Imperial College London

Relativity – Lecture 8

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Key concepts of lecture 7

The relativistic Doppler effect is caused by:

1. The source ‘catching up’ to the emitted waves (classical Doppler effect).
2. Time dilation.

Four-vectors

A four-vector is a vector with four elements, and transforms under the Lorentz transformations between inertial reference frames.

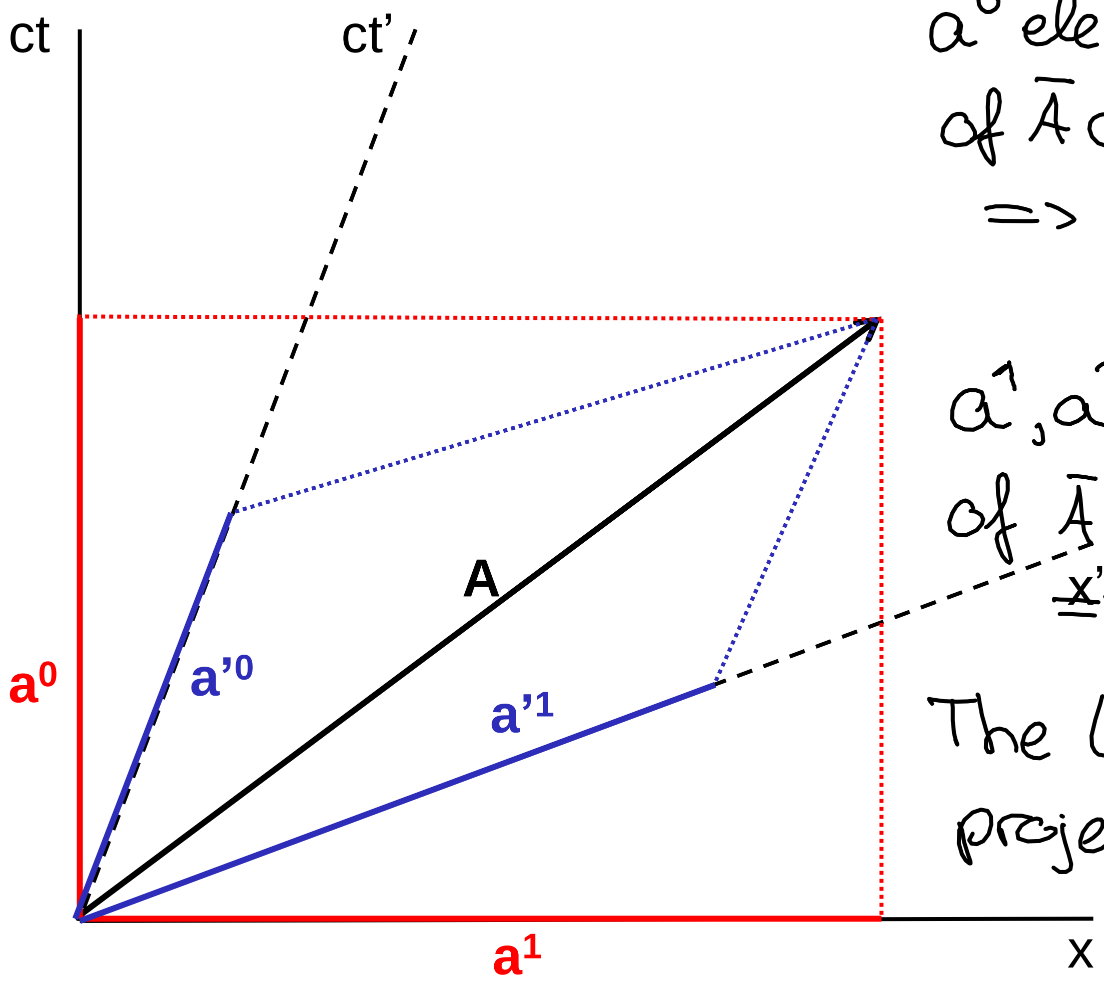
In this course:

Notation: $\boldsymbol{F} = (f^0, f^1, f^2, f^3) = (f^0, \boldsymbol{f})$.

OR: $\bar{\bar{F}} = \begin{pmatrix} f^0 \\ f^1 \\ f^2 \\ f^3 \end{pmatrix}$ (Use this form if multiplying by LT matrix)

Example: four-position $\boldsymbol{X} = (ct, x, y, z)$.

Four-vectors in spacetime



a^0 element is projection of $\bar{A}$ onto ct-axis
$\Rightarrow$ timelike element

a^1, a^2, a^3 are projections of $\bar{A}$ onto x, y, z axes
$\Rightarrow$ spacelike elements

The Lorentz Transformations project the same vector onto new ct' and x', y', z' axes.

