Nuclear & Particle Physics (Yr 2)

Prof. Henrique Araújo

h.araujo@imperial.ac.uk

Blackett Laboratory (B510)

Imperial College London

Second & Third Terms (2019-20)

PART I – PARTICLE PHYSICS

NPP Lecture 1 – Overview of the Standard Model

1.1 Introduction

These lectures will cover the nuclear and particle physics topics of the 2^{rd} year Atomic, Nuclear & Particle Physics course. Although, historically, nuclei were discovered before the elementary particles, the material will be presented in the reverse order, as the properties of nuclei are best understood in the context of the particles and forces which comprise them. Particle physics is the study of the most fundamental objects in nature and the forces between them. Our understanding is encompassed in the so-called "Standard Model" (SM) and we will start to explore this in this course, with much more detail to follow in the 3^{rd} and 4^{th} year courses (Physics of the Universe and Advanced Particle Physics, respectively). *Besides being introductory in nature, this course will focus on the the interplay between theory and experiment, and actually look at some of the key results that drove the theoretical development.*

Particle physics is a modern name for an age-old idea. For centuries, it was thought that atoms (from the Greek *atomos* meaning "not divisible") were the most basic constituents of matter. Unlike the ancient Greeks, you all know that atoms are not fundamental but are made of electrons orbiting a nucleus. The electrons, we believe, *are* fundamental (although maybe someone giving this course in the 22^{nd} century will start by saying "In the last century, physicists believed that electrons were fundamental but of course now we know...".) However, the nuclei we already know are not fundamental.[1] Nuclei are made of protons and neutrons and even these are not fundamental themselves; they in turn are made of "quarks" – see Figure 1.1. Specifically, there are three quarks in a proton or neutron; protons contain two 'up' quarks (u) and one 'down' quark (d), while neutrons contain one u- and two d-type quarks. In the same way as for electrons, we currently believe quarks are not made of anything else.

There is another fundamental particle which occurs abundantly in nature: the "neutrino". This is not found as a part of atoms because the force it feels is so weak that it is not bound – but neutrinos exist in great numbers throughout the universe.

[1] To be clear, by 'fundamental' we mean that particles have no size (they are infinitely small), have no substructure (they are not made of anything else), and cannot be excited or broken up. Obviously, all these things can only be experimentally shown to be true to some limit.

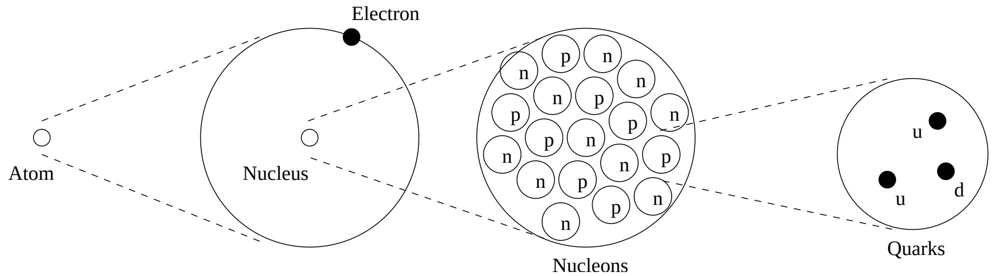


Figure 1.1: Schematic of the atom.

Over the last hundred or so years the combination of the special theory of relativity and quantum mechanics, along with the discovery of new particles, has led to the Standard Model, a crowning achievement of 20^{th} century science. It is remarkable that so large a number of natural phenomena[2] that characterize the world around us can be described in terms of underlying principles of such simplicity and elegance. The SM is built upon those principles, more mathematically upon the ideas of symmetries (and in particular gauge invariance), conserved quantities, and unification. It is a (relativistic) quantum field theory – allowing us to consider matter and forces in very similar ways – but here we will study things at a simpler level. *The SM comprises quarks and leptons as the building blocks of matter, and describes their interactions through the exchange of force carriers: the photon for electromagnetic interactions, the W and Z bosons for weak interactions, and the gluons for strong interactions.*

We know that the SM cannot be the whole answer; besides not incorporating gravity, questions remain about the nature of dark matter, why there are three families of particles, etc. The Large Hadron Collider (LHC) was designed to elucidate the origins of mass, and probe the energy regime where we know the SM must breakdown. The discovery of the Higgs boson at the LHC in 2012 was widely expected to be a portal to new physics beyond the SM. Physicists at the LHC are still eagerly looking forward to establishing the exact nature of the new boson and to the higher-energy running of the LHC, to find clues or answers to some of the other fundamental open questions in particle physics and cosmology. In parallel, advances in flavour physics will also help us understand what comes after the SM, and help elucidate questions such as CP violation in the lepton sector (and indeed why the universe is made up of matter rather than anti-matter). There is increasing synergy with astroparticle physics and precision cosmological studies probing such issues as dark energy and dark matter – which may well, at some fundamental level, relate to the Higgs field. Such programmes of work are likely to take several decades and will surely change our understanding of fundamental physics.

In the SM, all the "matter" particles have spin 1/2 and hence are 'fermions' (the latter term includes more generally particles with half-integer spin). In

[2]Almost no experimental measurement is in quantitative disagreement with its predictions; one exception is the recent data indicating neutrinos have mass (which we will look at).

contrast, the three forces in the SM are due to particles with spin 1, which are 'bosons' (the term for any integer spin particles). The only other type of particle in the SM, the Higgs boson, does not really fall into either category and it has spin 0 (we call this a "scalar" particle, in contrast to the spin-1 "vector" bosons). The Higgs is responsible for the "drag" on particles, even when in vacuum, which we perceive as mass.

1.1.1 The Standard Model as a quantum field theory

Expressing the SM as a quantum field theory allows us to treat matter and forces in a very similar way. Here, particles can be thought of in terms of quantised fields, which is an extension of the quantum mechanics which you have studied before. The frequency of the wave in the field characterises the type of particle, and the amplitude of the wave represents their number – e.g., the first excitation at a particular frequency gives one particle, a further excitation of the amplitude for the same frequency corresponds to two particles, etc. Clearly, we also could excite two different frequencies both to their first excited state to make two different frequency particles. Hence, the concept of a quantum field, unlike normal quantum mechanics, allows for an arbitrary and changeable number of particles to exist. This is necessary since (as we shall see) we can create and annihilate particles in reactions and decays.

Quantum field theory actually says that there is only one electron quantum field for the whole Universe and every electron which exists is due to excitations of this field. Hence, all electrons are "identical" in the QM sense, as they all arise from the same field. As you may recall from the atomic part of the course, there are particular properties for the resulting wavefunctions, namely their symmetries under the exchange of (identical) particles, with the actual symmetry depending on whether the particle is a fermion or a boson. Recall that for fermions this gives rise to the Pauli exclusion principle. Bosons, on the other hand, show the opposite effect, and they can happily share the same state.

1.2 The Matter Particles

The fundamental particles are organised into "families", also known as "generations", as shown in Table 1.1. The first generation contains the matter particles we have already met; the next two columns are the further generations which are heavier copies. Experimentally, there seems to be no difference in the properties of any of the particles of the same type between different generations except for their masses; for example:

$$\begin{aligned} m_e &= 9.11\times10^{-31}\ \text{kg} & m_\mu &= 1.88\times10^{-28}\ \text{kg} & m_\tau &= 3.17\times10^{-27}\ \text{kg} \\ &= 0.511\ \text{MeV/c}^2 & &= 105.7\ \text{MeV/c}^2 & &= 1777\ \text{MeV/c}^2. \end{aligned}$$

The principle of universality postulates that indeed all other properties are equal (e.g. charge). Naturally, we need to confirm experimentally if this is correct.

Table 1.1: The matter particles; spin 1/2.

	generation			charge	feels force		
	1[st]	2[nd]	3[rd]	(units of e)	Strong	EM	Weak
u-type quarks ($\times 3$ colours)	u	c	t	$+2/3$	Y	Y	Y
d-type quarks ($\times 3$ colours)	d	s	b	$-1/3$	Y	Y	Y
Charged leptons	e	μ	τ	-1	N	Y	Y
Neutral leptons (neutrinos)	ν_e	ν_μ	ν_τ	0	N	N	Y

All the spin-1/2 particles obey the same quantum mechanical wave equation, known as the Dirac equation. Though this equation is beyond the level of this course, one of its predictions, namely the existence of antiparticles, is not. Indeed, for every one of the above particles, there is a corresponding antiparticle with identical mass but opposite charge (and opposite other quantum numbers also). At some level, antiparticles can be thought of as particles going backwards in time, as we shall see later.

In the SM, the matter particles do *not* interact with each other directly, but rather with force fields. For example, the repulsion between two electrons should be thought of as the interaction of an electron with a photon (the force particle carrying the electromagnetic field) which in turn interacts with the other electron.

1.3 The Force Particles

The three forces in the SM are the electromagnetic, strong and weak forces.[3] The electromagnetic force is well known and is responsible for light, electricity, magnetism, atomic structure, etc. The strong force holds the protons and neutrons together in the nucleus and is clearly stronger than the electromagnetic repulsion of the positively charged protons, hence being called strong. The weak force is less obvious at the macroscopic level for the simple reason that it is less strong, but is responsible for the critical first step of the reactions which power the Sun as well as causing beta decay of nuclei.

Matter particles (fermions) interact via the exchange of the force carriers (bosons), which are the quantisation of the force fields; see Table 1.2. In contrast to the matter particles, each force particle has a different wave equation, and it is this fact which gives each force a different characteristic, e.g. different range or strength.[4] The EM force particle, the photon, has a wave equation which comes from Maxwell's equations. The strong and weak force equations are more

[3] The SM does *not* include the fourth known force, gravity. In practical terms, this is fine because it has a completely negligible effect on any particle physics experiment, since the gravitational force between particles is so much weaker than even the weak force.

[4] In fact, we believe that the different forces, weak, electromagnetic and strong (and even eventually gravity) are low energy manifestations of a single fundamental interaction.

complicated, not least because they carry the strong and weak force charges themselves, i.e. they "self-interact". This would be equivalent to the photon being charged, and it makes things much more entertaining.

Table 1.2: The force particles; spin 1.

Force	name	symbol	number	EM charge	mass
Strong	gluons	g	8	0	0
EM	photon	γ	1	0	0
Weak	W and Z	$W^{\pm}$, Z^0	3	± 1, 0	80 & 91 GeV/c^2

What you have probably conceived of as a force so far is something which causes bodies to change their motion, as per Newton's second law. In fact, this is true only in the classical limit of the systems which we call forces. When we combine the idea of a force (or potential) field with both relativity and quantum mechanics, we unveil other behaviour. The bosons carry energy and momentum and so, to conserve these quantities, the energy and momentum of the matter particles must also change. This gives rise to the change of motion we observe classically. An example is the scattering of an alpha particle (a helium nucleus) from a larger nucleus, so-called Rutherford scattering, which led to the discovery of the structure of the atom. Both nuclei are positively charged and so, classically, we think of them as repelling each other. At a fundamental level, this repulsion comes about because each is radiating off photons, the force carriers, or quanta, of the electromagnetic field. When a photon is radiated from one nucleus and absorbed by another, then they are pushed apart. One loose analogue is ships firing cannonballs at each other: the recoil pushes the ship which is firing in the opposite direction (because the cannonball carries away momentum, which must be balanced to conserve it) and the impact pushes the other ship in the opposite direction as it absorbs the cannonball and its momentum. However, note that forces can also be attractive, as would be the case for a positive and a negative charge; there is no easy classical picture for this case, but the momentum of the exchanged boson is directed in the opposite direction. In addition, we will also see that this emission of bosons actually allows particles not only to change their motion, but also their nature, i.e. they can decay or react because of the same forces. This clearly has no classical analogue.

1.3.1 Range of a force

All the relevant information about a force is contained in the potential function since the spatial derivative of the potential leads to the expression for the force. From the knowledge of the potential, derived in part from experiment, the motion of the particles affected by the interaction can be calculated.

An important type of 'screened' Coulomb potential much used in atomic and particle physics is the Yukawa potential:

$$\phi = \frac{g}{r} e^{-mcr/\hbar} = \frac{g}{r} e^{-r/r_o},$$

where m is the mass of the force carrier and r_o is a characteristic range. Note that this is essentially a Coulomb potential 'killed off' by the exponential term for large distances. We might also look at the Yukawa potential as consisting of (i) a term which represents the "strength" of the interaction (more precisely it gives the probability of emission or absorption of a quantum from the particle carrying the charge that generates the field under question) and (ii) a term for the range of the interaction which gives the variation in strength of the interaction as a function of the separation of the interacting particles.

The exchanged particle cannot be real (to conserve energy and momentum) and is instead said to be *virtual*. From the Heisenberg uncertainty principle, one can borrow a minimum energy Δmc^2 to create a virtual particle provided that we pay it back within a time

$$\Delta t \sim \frac{\hbar}{mc^2}.$$

During such a time, the particle can travel a maximum distance given by

$$\Delta r \sim c\Delta t = \frac{\hbar}{mc} = r_o.$$

The Yukawa potential is sketched in Figure 1.2.

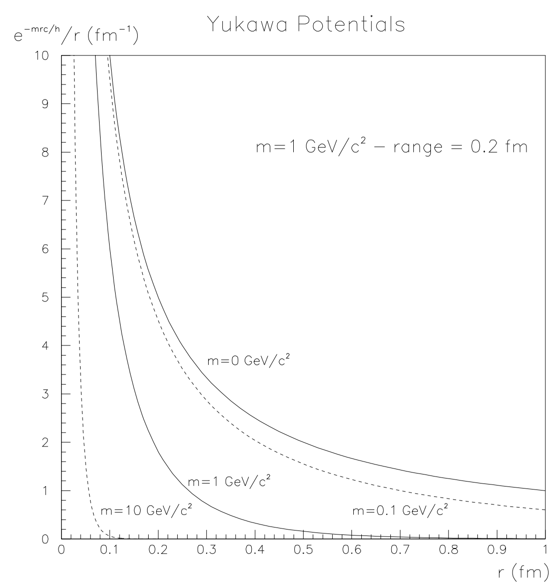


Figure 1.2: Yukawa potentials for various mediator masses.

How does this compare with the massless case? For $r \ll r_o = \hbar/mc$ they look very similar while for larger values of the radius the potential is much reduced, as expected: the potential cuts off for radii greater than the "range" given by $\hbar/mc$; the larger the mass, the shorter the range of the force. To set the scale, a mass of 1 GeV/c^2 gives a range of 0.197 fm.

While the photon is thought to be exactly massless and hence has an infinite range, this is not the case for the *weak* interaction bosons, the $W^\pm$ and the Z^0. These have masses of around 80 GeV/c^2 and 91 GeV/c^2, respectively, which correspond to ranges of around 0.002 fm $= 2 \times 10^{-18}$ m. As we shall see in more detail later, this extremely short range is what makes the weak interaction appear so weak; the actual equivalent of charge for the weak interactions (which we call "weak charge") is in fact a little larger than for electromagnetism, but it is masked by the mass effect at energies less than $M_{W,Z}c^2$. Note that most processes we will consider in this course are at much lower energies than this, so you should think of the weak force as being very weak for the reactions and decays we will consider, i.e.

$$\text{Strong} \gg \text{Electromagnetic} \gg \text{Weak}.$$

The *strong* force bosons are called "gluons" and they are massless, like photons. This means that the strong force has, in principle, an infinite range. However, the gluons themselves carry the equivalent of the charge for the strong force, known as "colour". Therefore, gluons can radiate and absorb other gluons and this complicates the picture. In fact, the range of the force is effectively limited to $\sim 1\,\text{fm}$, i.e. nuclear sizes. Also, we never see "bare" colour charges but only the equivalent of uncharged combinations, called hadrons. We discuss these aspects of the strong force later.

1.3.2 Feynman diagrams

Feynman diagrams are graphical representations of the mathematical expressions underlying the behaviour of elementary particles. The derivation and calculation of Feynman diagrams is beyond the scope of this course, but they are nonetheless extremely useful in understanding particle interactions and so we will use them in a simplistic way. Just keep in mind that behind every apparently simple diagram lies a lot of rigorous mathematics that describes fully a particular process – from simple elastic scattering to any reaction or particle decay.

To calculate the amplitude (and hence the probability) of a reaction, e.g. the scattering of two electrons, we perform a perturbative calculation. It turns out that each of the mathematical terms in the perturbation series can be represented as a diagram, where each part indicates a particular factor in the calculation.

Straight lines in Feynman diagrams represent particles with well-defined energy and momentum, i.e. in a defined quantum eigenstate. These can meet at points ("vertices") where the actual interactions take place. The mathematical expressions for the forces, from which the diagrams arise, define precisely what sorts of vertices are allowed. Once you know these allowed vertices (and there are not so many) you can describe any process in particle physics.

For example, in quantum electrodynamics (QED), the relativistic quantum theory of electromagnetism involving charged particles and photons, the only allowed vertex is one with two charged fermion lines (e.g. electrons) and one photon line, see Figure 1.3, which represents an electron radiating (or absorbing) a photon and changing to another state.

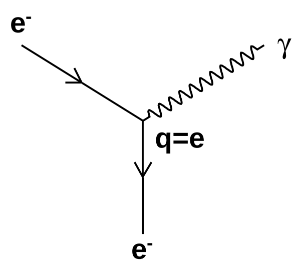


Figure 1.3: Basic QED vertex.

All Feynman diagrams representing QED processes are constructed purely from these basic vertex units. For example, the elastic scattering between two electrons uses two such vertices as shown in Figure 1.4.

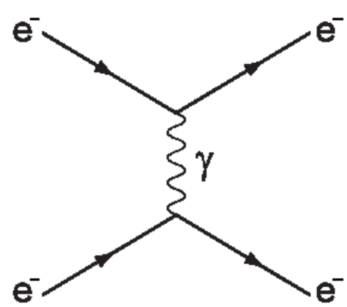


Figure 1.4: Feynman diagram for electron-electron scattering.

Note the factor of the charge q in Figure 1.3, which is equal to e here (and which we denoted generically for any 'type' of charge as g in the previous section) written by the vertex; it is the fact that the electron is charged which enables it to interact with the photon. The amplitude for this interaction to happen is directly proportional to the charge. Hence, diagrams with n vertices have a factor of e^n in the amplitude and hence e^{2n} in the probability (as the probability is the square of the amplitude). The dimensionless number which gives some idea of whether the electric charge is large or not is the fine structure constant:

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}.$$

Hence, for n vertices there will be a factor of α^n in the probability, and large n will have a lower probability: diagrams with fewer vertices are more important.

We said previously that antiparticles act like particles going backwards in time; this statement only really makes sense in the context of Feynman diagrams. Explicitly, the same vertex can represent various other combinations of electrons and/or positrons by reversing their directions and replacing them with their antiparticles – as shown in Figure 1.5 for radiation (or absorption) of a photon by an electron, e^+e^- annihilation, radiation (or absorption) of a photon by a positron, and e^+e^- pair creation, respectively.

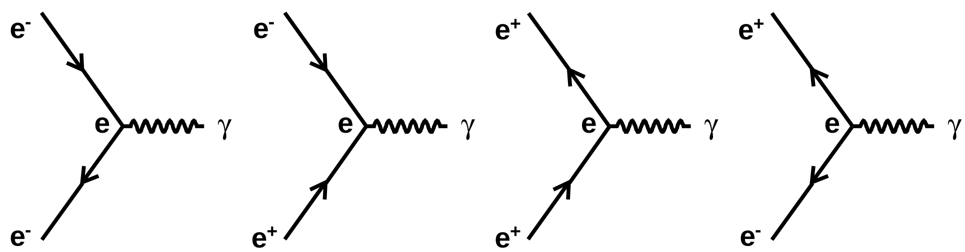


Figure 1.5: Basic QED vertices.

A reaction with only one vertex in the diagram, e.g. $e \to e\gamma$, cannot occur as it cannot conserve energy and momentum (as is easily seen by considering the initial electron rest frame). Hence, an actual reaction needs more than one vertex, as we mentioned above. For example, in $e^-\mu^-$ scattering the lowest order diagram, which has two vertices, would be as shown in Figure 1.6.

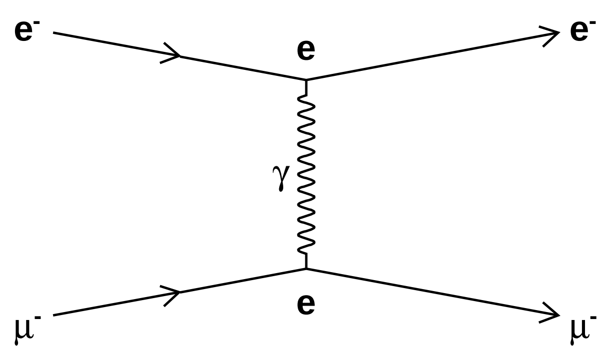


Figure 1.6: Feynman diagram for $e^-\mu^-$ scattering.

This one diagram covers both emission by the electron followed by absorption by the muon and vice-versa. Note, sometimes time and space axes are added to these diagrams to remove any ambiguity about what reaction is represented.

Only the particles at the edges of the diagram (the original and final e^- and μ^- in this case) are observed experimentally. The photon which causes the scatter is never seen directly – it is "virtual". It is not a free particle and so it does not obey $E^2 = p^2c^2 + m^2c^4$. The 'external' lines (the initial and final state electron and muon) must conserve energy and momentum.

The above diagram has two vertices, and so it gives an amplitude proportional to e^2. Hence, the probability for this reaction is proportional to e^4, or α^2.

Finally, one last word about the conventions used in Feynman diagrams. Traditionally, these were representations in spacetime plots, with a vertical time axis and space running positive to the right – like those you used in your Special Relativity course. Particle lines have arrows pointing up, while antiparticles point down (as if travelling back in time). Some physicists, and certainly many textbooks, still use this convention,[5] but more modernly we tend to represent time flowing to the right instead, i.e. an initial state on the left evolves into a final state on the right; in this case the arrows point to the right for particles and to the left for antiparticles – so far so good; this is the convention we will try to adopt consistently in this course. However, you will find many instances where the initial-state arrows point to the vertex for both particles and antiparticles, and similarly for the final state. We will not worry too much about this: everything should be clear as long as all particles and antiparticles are clearly indicated. In the end the mathematics behind the diagram is what is important and, for our purposes here, we will focus to a great extent on the properties of the vertex itself, which do not care about such distinctions.

1.4 The Higgs Particle

The final piece of the SM is the Higgs particle, which has spin 0 and has no charge — see Table 1.3. Without this particle, the SM would be inconsistent, as all the matter and force particles would need to be massless. Until July 2012 no experimental evidence for it had been found. The results from the Tevatron experiments in the US, and in particular the ATLAS and CMS experiments at LHC are consistent, within uncertainties, with the expectations for the SM Higgs boson. For example, the new particle couples to other fundamental particles in the exact proportion predicted by the SM (i.e. for fermions (f) with a rate proportional to m_f^2 and for bosons (V) with a rate proportional to m_V^4).

Table 1.3: The Higgs particle; spin 0.

Name	symbol	number	EM charge	mass
Higgs	H	1	0	125 GeV/c^2

[5] D. Kaiser's '*Physics and Feynman's Diagrams*, Sci. Am. 95 (2005) offers an interesting historical account of the use of Feynman diagrams and some of the conventions used.

NPP Lecture 2 – Tools of the Trade

2.1 Introduction

In this lecture we will look at various basic concepts and formulæ used in particle physics to enable us to understand the fundamental building blocks of nature and, in particular, the forces that govern them. As we said previously, the SM is built upon symmetries (or conserved quantities), and several topics covered today concern these. You will have seen examples of these before, e.g. energy, momentum, total angular momentum, and today we will introduce parity and charge conjugation. In addition to such quantities, there are three basic properties of forces which can be experimentally determined:

1. The masses (or energies) of bound states, i.e. the *mass spectrum*. This is only useful in practise for the strong force as bound states of the weak interaction are too feebly bound to be seen, and we already know the EM force law. To form a bound state involves many boson exchanges.

2. The decay rates or *widths* of unstable particles. QM relates the lifetime τ to the width Γ by $\Gamma = \hbar/\tau$, as will be discussed below. Decays usually involve only a few boson exchanges.

3. The reaction rates, usually expressed as *cross sections* (again discussed below). As for decays, these usually involve only a few exchanges.

2.2 Energy and Momentum

In particle physics we often deal with particles travelling close to the speed of light – photons, of course, always do travel *at* the speed of light. Hence, we need to review the formulæ for relativistic kinematics, and the appropriate formula relating energy and momentum is $E^2 = p^2c^2 + m^2c^4$. For a massless particle this becomes $E = |\boldsymbol{p}|c$.

Recall that E and $\boldsymbol{p}c$ form a "four-vector", and so transform under a Lorentz transformation in the same way as ct and $\boldsymbol{r}$. Hence, for a boost by v_b along, say, the z direction, the mass is invariant, meaning it does not depend on the Lorentz frame (see Problem Sheet #1). This is the "correct" way to consider mass; concepts such as "rest mass" are less useful.

2.2.1 Multiparticle systems

In a collision or decay, there will be more than one particle involved. The total energy $E_T = \sum_i E_i$ and total momentum $\boldsymbol{p}_T = \sum_i \boldsymbol{p}_i$ are always conserved; this is a very fundamental principle. Although their actual values depend on the frame, their conservation clearly holds in all frames. We can define a total mass of the system through

$$m_T^2 c^4 = E_T^2 - p_T^2 c^2 \, ,$$

which, in the same way as shown above, will be the same – i.e. it is invariant – in all frames.

The *centre-of-mass* frame is defined to be the frame where $\boldsymbol{p}_T = 0$, so in this frame $m_T c^2 = E_{\mathrm{CM}} = \sqrt{s}$. Hence, the total mass of the system is often called the centre-of-mass energy and its square is usually denoted by s. The invariance of this quantity is very useful.

2.3 Spin

If all particles were spin 0, then the above would cover all the important quantities. However, spin is an additional complication which we need to consider. In quantum mechanics, the spin vector $\boldsymbol{S}$ (or in fact any angular momentum) is quantised both in terms of its length and its components. For a particle with spin quantum number s, where s is an integer or half-integer, the total length of the spin angular momentum vector is $\sqrt{s(s+1)}\hbar$. Note, it is meaningless to consider a negative length; $s \geq 0$ always. For the components along any axis, e.g. z, quantum mechanics says that the component eigenvalues can be:

$$s_z = \, -s\hbar, \, -(s-1)\hbar, \, -(s-2)\hbar, \, \ldots, \, +(s-2)\hbar, \, +(s-1)\hbar, \, +s\hbar \, ,$$

which, therefore, has $2s+1$ possible values. The immediate question which arises is: which axis is a sensible choice? A common choice is to resolve the spin along the momentum axis. Hence, we define the quantity *helicity* as:

$$h = \frac{\boldsymbol{S}.\boldsymbol{p}}{|\boldsymbol{p}|}.$$

For example, for a spin-1/2 particle the only possible values of helicity are $h = \pm 1/2$. Helicity (and indeed momentum) are constant for free particles. In a

Figure 2.1: Helicity for a spin-1/2 particle.

reaction, particles can (and do) change their momentum and/or helicity. One of the things we study in particle physics is how reactions change these quantities.

2.4 Symmetries and Conservation Laws: Parity and Charge Conjugation

There is a very deep connection in physics between symmetries and conserved quantities. For example, we can perform an experiment today or tomorrow. Assuming we remove all the external factors which depend on when we do it, we should obtain the same answer. This is because there is a time translation symmetry: there is no absolute time scale. This symmetry actually leads to conservation of energy. Similarly, symmetry under translation in each of the three directions of space leads to conservation of the three components of momentum. Symmetry of direction in space (as space is isotropic) leads to conservation of angular momentum.

Parity and charge conjugation are further examples. Parity is an operation which takes a (polar) vector, e.g. the spatial position $\boldsymbol{r}$, and reflects it through the origin to make it $-\boldsymbol{r}$. Charge conjugation takes every particle and replaces it with its antiparticle (and vice versa). It seems obvious from everyday experience that things will work the same after a parity operation: a mirror-image of any machine will function in the same way. It maybe not be so obvious that we would expect systems to be the same after swapping particles and antiparticles. However, if you think about the EM force between any pair of particles, it goes as Q_1Q_2 – so swapping the signs of Q_1 and Q_2 in fact keeps all the forces the same and, since the masses are the same, this has no effect.

Parity and charge conjugation are somewhat different from the other symmetry cases because they are discrete, rather than continuous symmetries. Take the example of time translation invariance which leads to energy conservation. Because any translation is possible, then any energy is allowed. It turns out that for a discrete transformation symmetry, then there is a discrete quantum number which is conserved. Hence, there is still a connection between a symmetry and a conserved quantity, even for the discrete case.

What are these discrete eigenvalues for parity and charge conjugation? Consider the parity operator, $\hat{P}$, which reflects polar vectors through the origin: $\boldsymbol{r} \to -\boldsymbol{r}$. For an eigenstate of parity, where

$$\hat{P}\psi = P\psi\,,$$

it is obvious that applying the parity operation a second time must return everything to its original state:

$$\hat{P}^2\psi = P\hat{P}\psi = P^2\psi\,,$$

and since this must be just ψ again, then $P^2 = 1$ and the eigenvalues of parity must be ± 1. The same argument holds for the charge conjugation operator; this changes particles to antiparticles so a second application must return everything back to the original state. Again, the eigenvalues of $\hat{C}$ must be $C = \pm 1$.

Hence, we can look at parity (or charge conjugation) conservation either in terms of whether there is a symmetry or in terms of whether the quantum number of P (or C) is conserved. What we find is that Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD) both conserve P and C or, equivalently, respect $\hat{P}$ and $\hat{C}$ symmetries, whereas – remarkably – the weak force does not.

2.5 Decays

2.5.1 Lifetimes

A decay, the change from one quantum state (the initial particle) to another (the final or "daughter" particles), is characterised by the decay rate (or transition), λ, and has units of s^{-1}. It gives the probability per unit time for the state to decay. The probability that the particle will survive at least until time t after it is known to exist at time $t = 0$ is given by the famous exponential decay law,

$$S(t) = e^{-\lambda t},$$

or, in terms of the lifetime,

$$S(t) = e^{-t/\tau}.$$

Considering many particles, with initial number N_0, then the number which exist at any later time is given by

$$N(t) = N_0 S(t) = N_0 e^{-\lambda t}.$$

Hence, the number is reduced to a fraction $1/e \approx 0.368$ of the initial value after one lifetime. Note that

$$\frac{dN}{dt} = -\lambda N_0 e^{-\lambda t} = -\lambda N\,,$$

showing that λ is indeed the rate of decay per nucleus.

Another quantity often used in nuclear physics is the half-life, $T_{1/2}$, which is defined as the time taken for 1/2 of the initial sample to decay; at such time,

$$N(t = T_{1/2}) = \frac{N_0}{2} = N_0 e^{-\lambda T_{1/2}}.$$

Hence,

$$e^{\lambda T_{1/2}} = 2\,,$$

so

$$T_{1/2} = \frac{\ln 2}{\lambda} = \tau \ln 2 = 0.693\,\tau.$$

The half-life is therefore $\approx 31\%$ shorter than the mean lifetime, as shown below.

Note that these quantities are all independent of the "real" start time of the material, i.e. when it was actually created. If we observe the particle is alive at $t = 0$, everything after that is independent of anything that happened previously, i.e. of the particle's history.

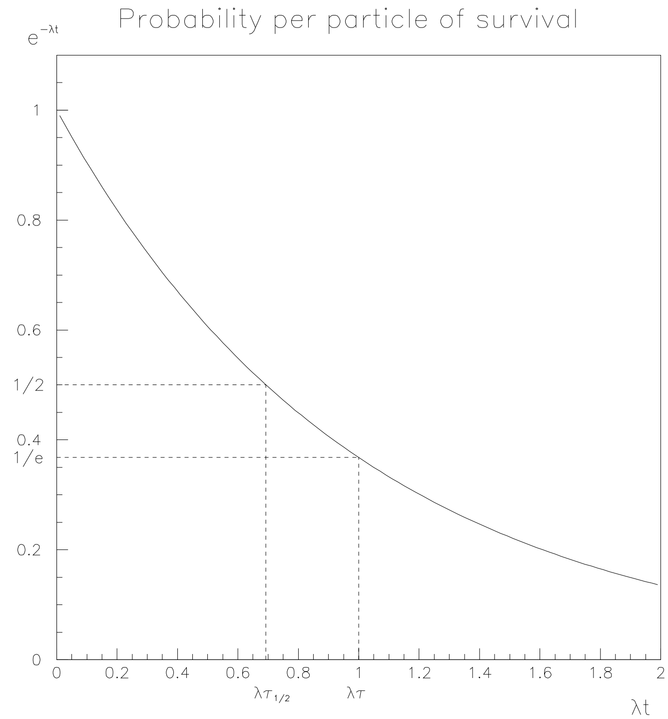


Figure 2.2: Exponential decay law.

2.5.2 Widths

Any state which is unstable (i.e. has a finite lifetime) does not have an exact energy – or, equivalently, an exact mass if you consider the rest frame of the decaying particle.

You may think of this as application of Heisenberg's uncertainty principle:

$$\Delta t \, \Delta E \sim \hbar.$$

Hence, for a particle with a finite lifetime, the mass is uncertain and this is quantised by the width Γ:

$$\Delta mc^2 = \Gamma = \frac{\hbar}{\tau} = \hbar\lambda.$$

Note that very short lifetimes correspond to large widths, while very long-lived particles have small widths. Stable particles, such as the proton and electron, have zero width.

