Imperial College London

Relativity – Lecture 9

Dr Caroline Clewley

Key concepts of lecture 8 - 1

- A physical vector quantity is represented by a four-vector in Special Relativity.

- A four-vector transforms between inertial frames under the Lorentz transformations.

- The norm of a four-vector is invariant.

- The four-velocity is $\boldsymbol{U} = \gamma_u(c, \boldsymbol{u})$.

Key concepts of lecture 8 - 2

- The energy-momentum four-vector is
  $\boldsymbol{P} = (E/c, \mathbf{p})$.

- Here $\mathbf{p} = \gamma_u m u$, and $E = \gamma_u m c^2$
  is the total energy of the particle.

- The norm of $\boldsymbol{P}$ is m^2c^2. So $(mc^2)^2 = E^2 - (pc)^2$.

- The (rest-) mass is therefore invariant.

- The kinetic energy is $T = E - mc^2$.

Example: rest energy of an electron

$m_e = 9 \times 10^{-31}$ kg $\quad E = mc^2 = 9 \times 10^{-31} \times 9 \times 10^{16} = 8.1 \times 10^{-14}$ J

awkward unit for particle physics

$1\ \mathrm{eV} = 1.6 \times 10^{-19}$ J

so $m_e c^2 = \dfrac{8.1 \times 10^{-14}}{1.6 \times 10^{-19}} = 0.511$ MeV

$[E] =$ MeV (eV, keV, MeV, GeV, TeV)

$[p] = \dfrac{\mathrm{MeV}}{c}$

$[m] = \dfrac{\mathrm{MeV}}{c^2}$

Example: radioactive decay

$$^{235}_{92}U + n \rightarrow {}^{144}_{56}Ba + {}^{89}_{36}K + 3n$$

Energy difference is 175 MeV.

Nuclear binding energy is large compared to particle masses!

All forms of energy contribute to the rest mass: electrostatic, nuclear, thermal, etc.

Energy & momentum conservation

It has been shown experimentally that energy-momentum is conserved.

$$\sum \bar{p}_i = \sum \bar{p}_j \qquad \text{e.g. } \bar{p}_1 + \bar{p}_2 = \bar{p}_3 + \bar{p}_4$$

E-p before = E-p after

Therefore, in a particular frame E and p are separately conserved (e.g. in a collision), just as in classical mechanics.

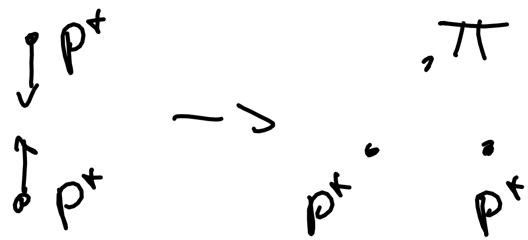
$$E_{before} = E_{after} \quad , \quad \vec{p}_{before} = \vec{p}_{after}$$

↑ 3-vector relativistic momentum

Example: energy & momentum conservation

Two protons (m = 1GeV/c^2) collide to form a pion (m = 140 MeV/c^2).

If all particles are at rest after collision, what was the initial velocity?



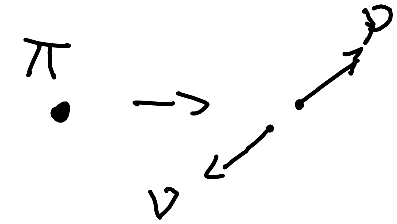
$$p^+ + p^+ \longrightarrow p^+ + p^+ + \pi$$

Cons. of energy: $\underbrace{2\gamma_u m_p c^2}_{\text{Total initial proton energy}} = \underbrace{2m_p c^2}_{\text{rest energy } p^+} + \underbrace{m_\pi c^2}_{\text{rest energy } \pi}$

Solve for $\gamma_u = 1 + \dfrac{m_\pi}{2m_p} = 1.07$ $\left(\beta = \sqrt{1 - \dfrac{1}{\gamma_u^2}}\right)$