Four-vector algebraic rules

The sum of two four-vectors is a four-vector:

$$\bar{A} + \bar{B} = (a^0 + b^0, a^1 + b^1, a^2 + b^2, a^3 + b^3) = \bar{C}$$

[like regular 3-vectors]

The inner product of two four-vectors is invariant:

$$\bar{A} \cdot \bar{B} = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = C = \bar{A}' \cdot \bar{B}'$$

(Because all valid 4-vectors transform under LT)

So the norm of two four-vectors is invariant:

$$\bar{A} \cdot \bar{A} = a^{0^2} - a^{1^2} - a^{2^2} - a^{3^2} = \bar{A}' \cdot \bar{A}'$$

e.g. $\bar{X} \cdot \bar{X} = (ct)^2 - x^2 - y^2 - z^2 = s^2$

(Note that this is not a conservation law!)

The four-velocity

~~***U*** = d***X*** / d*t*?~~ ***U*** = d***X*** / dτ!

$u' = \frac{dx'}{dt'}$ needs LT of both dx and dt!

proper time

$$\frac{dx^1}{d\tau} = \frac{dx}{d\tau} = \gamma_u \frac{dx}{dt} = \gamma_u u_x$$

What is γ? $\gamma = \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}} \equiv \gamma_u$

(So γ is Lorentz factor of particle, not of Reference frame!)

$$\frac{dx^0}{d\tau} = \frac{dct}{d\tau} = \gamma_u c$$

So $\bar{U} = \gamma_u (c, \bar{u})$

The norm of the four-velocity

What is the norm of the four-velocity?

$$\bar{U}\cdot\bar{U} = \gamma_u^2 c^2 - \gamma_u^2 u^2 = \frac{1}{1-\left(\frac{u}{c}\right)^2}\left(c^2-u^2\right) =$$

$$= c^2 \frac{1-\frac{u^2}{c^2}}{1-\frac{u^2}{c^2}} = c^2$$

(Magnitude of four-velocity is c)

$\Rightarrow$ Invariant, as expected.

=> invariant, as expected.

The energy-momentum four-vector

What about momentum?

Try $\boldsymbol{P} = m\boldsymbol{U}$: $\bar{P} = \gamma_u (mc, m\bar{u}) = (\gamma_u mc, \bar{p})$

where $\bar{p} = \gamma_u m\bar{u}$ is the relativistic momentum.

What about P^0? $\gamma_u mc = \dfrac{mc}{\sqrt{1-\left(\frac{u}{c}\right)^2}} =$

Bin. Expansion

$= mc\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \ldots\right) = mc + \frac{1}{2}m\frac{u^2}{c}$

$\times c$: $mc^2 + \frac{1}{2}mu^2 = \gamma_u mc^2 \rightarrow$ total energy of particle

rest energy cl. kinetic energy So $\bar{P} = \left(\frac{E}{c}, \bar{p}\right)$

P is the energy-momentum four-vector.

The norm of the energy-momentum four-vector

What is the norm of ***P***?

$$\bar{P}\cdot\bar{P} = \left(\frac{E}{c}\right)^2 - p^2 = \gamma_u^2 m^2 c^2 - \gamma_u^2 m^2 u^2$$

$$= m^2\left(\frac{c^2-u^2}{1-\left(\frac{u}{c}\right)^2}\right) = m^2c^2.$$

(magnitude is mc)

This must be invariant!

$$\text{So } (mc^2)^2 = E^2 - (pc)^2$$

So *E* and ***p*** are frame-dependent, but they combine into a frame-independent quantity ***P***, whose invariant length is the mass.

A note on mass

The invariant mass *m* we use here is also called the "rest-mass", or "invariant mass", m_0.

Some people use $m = \gamma m_0$, and call *m* the relativistic mass. The relativistic mass of an object increases as its velocity increases.

Momentum of light

Light has no mass, so we cannot derive momentum in the usual way.

Instead, use $E^2 = (mc^2)^2 + (pc)^2$.

$$m = 0 \Rightarrow E^2 = (pc)^2$$

$$E = pc$$

Remember $E = h\nu = \frac{hc}{\lambda} \Rightarrow p = \frac{h}{\lambda}$

Kinetic energy

Kinetic energy is the difference between the total energy and the rest energy.

Remember: $E = mc^2 + \frac{1}{2}mu^2 + \ldots$ ← higher order relativistic corrections

Total energy: γmc^2 — rest energy — classical kinetic energy

So $T = E - mc^2 = \gamma_u mc^2 - mc^2 = (\gamma_u - 1)mc^2$

Or $T = E - \sqrt{E^2 - pc^2}$

Summary - 1

- A physical vector quantity is represented by a four-vector in Special Relativity.

- A four-vector transforms between inertial frames under the Lorentz transformations.

- The norm of a four-vector is invariant.

- The four-velocity is $\boldsymbol{U} = \gamma_u(c, \boldsymbol{u})$.

Summary - 2

- The energy-momentum four-vector is $\boldsymbol{P} = (E/c, \mathbf{p})$.

- Here $\mathbf{p} = \gamma_u m u$, and $E = \gamma_u m c^2$ is the total energy of the particle.

- The norm of $\boldsymbol{P}$ is m^2c^2. So $(mc^2)^2 = E^2 - (pc)^2$.

- The (rest-) mass is therefore invariant.

- The kinetic energy is $T = E - mc^2$.