A complication of the above is that it is often the case that particles can decay in several different ways. Each of these decay modes will happen independently of the others, and so we can consider each to have a separate decay rate constant λ_i and hence we can define *partial widths*:

$$\Gamma_i = \hbar\lambda_i \, .$$

The total decay rate is clearly the sum of the separate rates:

$$\lambda = \sum_i \lambda_i \,,$$

and the total width, which is the actual physical uncertainty on the mass, is then

$$\Gamma = \hbar\lambda = \sum_i \hbar\lambda_i = \sum_i \Gamma_i .$$

The proportion of decays to a particular mode is called the *branching ratio* or *branching fraction*:

$$\mathcal{BR}_i = \frac{\Gamma_i}{\Gamma} = \frac{\lambda_i}{\lambda} \,,$$

and clearly

$$\sum_i \mathcal{BR}_i = 1.$$

The lifetime is, as before

$$\tau = \frac{\hbar}{\Gamma} \,,$$

as there is no physical meaning to a "partial lifetime".

2.6 Reactions

What about reactions, whereby two interacting particles result in a number of new particles in the final state? We can clearly measure the rate of a reaction if we fire a beam at some material containing the target particles. However, this rate will depend on the rate at which the beam particles enter the material and the density of the target particles in the material, as well as the properties of the fundamental force we are trying to measure. We need a basic property which is independent of all the other factors which might vary from one experiment to another. This property is the *cross section*, which is the effective target area presented to the incoming particle for it to cause that particular reaction.

Consider a thin piece of material with thickness d containing target particles with number density n, as illustrated in Figure 2.3. Each target particle presents an area, the cross section σ, for the reaction. For a front surface area A, there are Adn targets, which therefore have a total target area $Adn\sigma$. Hence, the probability of an incoming particle hitting one of the targets is $Adn\sigma/A = dn\sigma$. Therefore, in this 'thin target approximation' the reaction rate per unit area of target is simply:

$$R = \phi \, d \, n \, \sigma \,,$$

where ϕ is the incident particle flux (particles per unit area per unit time); the point is that d and n can be different for different experiments, but σ is the physical property we compare.

Note that this only works for thin materials, meaning d is small enough that the probability of a reaction $dn\sigma \ll 1$. Otherwise, every incoming particle will see

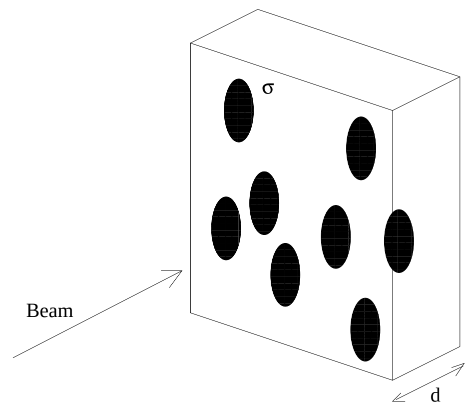


Figure 2.3: The concept of reaction cross section.

several targets and have multiple interactions. In this situation, a more appropriate measure is the mean free path, ℓ, of the particle in the material. Clearly, on average the particle will move this distance before interacting (corresponding to a probability of 1) so the mean free path is given by $\ell n\sigma = 1$ or

$$\ell = \frac{1}{n\sigma}\,.$$

Another useful measure is the mean time between collisions; since $\ell = vt$, this is clearly $1/vn\sigma$.

In addition to the total cross section we may well be interested in so-called differential cross sections. These may reflect the probability for scattering at a particular angle or at a particular energy, to give two examples. We illustrate this idea using a rather non-physical model depicted in Figure 2.4, which is meant to convey the general concept. Consider a model for nuclei as being hard spheres, like billiard balls. (We are ignoring any electromagnetic interaction here.) Consider an incoming light nucleus of radius r_1, and a heavy target nucleus of radius r_2, where $m_2 \gg m_1$ so we can ignore the motion of the target nucleus. The impact parameter b is defined as the distance of closest approach of the centres of the nuclei if no interaction happened.

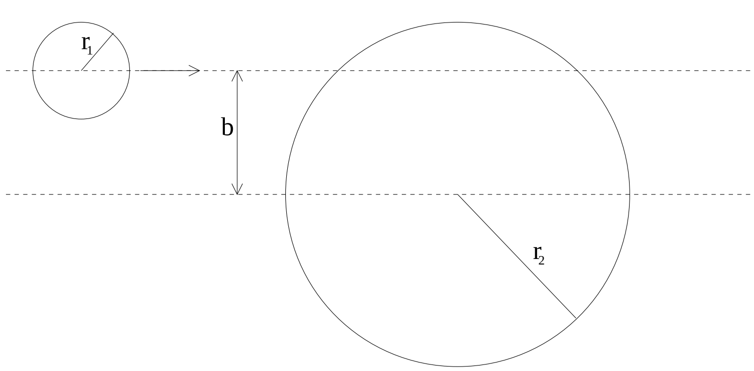


Figure 2.4: Scattering of hard spheres.

Clearly, a collision happens whenever $b < r_1 + r_2$, so the cross section is simply given by $\pi(r_1 + r_2)^2$. Note, as is generally the case, the cross section depends on both the particles taking part in the reaction.

We might also be interested in the angular distribution of the scattered nuclei after the reaction, in which case we want the cross section to scatter to a particular angle. This hard scattering can be shown to have a distribution

$$\frac{d\sigma}{d\theta} = \frac{\pi}{2}(r_1 + r_2)^2 \sin\theta\,,$$

or, equivalently, since it is uniform in azimuth

$$\frac{d\sigma}{d\theta d\phi} = \frac{1}{4}(r_1 + r_2)^2 \sin\theta.$$

In terms of the solid angle element $d\Omega = \sin\theta\, d\theta\, d\phi$, which can also be written as $d(\cos\theta)\, d\phi$, then this becomes

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d(\cos\theta)\, d\phi} = \frac{(r_1 + r_2)^2}{4}.$$

This shows that the cross section per unit solid angle is independent of θ or ϕ. By definition, this means the scattering is *isotropic*, i.e. the scattered particles are emitted equally in all directions. Note the use of $d/d(\cos\theta)$; in terms of $d/d\theta$ we have a $\sin\theta$ dependence – but this is what is needed for an isotropic distribution as θ does not map trivially onto an isotropic distribution. Figure 2.5 illustrates this point.

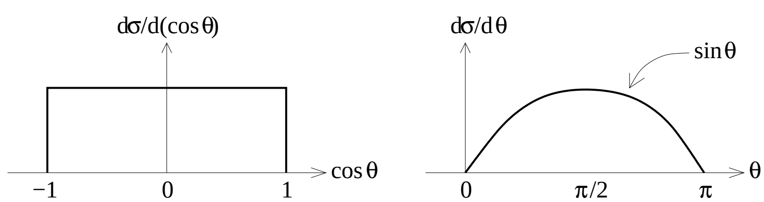


Figure 2.5: Angular distribution for isotropic scattering.

2.7 Units

Finally, a brief word on units. As is common in many branches of physics, the areas of particle and nuclear physics use convenient units which are not SI units.

It is common to measure energies in electron volts, eV, which is the energy acquired by a charge e in moving through a potential difference of 1 V, i.e. 1.602×10^{-19} J. In fact, in nuclear and particle physics, more common units are its higher multipliers of keV (10^3 eV), MeV (10^6 eV), GeV (10^9 eV) or even TeV (10^{12} eV).

Einstein's relation linking energy, mass and momentum is given by $E^2 = p^2c^2 + m^2c^4$, and thus masses are often given in units of MeV/c^2 and momentum can be given in MeV/c.

For example, the electron mass of 9.11×10^{-31} kg corresponds to $(9.11 \times 10^{-31}) \times (2.998 \times 10^8)^2/1.602 \times 10^{-19} = 5.11 \times 10^5$ eV/c^2 = 0.511 MeV/c^2. Similarly, the proton mass of 1.67×10^{-27} kg is usually written as 938.4 MeV/c^2. One other mass unit commonly used in nuclear physics is the atomic mass unit (amu), which is defined to be 1/12 of the mass of a $^{12}_{6}$C atom and corresponds to 931.5 MeV/c^2 or 0.9315 GeV/c^2. It is useful to remember that the proton, neutron and amu masses are all around 1 GeV/c^2.

Particles and nuclei are small, so short distance scales are common. A frequently used unit is the femtometre or Fermi, 1 fm = 10^{-15} m. Areas are often given in barns, where 1 barn = 10^{-28} m^2 = 10^{-24} cm^2 = 100 fm^2.

Caution: Most text books will use "natural units", in which the fundamental constants are defined to be $\hbar = c = \epsilon_0 = 1$ so they do not appear in any equations. This simplifies most calculations and, by dimensional analysis, the correct factors can be put back at the end. We will not use this convention in this course, but it is inevitable that we will refer to energies, momenta and masses in GeV or similar energy units by force of habit – but we will otherwise try not to use natural units in our calculations.

NPP Lecture 3 – The Electromagnetic Force

3.1 Introduction

Although we know a lot about the electromagnetic force from macroscopic experiments, it is worth looking at it in the context of particle physics. We will use the concepts that we introduced in the previous lectures, and then look at the strong and weak forces in terms of the same observables later.

As we have already seen, the force carrier is the photon, a spin-1 boson which has no charge and no mass – and hence infinite range. There is only one type of vertex allowed for QED – that shown in Figure 1.3 on page 10 – and all Feynman diagrams must be drawn using this. How would we study this force in terms of bound states, decays and reactions?

3.2 Bound States

Bound states of particles held together by electromagnetism are very familiar; the simplest is of course the hydrogen atom. By measuring the energy spectrum we can find the form of the potential, and hence the nature of the force. Obviously, for electromagnetism the potential was historically discovered before quantum mechanics but, in principle, if Schrödinger's equation had come first, then the $1/r$ form of the potential could have been found by studying the hydrogen atomic spectrum, and this idea is illustrated below. (We can clearly apply this in the case of the strong force, where the potential is not known *per se.*) Solving the time-independent Schrödinger equation, ignoring the electron and proton spin, namely:

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi - \frac{e^2}{4\pi\epsilon_0 r}\psi = E\psi$$

where μ is the reduced mass

$$\mu = \frac{m_e m_p}{m_e + m_p} = 0.9995 m_e\,,$$

and r is the distance between the electron and proton, gives energy eigenvalues

$$E_n = -\left(\frac{e^2}{4\pi\epsilon_0\hbar c}\right)^2 \frac{\mu c^2}{2n^2} = -\frac{\alpha^2\mu c^2}{2n^2}.$$

Clearly, the energy spectrum depends on the functional form of the potential. A different choice, such as the simple (3D) harmonic potential $m\omega^2 r^2/2$, would give very different energy values:

$$E_n = \hbar\omega \left(n + \frac{3}{2} \right) .$$

Both cases are depicted in Figure 3.1.

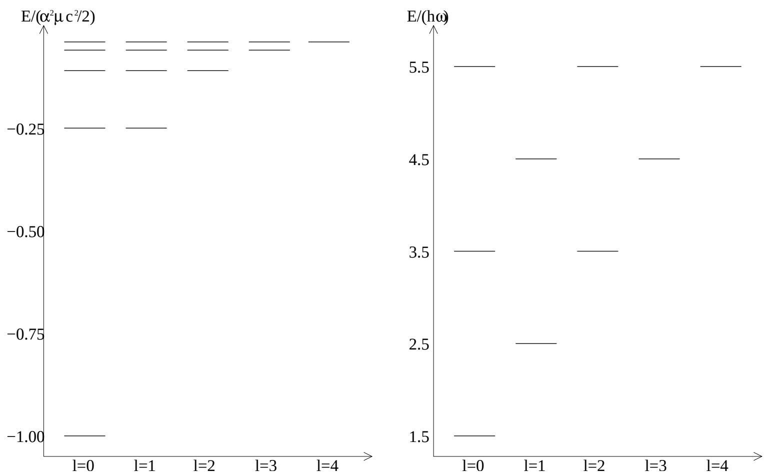


Figure 3.1: Energy levels for the Coulomb potential (left) and for the simple harmonic oscillator (right).

We can consider other systems, e.g. positronium, which are similar – or indeed any of the more exotic atoms that have been made, e.g. with negative muons (the second generation, heavier version of the electron) and pions (the quark-antiquark bound state of $d\overline{u}$) being captured by nuclei into atomic orbits. Allowing for the difference in the reduced mass, these all give the same spectrum. Hence, we believe that electromagnetism works the same no matter which particle the charge is on; this is a non-trivial result.

3.3 Decays

The fundamental interactions of QED can all be drawn in terms of the Feynman diagram vertex which we discussed previously. At this vertex, there is a charged fermion (or antifermion) radiating or absorbing a photon. The fermion changes its energy and momentum state in general, but does not change its nature; e.g. a muon never changes into an electron. Hence, there are very few types of decay which can occur through electromagnetism. One type is from an excited state to a ground state of a composite particle; this is again familiar from atomic physics, but also occurs for particles and nuclei. For example, the strange baryon Σ^0 decays to a Λ^0 baryon (which has the same quark content, uds) via the electromagnetic decay $\Sigma^0 \to \Lambda^0\gamma$, with one of the charged quarks emitting a photon.

A second type of decay occurs because we can reverse one of the fermion lines at the vertex, meaning that electron-positron annihilation is possible, as we saw previously. This is how positronium is able to decay. This makes it fundamentally different from hydrogen. We will use this decay process to illustrate, in a very general way, how decays can reveal some properties of the force involved. The only particles which interact electromagnetically and are lighter than the electron and positron are photons themselves, so the decay must be to photons (at least two, to conserve energy and momentum) and can be drawn with Feynman diagrams as in Figure 3.2.

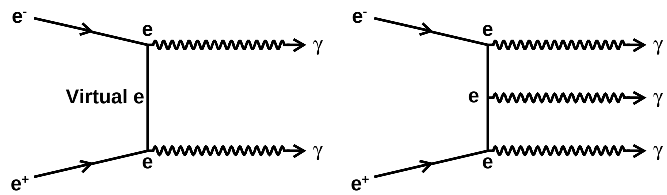


Figure 3.2: Electromagnetic decay of positronium into two or three photons.

Note that the intermediate electron is not seen; fermions can also appear in Feynman diagrams as virtual particles.

In fact, positronium has its ground state split by a hyperfine (spin-spin) interaction into two very closely spaced states which have different charge conjugation (C) quantum numbers. Due to conservation of this quantum number, the ground state has to decay to only even numbers of photons (mostly two), while the higher state can only decay to odd numbers (mostly three). The lifetimes of these two states are 1.2×10^{-10} s and 1.4×10^{-7} s, respectively. The second is slower by a factor of around 1000 and agrees well with QED calculations. Most of this factor is easily understood as being simply the power of α due to the above diagrams: the first goes as α^2, while the second goes as α^3, which is down by over a factor of 100. Hence, the decay rate tells us about the force, albeit in a less direct way.

Let us think a little more about the charge conjugation issue here. Given that charge conjugation on any fermion (f) and antifermion of the same type gives $\hat{C}|f\overline{f}\rangle = (-1)^{(L+S)}|\overline{f}f\rangle$, and that the two different ground states both have L=0, but S=0 and 1, respectively, then they will have $C = 1$ and $C = -1$, respectively. Additionally $\hat{C}|\gamma\rangle = -|\gamma\rangle$, and thus a system of n photons has an overall C value of $(-1)^n$. Hence, two (or any even number of) photons have $C_{\gamma\gamma} = +1$ while three (or any odd number of) photons have $C_{\gamma\gamma\gamma} = -1$. And thus, as C is conserved in the EM interaction, we would expect the $S = 0$ state to go to two photons, and the $S = 1$ state to go to three photons, with a relative rate as above – which indeed is what we see.

3.4 Reactions

Reaction rates can also be used to study forces. We will cover this topic in some detail as it covers some new concepts and is the basis for a lot of what we will do later. Again, because the Feynman diagrams for QED do not change the particle type, then there are basically only two types of reaction: scattering of particles from each other, and particle-antiparticle annihilation to produce something else.

3.4.1 Scattering

Scattering was historically very important. This was how the atomic structure of a central heavy nucleus surrounded by light electrons was discovered in the Rutherford scattering experiment. Rutherford (or actually his two PhD students) fired alpha particles (helium nuclei, $Z_1 = 2$) at a thin gold foil ($Z_2 = 79$) and measured the deflection angle of the alphas which emerged.

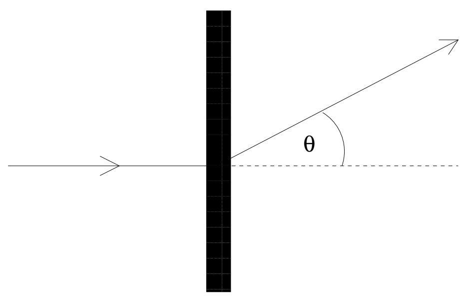


Figure 3.3: Scattering angle.

Knowing the structure of an atom, i.e. that it has a central nucleus, we can calculate the distribution in the angle θ that we expect. This can be done with a classical calculation, although it turns out that a first order perturbation quantum calculation gives the same answer. The details are omitted but the resulting differential cross section turns out to be:

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 m v^2}\right)^2 \frac{1}{4\sin^4(\theta/2)} = \left(\frac{Z_1 Z_2 \alpha\hbar c}{2mv^2}\right)^2 \frac{1}{\sin^4(\theta/2)}$$

In contrast to the hard-sphere case, this is very definitely not isotropic; instead, it falls very rapidly with increasing scattering angle. The very rapid rise of the cross section in the forward direction says the angle of scatter is usually very small. This is because the force is small at large distances and there is more target area far from the nucleus than close to the nucleus. It was the agreement between the angular distribution from Rutherford's scattering experiment and this formula which provided the first evidence for compact nuclei at the core of atoms.

The above ignores the spin of any particles involved. If they have spin, then the force will be different; a spinning charged particle has a magnetic moment which will interact with the magnetic field resulting from the other particle's motion (and the other particle's magnetic moment if it has one). To do the calculation with spin requires the full apparatus of QED. For $e^-\mu^- \to e^-\mu^-$ (see Figure 3.4), where $Z_1 = Z_2 = 1$, the result in the centre-of-mass frame is:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2\hbar^2c^2}{2s}\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)}\,.$$

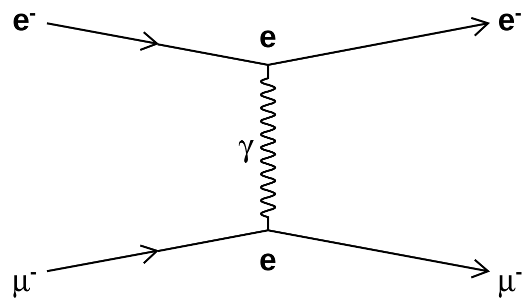


Figure 3.4: Electron-muon scattering.

Note, the diagram has two vertices and we get α^2 as expected. The angular dependence is very similar to Rutherford scattering but it has an extra $\cos^4(\theta/2)$ term due to the spin. The cross section for these two as a function of angle of the scattered electron is shown in Figure 3.5.

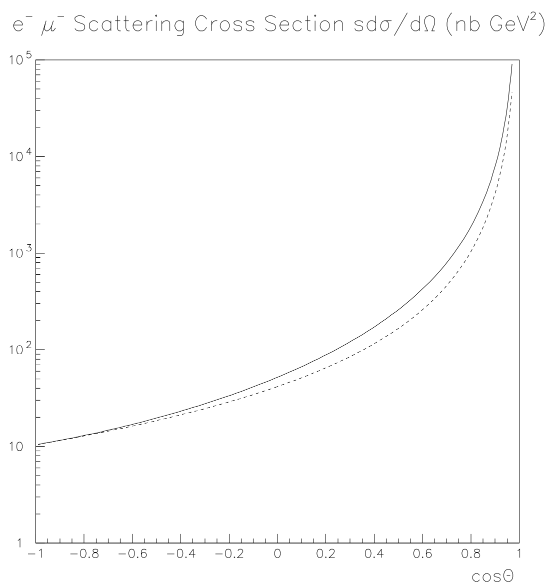


Figure 3.5: Angular cross section for electron-muon scattering with (solid line) and without (dashed line) spin effects.

It turns out this cross section has not been thoroughly measured because of experimental difficulties with muon beams. This is not true of the second type of QED reaction we will look at, namely e^+e^- annihilation.

3.4.2 Annihilation

Consider $e^+e^- \to \mu^+\mu^-$ for which there is copious experimental data; the Feynman diagram for this reaction is shown in Figure 3.6.

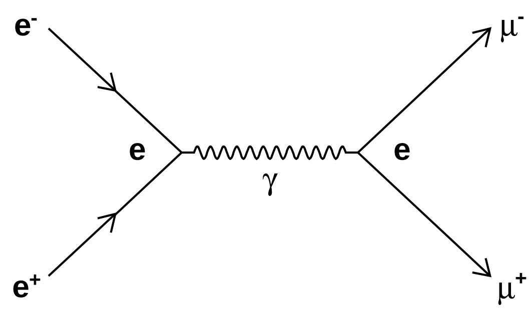


Figure 3.6: Electron-positron annihilation into muons.

This gives a cross section:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2\hbar^2c^2}{4s}(1+\cos^2\theta)\,.$$

Note again, two vertices give a cross section proportional to α^2. In terms of $\cos\theta$, this is just a quadratic.

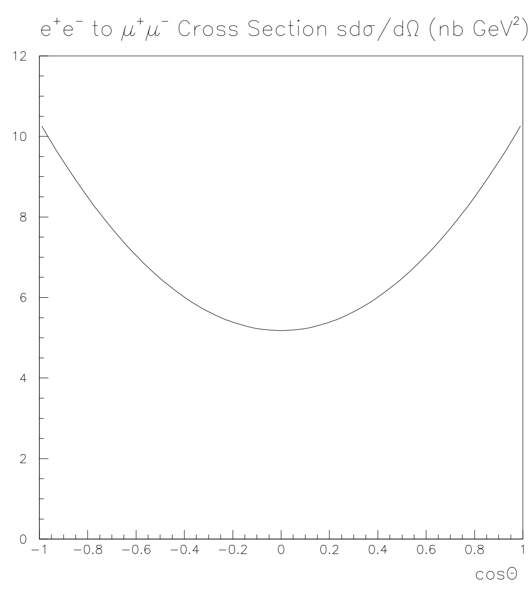


Figure 3.7: Electron-positron angular cross section.

This has been measured both for muons and taus – see Figure 3.8. The angular dependence is in reasonably good agreement with the QED calculation (the small discrepancy is understood in terms of weak force effects not included

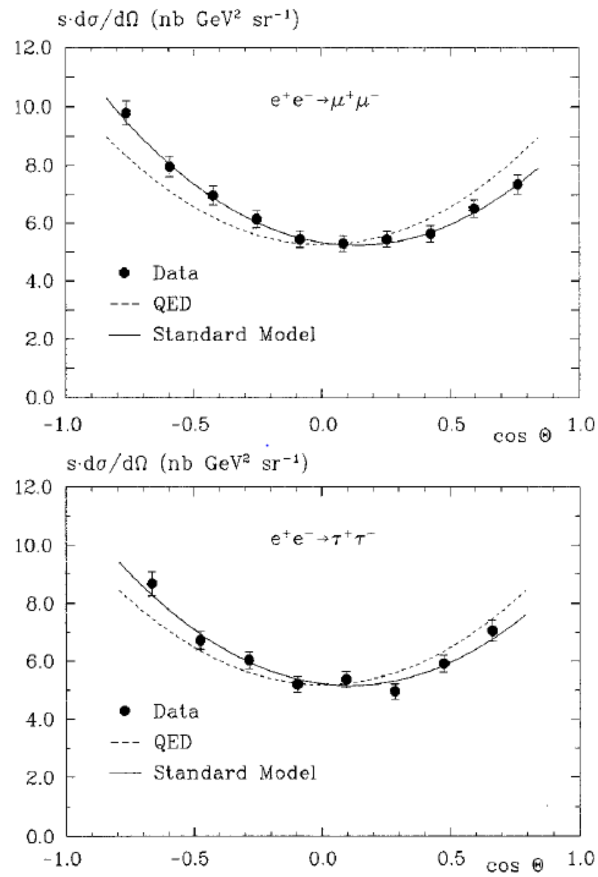


Figure 3.8: Angular cross section data for electron-positron annihilation into muons and taus (JADE experiment at PETRA).

here, but the Standard Model lines in the figure account for those effects). In addition, the magnitude shows that the muon and tau have the same charge as the electron. These are significant results.

Is this shape actually dependent on our assumptions? [Again, here the concept, not the detail, is important.] It turns out that we are sensitive to the spin of both the photon and the muon (or tau) here. If the photon were spin 0, there could not be a preferred direction for the outgoing muons so they would have to come out isotropically, resulting in a flat distribution in $\cos\theta$. Alternatively, it turns out that having spin-0 muons instead would give a $\sin^2\theta$ distribution, as represented in Figure 3.9.
The data show neither of these possibilities is true, and hence proves that it is indeed a photon which is being exchanged here and also that the muon and tau are both spin-1/2, like the electron. This supports the previous statement that the muon and tau are heavier versions of the electron.

Finally, note the $1/s$ behaviour; the cross section becomes very small at high energies. Conversely, the cross section gets very large as the energy is reduced; hence, at rest matter and antimatter annihilate very easily.

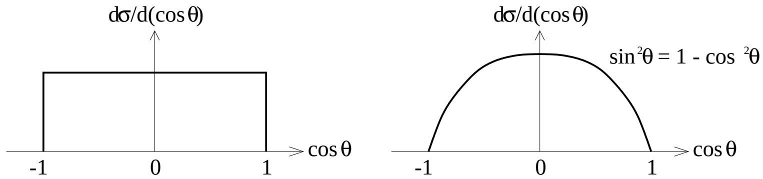


Figure 3.9: Angular distributions for isotropic scattering, which is flat in $\cos(\theta)$, and a $\sin^2(\theta)$ distribution.

Appendix I – non-examinable material
Electron-Positron Scattering

You may be wondering why we have not looked at the reaction $e^+e^- \to e^+e^-$ (known as Bhabha scattering, after the Indian physicist H. J. Bhabha). This is actually more complicated because there are two lowest order diagrams that contribute in this case: one for scattering like in $e^-\mu^- \to e^-\mu^-$, and one for annihilation followed by pair production as in $e^+e^- \to \mu^+\mu^-$. The relevant Feynman diagrams are shown in Figure 3.10.

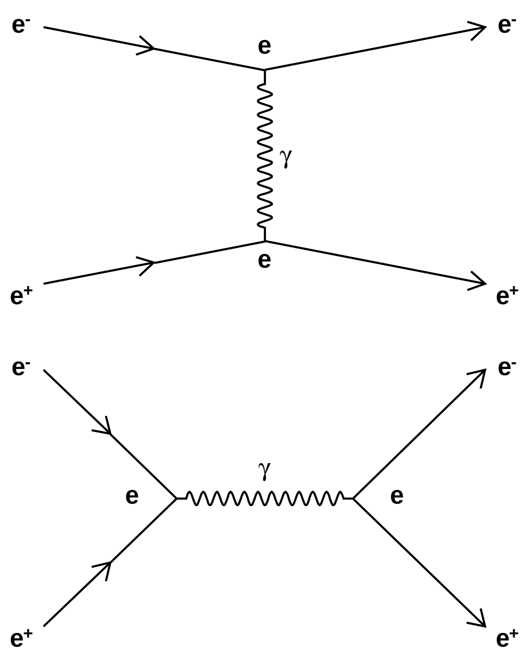


Figure 3.10: Diagrams contributing to $e^+e^- \to e^+e^-$ reaction.

Now we must interfere them as they are quantum mechanically indistinguishable, even in principle. The result is similar to $e^-\mu^- \to e^-\mu^-$ as the upper Feynman diagram gives a much larger contribution than the lower diagram. The measured cross section is shown in Figure 3.11, and has a similar shape to that in Figure 3.5.

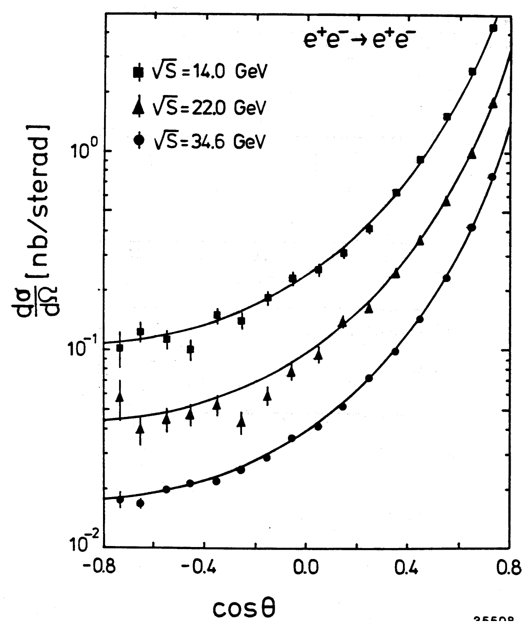


Figure 3.11: Differential cross section for $e^+e^- \to e^+e^-$ reaction (JADE experiment at PETRA).

NPP Lecture 4 –
The Strong Force (QCD) Part I

4.1 Introduction

We now begin to consider the first new force we will look at, namely the strong force. This has some similarities to, and some differences from, QED. The SM theory of the strong force is called Quantum Chromodynamics (QCD). It is called "chromo" (from the Greek for colour) because the strong charge equivalent is called colour and is labelled by red, blue and green, although these have *nothing* to do with colours of light; they are simply convenient labels.

4.2 Gluons

The bosons of the strong field, equivalent to the photons of QED, are called 'gluons' and there are eight distinct gluon particles, compared to only one photon in QED. All eight are massless, like the photon, and so (in principle) they have infinite range, although in practice things are more complicated, as we shall see.

Of the twelve fundamental fermions, the gluons only interact with the six quarks and not the six leptons. Another way to say the same thing is that the quarks have strong charges (i.e. colour), whereas the leptons are colour uncharged (or 'colour neutral'). In some ways, QCD is similar to QED, and one similarity is in the allowed Feynman diagram vertices. QCD says there is only one type of quark-gluon interaction vertex allowed and it looks just like QED: which is a

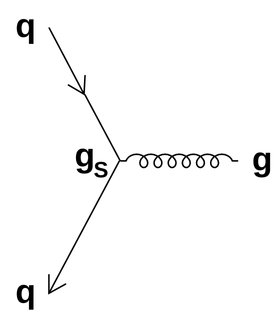


Figure 4.1: Quark-gluon vertex allowed in QCD.

quark radiating or absorbing a gluon. The quarks in this diagram can be time-reversed, just as we did for the electrons in QED (as this is a fermion property coming from the Dirac equation and is nothing to do with any particular force) to get $q\overline{q}$ annihilation and production from the same vertex term. Note, the vertex strength, i.e. the charge equivalent, is labelled by g_S rather than e; this is the standard notation for the strong charge value of a quark. In addition, in the same way that the fine structure constant was the relevant dimensionless parameter to assess the strength of the QED coupling,

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}\,,$$

we can form a strong force equivalent:

$$\alpha_S = \frac{g_S^2}{4\pi\hbar c} \sim 0.1 \gg \alpha\,,$$

where the exact value is not well defined, but is generally much larger that α; this is why the strong force is called "strong".

4.3 Colours

So far, everything looks very similar to QED. However, QCD is in fact very different as a simple look around shows: we do not have gluon lights and strong force power stations. This is because the nature of the QCD charge is very different to that in QED. In QED, charge can be positive or negative and these add in a simple and obvious way to give a total charge which is always conserved. However, in QCD there are three types of charge, not one; these are labelled by red, blue and green as mentioned above. Each type of charge is separately conserved, but adding these is much more complicated. Each quark actually comes with one g_S unit of one of the three types of charge and each quark exists with all three types, e.g. u_r, u_b and u_g, so in fact there are three u quarks, three d quarks, etc. All three of each flavour of quark have exactly the same mass. Hence, there are really not 6 fundamental quarks, but 18. The resulting three antiquarks come with the opposite types of colour (the equivalent of negative), e.g. $\overline{u}_{\overline{r}}$, $\overline{u}_{\overline{b}}$ and $\overline{u}_{\overline{g}}$.

This would still not make QCD so different from QED until we consider the gluons. These also carry colour charge; in fact, each gluon carries one g_S unit of charge of one colour and one g_S unit of charge of an anticolour. For example, one of the eight gluons is $g_{r\overline{b}}$ and its antiparticle is $g_{b\overline{r}}$, which is another of the eight. Different combinations of colour and anticolour give the eight gluons mentioned previously.

4.4 Self Interactions: Confinement and Asymptotic Freedom

We now have the situation where we have said gluons carry colour charge, which would be equivalent to photons being electrically charged. Gluons can therefore act like quarks and radiate other gluons; this means there are more allowed Feynman diagrams, such as:

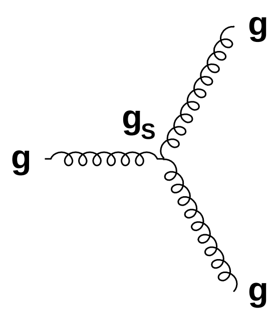


Figure 4.2: Gluon-gluon vertex allowed in QCD.

This ability to 'self-interact', coupled with the fact that gluons are massless, means that the gluon force fields have very different properties to electromagnetic force fields, and leads to the important concepts of "confinement" and "asymptotic freedom".

Confinement means that we never see single quarks or gluons, but only the bound states called 'hadrons', i.e. the mesons and the baryons. This is why giving an exact value to α_S is not possible: we never have a state with a non-zero strong charge on it. Consider how difficult it would be to measure e if only neutral atoms were available.