$\dfrac{u}{c} = 0.36$

Example: pion decay $\pi \to \mu + \nu_\mu$



Express p_μ in m_π and m_μ

$$E_{before} = m_\pi c^2 \,, \quad p_{before} = 0$$

$$E_{after} = E_\mu + E_\nu \,, \quad \bar{p}_{after} = \bar{p}_\mu + \bar{p}_\nu$$

cons. of p: $\bar{p}_\mu + \bar{p}_\nu = 0 \Rightarrow \bar{p}_\mu = -\bar{p}_\nu$ (1)

cons. of E: $E_\mu + E_\nu = m_\pi c^2$ (2)

$E_\nu = p_\nu c$ ($m_\nu = 0$!), $E_\mu = \sqrt{(m_\mu c^2) + (p_\mu c)^2}$ (using $E^2 = (mc^2)^2 + (pc)^2$)

So $\sqrt{(m_\mu c^2)^2 + (p_\mu c)^2} + \underbrace{p_\nu c}_{= p_\mu c \text{ (from 1)}} = m_\pi c^2$ (from 2)

Example pion decay.

Squaring gives: $(m_\mu c^2)^2 + (p_\mu c)^2 = (m_\pi c^2 - p_\mu c)^2$

$$(m_\mu c^2)^2 + \cancel{(p_\mu c)^2} = m_\pi^2 c^4 - 2 m_\pi p_\mu c^3 + \cancel{(p_\mu c)^2}$$

Isolating p_μ gives: $p_\mu = \dfrac{c^4(m_\pi^2 - m_\mu^2)}{2 m_\pi c^3}$

$$= \frac{c(m_\pi^2 - m_\mu^2)}{2 m_\pi}.$$

Useful relations

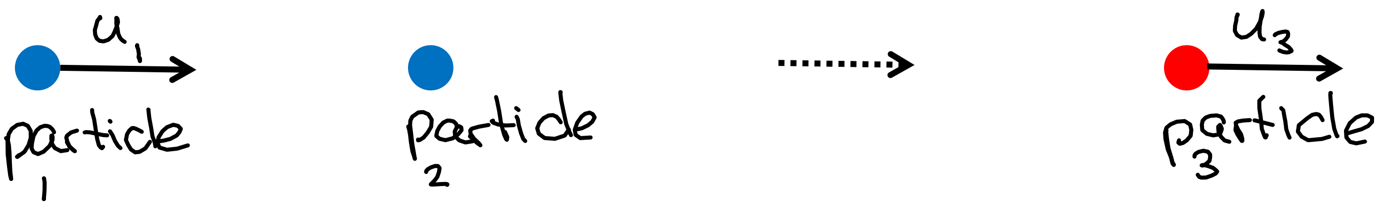
We know: $p = \gamma_u m u$, $E = \gamma_u m c^2$

Solve for u, γ_u:

u: $\quad u = \dfrac{p}{\gamma_u m} = \dfrac{pc^2}{E}$ or $\beta_u = \dfrac{pc}{E}$

γ_u: $\quad \gamma_u = \dfrac{E}{mc^2}$

Example: particle collision



Particle 1 moves with speed u_1 = 15/17 c along the x-axis, and collides with stationary particle 2 to produce particle 3.

Particles 1 and 2 have masses $m_1 = m_2 = 8/c^2$ units.

$$\beta_1 = \frac{15}{17} \Rightarrow \gamma_{u_1} = \sqrt{\frac{1}{1-\beta^2}} = \frac{17}{8}$$

$$\text{So } E_1 = \gamma_{u_1} m_1 c^2 = \frac{17}{8} \times \frac{8}{c^2} \times c^2 = 17$$

$$p_1 = \gamma_{u_1} m_1 u_1 = \frac{17}{8} \times \frac{8}{c^2} \times \frac{15}{17} c = \frac{15}{c}$$

Example: particle collision

Particle	$\boldsymbol{P}_i = (E_i/c, \boldsymbol{p}_i)$	β_i	m_i
1	$(17/c,\ 15/c)$	$15/17$	$8/c^2$
2	$(8/c,\ 0)$	0	$8/c^2$
3	$(25/c,\ 15/c)$	$3/5$	$20/c^2$

Particle 2: $E_2 = m_2c^2$ (at rest!) and $p_2 = 0$

Particle 3: $\frac{E_3}{c} = \frac{E_1}{c} + \frac{E_2}{c} = \frac{17}{c} + \frac{8}{c} = \frac{25}{c}$ cons. of E

$p_3 = p_1 + p_2 = \frac{15}{c} + 0 = \frac{15}{c}$ cons. of p.

