write

$$\begin{aligned}
P_{\text{in}}^2 &= P'^2_{\text{out}} \\
\left(\frac{E + mc^2}{c}\right)^2 - \boldsymbol{p}^2 &= (4mc)^2 \\
\frac{E^2}{c^2} + 2Em + m^2c^2 - \frac{E^2}{c^2} + m^2c^2 &= 16m^2c^2 \qquad \text{(using Eq. 28)} \\
2Em &= 14m^2c^2 \\
E &= 7mc^2.
\end{aligned}$$

One needs to accelerate a proton to a kinetic energy of six proton masses to produce one anti-proton.

You can redo the calculation in the case both incoming protons are colliding head-on at the same energy and would find you need only $K = mc^2$ for each proton. In the fixed target case the energy of four proton masses is wasted in useless momentum in the fixed target experiment. This is why modern particle physics experiments all run using colliders.

References

[1] Hugh D. Young and Roger A. Freedman. *University Physics*. Pearson, 2007. ISBN 978-0-321-50130-1.

[2] A.P. French. *Special Relativity*. The M.I.T. Introductory Physics Series. Chapman and Hall, 1968. ISBN 0-412-34320-7.

[3] JOHN B. KOGUT. *Introduction to Relativity*. Complementary Science Series. Harcourt Academic Press, 2001. ISBN 978-0124175617.

[4] EDWIN F. TAYLOR AND JOHN ARCHIBALD WHEELER. *Spacetime Physics*. W.H. Freeman and Company, 1991. ISBN 978-0716723271.

[5] RICHARD P. FEYNMAN, ROBERT B. LEIGHTON, AND MATTHEW SANDS. *Feynman Lectures on Physics*. Addison Wesley Longman, 1970. ISBN 978-0201021158.

[6] GALILEO GALILEI. *Dialogue Concerning the Two Chief World Systems*. University of California Press, 1953. Translated by Stillman Drake. Original version published in 1632.

[7] ALBERT EINSTEIN, Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt. *Annalen Phys.*, 17:132, 1905.

[8] A.A. MICHELSON. *Am. J. Sci.*, 122:120, 1881.

[9] A.A. MICHELSON AND E.W. MORLEY. *Am. J. Sci.*, 134:333, 1887.

[10] ALBERT EINSTEIN, Zur Elektrodynamik bewegter Körper. *Annalen Phys.*, 17:891–921, 1905.

[11] CERN. Welcome to CERN. http://public.web.cern.ch/Public/.

[12] C. AMSLER ET AL., *(Particle Data Group)*, Review of particle physics. *Phys. Lett.*, B667:1, 2008.

[13] ISAAC NEWTON. *PhilosophiæNaturalis Principia Mathematica*. London Reg. Soc. Praeses, 1687.

[14] HENDRIK ANTOON LORENTZ, Electromagnetic phenomena in a system moving with any velocity less than that of light. *Proc. Acad. Science Amsterdam*, IV:669–678, 1904.

[15] ALBERT EINSTEIN, Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig? *Annalen Phys.*, 18:226–228, 1905.

[16] M.-W. YAO ET AL., *(Particle Data Group)*, Review of Particle Physics. *Journal of Physics*, G(33), 2006.

[17] G. D. ROCHESTER AND C. C. BUTLER, EVIDENCE FOR THE EXISTENCE OF NEW UNSTABLE ELEMENTARY PARTICLES. *Nature*, 160:855–857, 1947.