Asymptotic freedom says that the QCD field has totally the opposite behaviour to QED at short distances also, namely the force goes to zero in this case. This means that if a quark is hit, e.g. by a photon from outside the hadron, for a short time such that it does not move very far, then it acts as if it is not in a bound state at all. It is called asymptotic freedom as it only holds exactly for infinitesimally small times or distances and freedom as the quarks act as if free, not bound. One analogy often used is the "balloon" model, where the quarks are pictured as being within an infinitely strong balloon. When they are moving around the middle, there is no force on them and they act as if free. However, if given a big kick (e.g. by a photon) then they hit the balloon wall and get bounced back in; they can never escape, as illustrated in the sketch below.

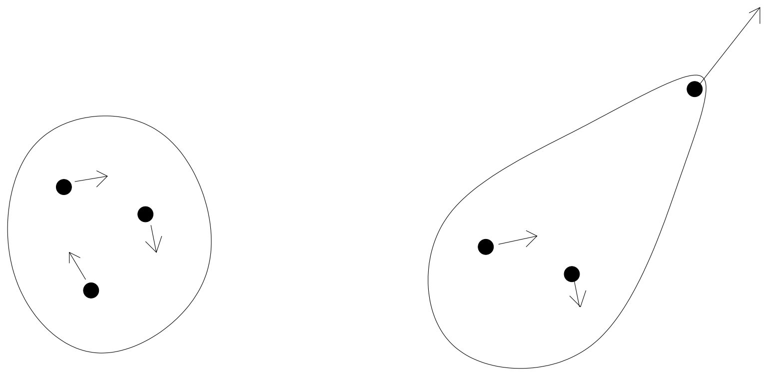


Figure 4.3: Illustration of asymptotic freedom for a proton with three quarks.

4.5 Sanity Check: Gauge Invariance

You are probably thinking this is all very weird and complicated. You are right: it is. However, the whole structure actually results from a single change between QCD and QED, made within the overall principle of "gauge invariance". This is beyond the scope of this course, but it is a basic physical principle underpinning all three of the forces we will consider. Its most commonly known manifestation is that there cannot be an absolute value of EM potential, by definition; we can only be sensitive to differences in the potential. However, this principle applied to QCD gives all of the above features; they are not arbitrarily added in *ad hoc* to make the theory agree with experiment.

In fact, gauge invariance requires one more feature, which is the *universality* of the colour charge. In QED, the charge on a fundamental particle could in principle take any value (although we in fact see only $\pm e/3$, $\pm 2e/3$ and $\pm e$). However, gauge invariance in QCD says the gluons must have the same colour charges as the quarks, or it does not work. This then means that all quarks (and gluons) must have the same charge g_S, not $g_S/3$ or any other value. We will see the results of this in the next couple of lectures.

4.6 Hadrons

The most obvious result of confinement is that free quarks have never been observed. We only ever see hadrons, which are bound states of $q\bar{q}$ (mesons) or qqq (baryons). Indeed, the fact that we only see these two types of hadron, and furthermore not all possible qqq combinations for the baryons, is very powerful evidence for the concept of colour, and of QCD as a whole.

4.6.1 Colour singlets

The hadrons themselves must be uncharged with respect to the strong force, or confinement would hold for them also and we would not observe such states as free particles. This means they must be the equivalent of neutral for the colour charge. Because this charge comes in three types (r, b and g) rather than one type as in electromagnetism, then we need to find the colourless or colour singlet combinations. There are in fact two ways to achieve this, and these are give them below without proof (and of course their detailed composition is non-examinable), namely:

$$C = 0: \quad (r\overline{r} + b\overline{b} + g\overline{g})/\sqrt{3}\,,$$

and

$$C = 0: \quad (rbg - rgb + grb - gbr + bgr - brg)/\sqrt{6}\,.$$

This can clearly be formed using the $q\overline{q}$ and qqq combinations of mesons and baryons, so we would expect mesons and baryons to exist. Additional details are given in Appendix I if you want to find out more. [As an aside: hence the name "colour", as the rbg combination gives something colourless, i.e. "white".]

4.6.2 Observed baryons

Not only does the fact that we see baryons at all give evidence for colour, but so does the fact that we only see certain baryons i.e. certain combinations of quarks, as illustrated below. When considering which states are allowed, there is a complication here which is not present in the mesons, which is that identical particles can appear in these combinations. For example, the proton is uud, and so some of these states for those cases are forbidden by the Pauli exclusion principle. Recall that this requires the overall wavefunction to be antisymmetric under exchange. To see whether the overall wavefunction is antisymmetric we need to consider what happens to the constituent parts of the wavefunction. If were to consider it made up of just a spatial part and a spin part, then 9 combinations (for the u, d and s quarks) would be allowed, whereas if we were to consider it made up of a spatial part, a spin part AND a colour part, then 18 combinations would be allowed. And indeed 18 combinations is what is seen. Additional details are given in Appendix II (non examinable).

4.6.3 Meson and baryon masses

For now, let us consider the simpler meson case and, in fact, let us restrict ourselves to the u, d and s quark "flavours". There are then nine possible flavour combinations (a "nonet") of $q\overline{q}$. Each forms a meson with a ground state ($J = 0$) and excited states ($J = 1$). All the nine ground states have the same quantum numbers as all the first excited states; the wavefunctions only differ because of the different masses of the quarks which each meson contains. This is completely analogous to comparing the spectra of hydrogen and positronium when allowing for the different reduced mass.

These combinations and their ground and first excited states are listed in Table 4.4 below, where the last column will be explained later this lecture. We call these ground states and first excited states separate (composite) particles, although in the analogy with the hydrogen atom, they correspond to just the excited states.

Table 4.4: The observed mesons with u, d and s quarks.

State	Meson	Measured Mass (GeV/c^2)	Quark Pair	Predicted Mass (GeV/c^2)
$J^P = 0^-$	$\pi^\pm$	0.1396	$u\overline{d}$, $d\overline{u}$	0.1395
	$K^\pm$	0.4937	$u\overline{s}$, $s\overline{u}$	0.4938
	K^0, $\overline{K}^0$	0.4977	$d\overline{s}$, $s\overline{d}$	0.4980
$J^{PC} = 0^{-+}$	π^0	0.1350	$u\overline{u}$	0.1340
	η	0.5475	$d\overline{d}$	0.1449
	η'	0.9578	$s\overline{s}$	0.7871
$J^P = 1^-$	$\rho^\pm$	0.7669	$u\overline{d}$, $d\overline{u}$	0.7702
	$K^{*\pm}$	0.8916	$u\overline{s}$, $s\overline{u}$	0.8918
	K^{*0}, $\overline{K}^{*0}$	0.8961	$d\overline{s}$, $s\overline{d}$	0.8932
$J^{PC} = 1^{--}$	ρ^0	0.7691	$u\overline{u}$	0.7692
	ω	0.7819	$d\overline{d}$	0.7713
	ϕ	1.0194	$s\overline{s}$	1.0365

To what extent can we understand the mass spectrum of these mesons? Normally, for a bound state, we would say:

$$m_{BS} = m_q + m_{\overline{q}} - \frac{E_B}{c^2}\,,$$

where E_B is the binding energy which must be positive to obtain a stable state.

In atoms, splitting between levels arises, in part, through $\boldsymbol{S}_N.\boldsymbol{S}_e$ couplings (the hyperfine structure) due to the magnetic dipole moment of the circulating electron coupling to the nuclear magnetic dipole moment $\Delta E \propto \boldsymbol{\mu}_e.\boldsymbol{\mu}_N$. Since the basic quark-gluon vertex for QCD is very similar to QED, then we might expect a magnetic-field equivalent in QCD and for particles to have QCD-magnetic moments. Since we know from the Dirac equation that for fundamental fermions

$$\boldsymbol{\mu} \propto \frac{\boldsymbol{s}}{m}\,,$$

then we can say a QCD hyperfine equivalent interaction would have a term like

$$\propto \frac{\boldsymbol{s}_q.\boldsymbol{s}_{\overline{q}}}{m_q m_{\overline{q}}}\,,$$

with the binding energy being the expectation value

$$\frac{E_B}{c^2} = -K\frac{\langle \boldsymbol{s}_q.\boldsymbol{s}_{\overline{q}}\rangle}{m_q m_{\overline{q}}}\,,$$

for a constant of proportionality K and where the negative sign is chosen to make K positive. The $\boldsymbol{s}_q.\boldsymbol{s}_{\overline{q}}$ term might look difficult but, since $L = 0$, then

$$\boldsymbol{J} = \boldsymbol{s}_q + \boldsymbol{s}_{\overline{q}}\,,$$

so

$$J^2 = s_q^2 + s_{\overline{q}}^2 + 2\boldsymbol{s}_q.\boldsymbol{s}_{\overline{q}}\,,$$

which means

$$\langle \boldsymbol{s}_q.\boldsymbol{s}_{\overline{q}} \rangle = \frac{1}{2}\langle J^2 - s_q^2 - s_{\overline{q}}^2 \rangle = \frac{1}{2}\left[J(J+1) - s_q(s_q+1) - s_{\overline{q}}(s_{\overline{q}}+1)\right]\hbar^2\,.$$

Clearly, $s_q = s_{\overline{q}} = 1/2$ and so, for the ground states with $J = 0$, we would get:

$$m_{GS} = m_q + m_{\overline{q}} - \frac{3}{4}\frac{K\hbar^2}{m_q m_{\overline{q}}}\,,$$

and for the first excited state, with $J = 1$,

$$m_{FES} = m_q + m_{\overline{q}} + \frac{1}{4}\frac{K\hbar^2}{m_q m_{\overline{q}}}\,,$$

where the constant of proportionality K is to be determined. For this model, we have six useful values of the masses (ground and excited states of the different flavour pairs) and four parameters (m_u, m_d, m_s and K). Good agreement for all six is obtained with $m_u = 0.3052$ GeV/c^2, $m_d = 0.3075$ GeV/c^2, $m_s = 0.4871$ GeV/c^2 and $K = 0.0592$ GeV$^3/\hbar^2$c^6. This then gives predictions of the mesons containing the $u\overline{u}$, $d\overline{d}$ and $s\overline{s}$. The result is shown in Table 4.4. There is good agreement except for the η and η' were the difference is well understood (but beyond the scope of this lecture).

One can generalise the idea above to predict the mass of the baryons:

$$M = \sum_i m_i + \sum_{i>j} K\frac{\langle \boldsymbol{s}_i.\boldsymbol{s}_j \rangle}{m_i m_j}\,.$$

This is a little trickier than for mesons as there are three spin-spin terms which, in general, have different mass denominators. The easiest case is when all masses are approximately equal, e.g. the uud case with the approximation $m_d = m_u$, where it simplifies to:

$$M = 3m_u + \frac{K}{m_u^2}\langle \boldsymbol{s}_1.\boldsymbol{s}_2 + \boldsymbol{s}_1.\boldsymbol{s}_3 + \boldsymbol{s}_2.\boldsymbol{s}_3 \rangle$$

and the derivation then proceeds much like the one above.

Doing this again gives good agreement with data. However, we do find that the actual mass values (and K values) we get are a little different for the meson and baryon cases, although the mass differences are reasonably similar, with $m_d - m_u \sim 3$ MeV/c^2. This tells us we are not determining just the mass here but in fact what we parametrise as mass depends in fact on the bound state wavefunctions, etc. Some of what we call mass is actually a cloud of quarks and gluons which move along with the bare quark, due to the large value of α_S. How much of the above values is "real" mass? It turns out only to be around 1% for the u and d quarks! The effective masses we see here are called "constituent" masses as they are appropriate to use when considering the quarks as constituents of hadrons.

$$m_u \sim 300 \text{ MeV/c}^2, \qquad m_d \sim 300 \text{ MeV/c}^2, \qquad m_s \sim 500 \text{ MeV/c}^2 \,.$$

However, when we interact with quarks using, e.g. a high energy photon, then the masses we measure for the quarks from kinematics, called the "current" masses, are quite different:

$$m_u \sim 1 \text{ MeV/c}^2, \qquad m_d \sim 4 \text{ MeV/c}^2, \qquad m_s \sim 200 \text{ MeV/c}^2 \,.$$

Here, the quarks act as if free (asymptotic freedom) and so appear not to be dragging the large mass of field around (which dominates at low energies).

The strong charge (colour) of each quark must be identical for the theory of QCD to work; this is referred to as *universality*. It is not true for QED, where we see several different charges on the fundamental fermions, i.e. $e/3$, $2e/3$ and e. Because of QCD universality, both the u and d quark have effectively the same strong field attached to them. Because this dominates, they appear to have very similar constituent masses. Hence, the reason why the meson states are so close in mass is not because the u and d masses are similar but because the masses are very small compared with the associated field energy and because their QCD charge is the same.

4.7 Meson Decays: Excited State Decays

The excited states can decay to ground states, as would be expected. This is possible through photon emission in a similar way as in atoms, e.g. $\rho^+ \to \pi^+\gamma$, but is in fact very rare with a branching fraction of only 4×10^{-4}. This is because it is (clearly) an electromagnetic force decay and there are alternative decay modes which use the strong force and, since $\alpha_S \gg \alpha$, these can proceed much faster. As $m_\rho > 2m_\pi$, then in fact the ρ^+ mostly decays as $\rho^+ \to \pi^+\pi^0$. This requires a new $q\overline{q}$ pair to be created at a gluon vertex, as shown in the figure.

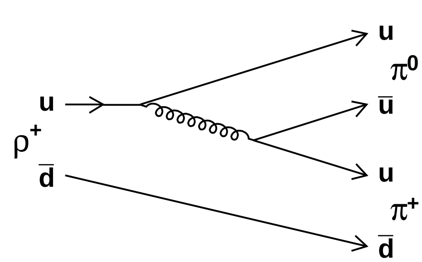


Figure 4.4: Hadronic decay of the ρ^+ meson into two pions.

Because this is a strong force decay with only two vertices, then it goes very fast and the lifetime of the ρ^+ is around 10^{-23} s. This gives a width of 150 MeV, which is not negligible compared with its mass of 770 MeV/c^2 (and this can be measured). The ρ^0 decays as $\rho^0 \to \pi^+\pi^-$ and again has a similar lifetime and width. These are typical orders of magnitude for strong decay particle lifetimes.

Because the strong force does not change flavour, the K^* mesons must decay to produce a K meson so as to preserve the s quark, e.g. $K^{*+} \to K^+\pi^0$ or $K^{*+} \to K^0\pi^+$ – see Figure 4.5. The decay rates are again of the same order and

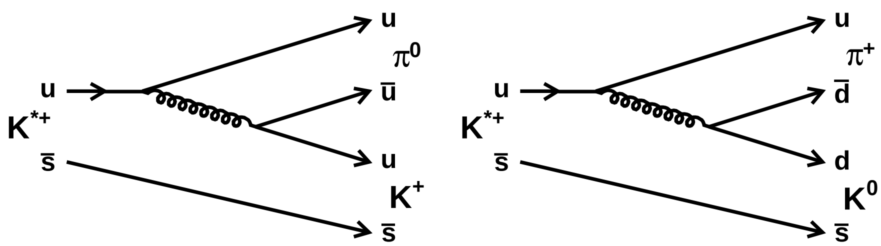


Figure 4.5: Hadronic decays of the K^{*+} meson.

the K^* widths are around 50 MeV. Slightly smaller widths are found for the ω and ϕ, but only because of a more restricted set of final states which they can decay to; they all decay strongly.

4.7.1 Heavy quarks

There are also three other flavours of quarks, charm c, bottom b and top t. Of these, top quarks decay too fast to form hadrons, but mesons containing charm ($m_c \sim 1.6$ GeV/c^2) and bottom ($m_b \sim 4.9$ GeV/c^2) quarks are also seen. For example, singly-charmed mesons such as $c\overline{u}$ (D^0 and D^{*0} for the ground and first excited states) and $c\overline{d}$ (D^+ and D^{*+}) have been observed. These will behave like the K mesons discussed above, though the ratio of the strong to EM decays for the excited states will be slightly different as the phase-space is smaller. So called "hidden-charm" (i.e. no net charm) mesons containing $c\overline{c}$ (η_c and J/ψ) have also been observed. Similar combinations of bottom quarks in mesons have also been observed.

4.8 Appendix I – non-examinable material: Counting Colours

We mentioned previously that there are *eight* gluons; however, all combinations of r, b and g with $\overline{r}$, $\overline{b}$ and $\overline{g}$ would imply nine. Six of these are the combinations like $r\overline{b}$ above, and are straightforward. However, there are three combinations ($r\overline{r}$, $b\overline{b}$ and $g\overline{g}$) which we might think have no total colour charge. However, adding colour charge is not quite so simple as this. Let us use an analogy: with two spin-1/2 particles, each with $s_z = \pm 1/2$, there are four combinations, namely:

$$|{+1/2},{+1/2}\rangle \equiv \uparrow\uparrow, \quad |{+1/2},{-1/2}\rangle \equiv \uparrow\downarrow, \quad |{-1/2},{+1/2}\rangle \equiv \downarrow\uparrow, \quad |{-1/2},{-1/2}\rangle \equiv \downarrow\downarrow \ .$$

However, they do not all correspond to definite total spin values, i.e. total spin eigenstates, which can be $S = 0$ or $S = 1$. For these we have to take particular combinations of the above, specifically:

$$\begin{array}{lll}
|S=1, S_z=+1\rangle & = |+1/2, +1/2\rangle & = \uparrow\uparrow \\
|S=1, S_z=\ \ \ 0\rangle & = \frac{1}{\sqrt{2}}(|+1/2,-1/2\rangle + |-1/2,+1/2\rangle) & = \dfrac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\
|S=1, S_z=-1\rangle & = |-1/2, -1/2\rangle & = \downarrow\downarrow \\
|S=0, S_z=\ \ \ 0\rangle & = \frac{1}{\sqrt{2}}(|+1/2,-1/2\rangle - |-1/2,+1/2|\rangle) & = \dfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\,.
\end{array}$$

Hence, although the states $|+1/2, -1/2\rangle$ and $|-1/2, +1/2\rangle$ have $S_z = 0$, they still have some amount of total $S = 1$ as well as $S = 0$. By analogy, the states $r\overline{r}$, $b\overline{b}$ and $g\overline{g}$ have zero individual colour charges, but we can still have non-zero total colour if they are combined correctly. Explicitly, the combinations needed are:

$$C=1:\ \frac{1}{\sqrt{6}}(r\overline{r}+b\overline{b}-2g\overline{g}), \qquad C=1:\ \frac{1}{\sqrt{2}}(r\overline{r}-b\overline{b}), \qquad C=0:\ \frac{1}{\sqrt{3}}(r\overline{r}+b\overline{b}+g\overline{g})\,.$$

Hence, of these three, there is one total colour zero combination $C = 0$, the so-called “colourless” state, and two (plus the previous six) states with total colour of one unit, called “coloured” states. For some reason, the universe is built so this one colourless gluon does not interact, i.e. the strong force requires $C = 1$ not $C = 0$. Hence, there are only eight interacting gluons, not nine. We have actually already met this colourless combination: it corresponds to the mesons (i.e. $q\overline{q}$ states).

4.9 Appendix II – non-examinable material: Counting Spin

Again let us just consider u, d and s flavour quarks to make the point. There are ten possible combinations of flavours for three quarks, hence we would naively expect ten ground states, ten excited states, etc, where the ten different baryons in each case have the same wavefunctions but just different flavours of quarks. But we have to consider the Pauli exclusion principle as we said.

We will first consider the one case where there are no identical particles, namely uds states. Again, the lowest mass states have $L = 0$ so we only need to consider spin. Two quarks can be combined to give $S = 0$:

$$|0,\quad 0\rangle = \sqrt{\frac{1}{2}}\,(\uparrow\downarrow - \downarrow\uparrow)\;,$$

or $S = 1$

$$
\begin{aligned}
|1,+1\rangle &= \uparrow\uparrow \\
|1,\ \ 0\rangle &= \sqrt{\frac{1}{2}}\,(\uparrow\downarrow + \downarrow\uparrow) \\
|1,-1\rangle &= \downarrow\downarrow\ .
\end{aligned}
$$

Adding the third quark to $S = 0$ gives $S = 1/2$:

$$
\begin{aligned}
|1/2,+1/2\rangle &= |0,\ \ 0\rangle \uparrow = \sqrt{\frac{1}{2}}\,(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \\
|1/2,-1/2\rangle &= |0,\ \ 0\rangle \downarrow = \sqrt{\frac{1}{2}}\,(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)\ ,
\end{aligned}
$$

while adding it to $S = 1$ can give $S = 1/2$:

$$
\begin{aligned}
|1/2,+1/2\rangle &= \sqrt{\tfrac{2}{3}}|1,+1\rangle \downarrow - \sqrt{\tfrac{1}{3}}|1,\ \ 0\rangle \uparrow = \sqrt{\frac{2}{3}}\uparrow\uparrow\downarrow - \sqrt{\frac{1}{6}}\uparrow\downarrow\uparrow - \sqrt{\frac{1}{6}}\downarrow\uparrow\uparrow \\
|1/2,-1/2\rangle &= \sqrt{\tfrac{1}{3}}|1,\ \ 0\rangle \downarrow - \sqrt{\tfrac{2}{3}}|1,-1\rangle \uparrow = \sqrt{\frac{1}{6}}\uparrow\downarrow\downarrow + \sqrt{\frac{1}{6}}\downarrow\uparrow\downarrow - \sqrt{\frac{2}{3}}\downarrow\downarrow\uparrow\,,
\end{aligned}
$$

or $S = 3/2$:

$$
\begin{aligned}
|3/2,+3/2\rangle &= |1,+1\rangle \uparrow = \uparrow\uparrow\uparrow \\
|3/2,+1/2\rangle &= \sqrt{\tfrac{1}{3}}|1,+1\rangle \downarrow + \sqrt{\tfrac{2}{3}}|1,\ \ 0\rangle \uparrow = \sqrt{\frac{1}{3}}\,(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow) \\
|3/2,-1/2\rangle &= \sqrt{\tfrac{2}{3}}|1,\ \ 0\rangle \downarrow + \sqrt{\tfrac{1}{3}}|1,-1\rangle \uparrow = \sqrt{\frac{1}{3}}\,(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow) \\
|3/2,-3/2\rangle &= |1,-1\rangle \downarrow = \downarrow\downarrow\downarrow\ .
\end{aligned}
$$

Note, these two $S = 1/2$ states have different internal quark spin structures: they are not the same state, as one has the first two quarks in $S = 0$ and the other has them in $S = 1$. Hence, of the ground state and first two excited states, we would expect there to be two spin 1/2 and one spin 3/2. The order turns out to be that the ground state is the $S = 1/2$ built on $S = 0$, while the first excited state is the $S = 1/2$ built on $S = 1$ and the second excited state is $S = 3/2$. These are observed as the Λ, the Σ^0 and the Σ^{*0}, respectively.

For the six flavour combinations with two identical quarks, then we need to look at the symmetry under interchange of the spin states. Firstly, any orbital angular momentum state has an exchange symmetry of $(-1)^L$; since $L = 0$ here, this is symmetric for any interchange. Earlier, we saw the uncoloured rbg combination is totally antisymmetric:

$$
C = 0: \quad (rbg - rgb + grb - gbr + bgr - brg)/\sqrt{6}\,,
$$

so we require the spin part to be *symmetric* under interchange. This is tedious in general, but looking at the two-particle states of $S = 0$ and 1, it is clear the

$S = 1$ are symmetric and hence all the total states which use those will have a positive exchange symmetry for at least the first two quarks. This is not true for the $S = 0$ so only the states built on $S = 1$ are allowed for two identical quark flavours. Hence, only two states, one $S = 1/2$ and one $S = 3/2$, exist for this case; e.g. for uud, these are the proton and the Δ^+. Note, the wavefunctions of the ground and first excited states here actually correspond to the first and second excited states of the uds case.

Having three quarks all identical is even more restrictive and, indeed, for these three cases, only the spin-3/2 state is symmetric under all three interchanges. Hence only one state (the ground state) is seen and it has $S = 3/2$; e.g. for uuu, the only state allowed is the Δ^{++}, which has the wavefunction of the second excited state of the uds case.

Hence, we see only one particle (a "singlet") in the lower of the spin-1/2 states (uds), seven particles (a "septet") in the higher of the two, and all ten (a "decuplet") in the spin-3/2 state. As they have the same J^P values, all the spin-1/2 states are sometimes considered together and are referred to as an "octet". Note, the absence of colour would give a very different result as then we would need the antisymmetric spin states. Only the ground state would be allowed for the uud combination and no states at all are allowed for the uuu combination, so we would have 9 particles in total, not 18 – as shown in the diagram in Figure 4.6.

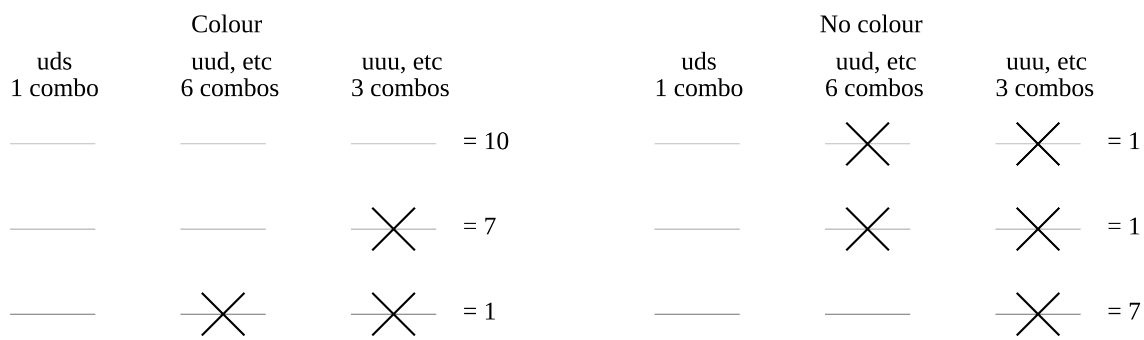


Figure 4.6: Allowed baryon states with (18) and without (9) quark colour.

NPP Lecture 5 – The Strong Force (QCD) Part II and The Weak Force Part I

5.1 Introduction

We finish looking at QCD in this lecture by considering electron-positron annihilation to hadrons. In this case, asymptotic freedom dominates – in contrast to the hadron bound states where confinement dominates. We will be using QED to probe QCD.

After this, we start discussing the weak force.

5.2 Electron-Positron Annihilation into Quarks

If we assume that the quarks produced are completely free, then the lowest order Feynman diagram for $e^+e^- \to q\bar{q}$ is that shown in Figure 5.1.

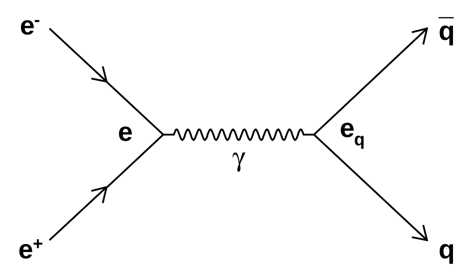


Figure 5.1: Electron-positron annihilation into quarks.

This is very similar to the diagram we have already seen for $e^+e^- \to \mu^+\mu^-$, the only difference being that the quark has an electric charge e_q rather than e. Hence:

$$\text{Amplitude}(\mu\mu) \propto e^2, \qquad \text{Amplitude}(q\bar{q}) \propto ee_q\,,$$

so the cross section goes as

$$\sigma_{\mu\mu} \propto e^4 \propto \alpha^2, \qquad \sigma_{q\bar{q}} \propto e^2 e_q^2 = e^4(e_q/e)^2 \propto \alpha^2(e_q/e)^2\,.$$

We never see free quarks, which means that these quarks must (somehow) always become hadrons – we will return to this later. Also, there are several types of quarks, all of which must result in hadrons, so the cross section for hadron production, which is what we actually observe, is the sum over all quarks:

$$\sigma_H \propto \alpha^2 \sum_q \left(\frac{e_q}{e}\right)^2 .$$

This is often expressed as the ratio[6] of the hadron-to-muon cross sections, where all the other common factors cancel:

$$R = \frac{\sigma_H}{\sigma_{\mu\mu}} = \sum_q \left(\frac{e_q}{e}\right)^2 .$$

For example, for centre-of-mass energies between $2m_b c^2 \sim 10$ GeV and $2m_t c^2 \sim 340$ GeV, then all quarks except top can be produced, so

$$R = \frac{4}{9} + \frac{1}{9} + \frac{1}{9} + \frac{4}{9} + \frac{1}{9} = \frac{11}{9},$$

which is *not* in good agreement with experiment. In fact, we have forgotten about the fact that each flavour of quark comes in three colours, one of each of r, b and g, so these each contribute, making $R = 11/3$. Below the b quark threshold this becomes $R = 10/3$ and then, below the c threshold, $R = 6/3$. This is in much better agreement (with the difference being understood from higher order diagrams – see below) and is strong evidence for colour.

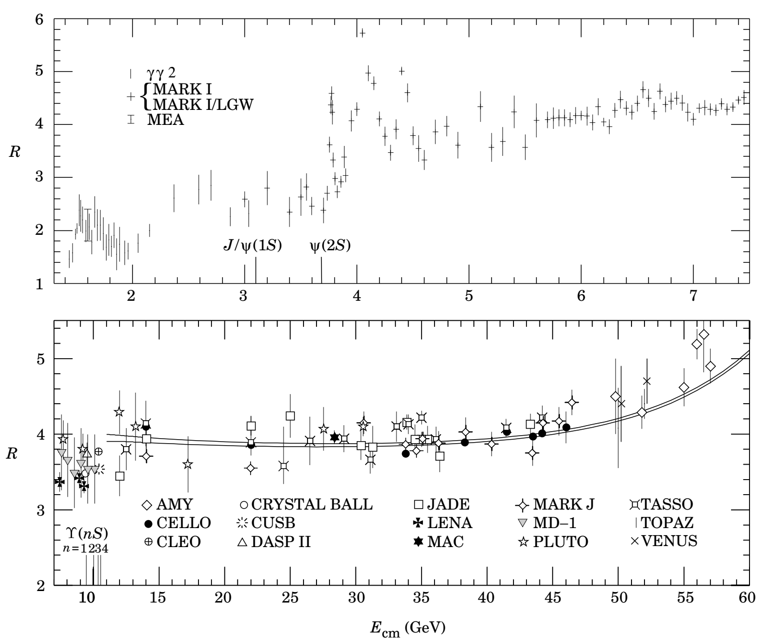


Figure 5.2: Ratio of hadron-to-muon cross sections for e^+e^- annihilation.

[6] Ratios are often easier to calculate and measure than individual cross sections as many common factors and systematic errors cancel.

5.2.1 Higher order effects – gluon radiation

We have been assuming that the gluon fields act to change the quarks into hadrons in small energy interactions, so the jets follow the original quark directions. However, occasionally there will be a high energy gluon radiated off one of the quarks. This process of $e^+e^- \to q\overline{q}g$ is calculable from the next order Feynman diagram, and shown overleaf. The matrix element of these events is clearly proportional

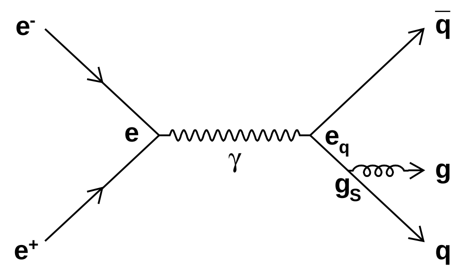


Figure 5.3: Final state radiation: gluon emission by a quark final state.

to g_S, so the rate is proportional to α_S; they add to the total cross section such as to give an extra term:

$$\sigma_H = \sigma_H^{L0}\left(1 + \frac{\alpha_S}{\pi}\right) .$$

For $\alpha_S \sim 0.1$, this gives a 3% correction to R. In fact, this makes up a large part of the discrepancy with the lowest order 11/3 value. In principle, this allows a measurement of α_S.

5.3 Hadronisation

A lot more information is available from the interactions we discussed in this lecture than just the total cross section. What do these events look like? Do all the hadrons come out isotropically? What types of hadron are produced? Looking at pictures of these events shows that the production is not isotropic, but the hadrons tend to be highly collimated into "jets" that closely follow the the original quark direction. Hence, a measurement of the jet angular direction can be done, although q and $\overline{q}$ cannot be easily distinguished and so only the magnitude of $|\cos\theta|$ can be measured. However, this clearly agrees with the $1 + \cos^2\theta$ distribution we would expect for spin-1/2 production. The process of hadronisation cannot be calculated perturbatively as the eventual production of hadrons must involve very many gluon interactions. It is therefore very difficult to understand this process from Feynman diagrams, but a phenomenological model has been produced which reproduces all the main features. A detailed description is beyond the scope of this course, but a brief explanation is given in the non-examinable Appendix at the end of this lecture.

5.4 The Weak Force – Introduction

When we started looking at QCD, we went through the properties of the gluons and quarks. We must now do the same for the weak force. In some ways, it is easier than for QCD, e.g. there is no confinement or asymptotic freedom, but there are other complicating factors which make it more difficult. We will see that it does not conserve $\hat{P}$ and $\hat{C}$, and in some cases even $\hat{CP}$.

All fundamental fermions carry weak charge; in particular it is the only charge on the neutrinos. Hence, all quarks, charged and neutral leptons feel this force although we will concentrate on electrons and electron neutrinos for now.

5.5 The Weak Force Bosons

Firstly, let us look at the force bosons. There are three, namely the $W^{\pm}$ and the Z^0, where the W^+ and the W^- are antiparticles of each other and the Z (like the photon) is its own antiparticle. These are the equivalents of the eight gluons. However, the $W^{\pm}$ interactions have a critical difference from all the forces we have seen so far: they *change* the type of particle, e.g. an electron becomes an electron neutrino.

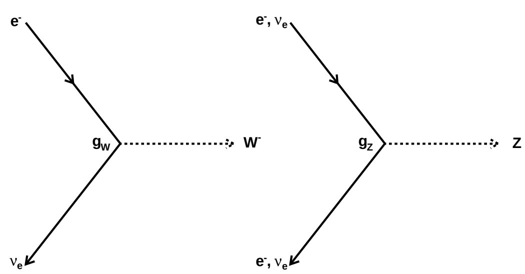


Figure 5.4: Vertices of the weak force bosons W and Z.

