Imperial College London

Relativity – Lecture 9

Dr Caroline Clewley

Key concepts of lecture 8 - 1

- A physical vector quantity is represented by a four-vector in Special Relativity.
- A four-vector transforms between inertial frames under the Lorentz transformations.
- The norm of a four-vector is invariant.
- The four-velocity is $\boldsymbol{U} = \gamma_u(c, \boldsymbol{u})$.

Key concepts of lecture 8 - 2

- The energy-momentum four-vector is $\boldsymbol{P} = (E/c, \mathbf{p})$.
- Here $\mathbf{p} = \gamma_u m u$, and $E = \gamma_u mc^2$ is the total energy of the particle.
- The norm of $\boldsymbol{P}$ is m^2c^2. So $(mc^2)^2 = E^2 - (pc)^2$.
- The (rest-) mass is therefore invariant.
- The kinetic energy is $T = E - mc^2$.

Page 3

Example: rest energy of an electron

Page 4

Example: radioactive decay

Energy difference is 175 MeV.

Nuclear binding energy is large compared to particle masses!

All forms of energy contribute to the rest mass: electrostatic, nuclear, thermal, etc.

Energy & momentum conservation

It has been shown experimentally that energy-momentum is conserved.

Therefore, in a particular frame E and p are separately conserved (e.g. in a collision), just as in classical mechanics.

Example: energy & momentum conservation

Two protons (m = 1GeV/c^2) collide to form a pion (m = 140 MeV/c^2).

If all particles are at rest after collision, what was the initial velocity?

Example: pion decay

Useful relations

We know: $p = \gamma_u mu$, $E = \gamma_u mc^2$

Solve for u, γ_u:

Example: particle collision

Particle 1 moves with speed u_1 = 15/17 c along the x-axis, and collides with stationary particle 2 to produce particle 3.
Particles 1 and 2 have masses $m_1 = m_2 = 8/c^2$ units.

Example: particle collision

Particle	$\boldsymbol{P}_i = (E_i/c, \boldsymbol{p}_i)$	β_i	m_i
1			
2			
3			

Points to note from particle collision example

- The two incoming particles have the same energy-momentum vector length.
- The mass of particle 3 is *not* $m_3 = m_1 + m_2$. (m is frame-invariant, but not conserved!)

Lorentz transform for E, p

$$\frac{E'}{c} = \gamma\left(\frac{E}{c} - \beta\, p_x\right)$$

$$p_x' = \gamma\left(p_x - \beta\frac{E}{c}\right)$$

Remember that E, p have γ_u factor in their definitions.
The γ in the transformation is γ_v.

The centre-of-momentum frame

It is often easier to solve problems in the "centre-of-momentum" frame, where total momentum is zero. In other words, $p_{before} = p_{after} = 0$.

The centre-of-momentum frame

Particle	$\mathbf{P'}_i = (E'_i/c, \mathbf{p'}_i)$	β'_i	m'_i
1			
2			
3			

Transforming frames: conclusion

Note that energy and momentum are conserved separately in any one frame.

However, when transforming frames, the energy and momentum change.

In other words, a Lorentz transformation changes energy into momentum, and vice versa.

However, the norm of the four-vector is invariant, so $E^2 = p^2c^2 + \left(mc^2\right)^2$ is always true.

Reminder: get the terminology right.

- **Conserved**: a quantity which is not changed by a physical process. This refers to one frame at a time, and a conserved quantity will typically have different numerical values in different frames.
- **Invariant**: a quantity which is not changed by a coordinate transformation. The term refers to more than one reference frame; an invariant quantity will not necessarily be conserved in a particular process.
- **Constant:** refers to a quantity which does not change in time, such as the mass of the Universe.

- The speed of light is conserved, invariant, and constant!

Tip: solving problems

Try to solve problems first by using energy conservation alone. Some problems require you to use both energy and momentum conservation.
You can eliminate one variable using

$$E^2 = p^2c^2 + \left(mc^2\right)^2$$

for example $p = \sqrt{\left(E/c\right)^2 - \left(mc\right)^2}$

For a massless particle, *E* = *pc*.

You can also leave out all of the c's and put them in at the end using dimensional analysis.