$\beta_3 = \frac{p_3 c}{E} = \frac{15}{25} = \frac{3}{5}$

$m_3 = \sqrt{\left(\frac{25}{c}\right)^2 - \left(\frac{15}{c}\right)^2} \times \frac{1}{c} = \frac{20}{c^2}$ $\left((mc)^2 = \bar{p} \cdot \bar{p}\right)$

Points to note from particle collision example

- The two incoming particles have the same energy-momentum vector length.

- The mass of particle 3 is *not* $m_3 = m_1 + m_2$. (m is frame-invariant, but not conserved!)

Lorentz transform for E, p

$$\frac{E'}{c} = \gamma\left(\frac{E}{c} - \beta\, p_x\right)$$

$$p_x' = \gamma\left(p_x - \beta\,\frac{E}{c}\right)$$

Remember that *E*, *p* have γ_u factor in their definitions.

The γ in the transformation is γ_v.

The centre-of-momentum frame

It is often easier to solve problems in the "centre-of-momentum" frame, where total momentum is zero. In other words, $p_{\text{before}} = p_{\text{after}} = 0$.

So $\bar{P}_3 = \Big(\underbrace{\gamma_{u_1} m_1 c + \gamma_{u_2} m_2 c}_{\frac{E_1}{c} + \frac{E_2}{c}},\ \underbrace{\gamma_{u_1} m_1 u_1 + \gamma_{u_2} m_2 u_2}_{p_1 + p_2}\Big)$

$$0 = p_3' = \gamma_v \left(p_3 - \beta \frac{E_3}{c}\right) = \gamma_v\big(\gamma_{u_1} m_1 u_1 + \gamma_{u_2} m_2 u_2 - \beta(\gamma_{u_1} m_1 c + \gamma_{u_2} m_2 c)\big)$$

$$\Rightarrow \beta = \frac{\gamma_{u_1} u_1 + \gamma_{u_2} u_2}{\gamma_{u_1} + \gamma_{u_2}} = \frac{3}{5}$$

The centre-of-momentum frame

Particle	$\boldsymbol{P'}_i = (E'_i/c, \mathbf{p'}_i)$	β'_i	m'_i
1	$(10/c, 6/c)$	$3/5$	$8/c^2$
2	$(10/c, -6/c)$	$-3/5$	$8/c^2$
3	$(\frac{20}{c}, 0)$	0	$20/c^2$

$$\frac{E'_3}{c} = \gamma_v\left(\frac{E_3}{c} - \beta p_3\right) = \frac{5}{4}\left(\frac{25}{c} - \frac{3}{5}\,\frac{15}{c}\right) = \frac{20}{c} \quad (LT)$$

$$\frac{E'_2}{c} = \gamma_v\left(\frac{E_2}{c} - \beta p_2\right) = \frac{5}{4}\left(\frac{8}{c} - 0\right) = \frac{10}{c}$$

$$p'_2 = \gamma_v\left(p_2 - \beta\frac{E_2}{c}\right) = \frac{5}{4}\left(0 - \frac{3}{5}\,\frac{8}{c}\right) = \frac{-6}{c}$$

$E'_1 = E'_2$ and $\bar{p}'_1 = -\bar{p}'_2$ $\quad (m_1 = m_2)$

Transforming frames: conclusion

Note that energy and momentum are conserved separately in any one frame.

However, when transforming frames, the energy and momentum change.

In other words, a Lorentz transformation changes energy into momentum, and vice versa.

However, the norm of the four-vector is invariant, so $E^2 = p^2c^2 + \left(mc^2\right)^2$ is always true.