In fact, as the W carries off EM charge (which is always conserved), the charges on the incoming and outgoing particles must always differ by one unit. The change of the type of particle therefore allows the fundamental particles to decay.

In contrast, the Z is uncharged and acts much more like the photon and the gluons, in that it does not change the fermion type. Therefore, the Z vertex has the same particles going in and out, e.g. electron goes to electron or electron neutrino goes to electron neutrino, as shown above. Note that the weak charge for the Z is g_Z, not g_W; the weak force equivalent of charge is (apparently at least) different for the two bosons. For completeness, the μ and τ and their neutrinos have identical interactions to the electron and the electron neutrino. They all carry the same universal weak charge strength g_W (or g_Z).

As for gluons, the $W^{\pm}$ and Z carry weak charge (and indeed the $W^{\pm}$ clearly also carry EM charge). Hence, there are again self-interaction vertices as shown in Figure 5.5.

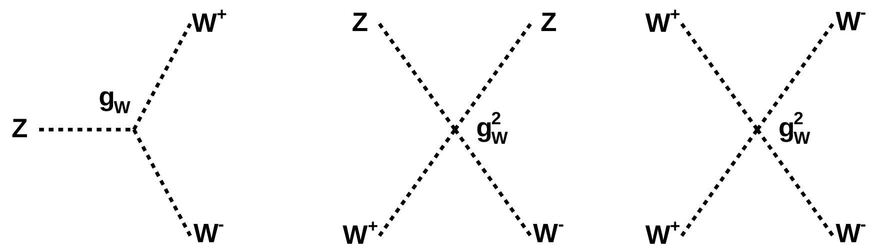


Figure 5.5: Weak force boson self interactions.

However, we see bare electrons so these cannot lead to confinement (or, it turns out, to asymptotic freedom) because the $W^\pm$ and Z bosons are very heavy, $m_W \approx$ 80 GeV/c^2 and $m_Z \approx 91$ GeV/c^2 and hence the force is very short ranged. (This restricted range prevents the field from remaining large at large distances which was the issue for QCD.) Thus, although self-interacting, the weak force is not confining.

Just as for QED and QCD, we can form a dimensionless constant for W interactions:

$$\alpha_W = \frac{g_W^2}{4\pi\hbar c}\,.$$

The value of this parameter was historically hard to measure. For interactions with energies well below $m_W c^2$, then only virtual Ws could be made. This meant they were always restricted to be internal in the relevant Feynman diagrams and begin and end at two vertices, e.g. for $\nu_e e$ scattering.

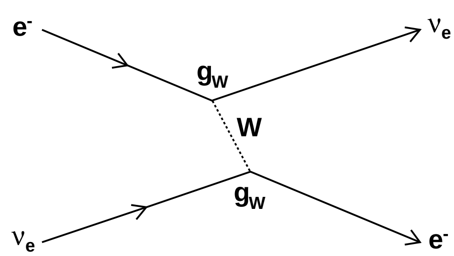


Figure 5.6: A virtual W exchanged in $\nu_e e$ scattering.

In the low energy limit (i.e. well below the W mass), the cross section term due to the W in these diagrams goes as $1/m_W^2$, so the amplitude was always proportional to g_W^2/m_W^2. Neither g_W nor m_W could be independently measured, so only the value of this combination was known. This was parametrised as the Fermi coupling constant $G_F = \sqrt{2}/8\,\hbar^3c^3(g_W^2/m_W^2)$, where $G_F/(\hbar c)^3 = 1.166 \times 10^{-5}$ GeV^{-2}. The weak force was labelled "weak" as this value is so small; at low energies, it gives decay rates and cross sections much smaller than for EM interactions, let alone strong ones.

This low-energy limit is effectively treating the vertices as being at a single point, i.e. an infinitely short range force as mediated by an infinitely massive boson or, equivalently, as a four-point coupling – see Figure 5.7. Indeed, it was Fermi's theory in terms of such a four-point coupling which originally gave rise to the constant G_F.

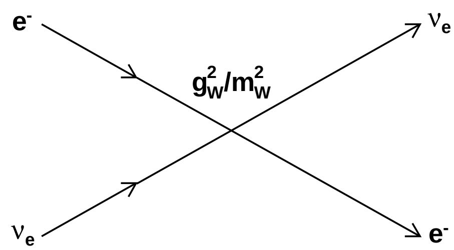


Figure 5.7: A four-fermion interaction involving a contact force with no range.

However, in the 1980s, real (as opposed to virtual) W particles were observed and their mass of 80 GeV/c^2 was measured. This allowed g_W (and hence α_W) to be calculated from G_F; it turns out that

$$\alpha_W \approx 0.034 \approx \frac{1}{29} > \alpha\,,$$

i.e. it is substantially larger than the fine structure constant! The weak force is really stronger than the EM force. For energies $\gg m_W c^2$, the mass can be neglected and the weak force acts just like the EM force, and its relative larger strength compared with the EM force becomes manifest.

5.6 Lepton Number Conservation

The above vertices have the $W^{\pm}$ coupling the e to the ν_e. As stated, we can add further generations for μ/ν_μ and τ/ν_τ but these will be independent; the $W^{\pm}$ does not couple e to ν_μ, for example, and so a diagram such as that in Figure 5.8 is forbidden as is any other combination except $e \leftrightarrow \nu_e$, $\mu \leftrightarrow \nu_\mu$ and $\tau \leftrightarrow \nu_\tau$. Each vertex has the same generation of lepton on both fermion lines so we can

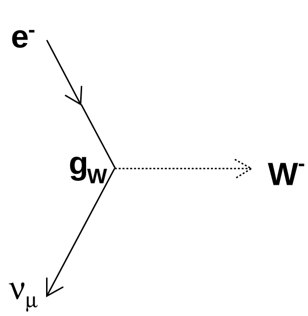


Figure 5.8: A forbidden weak coupling: the lepton flavour is not conserved.

introduce the concept of "lepton numbers" and obtain conserved quantities. We need three separate lepton numbers for each of e, μ and τ and their neutrinos. This number is the equivalent of particle type (or flavour) in the electromagnetic and strong interactions, as it is never changed by those forces.

Table 5.5: Lepton number values.

Lepton	L_e	L_μ	L_τ
e^-, ν_e	+1	0	0
e^+, $\overline{\nu}_e$	−1	0	0
μ^-, ν_μ	0	+1	0
μ^+, $\overline{\nu}_\mu$	0	−1	0
τ^-, ν_τ	0	0	+1
τ^+, $\overline{\nu}_\tau$	0	0	−1

Appendix I – non-examinable material

Hadronisation

Comparing opposite-sign electromagnetic charges to QCD charges, then the gluon lines self-interact and tend to collapse to a fixed 'radius'. This is often called

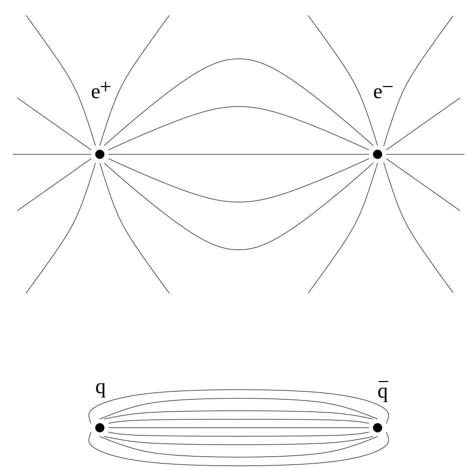


Figure 5.9: EM and QCD field lines between electrical and strong charges.

a 'string' (but has no connection with the string theory of extended quantum objects in 20 dimensions...). Because the string has a roughly constant width, ∼1 fm, it stores a constant amount of energy per unit length. This is why the force between the quarks does not die away with distance, but is constant at around 14 tonnes. Hence, as the quarks produced from the photon fly apart, they produce a long flux tube and would eventually grind to a halt and come flying back. However, there is another process which occurs before this happens. The initial kinetic energy of the quarks is going into energy stored in the flux tube. If a $q\overline{q}$ pair is produced within the tube, so removing a section of the tube, then the remaining energy stored in the field is reduced. The cut-off piece forms a meson (in this case) and the rest of the string can break up further.

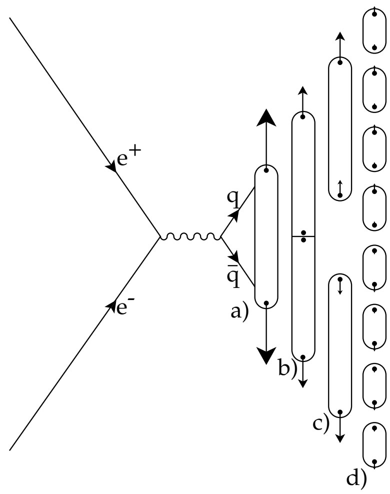


Figure 5.10: String fragmentation into mesons.

There is a subtlety here; if the $q\overline{q}$ are produced at a point, no field is removed and so no energy is liberated to make their masses. Hence, they need to be created a short distance apart to conserve energy. This requires them to tunnel

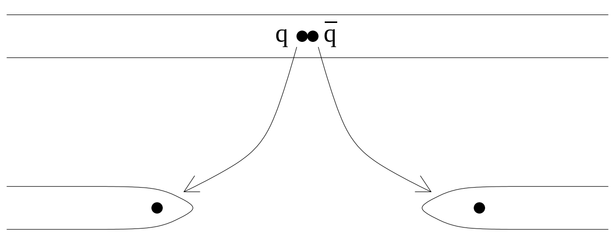


Figure 5.11: Tunnelling.

in order to be created, and this then depends strongly on the energy required. This tunnelling results in two effects:

- There is a big suppression for larger masses. The probability of $s\overline{s}$ is only 1/3 of that for $u\overline{u}$ or $d\overline{d}$ and the chance of getting $c\overline{c}$ (or $b\overline{b}$) is negligible. Hence, any c or b quarks observed must come from the primary photon production point. Furthermore, baryon production requires two $q\overline{q}$ pairs rather than one, so again it is suppressed, in this case by around 1/4.

- The quarks tend to be produced with small momentum, as this would otherwise increase the energy required. Because the force along the flux tube direction is large, they quickly pick up longitudinal momentum, but the transverse momentum remains small. The Heisenberg uncertainty principle says that because they are created within a flux tube of transverse size $\sim$1 fm, then they will have a spread in transverse momentum of $\sim$200 MeV/c.

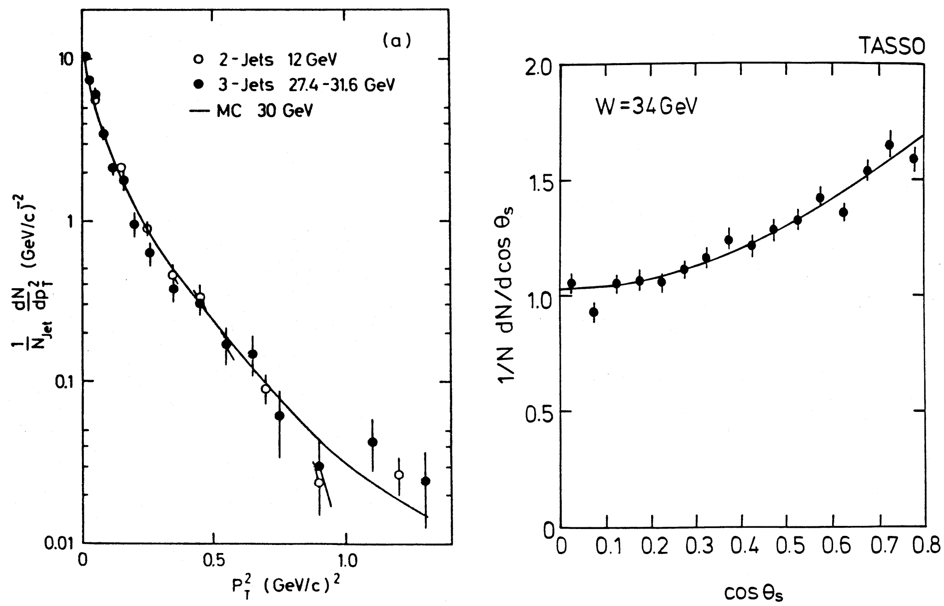


Figure 5.12: Jet transverse momentum (left) and angular cross section (right). From the TASSO experiment at DESY's PETRA collider, where the gluon was discovered in 3-jet events.

With large longitudinal momentum and limited transverse momentum to the original quark direction, then the jet direction is clearly the direction of the flux tube and hence very close to the original quark direction. Therefore, a measurement of the jet angular direction can be done, although q and $\overline{q}$ cannot be easily distinguished (only the magnitude of $|\cos\theta|$ can be measured). This agrees with the $1 + \cos^2\theta$ distribution we would expect for spin-1/2 production, as exemplified in Figure 5.12 (right).

NPP Lecture 6 – The Weak Interaction Part II – Helicity & Handedness, and Muon & Tau Decay

6.1 Introduction

In this lecture we examine helicity / handedness and hence why the weak force is called 'left-handed', and then we look at leptonic weak decays.

We have seen the Feynman diagram vertices for weak 'charged current' interactions (i.e. those involving $W^{\pm}$ exchange) and now we want to apply them. There are extra complications with quarks which we will discuss later, so first we will look at purely leptonic interactions and, in particular, muon and tau decay. In the absence of charged current weak interactions, these leptons would be stable, so their decays provide a good test of the weak theory.

6.2 Helicity and Handedness

Besides the bosons having mass, there is one more major difference between the weak force and the other two, and this is what leads to $\hat{P}$ and $\hat{C}$ violation. The weak vertex coupling strength actually depends on the helicity of the fermion. Recall that we defined helicity as the spin resolved along the momentum direction and so, for a spin-1/2 particle, the helicity can be $h = \pm 1/2$.

Figure 6.1: Helicity of a fermion.

Bizarrely, the W bosons only interact with leptons and quarks with $h = -1/2$ and not with ones with $h = +1/2$. Actually, strictly speaking, the exact quantity is something called handedness (which has no classical analogue) but in the limit of $v \to c$, handedness and helicity eigenstates coincide. Specifically, as $v \to c$, "right-handed" (RH) becomes helicity $+1/2$, and "left-handed" (LH) becomes helicity $-1/2$. The weak force is therefore often called a left-handed interaction.

Conversely, it turns out that antifermions only interact if they are right-handed, i.e. $h = +1/2$ in the relativistic limit. The actual amount of RH and LH in a helicity state is given by $(1 \pm v/c)/2$. Note, for very slow particles, with $v \ll c$, then they are approximately equally RH and LH. As most reactions we consider are highly relativistic, then using helicity instead of handedness is a good approximation. This behaviour of the fermions "losing their weak charge" if their spin flips over goes against all our intuition. However, it explains directly why $\hat{P}$ and $\hat{C}$ are violated.

Consider any weak reaction which is started by an incoming neutrino, which is almost massless and so has a velocity very close to $v = c$.

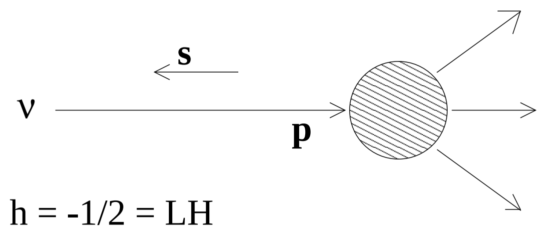


Figure 6.2: A neutrino reaction.

Under a parity operation, all momenta (being polar vectors) change sign but the spin (axial vector) does not. Hence, this system transforms as shown in Figure 6.3. This reaction does not happen as the neutrino is now right-handed; the rates of any weak reaction and its parity inverted equivalent are not equal. Indeed, one of the two rates is always actually zero in the relativistic limit.

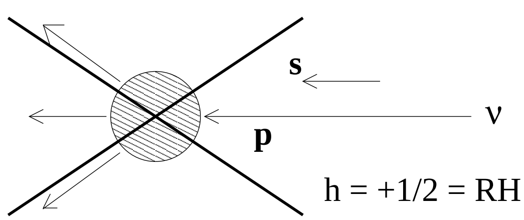


Figure 6.3: Not observed for neutrinos: $\hat{P}$ operation.

The same effect is also responsible for $\hat{C}$ violation; applying a $\hat{C}$ operation changes the neutrino to an antineutrino but does not effect the momentum or spin (Figure 6.4), which again does not happen as the antineutrino is left-handed.

Note that applying a $\hat{P}$ operation to this $\hat{C}$ inverted system (or equivalently a $\hat{C}$ operation to the $\hat{P}$ inverted system above) gives Figure 6.5. This works fine; it involves a right-handed antineutrino. In fact, this will have exactly the same rate as the original reaction. Hence, $\hat{P}$ and $\hat{C}$ are violated because of this weird handedness behaviour, but the combined $\hat{P}\hat{C}$ (or $\hat{C}\hat{P}$) operation is not. Figure 6.6 summarises which neutrinos are observed and which ones are not.

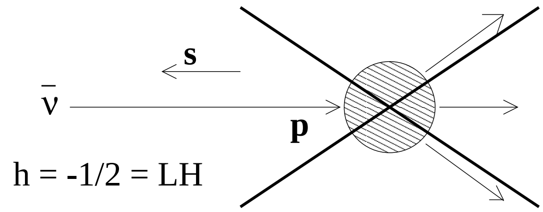


Figure 6.4: Not observed for neutrinos: $\hat{C}$ operation.

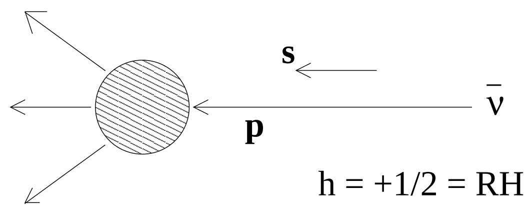


Figure 6.5: Combined $\hat{P}\hat{C}$ operations.

6.3 Muon Decay

Because the type of particle can change in a weak interaction, the heavier fundamental fermions can decay. The first example of this which we shall look at is the muon, mass 0.1057 GeV/c^2, which has only one significant decay mode:

$$\mu^- \to \nu_\mu e^- \overline{\nu}_e \, .$$

This couples μ to ν_μ and e to ν_e only and hence conserves each of the lepton numbers: the initial values are $L_\mu = +1$ and $L_e = 0$ and the final values are $L_\mu = +1$ and $L_e = (+1) + (-1) = 0$. Figure 6.7 shows this Feynman diagram.

6.3.1 Decay width and lifetime

This decay is structurally similar to the ones we have seen before. However, the final state is much more complicated because we have a three-body decay. In two-body final states, the overall energy and momentum conservation completely fixes both outgoing momentum magnitudes and only the overall θ and ϕ directions are left as free variables. Thus, the decay width is only a function of solid angle, $d\Gamma/d\Omega$, and the total width is given by integrating over the two angle variables.

It is straightforward to count the number of variables: each final state particle has three momentum components (from which the energy can be calculated using $E^2 = p^2c^2 + m^2c^4$). Hence, for a two-body final state, there are $2 \times 3 = 6$ variables needed to describe the final state. Total energy and momentum conservation give four constraints (one for energy and one for each momentum component) so the number of remaining free variables is $6 - 4 = 2$.

However, for a three-body decay, there are now $3 \times 3 = 9$ variables needed to describe the final state. The total energy and momentum conservation gives the same four constraints, so the number of free variables is $9 - 4 = 5$, rather

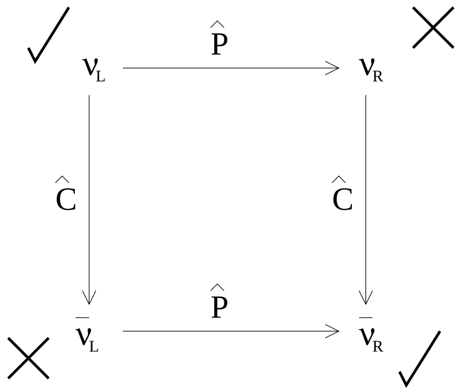


Figure 6.6: Summary of $\hat{P}$ and $\hat{C}$ operations.

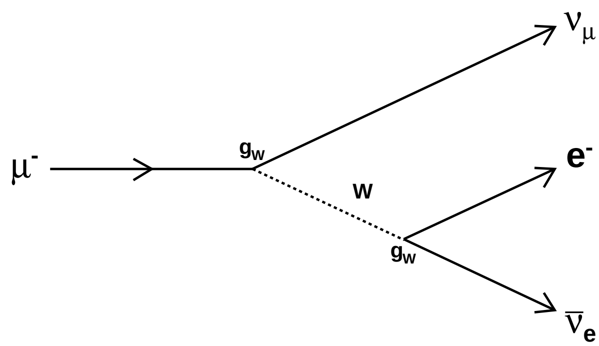


Figure 6.7: Feynman diagram for muon decay.

than 2. The decay rate will, in general, be a function of all five. Hence we need to integrate over five variables to get the total width. In particular, this tells us that the energy of the electrons from muon decay is not constrained to a single value (as would be the case for a two-body decay) but instead come out with a spread of energies, i.e. a continuous spectrum.
In fact, as the neutrinos are almost always unobserved in this decay, then often the only possible observables of the five are the three which give the electron energy and direction (θ and ϕ). Sometimes, e.g. when the muon is unpolarised, the only interesting variable *is* the electron energy. Hence, the usual calculation is to integrate over the other four and just express the width as a function of the electron energy. This is a laborious calculation, even ignoring the electron mass, but the result is:

$$\frac{d\Gamma}{dE_e} = 2\left[\frac{1}{8}\left(\frac{g_W}{M_W c^2}\right)^2\right]^2 \frac{2m_\mu^2 c^4 E_e^2}{(2\pi)^3}\left(1-\frac{4E_e}{3m_\mu c^2}\right) = \frac{G_F^2}{(\hbar c)^6}\frac{2m_\mu^2 c^4 E_e^2}{(2\pi)^3}\left(1-\frac{4E_e}{3m_\mu c^2}\right).$$

Note, this is in the low energy (i.e. four-fermion vertex) limit as we use G_F, which is perfectly valid here as all energies are $\sim$100 MeV or less (i.e. $\ll m_W$). This spectrum is called the Michel distribution and it peaks at the maximum allowed

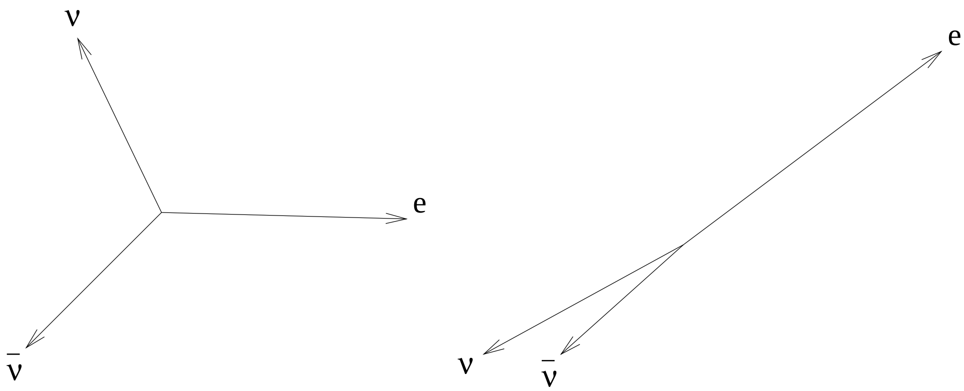


Figure 6.8: Illustrations of muon decay final states.

value of $E_e = m_\mu c^2/2$ (which you can show by differentiation). The predicted and measured spectra are shown in Figure 6.9.

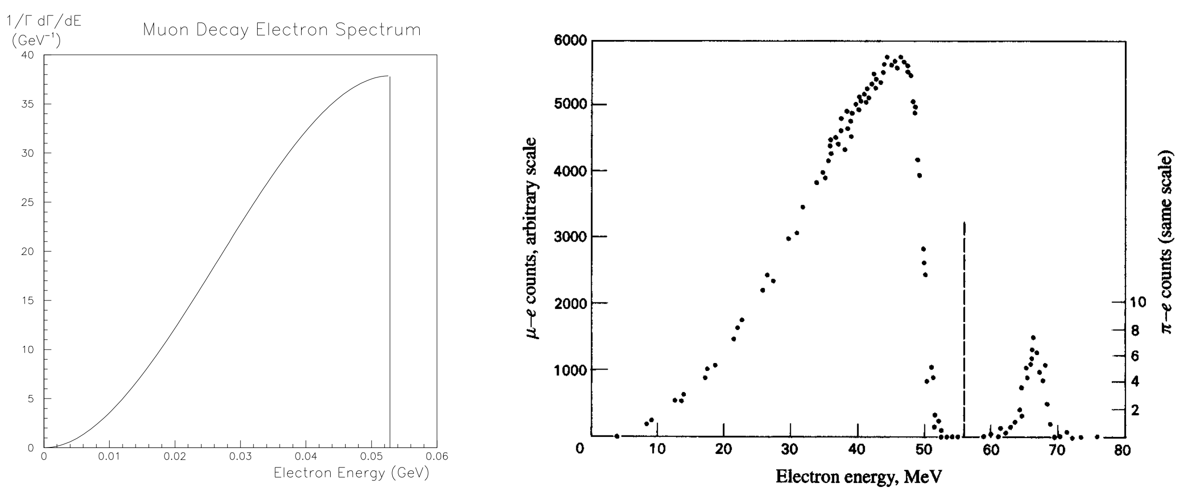


Figure 6.9: Calculated and observed Michel spectra for muon-decay electrons.

It is now straightforward to obtain the total width by integrating the above expression:

$$\begin{aligned}\Gamma &= \int_0^{E_{max}} \left(\frac{d\Gamma}{dE_e}\right) dE_e = \frac{G_F^2}{(\hbar c)^6} \frac{2m_\mu^2 c^4}{(2\pi)^3} \int_0^{m_\mu c^2/2} \left(E_e^2 - \frac{4E_e^3}{3m_\mu c^2}\right) dE_e \\ &= \frac{G_F^2}{(\hbar c)^6} \frac{(m_\mu c^2)^5}{192\pi^3} = 3.0 \times 10^{-19}\ \text{GeV}\,,\end{aligned}$$

which corresponds to a lifetime of $\tau_\mu = 2.2 \times 10^{-6}$ s, in good agreement with the experimental value of $(2.19703 \pm 0.0004) \times 10^{-6}$ s.

6.3.2 Decay length

We have seen a picture of $e^+e^- \to \mu^+\mu^-$ at Aleph, where the muons leave tracks in the detector. How come this is observed if they decay? The average time a muon lives for in its rest frame is $\langle t_0 \rangle = \tau_\mu$ so in a boosted frame, because of time dilation, this is lengthened to $\langle t \rangle = \gamma\tau_\mu$. In this time, it will go an average distance of $\langle d \rangle = v\langle t \rangle = \gamma\beta c\tau_\mu$. This is usually easiest to evaluate by noting $p = \gamma\beta mc$, so $\langle d \rangle = (p/mc)c\tau_\mu$. The combination $c\tau$ for any particle sets the scale of the decay lengths and for the muon, it is $c\tau_\mu = 660$ m. For the muons at Aleph, $p \sim 45$ GeV so $\langle d \rangle \sim 280$ km, which takes them all the way through the atmosphere, so they normally decay well after leaving the detector.

6.4 Tau Decay

The dependence of the decay width – and hence the lifetime – on the muon mass is striking: $\Gamma \propto m_\mu^5$. Where does this come from? The only masses in the problem are the muon and W masses and we know the amplitude is $\propto 1/M_W^2$ so the width $\propto 1/M_W^4$ Therefore, on dimensional grounds, the width (which has dimensions of energy) must be

$$\Gamma \propto \frac{m_\mu^4}{M_W^4} m_\mu c^2 \propto m_\mu^5 \,.$$

We can use this to look at the tau. One decay of the tau is

$$\tau^- \to \nu_\tau e^- \overline{\nu}_e \,,$$

which is exactly the same as for the muon, except the muon mass is replaced by the tau mass $m_\tau = 1.777$ GeV, i.e. $m_\tau/m_\mu = 16.8$. Hence, we would calculate the partial width for this decay mode to be:

$$\Gamma(\tau^- \to \nu_\tau e^- \overline{\nu}_e) = \frac{m_\tau^5}{m_\mu^5} \Gamma(\mu^- \to \nu_\mu e^- \overline{\nu}_e) = 4.0 \times 10^{-13} \text{ GeV} \,.$$

This is not the only decay mode for the tau, as the extra mass makes other modes accessible. There is a muon decay mode:

$$\tau^- \to \nu_\tau \mu^- \overline{\nu}_\mu \,,$$

which, ignoring the muon mass relative to the tau mass, will have the same rate as the electron decay. Since the tau is actually heavier than the pion, then there are also decays to hadrons, which obviously go via quarks as shown in Figure 6.10.

$$\tau^- \to \nu_\tau d\overline{u}$$

If we assume that asymptotic freedom holds then, from universality, the rate might be expected to be $e : \mu :$ hadrons $= 1 : 1 : 1$. However, this ignores colour, and just as the e^+e^- cross section gets enhanced by a factor of three, so does the

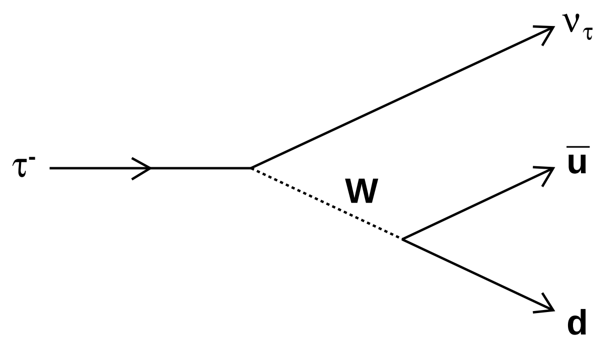


Figure 6.10: Feynman diagram for a semi-leptonic decay of the tau.

decay rate here. Hence, we might expect $e : \mu :$ hadrons $= 1 : 1 : 3$, i.e. 20% e, 20% μ and 60% hadrons. In fact, the measured rates are 17.8%, 17.4% and 64.8% so this is not a bad approximation. The total width is the sum of the partial widths and, in this approximation, it would be 5× the electron partial width:

$$\Gamma = 5\Gamma(\tau^- \to \nu_\tau e^- \overline{\nu}_e) = 2.0 \times 10^{-12} \text{ GeV}\,.$$

This gives $\tau_\tau = 3.2 \times 10^{-13}$ s, in reasonable agreement with the experimental value of $(2.906 \pm 0.011) \times 10^{-13}$ s.

How far does a tau go before decaying? Applying the above formula gives $c\tau_\tau = 87$ μm and for 45 GeV momentum at Aleph, then the average distance is 2.2 mm. Hence, all taus decay quite close to the beam collision point, well before they enter the tracking detectors; we do not usually see charged tracks due to taus themselves, only their decay products. So how can we measure the tau lifetime? With accurate tracking using a silicon vertex detector, the actual tau decay positions can be reconstructed.

NPP Lecture 7 – The Weak Interaction Part III – Neutrinos

7.1 Introduction

Neutrinos are special in many ways; they are much lighter than any other fermions (in the SM they are indeed assumed to be massless, although we now believe that to be false), they only feel the weak force (and so are very hard to detect) and they do not decay (a property shared only with the electron and maybe the proton).

Within the SM, being massless means that helicity and handedness correspond exactly for neutrinos — only $h = -1/2$ neutrinos (and $h = +1/2$ antineutrinos) can interact. The other helicities do not feel the weak force or any other force (except gravity), so there is effectively no way to know if they exist or not.

7.2 Neutrino Mixing

Over the last decade or so several experiments detected a phenomenon known as "neutrino mixing" (or "neutrino oscillations") which implies that the SM is exaggerating in saying that neutrinos are massless. This could be an important hint to the physics that lies beyond the SM, one of very few, and so it is an extremely hot research area right now. Mixing requires that the neutrinos have non-zero mass, albeit probably very small ≤ 0.1 eV/c^2, i.e. $10^{-7} m_e$. Such a small mass has not been measured directly so the only evidence for this is from mixing.

Putting massive neutrinos into the SM is, in fact, quite easy as all the other fermions have mass so the equations are well known. This would not be such a big deal; even the fact that the helicity states are not quite handedness states has no measurable effect, as the masses are so small that neutrinos travel always at $v \approx c$ anyway. However, mixing also requires a new interaction between the neutrinos which allows e.g. an electron neutrino to spontaneously change into a muon neutrino. This clearly violates lepton number conservation, for both L_e and L_μ in this example, and so requires us to have a new type of vertex. We do not know what this is; it must not allow processes such as $e^- \leftrightarrow \mu^-$ mixing or muon decays like $\mu^- \to e^- \gamma$ which have never been seen. It must be a very very small effect or it would have been seen previously, so what we have already learnt is still valid at some very good approximation.

7.3 Two-neutrino Mixing

The physics of mixing is fundamentally an application of basic quantum mechanics. It is a general QM result that a wavefunction which is not an energy eigenstate at $t = 0$ can always be decomposed into the complete set of energy eigenstates u_i

$$\psi(t{=}0) = \sum_i \alpha_i u_i \,.$$

The time dependence of the wavefunction is then given by

$$\psi(t) = \sum_i \alpha_i u_i e^{-iE_i t/\hbar} \,.$$

The general case of mixing involves all three neutrinos. This is somewhat complicated so we will consider the two neutrino case, e.g. only consider electron and muon neutrinos. The mixing phenomenon arises because the flavour states, ν_e and ν_μ, are not the energy eigenstates, which we shall label as ν_1 and ν_2. With only two states, the α_i can be most simply expressed as sines and cosines, so we can write:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \times \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} ,$$

in general, where θ is an arbitrary parameter to be determined. So, to be explicit,

$$\nu_e = \cos\theta\,\nu_1 + \sin\theta\,\nu_2, \qquad \nu_\mu = -\sin\theta\,\nu_1 + \cos\theta\,\nu_2 \,.$$

Note, these can be inverted to give

$$\nu_1 = \cos\theta\,\nu_e - \sin\theta\,\nu_\mu, \qquad \nu_2 = \sin\theta\,\nu_e + \cos\theta\,\nu_\mu \,.$$

When produced in a weak interaction, the neutrinos are put into the flavour states as that is what the weak vertex couples to. However, after that, they follow the above time dependence. We will work in the neutrino rest frame, where $E_i = m_i c^2$. If a state is initially created as ν_e at $t = 0$,

$$\psi(t{=}0) = \nu_e = \cos\theta\nu_1 + \sin\theta\nu_2 \,,$$

then, at a later time, it will be

$$\psi(t) = \cos\theta\,\nu_1\, e^{-im_1c^2t/\hbar} + \sin\theta\,\nu_2\, e^{-im_2c^2t/\hbar} \,.$$

The neutrinos can only be detected when they interact with something and they will do so through the weak interaction. This couples to the flavour states, so we need to know how much of each is present. Decomposing back to the ν_e and ν_μ states, then this is

$$\begin{aligned} \psi(t) &= \cos\theta(\cos\theta\,\nu_e - \sin\theta\,\nu_\mu)e^{-im_1c^2t/\hbar} + \sin\theta(\sin\theta\,\nu_e + \cos\theta\,\nu_\mu)e^{-im_2c^2t/\hbar} \\ &= (\cos^2\theta e^{-im_1c^2t/\hbar} + \sin^2\theta e^{-im_2c^2t/\hbar})\nu_e + \sin\theta\cos\theta(e^{-im_2c^2t/\hbar} - e^{-im_1c^2t/\hbar})\nu_\mu \,. \end{aligned}$$

The probability of observing the state as a ν_e is, therefore:

$$\begin{aligned} P_e &= |\cos^2\theta\, e^{-im_1c^2t/\hbar} + \sin^2\theta\, e^{-im_2c^2t/\hbar}|^2 \\ &= \cos^4\theta + \sin^4\theta + \sin^2\theta\cos^2\theta(e^{i\Delta mc^2t/\hbar} + e^{-i\Delta mc^2t/\hbar}) \\ &= \cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta\cos(\Delta mc^2t/\hbar)\,, \end{aligned}$$

where $\Delta m = m_2 - m_1$. Since $2\sin\theta\cos\theta = \sin 2\theta$, then this can be written as:

$$\begin{aligned} P_e &= \cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta - 2\sin^2\theta\cos^2\theta + 2\sin^2\theta\cos^2\theta\cos(\Delta mc^2t/\hbar) \\ &= (\cos^2\theta + \sin^2\theta)^2 - 2\sin^2\theta\cos^2\theta[1 - \cos(\Delta mc^2t/\hbar)] \\ &= 1 - \frac{1}{2}\sin^2 2\theta[(1 - \cos(\Delta mc^2t/\hbar)]\,. \end{aligned}$$

Similarly, the probability of observing the state as a ν_μ is:

$$\begin{aligned} P_\mu &= |\sin\theta\cos\theta(e^{-im_2c^2t/\hbar} - e^{-im_1c^2t/\hbar})|^2 \\ &= \sin^2\theta\cos^2\theta(1 + 1 - e^{i\Delta mc^2t/\hbar} - e^{-i\Delta mc^2t/\hbar}) \\ &= 2\sin^2\theta\cos^2\theta[1 - \cos(\Delta mc^2t/\hbar)] \\ &= \frac{1}{2}\sin^2 2\theta[1 - \cos(\Delta mc^2t/\hbar)]\,, \end{aligned}$$

so, clearly, the sum of the probabilities is one, as required. These probabilities are shown in Figure 7.1 for the case of $\theta = 30°$, i.e. $\sin^2 2\theta = 3/4$.

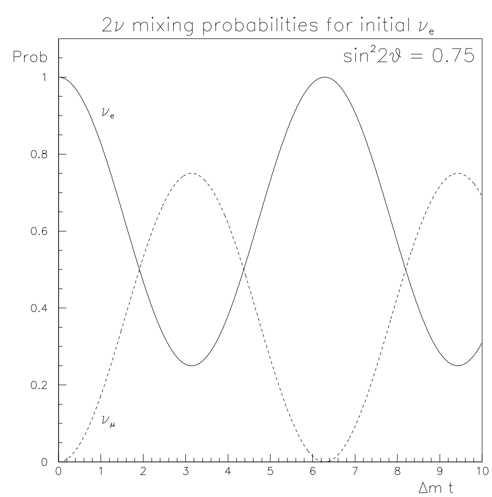


Figure 7.1: Probabilities P_e and P_μ assuming $\theta = 30°$ for an initial ν_e beam.

Physically, all that is happening is that the different masses which make up the two parts of the wavefunction oscillate with slightly different frequencies, so they go in and out of phase with each other with time. This is exactly equivalent to the phenomenon of beats for two close frequencies in acoustics. It requires a mass difference Δm which is non-zero (or the parts of the wavefunction will always stay in phase) so at least one of the neutrinos must have mass. It also requires an interaction which allows electron and muon neutrinos to change into each other and so gives a non-zero value of the mixing angle θ. For three neutrinos, there will be three angles and three mass differences, but the physical principles are the same.

The first evidence for neutrino mixing came from two sources; neither is from neutrinos produced in the laboratory. This is because Δm is small so neutrino mixing only happens slowly. Hence, the neutrinos need time to change and so have to travel a very long distance. This is hard to arrange within a laboratory. The two sources used for the first (and indeed some of the current) experiments are neutrinos produced in the atmosphere and neutrinos produced in the Sun.

7.4 Atmospheric Neutrinos

The upper layers of the atmosphere, at ∼50 km altitude, are continuously bombarded by cosmic rays from space — mostly protons, but also heavier nuclei and other particles. These collide with the nuclei in the atmosphere and produce hadronic interactions. The resulting jets create many hadrons, with those most copiously produced being the lightest, i.e. pions. The neutral pions decay to photons but, as we will see in the next lecture, the charged pions mainly decay as $\pi^- \rightarrow \mu^-\overline{\nu}_\mu$ (and the equivalent for the π^+). We know the muons themselves then decay as $\mu^- \rightarrow \nu_\mu e^-\overline{\nu}_e$. Hence, each pion should result in an electron (which is absorbed in the atmosphere), two muon-type (anti)neutrinos and one electron-type (anti)neutrino.

The neutrinos travel through the atmosphere and can be detected on Earth by their reactions on nuclei or atomic electrons, producing either electrons or muons as shown in the Feynman diagrams in Figure 7.2.

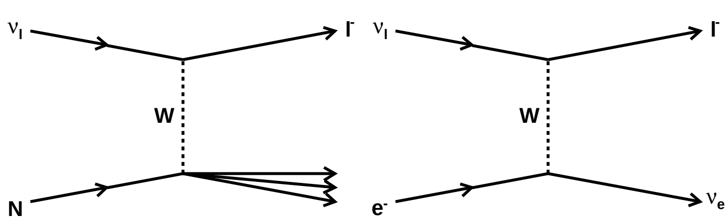


Figure 7.2: Interactions of atmospheric neutrinos with nuclei (left) and atomic electrons (right). The neutrino flavour is determined from the flavour of the outgoing lepton (l): an electron or a muon.

By identifying the outgoing particle in these reactions as an electron or a muon, the type of neutrino can be deduced. This was first done by the Super-Kamiokande experiment, which used a huge water-filled cavern to detect the light given off by the electron or muon as it travelled through the water.[7] The detector

[7]The lepton travels initially faster than the speed of light in water, leading to the emission

had to be so big as the neutrinos only have moderate energies ($\sim$1 GeV) and so the weak cross sections are very small $\sim 10^{-46}$ m^2, or 10^{-18} barns. In fact, most neutrinos can go through the whole Earth without interacting; only one in 10^{10} does so. Super-Kamiokande used this to see the mixing effect. For neutrinos coming directly from above the detector, i.e. travelling about 50 km, they saw rates which implied an original ratio of neutrino flavours of 2:1, as expected. However, for those coming from below, which have travelled around 12,000 km from the other side of the Earth, they saw a ratio of around 1:1.

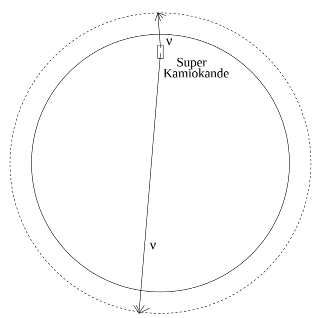


Figure 7.3: Detection of atmospheric neutrinos by Super-Kamiokande.

For intermediate directions, they reported values between these two. This is because 50 km is too short for neutrinos to have mixed significantly, while after 12,000 km there is a good probability that they will have changed into a different type, so the ratio becomes washed out. The Super-Kamiokande group obtained values of $|\Delta m|$ around 0.05 eV/c^2 and $\sin^2 2\theta = 1$, i.e. its maximum value.

7.5 Solar Neutrinos

As we shall see towards the end of the course, electron neutrinos with low energies are produced in the first stage of the nuclear reactions which power the Sun, through $p + p \to d + e^+ + \nu_e$, where d is the deuteron, a pn heavy hydrogen nucleus. The energy available to the protons is so small that this reaction never produces muons (let alone taus) so the neutrino source is purely ν_e. The observed power output of the Sun gives a very accurate measure of the reaction rate and hence of the number of neutrinos being emitted.

After travelling 150 million km, the neutrinos were detected by the SNO detector on Earth. The electron neutrinos can undergo similar W exchange (called 'charged current', or 'CC') reactions to the atmospheric case. However, they are low energy so any muon (or tau) neutrinos created through mixing do not have enough energy to make a muon (or tau). They can only be detected by their Z reactions (neutral current, NC, or electron scattering, ES), which do not distinguish type.

---

of Cherenkov radiation; the characteristics of this cone of light allow the separation of electron and muon interactions.

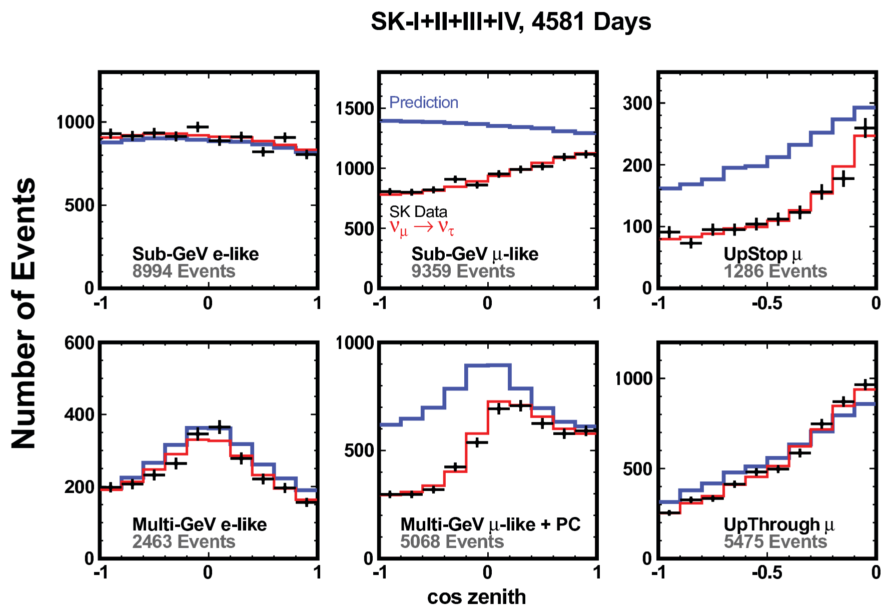


Figure 7.4: Atmospheric neutrino zenith angle distribution at Super-Kamiokande. The prediction without neutrino oscillations is shown in blue and the prediction including oscillations at the analysis best fit point is shown in red.

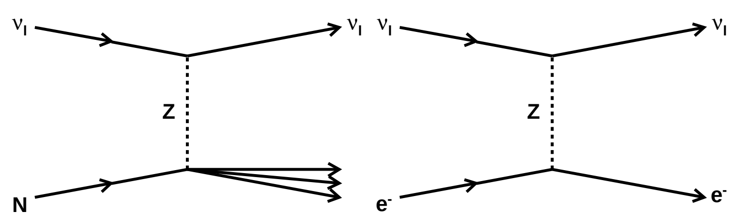


Figure 7.5: Detection of solar neutrinos via neutral current interactions with nuclei (left) or elastic scattering with atomic electrons (right).

This allows the electron neutrino rate and the total neutrino rate to be measured separately; any difference indicates that there must be some ν_μ or ν_τ being created. The measurements revealed an electron neutrino rate of about 1/3 of the total rate (see Figure 7.6), showing mixing must have occurred.

7.6 CP Violation in the Lepton Sector

As we shall see later, the fact that neutrinos have mass means that CP violation is possible in the lepton sector. This could help explain the matter-antimatter asymmetry in the universe, and is being investigated by the latest, and indeed the next generation, experiments.

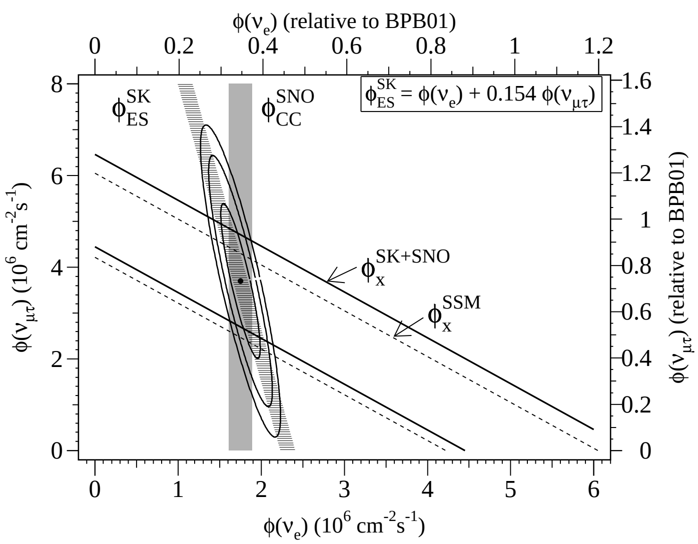


Figure 7.6: Neutrino oscillation results from the SNO experiment.

NPP Lecture 8 – The Weak Interaction Part IV – Meson Decays, the CKM Matrix and Electroweak Unification

8.1 Introduction

We complete our brief review of the weak force today by introducing a number of new ideas. Initially, we will look at the decay of the charged mesons introduced in lecture 4, and explore which new characteristics of the weak force this may reveal. This will naturally lead onto the so-called Cabibbo-Kobayashi-Maskawa (CKM) matrix. Finally, we introduce the idea of 'electroweak' force, the unification of the weak and electromagnetic forces, and we will mention the Higgs boson in this context.

8.2 Charged Pion Decay

The main decays of the charged pion are leptonic (i.e. they involve only leptons in the final state):

$$\pi^- \to e^-\overline{\nu}_e\,, \qquad \pi^- \to \mu^-\overline{\nu}_\mu\,.$$

In the following, the charge-conjugated decays — in this instance $\pi^+ \to e^+\nu_e$ — are also implied. This decay requires the quarks, e.g. $d\overline{u}$ in the case of a π^-, to have annihilated each other as there are (by definition for a leptonic decay) no quarks left. This therefore means that u and d quarks must have a weak interaction vertex just like the e and ν_e.

Note the d quark charge is $-e/3$ and the u quark charge is $+2e/3$, so they differ by e, just right to emit a W^-. Also, the u and d quarks do not have the same mass, just like the e and ν_e. Note also that this interaction does not conserve quark flavour: the weak interaction allows us to change the types of quarks. The Feynman diagram for pion decay is then relatively straightforward, as shown in Figure 8.2. The ratio of the number of decays to electrons versus muons also demonstrates a property of the weak interaction. We have stated previously that the muon is just a heavier version of the electron but is otherwise identical. Specifically, it has identical charges and, here, this means it has the same g_W weak charge. In fact, we saw the result of this in tau decays, where

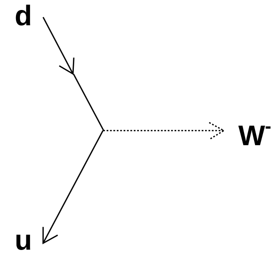


Figure 8.1: W$^-$-ud vertex, ignoring quark mixing.

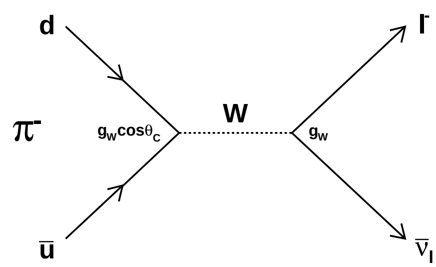


Figure 8.2: Feynman diagram for π^- leptonic decays.

the electron and muon branching fractions were practically equal (and were each one third of the hadron rate). This means that the electron and muon might be expected to have equal amplitudes in pion decay, and hence have equal rates here too. However, the muon mass (105.7 MeV/c^2) is not that much lower than the pion mass (139.6 MeV/c^2), so there is somewhat limited energy available for the muon decay and it would therefore be expected that the electron rate may be higher than the muon rate. A calculation shows this would lead to a difference in rates by a factor of around three. However, the muon rate is not only bigger than the electron rate, but it is around 8,000 times bigger! This is impossible to explain without considering the way the weak force interaction works.

We know the weak force only couples to LH fermions and RH antifermions. Also, in the relativistic limit, these correspond to $h = -1/2$ and $h = +1/2$, respectively, and when not at that limit they have $(1 \pm v/c)/2$ of each state. The antineutrino is almost massless compared to the charged leptons, so it must be at the relativistic limit to a very good approximation — see Figure 8.3. Hence, as it is RH, it must have $h = +1/2$. The pion has spin 0, so to balance angular momentum (recall $\mathbf{L}$=0), the charged lepton must also have $h = +1/2$.
If the charged lepton were also at the relativistic limit, this would not be allowed as it would be a RH fermion and so the decay could not occur. However, the charged leptons are not quite at $v = c$ and so when $h = +1/2$, they have a little LH state included, namely $(1-v/c)/2$. Hence, there is a suppression factor which gets larger as the velocity gets closer to c. What is the velocity of the charged

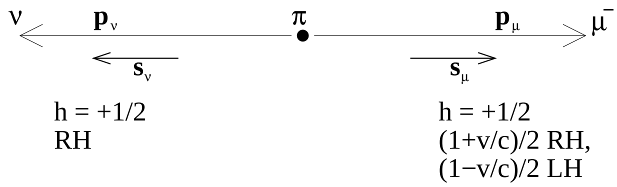


Figure 8.3: Helicity and handedness in charged pion decay (note that an antineutrino ($\overline{\nu}_\mu$) is emitted in this instance).

leptons? The standard two-body decay formula gives:

$$E_l = \frac{m_\pi^2 + m_l^2 - m_\nu^2}{2m_\pi} c^2 \, ,$$

which, in the approximation that the neutrinos are massless, is

$$E_l = \frac{m_\pi^2 + m_l^2}{2m_\pi} c^2 \, .$$

The momentum is then given by

$$p_l c = \sqrt{E_l^2 - m_l^2 c^4} = \frac{m_\pi^2 - m_l^2}{2m_\pi} c^2 \, ,$$

so that the velocity is

$$\frac{v_l}{c} = \beta_l = \frac{p_l c}{E_l} = \frac{m_\pi^2 - m_l^2}{m_\pi^2 + m_l^2} \, .$$

Putting in the relevant masses this evaluates to $\beta_e = 0.99997$ and $\beta_\mu = 0.272$, which is a ratio of $(1 - \beta_\mu)/(1 - \beta_e) = 28,000$. This explains the major part of the relative branching fractions for muons and electrons. Because of the small mass difference between the pion and the muon, the restricted energy available to the muon means it has a relatively low velocity. However, this means there is a lot more LH state mixed in and hence the muon rate is much higher.

The charged pion mass, $m_{\pi^\pm} = 139.6$ MeV/c^2, is slightly higher than that of the neutral pion, $m_{\pi^0} = 135.0$ MeV/c^2, so that $m_{\pi^\pm} > m_{\pi^0} + m_e$. So, a semileptonic decay is also possible, namely $\pi^- \to \pi^0 e^- \overline{\nu}_e$ (see Figure 8.4). However, as the energy available is so limited, the branching fraction is minute.

8.3 Charged Kaon Decay

The mere fact that kaons decay tells us something about the weak force and quarks which is new. The u and d form the first generation of quarks, like the e and ν_e form the first generation of leptons, and the weak force allows interactions between them. Hence, since c and s are the next generation, like μ and ν_μ, we would expect weak interactions between these two quarks also. However, c quarks are heavier than s quarks, so if s could only be turned to c, then the charged

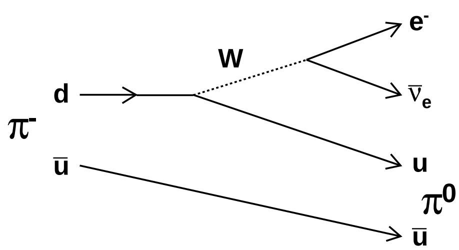


Figure 8.4: Semileptonic decay of the π^-

kaon, which is $s\overline{u}$ (not $s\overline{c}$) could not decay: all the hadrons with c are much heavier than the kaon. However, the kaon has leptonic decay modes just like the pion:

$$K^- \to e^- \overline{\nu}_e \,, \qquad K^- \to \mu^- \overline{\nu}_\mu \,,$$

and indeed its decay rate to muons is around 40,000 times higher than that to electrons, which is what we would expect from using the kaon mass rather than the pion mass in the previous calculation. Hence, we have to believe the s and $\overline{u}$ quarks are annihilating, so there must be a weak interaction vertex like that in Figure 8.5.

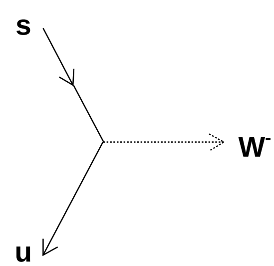


Figure 8.5: W^--us vertex, ignoring quark mixing.

This means that the decay diagram is just like that of the pion, as shown in Figure 8.6.

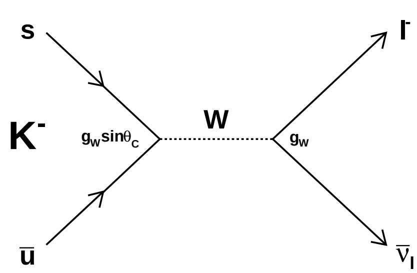


Figure 8.6: Leptonic decay of the charged kaon.

The implication of this decay is that there must be a cross-generational coupling for quarks, even though one is absent for leptons. Given that this exists, the other decays of the kaon are easy to understand; the semileptonic (leptons and hadrons produced) and hadronic decays (only hadrons produced) are shown in Figure 8.7.

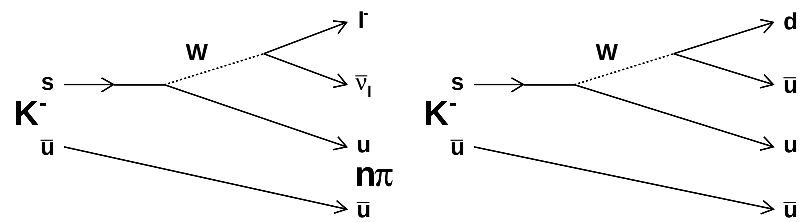


Figure 8.7: Semileptonic (left) and hadronic (right) decays of the charged kaon.

8.4 Charm Meson Decays: the CKM Matrix

We have found that there is a cross-generational coupling for quarks, where the u quark can connect to either the d or the s. The obvious question is whether the c quark can do the same thing, which would mean each could connect to the other two of the different EM charge. Looking at charm meson decays shows that this is indeed possible. The charmed D^+ meson, $c\overline{d}$, has been seen to decay (amongst other decays modes) semileptonically to both $l^+\nu_l\overline{K}^0$ and also $l^+\nu_l\pi^0$, given by diagrams like those in Figure 8.8. Hence, for quarks, any $+2e/3$ quark (u or c)

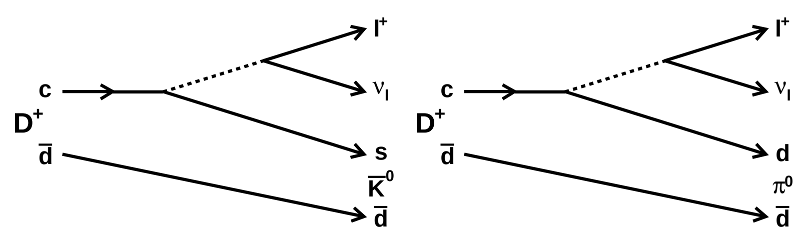


Figure 8.8: Semileptonic decays of the D^+ charmed mesons.

can connect to any $-e/3$ quark (d or s) in a weak interaction. The relative weak force strengths are usually described by the Cabibbo matrix V_{ij}

$$\begin{pmatrix} V_{ud} & V_{us} \\ V_{cd} & V_{cs} \end{pmatrix} = \begin{pmatrix} 0.975 & 0.225 \\ -0.225 & 0.975 \end{pmatrix} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix},$$

where $\theta_C \approx 13^\circ$ is called the "Cabibbo angle". Hence, it is close to, but not exactly, a unit matrix. This is in fact a *rotation* matrix between the flavour eigenstates d and s and the actual weak eigenstates of the quarks ("not quite d"

and "not quite s"): it means that every ud vertex has a multiplicative factor of $V_{ud} = \cos\theta_C$, etc.

Looking at the third generation of b and t quarks, this matrix has to be extended as we might expect, so any of u, c and t can connect to any of d, s and b. The resulting 3×3 Cabibbo-Kobayashi-Maskawa (CKM) matrix is:

$$\begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} \approx \begin{pmatrix} 0.975 & 0.225 & 0.004 \\ 0.225 & 0.975 & 0.040 \\ 0.009 & 0.040 & 0.999 \end{pmatrix}$$

The CKM has a very important property. While the general 2×2 Cabibbo matrix only requires one parameter and is real-valued, the general 3×3 CKM matrix needs four, of which one results in the matrix being complex. This is critical for our existence: it turns out that only by having a complex matrix for these connections can CP symmetry actually be violated. This must happen to obtain more matter than antimatter in the universe and hence allow us to exist. Therefore, with only two generations, the Cabibbo matrix would be real and no matter-antimatter difference would be allowed. Three generations is the minimum needed to allow us to exist. However, no one really knows why there *are* three generations; it is an experimental fact, and not a SM requirement. CP violation was first observed in the kaon system (mesons containing strange quarks) in 1964. Since, it has been seen in the B-system (mesons with b-quarks) with the first observation in 2001. There is some evidence in the charm system, but further data are needed. In all cases so far the size of the effect seen is in good agreement with the SM prediction, but far less than that needed to explain the observed matter-antimatter asymmetry in the universe, and hence there is much effort in looking for alternative sources e.g. in the lepton sector or in extensions of the SM. And, indeed, looking for CP violation is an excellent way of probing new physics.

8.5 W and Z Bosons and EW Unification

We have seen the weak interaction vertices for W and Z bosons from various decays, where the W and Z are virtual, but given a collider with sufficient energy then real W and Z bosons can be produced. And, naturally, it was through the study of such decays that the theory was confirmed, and indeed that the concept of electroweak (EW) unification, namely the unification of the electromagnetic and weak forces, was established. A key aspect of the latter is the recently discovered Higgs boson, which is responsible for the mass of the W, Z and all fermions.

Real Ws were first produced in $p\bar{p}$ collisions in the 1980s, through collisions of the quarks and antiquarks in protons, and were only identified in the leptonic decay modes. Detailed studies were carried out in the second half of the 1990s when the LEP collider ran at centre-of-mass energies above $2m_W c^2$ and so could produce real W^+W^- pairs. At LEP the much cleaner experimental environment allowed all the W decay modes to be detected. We have seen that the allowed vertices for a W are to a charged lepton and the associated (anti)neutrino or to a charge $2e/3$ quark (i.e. u or c but not t as it is too heavy) and a charge $-e/3$ quark (i.e. d, s or b) — see Figure 8.9. Observations were in good agreement with predictions.

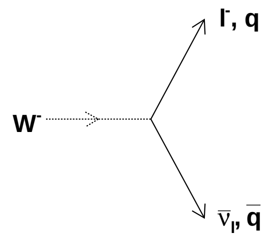


Figure 8.9: Decays of the W^- boson.

The LEP collider spent the first half of the 90s studying the reaction

$$e^+e^- \to Z \to f\overline{f}$$

at centre of mass energies close to m_Z. As a result, the Z sector of the weak interactions is much more precisely measured than the W sector. At these energies the Z production cross section is enormously enhanced over the competing photon cross section which dominates at lower energies, by a factor of $\sim$1,000. The Feynman diagram for these reactions is shown in Figure 8.10. The shape of

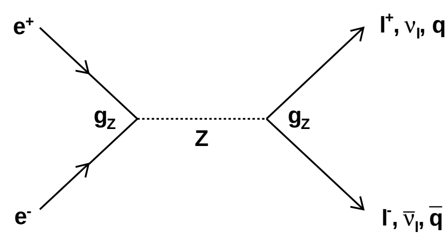


Figure 8.10: Electron-positron annihilation into quark or lepton pairs via Z bosons.

the cross section is sensitive to both M_Z and Γ_Z and these have been measured very accurately from this shape at LEP. From this we can even deduce that there are only three types of light neutrino — a topic for next year.

The SM makes precise predictions about the couplings of the fermions to the Z and these can also be accurately measured. Unlike for the W, the Z couplings are different for each type of different fermion (e, ν_e, u and d) and the overall Z charge g_Z is a little different from g_W. Within the SM, this is understood because the Z and the photon are actually mixtures of more fundamental particles, a photon-like particle and a neutral equivalent of the $W^\pm$. Because these two fundamental particles become mixed, the bosons we actually observe, the Z and the photon, have characteristics of both. This is why the Z is not quite the neutral equivalent of the W (e.g. their masses differ a little). This mixture is called the "electroweak" force and it was important to verify the details of this theory. The most sensitive measure is the angular distribution of the e^+e^- annihilation cross section, which has the form

$$\frac{d\sigma}{d(cos\theta)} \propto 1 + \cos^2\theta - C_w \cos\theta\,,$$

where the $\cos\theta$ term leads to a forward-backward asymmetry of the charges produced. This is the asymmetry mentioned briefly in one of your problem sheets, and is illustrated in Figure 8.11. Measuring the asymmetry for a particular final state allows the C_w parameter for that fermion to be determined. An asymmetry is useful experimentally as efficiencies cancel in the ratio so the effects of their uncertainties are reduced. All measurements were consistent with the electroweak theory.

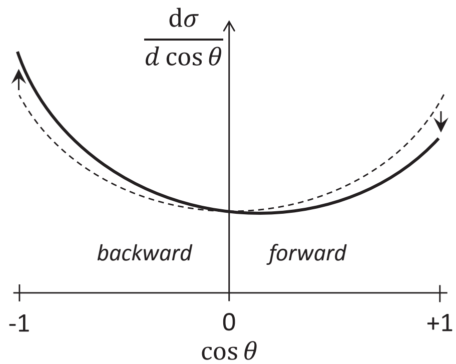


Figure 8.11: Differential cross section for e^+e^- annihilation into charged particles.

8.6 Higgs

The final part of the SM is the Higgs boson. Until 2012 this was a purely theoretical construct. As mentioned in passing before, the SM is built on symmetries, and in particular on the principle of "gauge invariance" — this is a generalisation of the fact that there is no absolute zero of the EM potential. Introducing mass terms for particles directly into the mathematical formulation of the various forces breaks this symmetry. Thus, the Higgs mechanism was added to the SM to explain the fact that the W and Z have non-zero mass (unlike the other force bosons, the photon and gluons), and in fact to also give mass to all the fermions. It was first postulated in the 1960s as part of the electroweak theory introduced above. This 'unification' is discussed in the later particle physics courses.

The Higgs is able to do this as it is a field which permeates all space at all times. Its ground state is unusual, in that the field is non-zero at the minimum of the potential energy. This means that even when there are no excited quanta of the field, i.e. no Higgs particles, there is still a field present which can give effects. In this theory, all particles are intrinsically massless as required by gauge invariance. The actual observed mass then arises as the non-zero Higgs field acts like drag and slows down the (otherwise massless) particles, making them appear like they have mass.

To enable this to happen in just the right way, the "charge" of each particle for interacting with the Higgs must be proportional to its mass. Clearly, without having observed the Higgs boson and checking it has all the properties we would expect from the theory, we cannot claim to have fully understood the fundamental forces. Hence, this search was the hottest topic in particle physics research for several years. First hints were seen at the Tevatron in the US and in particular observation was made at the LHC at CERN by the ATLAS and CMS Collaborations in July 2012. Since then, in fact very rapidly, a wealth of measurements of its characteristics e.g. coupling, spin, etc, indicate that, within the current uncertainties, what we have seen is the SM Higgs boson. Rather than the end of a journey, this discovery should be seen as the beginning — as a portal to beyond-the-SM (BSM) physics. The nature of the Higgs is defined by the BSM physics — for example, there may well be links to dark matter. It is the first time a scalar (spin-0) field has been observed, many believe this must link to the inflaton and inflation (i.e. the theory of the big bang). There are increasing connections between particle physics, astrophysics and cosmology.

PART II – NUCLEAR PHYSICS

NPP Lecture 9 – The Nuclear Force and the Semi-Empirical Mass Formula

9.1 Introduction

Protons and neutrons can actually bind together through the strong force tightly enough to form bound states. These bound states form the nuclei of atoms, and the sum of the proton charges is what holds the atomic electrons in their orbits.

In this lecture we examine the nuclear force, and then go on to derive the Semi-Empirical Mass Formula (SEMF). Understanding what makes certain nuclei stable is very important as we shall see. We will try to justify quantitatively the binding energy, BE, which relates to the nuclear mass, m_N:

$$m_{\mathrm{N}} = Zm_p + Nm_n - \frac{BE}{c^2}\,.$$

For a nucleus to be stable, then $BE > 0$, i.e. the mass of the stable nucleus is lower than the sum of the masses of its constituents.

9.2 Nuclei

A nucleus is described by the number of protons, Z, and the number of neutrons, N. The total number of protons and neutrons (collectively called nucleons) is denoted by the atomic mass number, A, which is, by definition $Z + N$. Nuclei with the same atomic number Z but differing values of N are called *isotopes*. Because they have the same electronic configuration, they are practically identical chemically, although their nuclear properties themselves are very different. Nuclei with the same N but differing Z are called *isotones* and nuclei with the same A but differing Z (and hence N) are called *isobars*; these are less common terms. Figure 9.1 illustrates this terminology.

You might think that if the baryons (and mesons) are colourless, i.e. have no net colour charge, then there would be no strong force between them. This is not actually true: the same argument in terms of EM charge would argue that there would be no forces between neutral atoms. This is clearly wrong; if true, all matter would be an ideal gas and no liquids or solids could form.

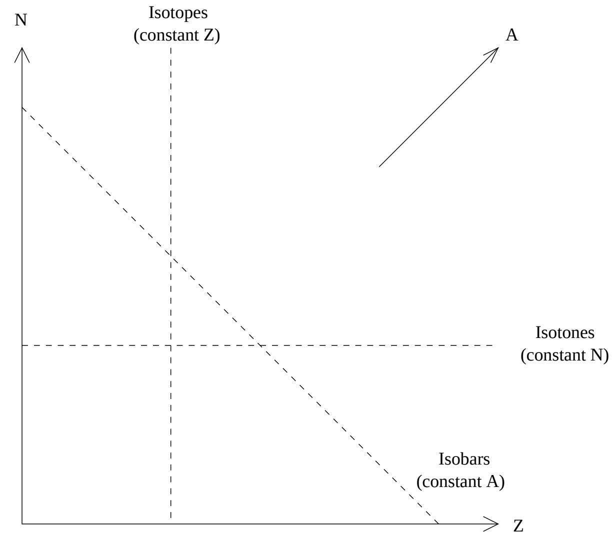


Figure 9.1: Nuclear isotopes, isotones and isobars.

9.3 Nucleon Forces

In the case of atoms, although they have no net charge, there are residual "multipole" electric fields which can interact, although these potentials fall off a lot quicker than $1/r$. These forces are normally referred to as Van der Waals forces, and are often parametrised by a potential of the form $a/r^{12} - b/r^6$. There is a similar effect for baryons: there is a residual Van der Waals-like strong force between them which, although much weaker than the full strong force, is still able to overcome the proton EM repulsion and bind nucleons together into nuclei. We will call this force the *nuclear force*. Like the Van der Waals force, it falls off quickly with r and so it is short range, $\sim$1 fm. Also like the Van der Waals force, it becomes large and repulsive at very short distances.

One important property of the nuclear force is that it is approximately independent of the nucleons involved. Both the proton and neutron are made of three quarks with very small masses compared to the mass of the QCD field surrounding them, which results in the proton and neutron masses being very close: $m_p = 938.3$ MeV/c^2 and $m_n = 939.6$ MeV/c^2, which only differ by 0.1%. All the quarks have the same strong charge and are in equivalent wavefunctions, so the residual force should be the same for both. Clearly, there is a repulsive EM force between protons which is absent for neutrons, but the nuclear force is effectively the same for both types of nucleon.

We can check that the nuclear force is independent of the nucleons involved most easily by comparing nuclei where all the protons and neutrons are interchanged. Such nuclei are called "mirror nuclei"; tritium and helium-3 are one example. Another example, specifically the excited states of $^{23}_{11}$Na (which has $Z = 11$ and $N = 12$) and $^{23}_{12}$Mg (which has $Z = 12$ and $N = 11$), is shown in Figure 9.2. The levels are seen to be very similar, with the differences explainable in terms of the EM force on the protons due to their charge.

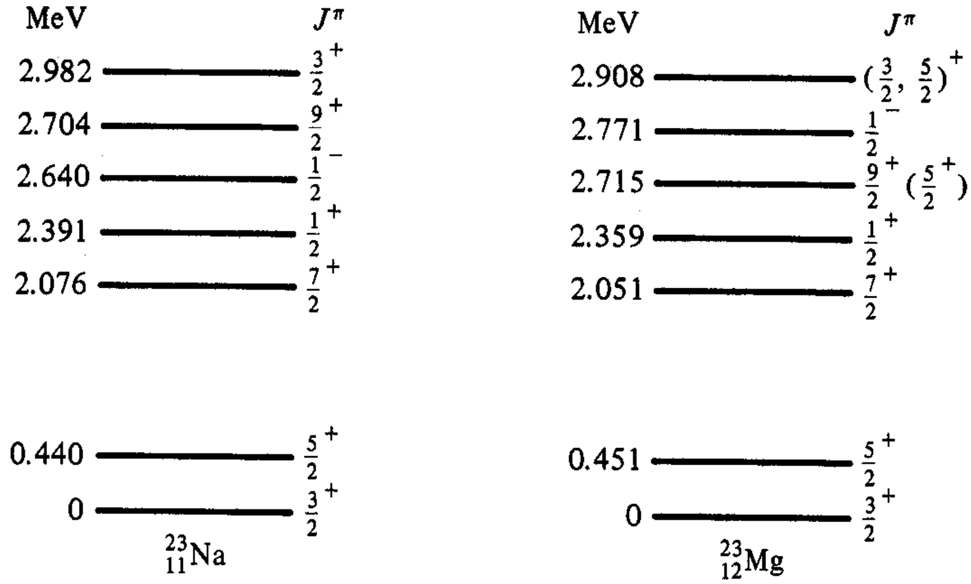


Figure 9.2: Nuclear energy levels of the $^{23}_{11}$Na and $^{23}_{12}$Mg mirror nuclei.

9.4 The Liquid Drop Model

We have seen that the nuclear force has a very short range ∼1 fm, which is only the size of a nucleon radius. It also saturates, i.e. it becomes very large and repulsive for short distances. This means that in a nucleus with many nucleons, they will not all crowd together at the origin but will spread out to occupy a finite volume each, liked packed spheres. In addition, the force each then imposes on the others is so short range that it is negligible for all but the nearest neighbours — see Figure 9.3.

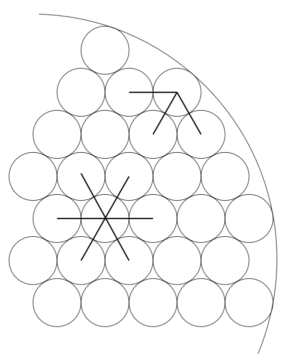


Figure 9.3: The liquid drop nuclear model.

This is physically similar to the saturation in a water drop, where the Van der Waals forces between the water molecules effectively make only the nearest-neighbour interactions significant. Because of this analogy, the resulting nuclear model is often called the "liquid drop" model. It gives a good description of the binding energies of the ground states of nuclei with many nucleons where, in practice, 'many' means $A \geq 20$.

There are two immediate consequences of this model. Firstly, the volume of the nucleus clearly would be expected to increase linearly with the number of nucleons, so $V \propto A = N + Z$. Since $V = 4\pi r^3/3$, the nuclear radius r should go as $r \propto A^{1/3}$. This is a good approximation and is usually expressed as $r = r_0 A^{1/3}$, where $r_0 = 1.2$ fm, roughly the radius of a nucleon. The second consequence is that binding energy for each nucleon would be expected to be constant. This is because each nucleon only binds to its nearest neighbours, so the contribution to the binding energy from each is a fixed value. Hence, we would expect $BE \propto A$, which is again found to be roughly true; as stated above, the binding energy for heavy nuclei is found to be roughly 8 MeV per nucleon.

However, there is an obvious problem with this approximation, and that is the nucleons on the surface of the nucleus. These will not have as many nearest neighbour bonds as the ones well within the nuclear volume as there are no nearest neighbours outside the nucleus. This is exactly the same effect which gives rise to surface tension in a water drop, so again the analogy holds. We would therefore expect that there will be a correction to the binding energy proportional to the sphere surface area, $4\pi r^2$. Since $r \propto A^{1/3}$, then our expression for the binding energy becomes

$$BE = a_v A - a_s A^{2/3} \,,$$

where a_v (the volume term constant) and a_s (the surface term constant) are parameters to be determined from data.

9.5 The Semi-Empirical Mass Formula (SEMF)

The liquid drop model reproduces the gross features of the binding energy but does not give the dependencies on the individual numbers of protons and neutrons. It predicts that the ground state binding energy only depends on A but is independent of Z or N, which is not correct. It ignores quantum effects such as the Pauli exclusion principle (and others) completely. To get a better agreement we need to add three more terms which explicitly depend on Z and N.

9.5.1 Coulomb repulsion

The first new term is easy to understand. We know there is an EM repulsive force between protons due to their charge, and so this will reduce the binding energy for nucleons with several protons. As we believe the nuclear force itself is independent of nucleon type, then the protons will on average be spread evenly throughout the nucleus, which means the charge density is uniform. It is a standard problem in electrostatics to calculate the energy required to assemble a sphere of uniform charge density, and the result is:

$$\frac{3}{5}\frac{Q^2}{4\pi\epsilon_0 r} \,.$$

For a nucleus with Z protons, this EM self-energy is, therefore,

$$\Delta BE = -\frac{3}{5}\frac{e^2}{4\pi\epsilon_0 r_0}\frac{Z^2}{A^{1/3}} = -a_c\frac{Z^2}{A^{1/3}}\,,$$

where $a_c = 0.72$ MeV. Note the dependence on Z^2 rather than Z; the EM force is long range and so every proton affects every other proton in the nucleus, not just its nearest neighbours. Contrast this with the short range nuclear force, where the nucleons only affect their nearest neighbours and the energy depends on A. In fact, although this Z^2 form is often used, strictly speaking it is not quite right. The energy given by the expression above is that needed to spread all the charge out throughout all space to an infinitely small density. However, the binding energy is defined as the energy need to break the nucleus into its constituent nucleons, i.e. to break it into neutrons and protons, but *not* to spread the individual proton charges out. Indeed, the equation says even one proton, i.e. $Z = 1$, gives a correction to the binding energy, even though there is nothing to repel it. This means the correction to the binding energy should not be quite as large. A better form is $a_c Z(Z-1)/A^{1/3}$, which is now zero for $Z = 1$. Clearly, for large Z, as found in large nuclei, these two are very similar. We now have

$$BE = a_v A - a_s A^{2/3} - a_c\frac{Z(Z-1)}{A^{1/3}}\,.$$

This by itself breaks the independence on Z, but clearly predicts that the largest binding energy for any A will be for $Z = 0$ or 1. We know this is not right as we have stable nuclei with all values of charge up to $Z > 100$ in the periodic table of the elements. We need to add two more terms to account for quantum effects.

9.5.2 Asymmetry term

The next term we will consider is called the "asymmetry term". The idea here is identical to the concept of a Fermi level in the physics of materials. The nucleons have energy levels in the nucleus and, being spin-1/2 particles, then each level can take two of each type of nucleon, as illustrated in Figure 9.4.

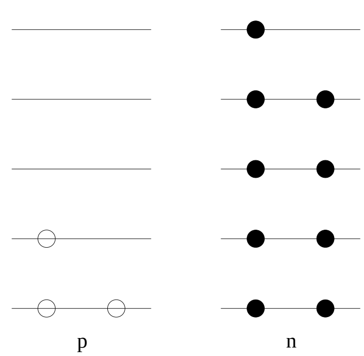


Figure 9.4: Filling up the nuclear states with two sets of identical particles.

If we really tried to form a nucleus purely from neutrons, as implied by the terms we have so far for the binding energy, they would have to be put into higher and higher energy levels and so would be less and less strongly bound, reducing the binding energy. Clearly, putting protons into the nucleus instead would be beneficial for the binding energy as they could go into the deepest empty proton levels. It is clear the best situation is when the two are evenly balanced with $N = Z$. The details of the exact energy levels and numbers per level will be messy and vary with A, but a reasonable parametrisation turns out to given by $\Delta BE \propto -(N-Z)^2$, i.e. the binding energy is reduced symmetrically for either $N > Z$ or $Z > N$. In fact, the spacing between states depends inversely on the size of the nucleus (as it does for a square well potential) so that larger nuclei have less of a binding energy loss if $N \neq Z$, hence, the full term used is:

$$\Delta BE = -a_a \frac{(N-Z)^2}{A} \, .$$

9.5.3 Pairing term

The final term is called the "pairing term". This occurs because of the different overlap of wavefunctions for pairs of nucleons in various states. For two identical nucleons in the same spatial state, with opposite spins to be antisymmetric as required, then the spatial wavefunctions are effectively identical and have maximal overlap. Because of the short-range force, this gives more of a binding energy for this particular pair. This effect occurs for all nucleons except potentially the ones in the highest occupied energy level for each type of nucleon, where there is either one or two nucleons of that type. Hence, the nucleus will be more strongly bound for ones with an even number of nucleons of either type. There are three cases:

1. Even-even, meaning an even number of both protons and neutrons, and hence even A. This has both pairs strongly bound.

2. Odd-odd, meaning an odd number of both protons and neutrons, and hence also even A. This is the least strongly bound.

3. Even-odd, meaning an even number of one type and an odd number of the other, and hence odd A. This has one strongly bound pair and so should be half way in between the previous two.

This is therefore simply parametrised by a form

$$\Delta BE = a_p \frac{1}{A^{1/2}} \, ,$$

where a_p takes a positive value for even-even nuclei, a negative value for odd-odd nuclei, and is zero for even-odd nuclei. Note, the pairing term implies even-even nuclei always have the spins of the nucleons in the same spatial state anti-parallel, so all such nuclei would be expected to have ground states with total spin zero; this is observed to be true.

9.5.4 Overall SEMF

Therefore, the total expression for the binding energy is

$$BE = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_a \frac{(N-Z)^2}{A} + a_p \frac{1}{A^{1/2}}\,,$$

and this is the semi-empirical mass formula. The best-fit parameters take values around $a_v = 15.8$ MeV, $a_s = 18.3$ MeV, $a_c = 0.72$ MeV, $a_a = 23.2$ MeV and $a_p = \pm 11.2$ MeV (or zero).

9.6 The Beta-Stability Curve

The semi-empirical mass formula is a function of only two variables, as $A = N+Z$. It gives the binding energy of the ground state of any nucleus, i.e. any values of Z and N. However, we have not observed nuclei with most of the combinations of Z and N which might be thought possible in the whole of the (Z,N) plane because the majority of them are highly unstable. The binding energy is largest in a specific region of the (Z,N) plane which is called the beta-stability valley ("valley" as the mass is at a minimum, "beta-stable" for reasons which will become clear in a later lecture).

It is easiest to analyse the binding energy along lines of *constant* A and we will look at it as a function of Z. Putting $N = A - Z$, then

$$BE = \left(a_v A - a_s A^{2/3} - a_a A + \frac{a_p}{A^{1/2}}\right) + \left(\frac{a_c}{A^{1/3}} + 4a_a\right) Z + \left(-\frac{a_c}{A^{1/3}} - \frac{4a_a}{A}\right) Z^2\,.$$

For odd A, when the pairing term a_p is zero, then this is a quadratic as shown in Figure 9.5.

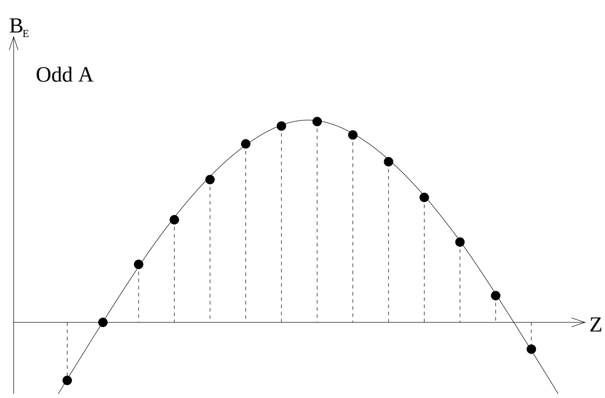


Figure 9.5: Binding energy as a function of Z for add-A nuclei.

There is a particular value of Z for which the binding energy is maximum but, away from this value, the binding energy falls and eventually goes negative at which point the nucleus is no longer bound. Even within the region of positive binding energy, the non-maximum values of Z are not necessarily stable; as we will see, beta decay allows them to change protons to neutrons and vice-versa, and so move along the curve to the maximum value. Hence, we only see the reasonably long-lived or stable nuclei which are at or near the maximum.

For even A, then as Z changes by one, then Z (and N, given that A is fixed and even) goes from even to odd or vice-versa. This means the pairing term changes sign. Hence, this shifts the quadratic curve up and down by $\pm a_p/A^{1/2}$ for the alternating even and odd Z values. However, the same stability arguments still hold; only nuclei near the peak live long enough to be seen.

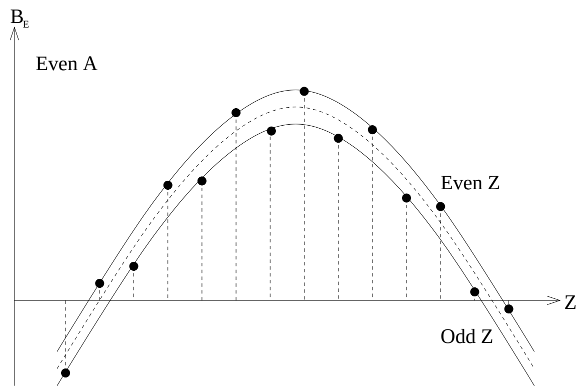


Figure 9.6: Binding energy as a function of Z for even-A nuclei.

We need to know how the Z position of the peak changes with A. In principle, we can approximate Z as continuous and take derivatives of the above quadratic function to find the maximum as a function of A (as in Problem Sheet 4). However, it is also useful to get an intuitive feel for what we would expect. The two terms driving this are the Coulomb and asymmetry terms: only a_c and a_a appear in the Z and Z^2 terms of the quadratic. If $a_c = 0$, then clearly $N = Z = A/2$ gives the maximum binding energy as the nuclei like to have equal numbers of levels filled. Conversely, if $a_a = 0$, then $Z = 0$ or 1 to minimise the Coulomb term and hence maximise the binding energy by limiting the amount of Coulomb repulsion. Hence, with both terms, we would generally expect that Z would be somewhat less than $A/2$ and N would be more than $A/2$. The relative size of these two terms is not the same for all A. The asymmetry term falls off as $1/A$ while the Coulomb terms falls off only as $1/A^{1/3}$. Hence the latter becomes more important as A increases. Conversely, for small A, particularly as $a_c \approx 0.7$ MeV $\ll a_a \approx 23.2$ MeV, the Coulomb term has little effect. Therefore, we expect $Z \approx A/2$ for small A but $Z < A/2$ for large A. The actual stable and observed nuclei which form the beta-stability valley are shown in Figure 9.7.

The line of the most stable nuclei is called the "beta-stability curve". Note, there can be several stable nuclei for a given A. Nuclei lying on the beta-stability curve are often studied. For example, the binding energy per nucleon of these as a function of A is also overleaf in Figure 9.8 and compared with the semi-empirical mass formula. The agreement is generally very good. Note, the binding energy per nucleon is reasonably constant, between 7.8 and 8.8 MeV for all nuclei with $A > 30$. It peaks around ${}^{56}_{26}$Fe, which means this is the most strongly bound nucleus. It is therefore energetically favourable both to break up heavier nuclei to bring them closer to ^{56}Fe and also to combine lighter nuclei. The former is called nuclear fission, and the latter nuclear fusion; the energy released in both cases can (and has) been used both for power generation and nuclear weapons. The small peaks showing disagreement are a result of quantum energy levels at particular values of Z and N and are the subject of the next lecture.

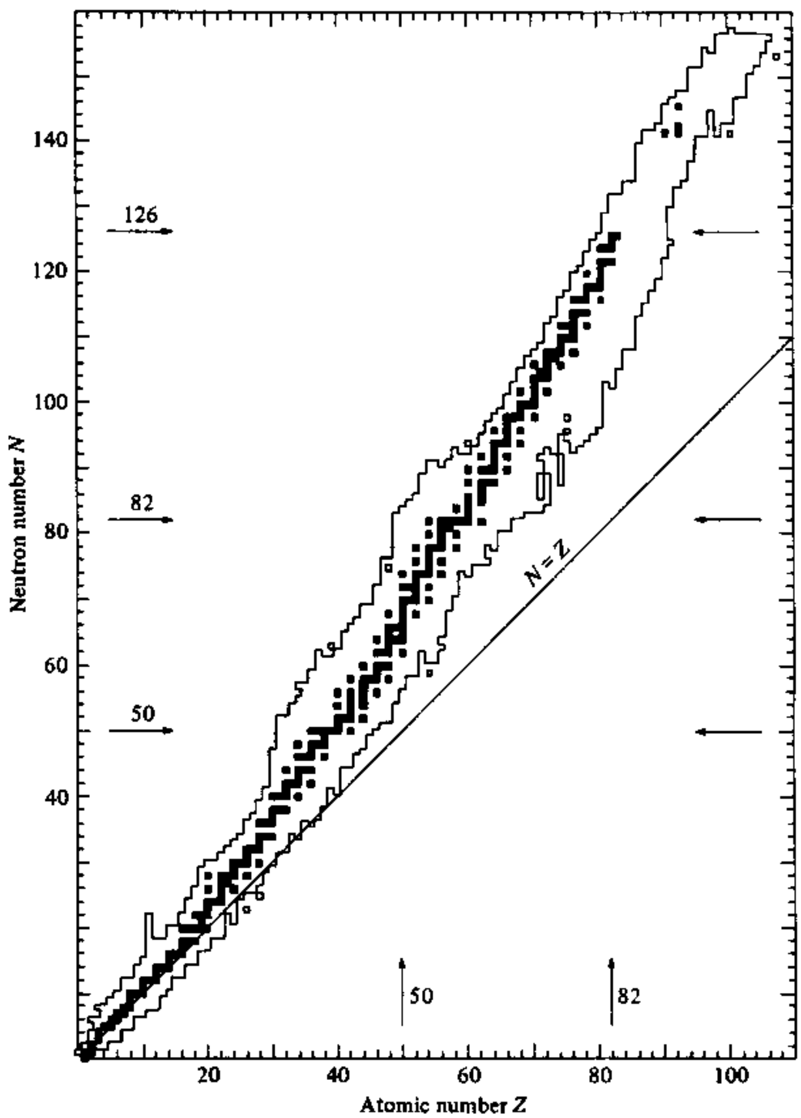


Figure 9.7: The nuclear stability valley: stable nuclei are shown in black and long-lived isotopes can be found within the two lines.

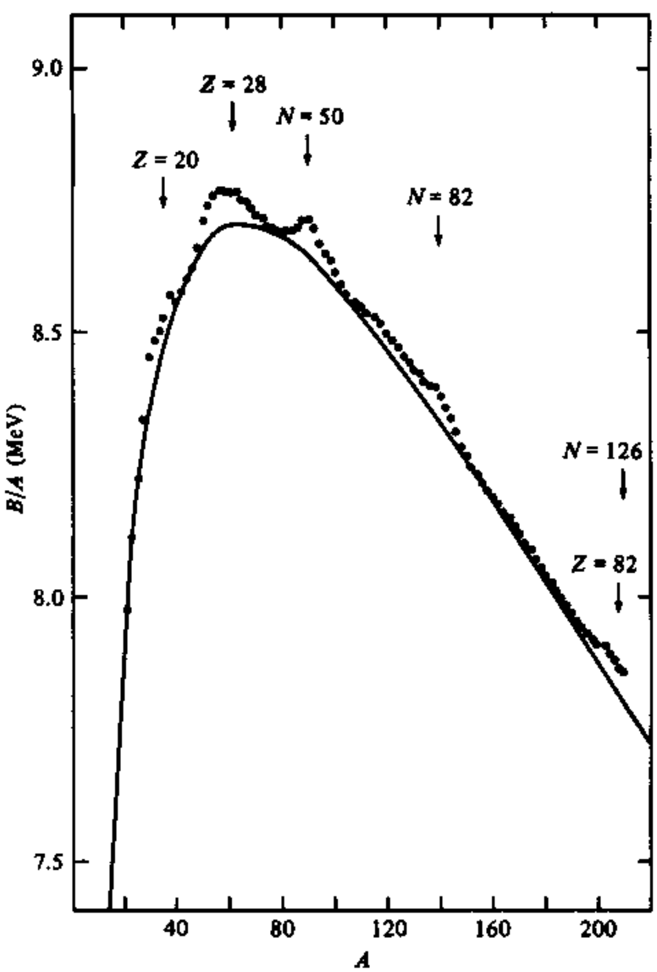


Figure 9.8: Binding energy per nucleon for stable nuclei.

NPP Lecture 10 – The Shell Model

10.1 Introduction

It is apparent that the semi-empirical mass formula does a good job of describing trends, but not the non-smooth behaviour of the binding energy. For this, we need to develop a very different model of the nucleus, one based on quantum energy levels. Recall from Quantum Mechanics that confining particles leads to the quantisation of their energy. It is surprising that such a radically different picture can describe the same physical system, but we shall see that several properties of nuclei are indeed well described by this model.

10.2 Magic Numbers

A closer look at the discrepancies from the semi-empirical mass formula is in order. We saw that there were particular values of Z and N for which the nuclei had a higher binding energy than would be expected. These strongly-bound states occur when Z or N have one of a set of so-called "magic numbers". In fact, even for $A < 20$, where the semi-empirical mass formula is not valid, it is apparent that certain nuclei, e.g. ${}^{4}_{2}\mathrm{He}$ with a binding energy of 28.3 MeV, are much more strongly bound that their neighbours, e.g. there are no bound $A = 5$ nuclei. The magic numbers which are observed over the whole range of nuclei are:

$$2, 8, 20, 28, 50, 82, 126\,.$$

Some nuclei have both Z and N at magic numbers, such as ${}^{4}_{2}\mathrm{He}$ ($Z = 2$, $N = 2$) and the most common isotope of lead, ${}^{208}_{82}\mathrm{Pb}$ ($Z = 82$, $N = 126$); these are called 'doubly-magic' and are correspondingly even more strongly bound.

The shell model says these magic numbers correspond to filling a quantum energy level, so giving a particularly well-bound nucleus. The magic nuclei are therefore equivalent to the inert gases (helium, neon, argon, etc.) in chemistry. While this provides a qualitative explanation, we still need to understand why the magic numbers have the values they do.

10.3 Nuclear Potentials

Ideally, we would write down the Schrödinger equation for the nuclear force potential and solve it to calculate the energy levels, as done for the hydrogen atom. However, this is not as simple as for hydrogen, for two reasons. Firstly, the potential energy for the nuclear force is much more complicated than the $1/r$ for hydrogen. Secondly, it is not a central potential in which the nucleons move independently; there is no central object corresponding to the proton in hydrogen but each nucleon feels the force from the others.

Hence, we need to make a physical guess for a reasonable potential and compare with the observed magic numbers. We will consider each nucleon as moving in a potential resulting from the *average* of the interactions with all the other nucleons. What would this potential look like? We already saw that the short-range force means that a nucleon is bound to all its nearest neighbours by an equal contribution to the binding energy for each nucleon. Inside the nucleus, the number of nearest neighbours is equal in all directions so the net force on any nucleon is in fact zero. Thus the effective potential is constant within the nucleus and the constant value must be negative to keep the nucleon bound. Outside the nucleus, more than a few fentometres away, the short range nuclear force will have died off, so again there will be no force and hence a constant effective potential, which we can take as zero. Finally, as stated previously, the nucleons near the surface have only nearest neighbour forces into the nucleus as they are missing the nearest neighbours outside. Hence, they do have a net inwards force and so a rising potential as the radius increases. This change to the potential takes place over a distance of order the nuclear force, so around 1 fm. Hence, we would guess an effective potential would look like that in Figure 10.1.

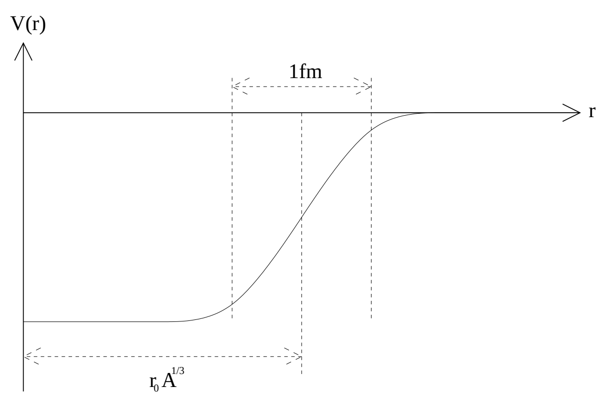


Figure 10.1: Woods-Saxon nuclear potential.

This is called the Woods-Saxon potential and is often expressed as:

$$V(r) = -\frac{V_0}{1 + e^{(r-a)/d}},$$

where $a \simeq r_0 A^{1/3}$ sets the nuclear radius (typically a few fm), and d sets the distance over which the potential rises (giving a nuclear 'skin' thickness $2d \simeq$

1 fm). While it is possible to solve the Schrödinger equation for this potential, it is not trivial. To give a feel for the results, we can look at some simpler cases, such as an infinite square well or a simple harmonic oscillator — see Figures 10.2 and 10.3.

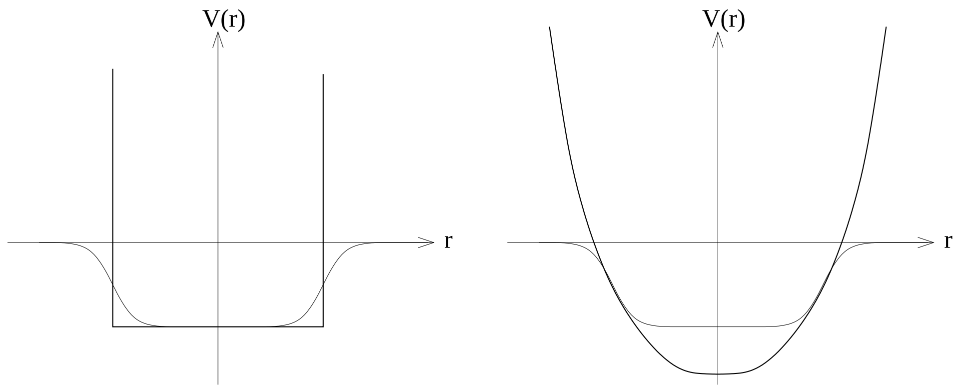


Figure 10.2: Comparison of the Woods-Saxon potential with the infinite square well (left) and the parabolic potential (right).

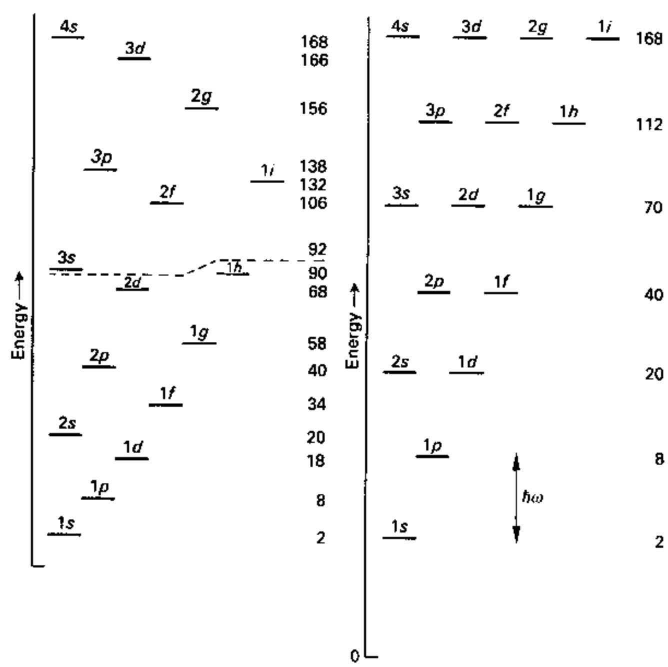


Figure 10.3: Energy levels for the 3D infinite square well and parabolic potentials; the numbers on the right are *cumulative* occupancies.

What do these predict for the magic numbers? Each state has $2l+1$ values of l_z and, due to the nucleon spin, each can take two protons (and also two neutrons) in the two s_z states. Hence, the number of protons (or neutrons) in an l state is $2(2l+1) = 4l+2$; we list these in Table 10.6.

Table 10.6: Multiplicity as a function of angular momentum quantum number.

l	$4l+2$
0	2
1	6
2	10
3	14

The first magic number of 2 corresponds to filling the first state in both cases. The next magic number is 8, which is the total number of nucleons which fills the first two states, again in either case. The other numbers given by completing the levels are shown in the diagrams above. They both give 20 but then start to disagree with the measured values for the magic numbers. Hence, we can reproduce the first few but not the higher values. You may think this is just a question of tweaking $V(r)$ to arrange the states to be just right, but it turns out it is not possible to get all the correct magic numbers by this method. We need to add the spin-orbit coupling as discussed in the next section.

Before moving on to this, a quick comment on the validity of our assumption that any nucleon moves independently in the average potential from the others. The nucleons are tightly packed and the nuclear force between them is strong. Hence, in a classical picture, a nucleon could not orbit in a trajectory independently of the other nucleons. However, our assumption and hence the shell model is valid because of quantum effects. To alter its motion, a nucleon would need to move to a different quantum state. Pauli exclusion prevents the nucleon going into an occupied state close in energy so the only states available to a nucleon to scatter into are much higher in energy. This is highly unlikely, so such scatterings are heavily suppressed.

10.4 Spin-Orbit Coupling

A new term is needed in the potential and this is a spin-orbit coupling, where the energy is $\propto \boldsymbol{l}.\boldsymbol{s}$, just as happens in atomic physics. This has the effect of splitting some of the $4l+2$ degeneracy and giving new energy levels. We previously said each l state has $2l+1$ values of l_z and $2s+1=2$ values of s_z. These could equally well be described by total angular momentum j and j_z, rather than l_z and s_z. For a given l, then there are two values of j, namely $l \pm 1/2$ and these have $2j+1 = 2(l \pm 1/2)+1 = 2l+2$ and $2l$ values of j_z, summing to $4l+2$ in total, as required. Without a spin-orbit coupling, both values of j have the same energy and so are totally degenerate. However, a spin-orbit coupling splits the two j values but leaves the j_z degeneracy in each one (as is required by the isotropy of space). To see how this works, we can use the same trick as we used for the hyperfine splitting in the mesons. The total angular momentum is

$$\boldsymbol{j} = \boldsymbol{l} + \boldsymbol{s},$$

so squaring gives

$$j^2 = l^2 + s^2 + 2\boldsymbol{l}.\boldsymbol{s}\,.$$

Rearranging, then

$$\boldsymbol{l}.\boldsymbol{s} = \frac{1}{2}\left[j^2 - l^2 - s^2\right]\,.$$

In terms of eigenvalues, this is

$$\langle \boldsymbol{l}.\boldsymbol{s}\rangle = \frac{\hbar^2}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]\,,$$

showing that this term does indeed depend on the value of j. Since $j = l \pm 1/2$, then for $l + 1/2$, this gives

$$\begin{aligned}\langle \boldsymbol{l}.\boldsymbol{s}\rangle &= \frac{\hbar^2}{2}\left[(l+1/2)(l+3/2) - l(l+1) - s(s+1)\right] \\ &= \frac{\hbar^2}{2}\left[l^2 + 2l + 3/4 - l^2 - l - 3/4\right] = \frac{\hbar^2}{2}l\,,\end{aligned}$$

while for $l - 1/2$, it gives

$$\begin{aligned}\langle \boldsymbol{l}.\boldsymbol{s}\rangle &= \frac{\hbar^2}{2}\left[(l-1/2)(l+1/2) - l(l+1) - s(s+1)\right] \\ &= \frac{\hbar^2}{2}\left[l^2 - 1/4 - l^2 - l - 3/4\right] = -\frac{\hbar^2}{2}(l+1)\,.\end{aligned}$$

The effect of applying this splitting to the Woods-Saxon potential is shown in Figure 10.4. Note that larger j states are pushed *down*: when spin is parallel to angular momentum ($j = l + 1/2$) the potential becomes more negative, i.e. the well is deeper and the state more tightly bound.

This model can also correctly predict the spins and parities of many nuclei where there is a single unpaired nucleon either alone in a state or missing from a completed state. It works particularly well for Z or N close to magic. All even-even nuclei are $J^P = 0^+$. If Z or N are both even and one corresponds to a completed level, then adding one extra of this type of nucleon means the total spin of the nucleus must be the angular momentum of this final nucleon, as must its parity. For example, ${}^{17}_{8}$O has $Z = 8$ and $N = 9$, so the final neutron must be in the next state above the level which gives the magic number 8. From the diagram overleaf, this is a $1d_{5/2}$ level and so has $j = 5/2$ and $l = 2$, which gives $P = (-1)^l = +1$. Hence, this nucleus would be expected to be $J^P = 5/2^+$, as observed. Similarly, removing one nucleon from a filled magic state gives a nucleus with total spin exactly opposite to the removed nucleon (as they sum to give zero), i.e. the same j value but opposite j_z, and also the same parity (as they multiply to give +1). This means it has the same quantum numbers as the unfilled state. Hence, for example ${}^{15}_{8}$O, with $Z = 8$ and $N = 7$, would have the properties of the $1p_{1/2}$ state, which has $l = 1$ and hence $P = (-1)^l = -1$, and so would be expected to be $J^P = 1/2^-$, again as observed.

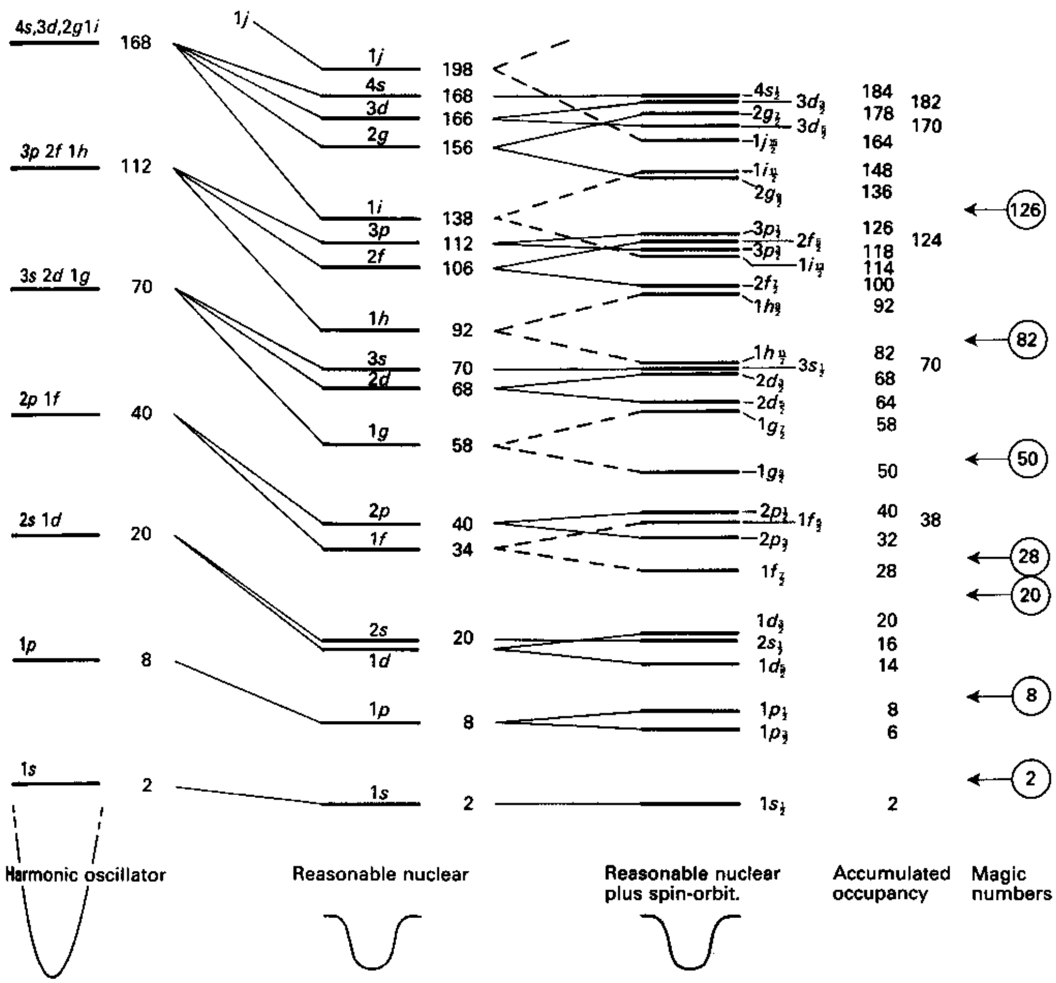


Figure 10.4: Spin-orbit level splitting for the Woods-Saxon potential.

NPP Lecture 11 – Gamma and Beta Decay

11.1 Introduction

How do nuclei decay? In general, all decays result in what we consider as 'radioactivity' and it is common knowledge that there is alpha, beta and gamma radiation. However, in fact, these decays are special cases of more general processes. We shall consider them in the reverse order: gamma and beta decay in this lecture, and alpha decay in the next.

11.2 Gamma Decay

We now know that the gamma-rays observed in nuclear radiation are simply high-energy photons, so gamma decays are EM decays. We know that the EM force does not allow us to change u to d quarks (or vice versa) so we cannot change the nucleus values of Z or N through these decays. Hence, only a very limited type of decay is possible: specifically, from an excited state to a ground state of the same nucleus. This is therefore the nuclear equivalent of atomic emission of light (fluorescence), but at much higher energies. In the same way, the energy differences between levels can be deduced from the energy spectrum of the photons seen. An example is the gamma spectrum for $^{177}_{72}$Hf shown in Figure 11.1, from which the level scheme shown in Figure 11.2 can be inferred.

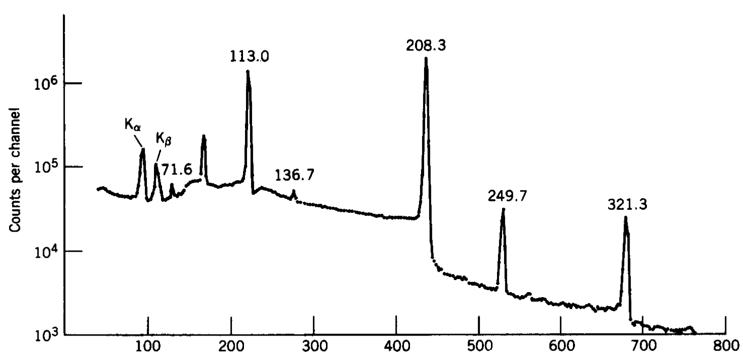


Figure 11.1: Gamma-ray spectrum from hafnium-177.

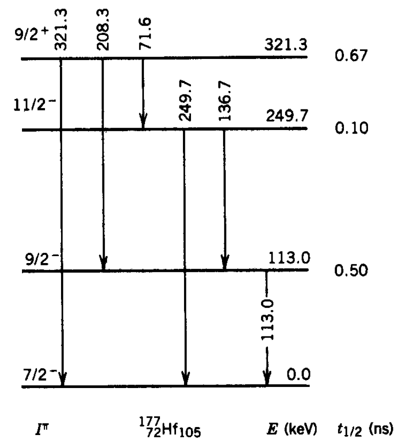


Figure 11.2: Decay scheme for hafnium-177.

Being an electromagnetic decay, the de-excitations tend to happen reasonably quickly as long as the change in angular momentum between the initial and final states is not too large. For photons of order 1 MeV between similar states, lifetimes tend to be around 10^{-16} s (a typical EM decay timescale), while for large changes in angular momentum $\Delta J \sim 4$ or 5, this can increase to 10^3 s. The suppression is due to the transition having to occur as a higher radiation multipole, just as in atoms. In all cases, gamma decay is extremely fast compared to the age of the Earth, so there are effectively no long-lived, naturally-occurring isotopes which emit gamma radiation. This does not mean it is not seen naturally, but only because other types of radiation can leave the resulting nucleus in an excited state that then deexcites by gamma emission.

There is a related process to gamma decay called *internal conversion*. Here, instead of a real photon being seen, a virtual photon is emitted and absorbed by one of the atomic electrons. This energy is usually much higher than its binding energy so the electron is ejected from the atom with a fixed energy (neglecting its original binding energy). Conversion electrons compete with gamma-ray emission for each transition (in the same way that 'Auger electrons' compete with x-ray fluorescence in atomic deexcitation).

11.3 Beta Decay

Beta decays have the ability to produce much more interesting behaviour as they are a weak-force process and they do change nucleons. The beta particle which is emitted is simply just an electron produced from a virtual W boson along with a (normally unobserved) electron antineutrino. The prototypical beta decay (β^-) is that of the isolated neutron:

$$n \to p + e^- + \overline{\nu}_e \,,$$

which we understand at a fundamental level as having a Feynman diagram as shown below.

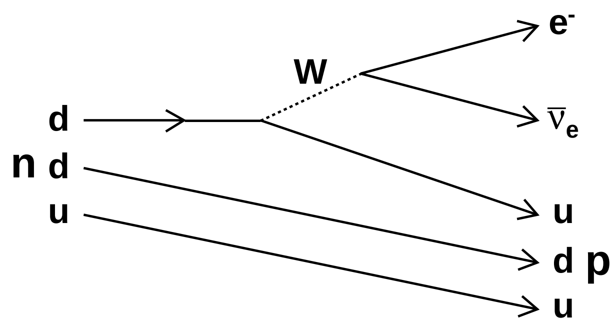


Figure 11.3: Feynman diagram for neutron decay.

This is only just energetically allowed: the mass of the neutron is 939.57 MeV/c^2 and that of the proton is 938.28 MeV/c^2, while the electron is 0.51 MeV/c^2, so the energy released, often labelled the 'Q-value', is $Q = 0.78$ MeV. (The neutrino masses are completely negligible at this level.) The small mass difference means that the neutron has a long lifetime compared with other hadron weak decays, such as the pion: the neutron has $\tau_n = 900$ s.

Note that this is a three-body decay. As we saw before, this means that the electron (and indeed the antineutrino) does not have a fixed energy, but is emitted with energies over a continuous spectrum – see the neutron decay beta spectrum in Figure 11.4. Indeed, it was that apparent violation of energy conservation in beta decays which led Pauli to first postulate the existence of an unseen particle, which we now know to be the neutrino (as discussed in the Radioactivity Experiment in 2nd year lab).

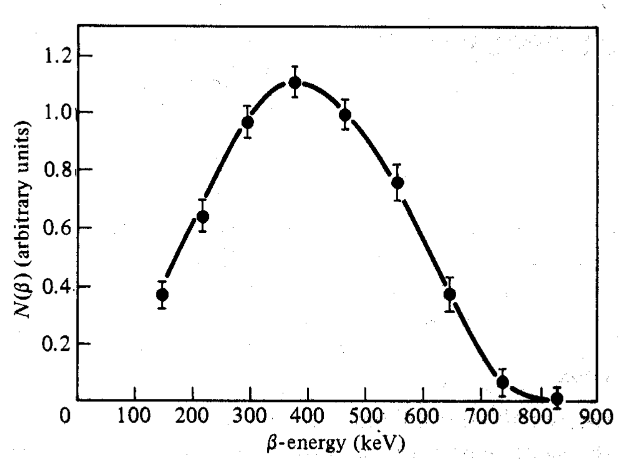


Figure 11.4: Measured beta spectrum of the neutron.

Note that, as far as the weak interaction is concerned, proton decay through

$$p \to n + e^+ + \nu_e \,,$$

with the very similar Feynman diagram shown in Figure 11.5, should be possible (β^+). In fact this does *not* happen purely because of energy conservation: the mass of the proton is smaller than the mass of the neutron plus the mass of the electron. Note also that conservation of baryon number implies a baryon ($B = 1$) in the final state, and so the decay to a lighter meson ($B = 0$) is forbidden.

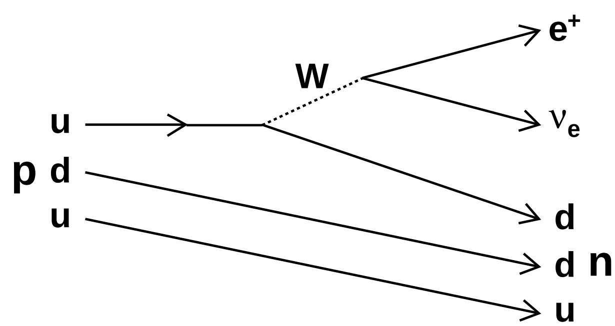


Figure 11.5: Feynman diagram for proton decay; this does not happen purely for energy conservation reasons: the neutron is heavier than the proton.

The obvious question then is why we can have nuclei with neutrons which survive for longer than 900 s? It is because once a neutron is bound into a nucleus, then changing it into a proton is basically changing $N \to N - 1$ and $Z \to Z + 1$ for fixed A. We have seen that this can change the nuclear binding energy quite substantially. Hence, this decay can actually increase the mass of the nucleus as a whole, so neutron decay within a nucleus can become energetically forbidden.

11.3.1 Odd-A nuclei

To see when this happens, let us look in more detail at odd A nuclei. Previously, we were interested in the binding energy, but now we need to know the total nuclear mass including the neutron-proton mass difference. Hence, plotting the mass as in Figure 11.6 looks inverted compared to the previous plots and gives a minimum in mass rather than a maximum in binding energy. The figure shows how β^- decay increases the Z of the nucleus at constant A, bringing it closer to the peak of the stability curve.

What about nuclei with too many protons? Here, the nuclear mass would be increased even further from the minimum if a neutron decayed to a proton, so this is not energetically allowed and the neutrons are stable. However, in the same way that nuclear masses become the important factor in determining whether neutrons can be stable even though free neutrons are not, then protons in nuclei

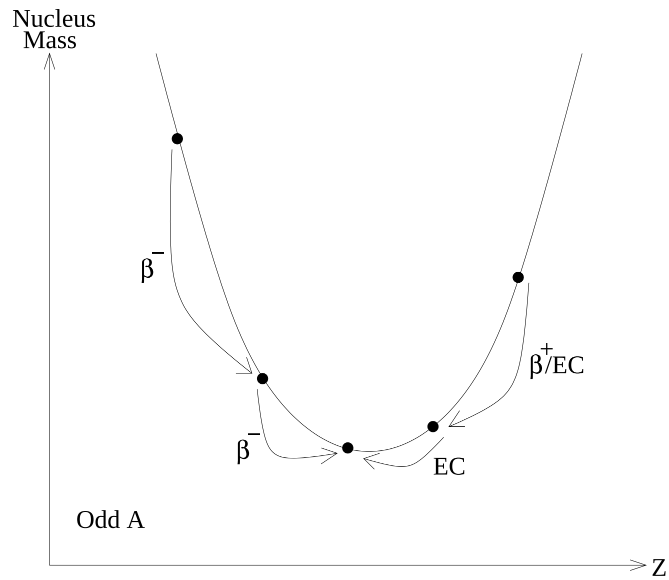


Figure 11.6: Beta decays increasing stability by changing Z at constant (odd) A.

can be energetically able to decay, even though free protons cannot. Hence, nuclei with Z greater than the minimum value can undergo positron emission as above, $p \to n + e^+ + \nu_e$, and so also obtain better stability. This process is called β^+ decay and is very similar conceptually to β^- decay, and again the three-body decay results in a spectrum for the positron (and neutrino).

There is a competing process to positron emission. Atomic electrons in an $l = 0$ state have a non-zero probability of being near the origin and hence being *inside* the nucleus (and, in fact, $|\psi(0)|^2 \neq 0$). This means that the process of electron capture (EC) can occur:

$$e^- + p \to n + \nu_e \, ,$$

which has an identical Feynman diagram to positron emission but with the positron reversed to make it an electron, as shown in Figure 11.7.

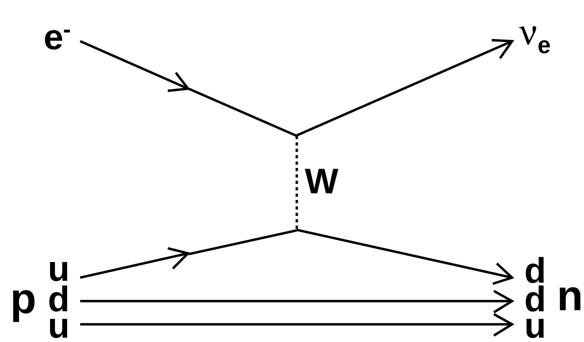


Figure 11.7: Feynman diagram for electron capture (EC).

In fact, this process can occur even when positron emission is energetically forbidden. The β^+ decay mode requires the nuclear masses to be $m_N(Z,N) > m_N(Z-1,N+1)+m_e$, whereas EC requires $m_e+m_N(Z,N) > m_N(Z-1,N+1)$, i.e. $m_N(Z,N) > m_N(Z-1,N+1) - m_e$. Hence, if the nuclei are different in mass by less than m_e, only electron capture is possible.

The equivalent process which could in principle compete with β^- decay, namely positron capture:

$$e^+ + n \to p + \overline{\nu}_e \,,$$

does not occur as there are no positrons circulating around the nucleus...

11.3.2 Even-A nuclei

Things are similar for even-A nuclei, but the pairing term adds a twist as illustrated in Figure 11.8.

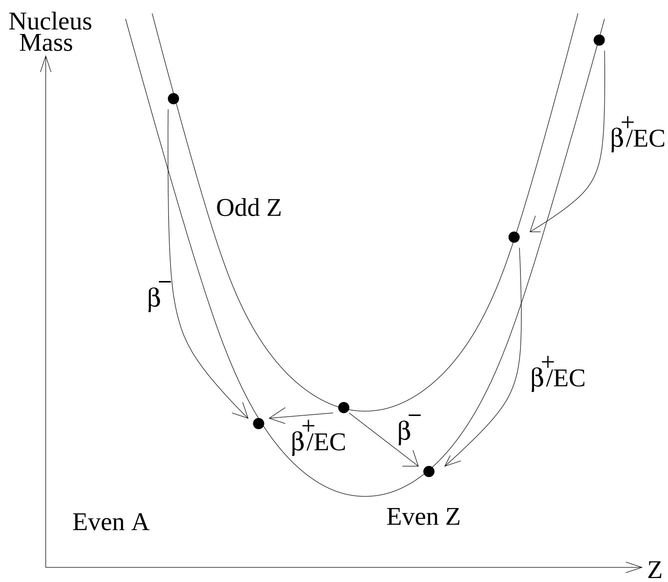


Figure 11.8: Beta decays increasing stability by changing Z at constant (even) A.

Beta decay from either side alternates between even-even (the lower mass curve) or odd-odd (the higher mass curve). Hence, it is perfectly possible to have an odd-odd nucleus which can decay both by β^- decay, β^+ decay and electron capture. The resulting two nuclei are both unstable against decay by those modes, even though one can be significantly heavier than the other. In fact, this raises the possibility of a process called 'double beta decay'. If the heavier even-even nucleus can convert two neutrons to protons simultaneously (or vice versa, depending on whether the lower Z nucleus is heavier or lighter), then it does not have to go via the heavier intermediate odd-odd state, but can go directly to the lower mass even-even nucleus:

$${}^A_Z\mathrm{X} \to {}_{Z+2}^{\;\;A}\,\mathrm{Y} + 2e^- + 2\overline{\nu}_e \,.$$

An example is ${}^{106}_{48}$Cd which can decay to ${}^{106}_{46}$Pd through double β^+ decay. Of course, it is very unlikely for two weak decays to happen at the same time so the rate is tiny and the lifetimes are correspondingly very long, up to 10^{22} years.

11.4 Neutrino Masses

While electron capture has a two-body final state, with the neutrino (and residual nucleus) having a single, fixed energy of emission, in the other β decays the electron and positron energies are not unique. The spectrum goes from zero right up to the difference in mass of the initial and final nucleus, ignoring the neutrino mass. If, however, the neutrino has a non-zero mass, then the maximum energy of the electron or positron is more limited.

This has been used to try to measure the neutrino mass; the upper portion of the electron spectrum should be truncated by an amount which depends on the neutrino mass. The most studied beta decay for this purpose is tritium, which decays to helium-3, ${}^3_1\mathrm{H} \rightarrow {}^3_2\mathrm{He} + e^- + \overline{\nu}_e$, with a very small energy release of only $Q = 18.6$ keV. This is shown in Figure 11.9. The small Q-value makes the observation of the effects of a non-zero neutrino mass somewhat easier to see. However, it is still a very hard experiment indeed, and while neutrino masses of more than $\sim$2 eV/c^2 have been excluded, it is very difficult to rule out neutrino masses at the levels allowed by the oscillations experiments.

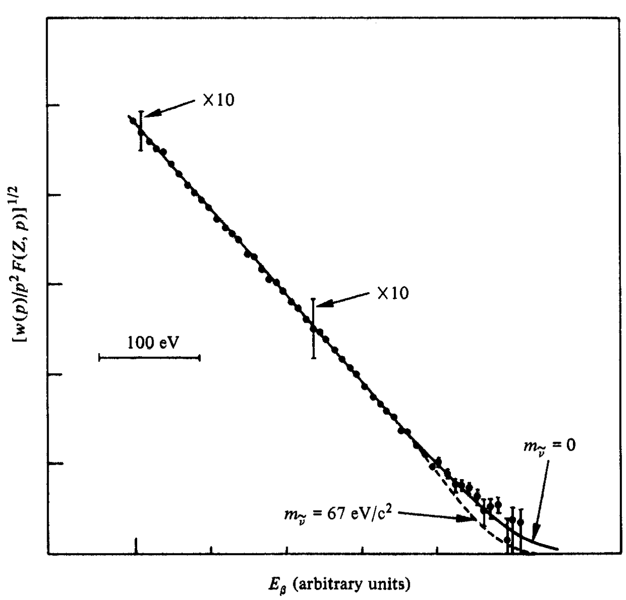


Figure 11.9: Endpoint of tritium beta spectrum, showing predictions for a massless neutrino compared with the effect of a (very) massive 67 eV/c^2 neutrino.

NPP Lecture 12 – Alpha Decay

12.1 Introduction

We have introduced gamma decays (due to the EM force) and beta decays (due to the weak force) and now we will examine alpha decays, which are due to the strong/nuclear force. In contrast to the previous decays which do not change A, alpha decay occurs by emission of nucleons, specifically an alpha particle, ${}^4_2\mathrm{He}$, is ejected. Generically,

$$ {}^A_Z\mathrm{X} \rightarrow {}^{A-4}_{Z-2}\mathrm{Y} + {}^4_2\mathrm{He}\,. $$

Nucleus Y clearly has a different number of nucleons to X. Figure 12.1 compares how alpha and beta decays move the nuclei around in the (Z,N) plane. Note that gamma decays cannot change Z, N or A.

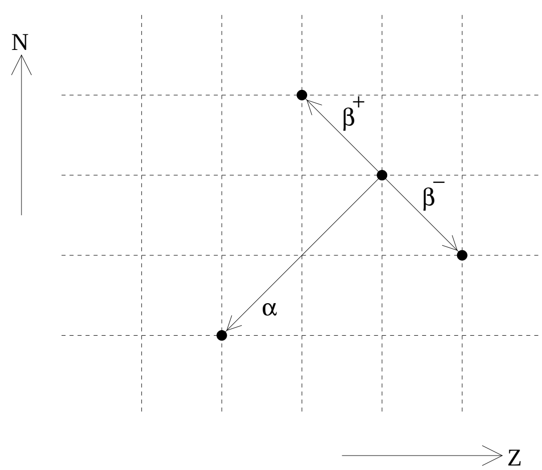


Figure 12.1: Change in nucleon numbers by $\beta^{\pm}$ and α decays.

Alpha decay is in fact only one specific case of a whole range of processes which involve nucleon emission. These range from single proton or neutron ejection up to splitting the nucleus into two roughly equal parts. Particularly in the latter case, these are called "fission" decays.

12.2 Alpha Decay Chains

Essentially, alpha decay is the most common form of fission. Why is this the case? The curve for binding energy per nucleon does not fall very steeply: it drops by only 10% over the whole A range above $^{56}_{26}$Fe. The slope is roughly 0.01 MeV/A. Hence, the energy release is quite small unless the products are particularly strongly bound. This is the case for $^{4}_{2}$He which is doubly magic and has $BE = 28.3$ MeV; although it has an A well below $^{56}_{26}$Fe, it still has ~ 7 MeV per nucleon. Even for alpha emission, this is only energetically possible for nuclei with $A \gtrsim 150$ and it needs significantly higher values of A than this for a reasonable energy release. Hence, alpha decay is seen mainly in heavy nuclei with large $A > 200$. However, such nuclei after alpha decay will leave a daughter nucleus with $A - 4$ which will also normally be above 150, and so it will itself be able to alpha decay. Hence a sequence of alpha decays is often seen that can be many decays long. They continue until they reach a nucleus which is stable (or semi-stable). One example, shown in Figure 12.2, is the decay sequence starting from $^{238}_{92}$U which ends at $^{206}_{82}$Pb. (Note that the Pb nucleus has Z at a magic number, hence being quite stable.)

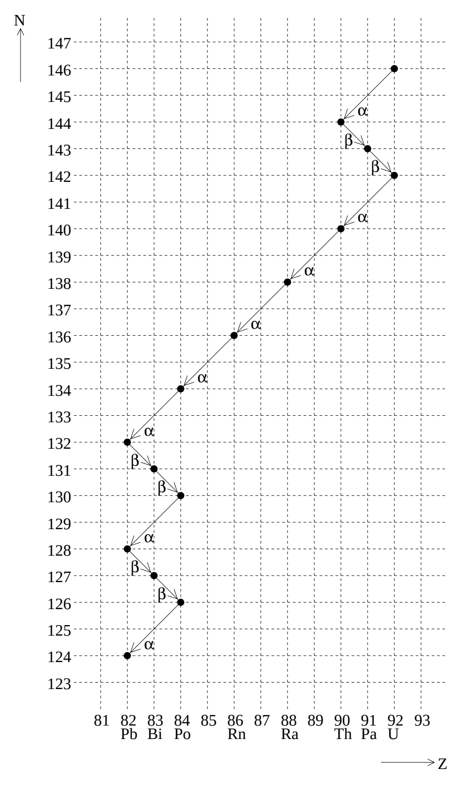


Figure 12.2: The uranium-238 radioactive decay chain.

Note that there are several beta decay steps here too. Alpha decays reduce Z and N equally, specifically decreasing them each by two per decay. However, the heavy, large A starting nucleus will have $N > Z$ as that is what is needed to be near the beta-stability curve. Hence, a pure alpha decay sequence would leave a lower A nucleus with a higher and higher fraction of neutrons, whereas the beta-stability curve requires the fraction of neutrons to become lower as A is reduced. Hence, the beta decays bring the intermediate nuclei back closer to the beta-stability curve. Note, there will always be too many neutrons, not too few, so positron emission or electron capture are basically not seen in these sequences. This is why, although radioactivity was discovered in the 19^{th} century, antimatter (specifically the positron) was not found until 1932.

One final point: beta decay does not change A, but alpha decay changes it by 4. Hence, all heavy nuclei with the same $A/4$ remainder (A modulo 4 or, writing $A = 4n + m$, then the value of m) will end up at the same nucleus. Hence, there are effectively only four such decay sequences.

12.3 Alpha Decay Rates

One obvious question is why do we see any of these sequences at all? This is a strong force decay and they have had around 5 billion years to decay. The answer is that some of the lifetimes for these decays are indeed in the billions of years — despite being due to the strong force. The range of lifetimes for alpha decay is found to vary by over 25 orders of magnitude, and this is effectively totally determined by the Z value and the size of the energy release, Q, as shown in Figure 12.3.

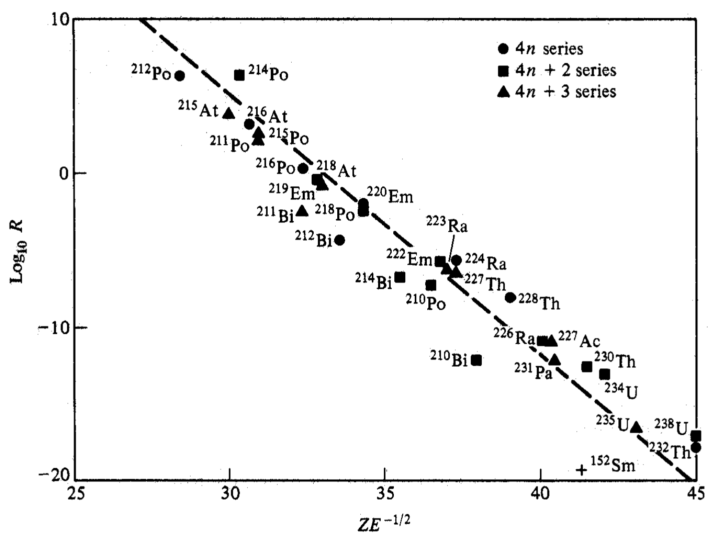


Figure 12.3: Alpha decay rate ($R = 1/\tau$ [s^{-1}]) as a function of $Z/\sqrt{Q}$.

This extremely strong exponential dependence on $Z/\sqrt{Q}$ can be understood by considering the alpha decay process in more detail. To be emitted, the alpha particle has to form outside the nucleus, as suggested by Figure 12.4.

We can consider a simple model where the alpha particle has an independent existence within the nucleus before the decay. What potential would it then feel? There are two forces: the nuclear and the EM force.

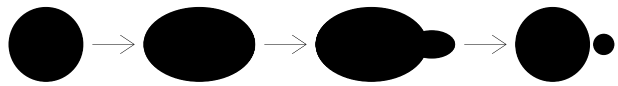


Figure 12.4: Formation of alpha particle from an oscillating nucleus.

The nuclear potential energy for the alpha will be effectively the same as for the nucleons, i.e. something like the Woods-Saxon shape which we discussed previously — see Figure 12.5 (left). There is also a large Coulomb repulsion due to the positive charges on both the nucleus and alpha particle, so the EM contribution to the potential looks like that in Figure 12.5 (right).

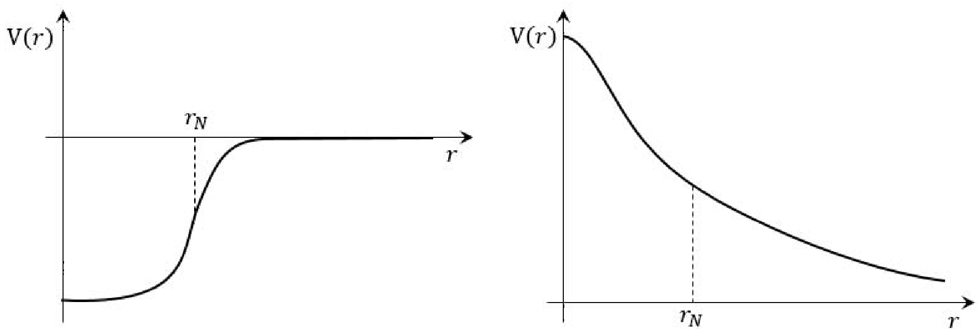


Figure 12.5: Nuclear (left) and EM (right) potentials experienced by alpha particle.

The total is then that shown in Figure 12.6, which shows that the alpha particle has a large potential barrier to overcome in its decay. If the energy

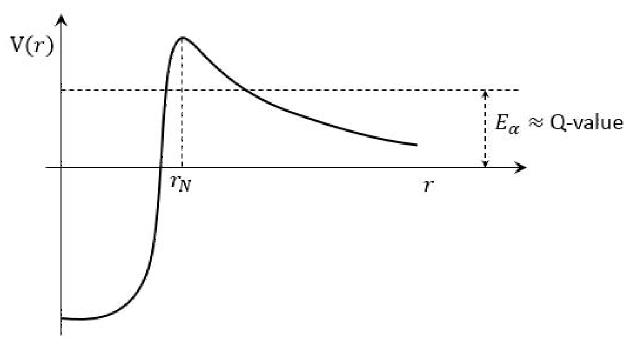


Figure 12.6: Alpha particle potential.

release Q is large, then its energy will be above the maximum of the potential and it can decay very quickly. However, the maximum of the potential will be approximately that given by the EM part:

$$V_{\max} \sim \frac{2(Z-2)e^2}{4\pi\epsilon_0 r_N} \sim \frac{2Ze^2}{4\pi\epsilon_0 r_0 A^{1/3}} = 2.4\frac{Z}{A^{1/3}}\ \text{MeV}\,.$$

For the nuclei which alpha decay, $Z/A^{1/3} \sim 15$ and so this barrier is several 10's of MeV, much larger than most observed alpha decay Q values.

Thus, as the Q value (the energy release) is usually well below this maximum, alpha decay can only occur by QM tunnelling. As you know, any QM tunnelling process drops exponentially as the barrier width or height increases. In this case the probability of transmission through the whole barrier is proportional to

$$e^{-2\int k(r)\,dr} = e^{-2G}\,,$$

where the Gamow factor G is given by:

$$G = \frac{\sqrt{2mQ}}{\hbar}\frac{Ze^2}{4\epsilon_0 Q} = \frac{e^2\sqrt{2m}}{4\epsilon_0\hbar}\frac{Z}{\sqrt{Q}} = 2.0\frac{Z}{\sqrt{Q}}\ \mathrm{MeV}^{-1/2}\,.$$

Appendix I shows this derivation if you want to explore further. The rate per nucleus R (equal to the inverse of the lifetime) is then expected to be:

$$R = \frac{1}{\tau} = ae^{-2G}\,,$$

so,

$$\log_{10} R = \log_{10} a - (2\log_{10} e)G = \log_{10} a - 1.7\frac{Z}{\sqrt{Q}}\,.$$

The plot in Figure 12.3 shows a slope of -1.7 which agrees well with most of the measured values. Hence, tunnelling is what causes the huge variation in lifetimes: for Q-values which only differ by 1 MeV, a difference of 10^5 is seen.

Appendix I – non-examinable material: Deriving the Gamow factor

For a rectangular barrier such as that depicted in Figure 12.7 the amplitude goes as e^{-kr}, where $k = \sqrt{2m(V_0 - E)}/\hbar$. This means the probability of transmission

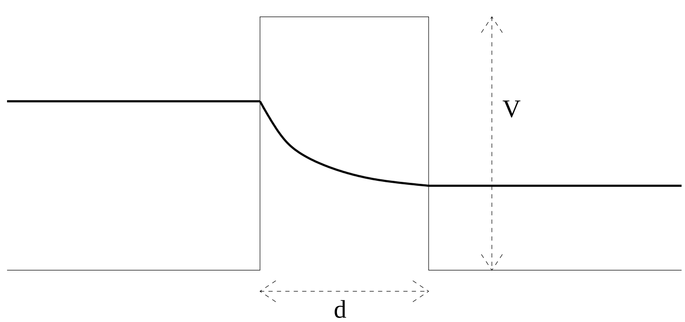


Figure 12.7: Rectangular potential barrier.

through the whole barrier is proportional to e^{-2kd}. Hence, a large d or small E give a very small probability. The above is for a constant V_0. However, the Coulomb barrier is a function of r, and so in this case the probability for an alpha getting through is proportional to

$$e^{-2\int k(r)\,dr} = e^{-2G}\,,$$

where the Gamow factor G is

$$G = \int k(r)\,dr = \frac{\sqrt{2m}}{\hbar}\int \sqrt{V(r) - Q}\,dr = \frac{\sqrt{2mQ}}{\hbar}\int \sqrt{\frac{2Ze^2}{4\pi\epsilon_0 Qr} - 1}\,dr\,.$$

The integral is over the range of radii for which the alpha energy is below the barrier, i.e. from the nucleus radius out to the radius where the alpha energy is greater than the potential. For small Q, this latter radius is much larger than the nuclear radius. This integral is not trivial (but can be found in most textbooks) but in the approximation of the upper limit being much larger than the lower, the integral is:

$$\int \sqrt{\frac{2Ze^2}{4\pi\epsilon_0 Qr} - 1}\,dr \approx \frac{2Ze^2}{4\pi\epsilon_0 Q}\frac{\pi}{2} = \frac{Ze^2}{4\epsilon_0 Q}\,.$$

This means the Gamow factor is:

$$G = \frac{\sqrt{2mQ}}{\hbar}\frac{Ze^2}{4\epsilon_0 Q} = \frac{e^2\sqrt{2m}}{4\epsilon_0\hbar}\frac{Z}{\sqrt{Q}} = 2.0\frac{Z}{\sqrt{Q}}\ \mathrm{MeV}^{-1/2}\,.$$

NPP Lecture 13 – Nuclear Fission – Part I

13.1 Introduction

We have seen how alpha decay can be understood as tunnelling through a Coulomb barrier when the alpha particle has less energy than the barrier height. Fission is the more general process of splitting a nucleus into two or more nuclear fragments. Although this clearly includes alpha decay, the term fission is more normally used to describe processes where the two daughter nuclei are much more even in size. We will look at the fission energetics, the types of fission and chain reactions, before going on to applications, such as nuclear power — over this lecture and part of the next.

13.2 Fission Energetics

As we now know the maximum binding energy per nucleon is reached around $^{56}_{26}$Fe, so we expect that fission is energetically possible for nuclei larger than around twice this size. It turns out that splitting a nucleus into two equal nuclei normally gives the largest energy release, Q:

$$Q = m_N(Z,N)c^2 - 2m_N(Z/2,N/2)c^2 = 2BE(Z/2,N/2) - BE(Z,N)\,.$$

This can be estimated from the semi-empirical mass formula. The volume terms clearly cancel, and ignoring the pairing term and approximating $Z(Z-1)$ to Z^2 gives:

$$\begin{aligned} Q = 2&\left[-a_s\left(\frac{A}{2}\right)^{2/3} - a_c\frac{Z^2/4}{(A/2)^{1/3}} - a_a\frac{(N/2-Z/2)^2}{A/2}\right] \\ &+ a_sA^{2/3} + a_c\frac{Z^2}{A^{1/3}} + a_a\frac{(N-Z)^2}{A}\,. \end{aligned}$$

This can be written as:

$$\begin{aligned} Q &= a_sA^{2/3}\left[1-2\left(\frac{1}{2^{2/3}}\right)\right] + a_c\frac{Z^2}{A^{1/3}}\left[1-2\left(\frac{1}{2^{5/3}}\right)\right] + a_a\frac{(N-Z)^2}{A}\left[1-2\left(\frac{1}{2}\right)\right] \\ &= a_sA^{2/3}(1-2^{1/3}) + a_c\frac{Z^2}{A^{1/3}}(1-2^{-2/3})\,. \end{aligned}$$

Energy is released when $Q > 0$, meaning

$$a_c \frac{Z^2}{A^{1/3}}(1 - 2^{-2/3}) > a_s A^{2/3}(2^{1/3} - 1)\,,$$

or

$$\frac{Z^2}{A} > \frac{a_s(2^{1/3} - 1)}{a_c(1 - 2^{-2/3})} = 0.702\,\frac{a_s}{a_c} \approx 18\,.$$

For nuclei on the beta-stability curve this is actually satisfied for $A > 100$, which gives $Z > 42$. Hence, like alpha decay, fission is only energetically possible for heavy nuclei.

Note, fission involves a much bigger change to A than alpha decay, and so gives a much larger shift closer to $^{56}_{26}$Fe in the binding energy per nucleon plot. Hence, the energy release in fission is significantly higher. We saw typical alpha decay energies are less than 10 MeV. Fission decays tend to release hundreds of MeV. This is one of the reasons why practical applications of nuclear energy use fission not alpha decay.

13.3 Spontaneous Fission

Classically, we can picture the fission process as the nucleus deforming and breaking up as represented in Figure 13.1.

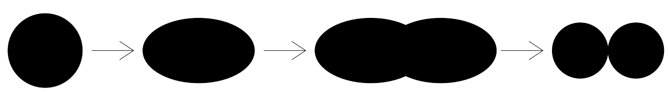


Figure 13.1: Representation of the nuclear fission process.

One critical issue is whether it takes energy or not to deform the nucleus in this way. The two terms which influence this are again the surface term and the Coulomb term. We have seen that nucleons on the surface are less strongly bound, due to missing nearest neighbours. For a given volume, a sphere has the smallest surface area and hence the biggest binding energy. Hence, deforming from a sphere to an ellipsoid with the same volume increases the surface area and so it requires energy. Explicitly, an ellipsoid can be defined in terms of a small deformation parameter δ, where the major axis is $r(1 + \delta)$ and the two minor axes are $r/\sqrt{1 + \delta}$. The surface area of the ellipsoid is then

$$4\pi r^2 \left(1 + \frac{2\delta^2}{5}\right)\,,$$

and so the change to the nucleus energy is $a_s(A^{2/3})(2\delta^2/5)$. However, a deformed nucleus has the protons on average further away from each other, and so the Coulomb energy is decreased. Hence, as the nucleus deforms, Coulomb energy is released. The approximate electrostatic energy of an ellipsoid for small deformations is:

$$\frac{3}{5}\frac{Q^2}{4\pi\epsilon_0 r}\left(1 - \frac{\delta^2}{5}\right)\,,$$

and hence the change to the nucleus energy from this effect is $-a_c(Z^2/A^{1/3})(\delta^2/5)$. For the Coulomb term to dominate, then

$$a_c \frac{Z^2}{A^{1/3}} \frac{\delta^2}{5} > a_s A^{2/3} \frac{2\delta^2}{5} \,,$$

which means

$$\frac{Z^2}{A} > \frac{2a_s}{a_c} \approx 51 \,.$$

For nuclei on the beta-stability curve, this corresponds to $A > 407$ which gives $Z > 144$. Clearly, no such nuclei exist in the periodic table. For these very large nuclei, the deformation actually reduces the nucleus energy so there is no barrier to this happening. This means they will spontaneously decay through fission and will do so extremely fast, $\sim 10^{-20}$ s. This sets an absolute upper limit to the periodic table.

For nuclei intermediate between these two values, i.e. $18 < Z^2/A < 51$, then fission overall is energetically allowed but for the deformation to start it requires energy and so, as for alpha decay, there is a potential barrier to tunnel through. In this case we cannot really consider one of the large daughter nuclei as forming and moving in a potential within the nucleus, so it is not so easy to draw such a potential well as for alpha decay. However, a qualitative picture can be obtained by looking at how the energy needed to deform the nucleus depends on the deformation. We know that, for nuclei in this range, it takes energy to deform the nucleus, but when the daughter nuclei are separated enough that the nuclear force between them is small, then the Coulomb force dominates, as for alpha decay. Hence, a rough idea of the energy as it deforms is that depicted in Figure 13.2.

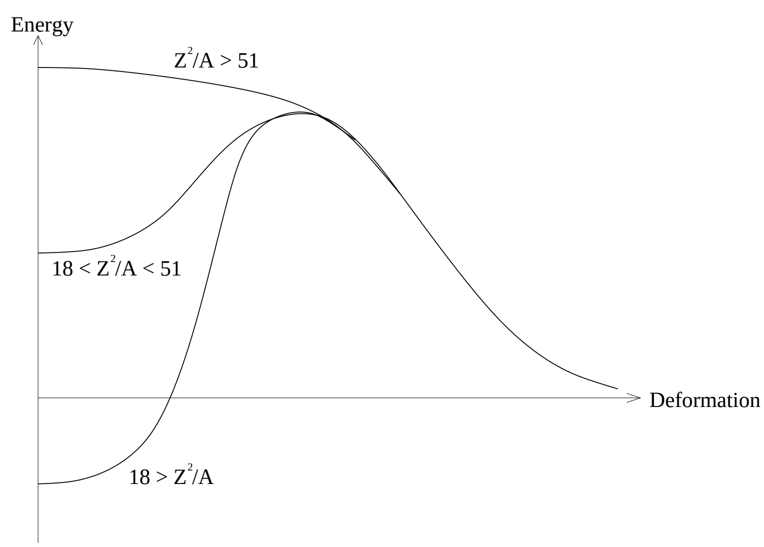


Figure 13.2: Potential energy of a deforming nucleus.

The fact that nuclei in this intermediate range have to tunnel gives again a very strong range of lifetimes and an exponential dependence on Z^2/A over 30 orders of magnitude — see Figure 13.3.

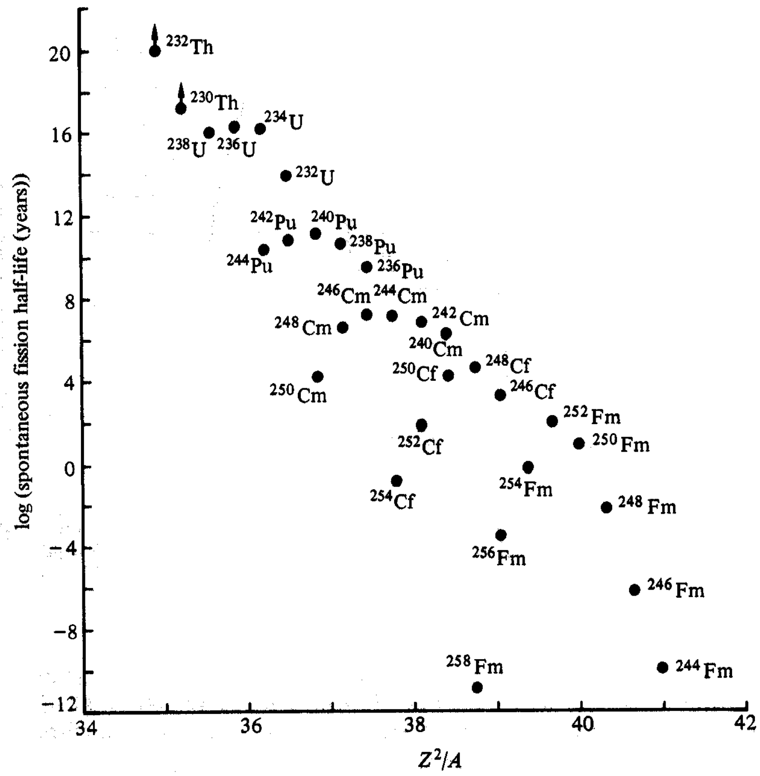


Figure 13.3: Dependence of fission half-life on Z^2/A parameter.

Spontaneous fission does not normally have a rate competitive with alpha decay; for example, $^{238}_{92}$U has a partial half life for alpha decay of around 10^9 years and for fission of 10^{16} years.

As is the case for alpha decay, the daughter nuclei tend to have too many neutrons to lie on the beta-stability curve. For alpha decays, where the changes are small, then beta decay is the usual way by which nuclei return to the beta-stability curve. However, in fission the daughter nuclei are usually highly excited and in any case often so far away from the curve that they often spontaneously emit neutrons, typically between one and four per fission. In fact, this subsequent neutron emission can itself be considered to be a further fission process, but as neutrons have no charge there is no Coulomb barrier to overcome.

13.4 Induced Fission

It is clear that for most heavy nuclei fission is difficult and hence slow. It can be speeded up enormously by exciting the nuclei. For practical uses of the fission energy, this is clearly essential. If enough energy can be added to a nucleus to raise its energy higher than the fission barrier, then it will fission within 10^{-20} s, the very fast rate associated with the strong force, as happens for the nuclei above the barrier, i.e. those with $Z^2/A > 51$.

There are several ways to excite nuclei to higher levels. The most obvious would be to bombard them with gamma radiation, as that is how excited nuclei can decay and indeed it is perfectly possible to excite them in this way. However, this is an EM reaction and so does not have as large a cross section as a strong force reaction. In addition, there are no simple ways to make intense gamma sources in practice.

A high fission rate requires a strongly interacting particle, such as a proton, neutron or alpha. Of these, neutrons are special: they are neutral and so, unlike protons and alphas, are not repelled from the nucleus by their EM charge. To react via the nuclear force, protons and alphas would have to either have a high enough energy to overcome the Coulomb barrier, or they would have to tunnel through the barrier, drastically reducing the rate. Neutrons face no such problem and so can react at any energy effectively down to zero. Cross sections tend to be largest at low energies as, roughly speaking, there is more time to react. Neutron absorption dominates at such energies, leaving an excited nucleus. Clearly, any excited nucleus has some probability of gamma decaying to the ground state and this competes with the fission decay. The ratio of gamma to fission decays depends strongly on how far above the fission barrier the excited nucleus is.

When a nucleus absorbs a neutron, the N and A values increase by one. Hence, if we have a nucleus ${}^{A}_{Z}\mathrm{X}$ in which we want to induce fission, then we have to start with its isotope ${}^{A-1}_{\ \ Z}\mathrm{X}$:

$$ {}^{A-1}_{\ \ Z}\mathrm{X} + n \rightarrow \ {}^{A}_{Z}\mathrm{X}^* \rightarrow \ \text{fission}\,. $$

Naturally, the neutron must possess enough energy to excite the nucleus over the fission barrier. This depends on the relative binding energies of the two isotopes. The total input energy of the reaction is:

$$ \begin{aligned} m_N(Z,N-1)c^2 + E_n &= Zm_pc^2 + (N-1)m_nc^2 - BE_i + m_nc^2 + T_n \\ &= Zm_pc^2 + Nm_nc^2 - BE_i + T_n\,, \end{aligned} $$

where T_n is the neutron kinetic energy. The final nucleus has an energy in its ground state of

$$ m_N(Z,N)c^2 = Zm_pc^2 + Nm_nc^2 - BE_f\,. $$

Hence, neglecting the small nucleus recoil energy, the excited nucleus will be formed at an energy above the ground state given by the difference of these:

$$ \Delta E = BE_f - BE_i + T_n\,. $$

To fission, ΔE must be greater than the fission barrier height. If the initial isotope is weakly bound and the final is strongly bound, so $BE_f - BE_i$ is large, then T_n can be small and even zero. Conversely, a small $BE_f - BE_i$ would demand a large neutron energy with a subsequent reduction in cross section.

The binding energy tends to vary quite smoothly (except near magic numbers) in the semi-empirical mass formula — with the exception of the pairing term. Clearly as A changes by one in neutron absorption, then one of the isotopes is even A and the other odd A. Hence, depending on whether the even-A is even-even or odd-odd, part of this difference will be given by $a_p/A^{1/2} \sim 1$ MeV (noting the larger contribution that comes from the volume term, as we are comparing nuclei with different A). Examples of nuclei which can fission after absorbing even a zero energy neutron are the odd A nuclei $^{233}_{92}$U, $^{235}_{92}$U, $^{239}_{94}$Pu and $^{241}_{94}$Pu. These all have even Z and odd N; the extra neutron then makes these nuclei even-even and so have a large BE_f. Even A nuclei which require a fast neutron include $^{232}_{90}$Th, $^{238}_{92}$U, $^{240}_{94}$Pu and $^{242}_{94}$Pu. In these cases, they are all even-even and so have a large BE_i to start with.

NPP Lecture 14 – Nuclear Fission – Part II Nuclear Fusion – Part I

14.1 Introduction

After examining spontaneous and induced fission, we will conclude this topic by looking at how nuclear fission has been exploited from the mid-20$^{\text{th}}$ century for both civilian and military ends.

Then we examine the process of *fusion*, whereby nuclei below ^{56}Fe can combine and release energy. After looking at the basics of fusion we present two examples, namely fusion power generation and nucleosynthesis.

14.2 Chain Reactions

We have concluded that fission results in nuclei with more neutrons than are needed to lie on the beta-stability curve. These are often ejected, and fission commonly results in between one and four neutrons being emitted. We have also seen that neutrons can excite nuclei and so speed up the fission rate enormously. Together, these two facts can allow a chain reaction to occur. If the neutrons produced in a nuclear fission can be used to cause further fissions, then a chain reaction occurs. This has been used to produce power from nuclear fission as a lot of energy is released. Atomic bombs and nuclear power stations work using chain reactions.

If the mean time required for a neutron to induce a fission event is τ_f, then the number of fissions in a short time δt is $n(\delta t/\tau_f)$ for n neutrons present. Taking the average number of neutrons from each fission which react further as m, then this number of fissions produces $mn(\delta t/\tau_f)$ new neutrons. Hence, the total change to the number of neutrons is:

$$\delta n = mn\frac{\delta t}{\tau_f} - n\frac{\delta t}{\tau_f} = (m-1)n\frac{\delta t}{\tau_f}\,.$$

Taking δt to be infinitesimally small, this gives:

$$\frac{dn}{dt} = (m-1)\frac{n}{\tau_f}\,,$$

which we can write in the form:

$$\frac{dn}{n} = (m-1)\frac{dt}{\tau_f}\,.$$

Integrating, we obtain

$$\ln n = (m-1)\frac{t}{\tau_f} + k\,,$$

for some constant of integration k, so

$$n = e^k e^{(m-1)t/\tau_f} = n_0 e^{(m-1)t/\tau_f}\,,$$

where n_0 is clearly the number of neutrons at time $t = 0$. Hence, the number of neutrons (and hence the fission rate) increases or decreases exponentially depending on the value of $m-1$. Clearly, when $m < 1$, there are not enough neutrons being produced to sustain the chain reaction indefinitely: such a situation is called 'subcritical'. Conversely, $m > 1$ means that more neutrons are produced at each step and so the reaction rate increases: this is called 'supercritical'. The dividing line, when $m = 1$, is said to be 'critical'.

We defined m as the average number of neutrons emitted from each fission which cause further fissions. This is clearly not necessarily the number emitted from each fission (i.e. the neutron 'multiplicity'). Some neutrons can react by other processes (e.g. absorption followed by gamma emission, rather than fission) or be lost from the surface of the material. One of the critical issues in sustaining a fission chain reaction is optimising the number of neutrons for the purpose required.

14.3 Uranium Fission

The most important material for fission chain reactions is uranium. Plutonium has similar properties with regard to fission and so can also be used for a chain reaction. However, all plutonium isotopes have lifetimes of around 10^5 years as opposed to 10^9 years for uranium, so there is no natural plutonium left. To be used, it has to first be manufactured in a nuclear reaction. Hence, for practical reasons, uranium is the most commonly used material.

The main fission isotope of uranium is $^{236}_{92}$U. This is an even-even nucleus and hence benefits from the extra pairing energy associated with this. The fission barrier height for this isotope is around 6.2 MeV above the ground state. It is made by neutron absorption from $^{235}_{92}$U, which is even-odd and so is less strongly bound. This results in the binding energy difference of these two nuclei being $\Delta BE = 6.5$ MeV, which is more than the barrier height. Hence, absorption of even a zero energy neutron by $^{235}_{92}$U is enough to give fissionable $^{236}_{92}$U. The A distribution of the two nuclei produced from this fission is shown in Figure 14.1. Note the nuclei tend to be unequal, despite the energy release being maximum for an equal division.

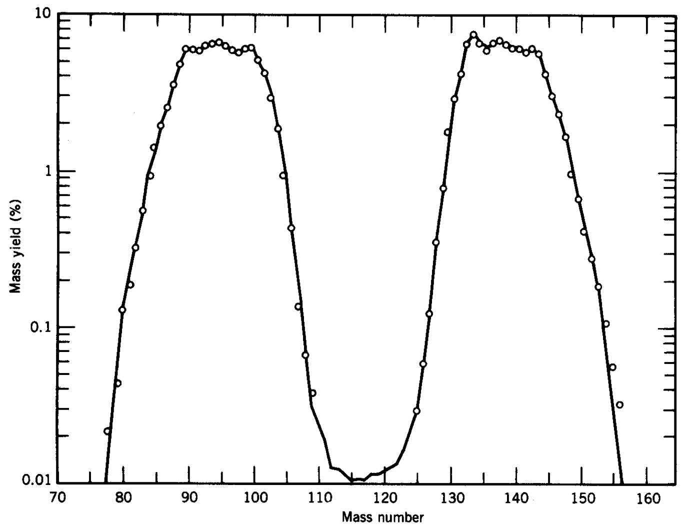


Figure 14.1: Mass distribution of fission fragments for neutron-induced fission of U-235.

Because these two fragments are excited and neutron rich, they rapidly give off neutrons. Each $^{236}_{92}$U fission event produces 2.5 neutrons on average, and so this nucleus has the possibility of producing a supercritical chain reaction if, again on average, at least one of these neutrons can be made to cause a further fission. The neutrons are emitted with an average energy of around 2 MeV each. The total immediate energy release per fission is around 180 MeV, although the daughter nuclei usually beta and gamma decay later on, giving roughly another 20 MeV at later times. This is a substantial amount of energy per fission event.

Natural uranium is found in two isotopes, the $^{235}_{92}$U needed for the above reaction and $^{238}_{92}$U. These have natural abundances of 0.72% and 99.28%, respectively, so the required isotope is actually a very small proportion of uranium ore. Would the other, more abundant isotope be useful for fission? $^{238}_{92}$U is an even-even nucleus and so relatively strongly bound. Neutron absorption by $^{238}_{92}$U gives $^{239}_{92}$U, which is even-odd. Hence, the binding energy difference in this case is quite a bit smaller than before, i.e. by $2a_p/A^{1/2} \sim 1.5$ MeV. The actual $\Delta BE = 4.8$ MeV, compared with a fission barrier height for $^{239}_{92}$U of 6.2 MeV, i.e. very similar to $^{236}_{92}$U, as would be expected as they have the same charge. Hence, in this case, neutrons of at least 1.4 MeV kinetic energy are needed to induce fission; lower energy ones will be absorbed and mainly produce gamma decays. The total and fission cross sections for neutrons on these two uranium isotopes are shown in Figure 14.2.

Although the fission cross section is a reasonable proportion of the total for $^{235}_{92}$U, it is not for $^{238}_{92}$U. The latter also has several resonances with large very cross sections below the fission threshold which will strongly absorb neutrons without fissioning. The main problem with getting a chain reaction from uranium is, therefore, that the fissionable isotope is such a small proportion of the total. Natural uranium is well below being critical, i.e. $m \ll 1$, due to the other neutron reactions in $^{238}_{92}$U.

14.4 Fission Reactors

Nuclear fission was discovered in the late 1930s and the obvious potential for vast energy release motivated a race to develop the first atomic bombs in the 1940s (see Appendix I). The more peaceful use of fission — in civil power generation — came only later. Here, the use of pure ${}^{235}_{92}$U is too expensive. Nuclear power stations use natural or slight enriched uranium and get around the problem of losing neutrons to the dominant ${}^{238}_{92}$U by using a 'moderator'. This is a light material, such as carbon or heavy water, which has a high likelihood of scattering neutrons (without capturing them) so they can lose energy quickly. Low Z nuclei are favoured as they are light and so the energy lost by the neutrons is larger. A common technique is then to have narrow uranium rods inserted into the moderator. The geometry is optimised so that the neutrons from fission, with energies around 2 MeV, have a good chance of escaping from the uranium rods into the moderator. They then scatter many times, losing energy each time, and usually become 'thermalised', meaning they have energies corresponding to the temperature of the moderator: in practise around 1000 K, or $k_BT \sim 0.1$ eV. These thermal neutrons then have some probability of scattering back into the uranium and, at such low energies, the most probable reaction is ${}^{235}_{92}$U fission as the very high cross section at such low energies more than compensates for the small proportion of this isotope.

The design of the reactor needs to balance the probabilities for losing neutrons altogether, having them re-enter the uranium too soon, with too high energies, and so be absorbed by the ${}^{238}_{92}$U resonances, having them react with the moderator, etc. — and achieve a value of $m = 1$ so the reaction continues at a constant level. Control is provided by inserting or withdrawing control rods made of a strongly neutron-absorbing material, such as cadmium or boron, which can soak up a lot of neutrons if needed and keep $m = 1$. Finally a cooling system is required to remove the heat (energy) produced by the fission; this heat is then converted to usable energy via a heat exchanger. Practical reactor design is extremely complex and there are many different ways in which a reactor can be built. A schematic of a nuclear power plant is shown in Figure 14.3.

One final comment is that the neutron-rich daughter nuclei tend to be highly radioactive, and this material — the radioactive waste — needs to be stored until safe. As some of the radioisotopes produced have lifetimes of millions of years, this is a very difficult issue and there is as yet no agreement on a long-term solution. What is clear, though, is that the use of other fission isotopes and more modern techniques, such as accelerator-driven systems, can very significantly reduce these issues. Increasing effort is finally going into such areas.

14.5 Fusion Reactions

The binding energy per nucleon curve is reproduced in Figure 14.4 below, showing that all nuclei below $A \approx 56$ are lower than the maximum.

Therefore, energy can be released through fusion. The generic fusion process is:

$$^{A_X}_{Z_X}\mathrm{X} + \ ^{A_Y}_{Z_Y}\mathrm{Y} \rightarrow \ ^{A_X+A_Y}_{Z_X+Z_Y}\mathrm{F} \,.$$

For energy release, the final nucleus must have less total mass than the initial two nuclei. Moreover, F must actually be in an excited state to conserve energy and momentum simultaneously. The released energy will therefore come out when it decays. For example, it can gamma decay to release the extra energy:

$$\mathrm{X} + \mathrm{Y} \rightarrow \mathrm{F}^* \rightarrow \mathrm{F} + \gamma \,.$$

We often need not consider F^* explicitly and, indeed, often no coherent nucleus can be considered to be made.

The simplest case of such a reaction is the creation of a deuteron ($d = \ ^2_1\mathrm{H}$) from a proton and neutron:

$$p + n \rightarrow d + \gamma \,,$$

where, for $E_p \approx E_n \approx 0$, the emitted photon energy is $E_\gamma = Q = (m_p + m_n - m_d)c^2 = 2.2$ MeV. This is clearly an electromagnetic reaction and so, in general, it does not have such a large cross section as a strong reaction.

Other possibilities for the energy to be released are to emit protons or neutrons in a strong interaction. An example of this is the so-called "D-T" reaction of a deuteron and tritium ($t = \ ^3_1\mathrm{H}$) to make helium and a neutron:

$$d + t \rightarrow \ ^4_2\mathrm{He} + n \,.$$

We will come across this reaction again when we discuss fusion power generation. It is also possible to emit weakly interacting particles; the most important reaction of this type is:

$$p + p \rightarrow d + e^+ + \nu_e \,,$$

but, being weak, the cross section is very small compared with the previous reactions. We will see this reaction again when we look at stellar nucleosynthesis (how the sun works).

14.6 Coulomb Barrier

There are no bound states made purely of neutrons even for $A > 2$ so, in fusion we will always consider combining nuclei containing some protons. This means they have positive charge and so in almost all cases there will be a Coulomb repulsive force to be overcome before the nuclei can be brought close enough together that the attractive nuclear force can take effect. This is effectively the same issue as for fission or alpha emission, but in reverse — see, e.g., Figure 12.5 on page 112. The barrier can, in principle, be overcome in two ways: either by having high enough energy for the two incoming particles, or by tunnelling. However, the latter gives such a small cross section that it is not done in practice. The barrier height is of order

$$\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 r_0 A^{1/3}} = a_c \frac{Z_1 Z_2}{A^{1/3}} = 0.72 \text{ MeV } \frac{Z_1 Z_2}{A^{1/3}} \,.$$

Clearly, we have accelerators which can speed up charged particles up to TeV, i.e. well beyond the Coulomb barrier height, so it is straightforward to do experiments on these reactions. However, for practical applications, this is not a feasible way to go and instead very high temperatures are used. This gives the nuclei a range of kinetic energies as per the Maxwell-Boltzmann distribution, so there are some at the higher end with enough energy to overcome the barrier. For a barrier of order 1 MeV, the temperature corresponding to this energy is $\sim 10^{10}$ K. Hence, even allowing a mean value well below this and using the high energy tail, still means extremely high temperatures are needed. Any matter is always a plasma at these temperatures as it is well above any atomic binding energies. Often the plasma is also compressed to very high pressures to increase the reaction rate. The technique of using such temperatures and pressures to overcome the Coulomb barrier is called 'thermonuclear' fusion.

14.7 Helium

Let us look at the binding energy per nucleon plot again in more detail. A clear feature is the large value for ${}^4_2$He, particularly compared with the values just above it. ${}^4_2$He is doubly magic and stands out as being much more strongly bound than its neighbours, with a value of $BE/A \approx 7.1$ MeV.

The maximum value of the binding energy per nucleon is around 8.7 MeV for nuclei close to iron. If we create ${}^4_2$He in fusion as a first step to building up to iron, we already get over 80% of the maximum possible energy release in this step alone.

Hence, all practical applications of fusion concentrate on combining hydrogen isotopes into ${}^4_2$He. This also has the advantages that the Coulomb barriers are smaller for these nuclei and that hydrogen and deuterium are readily available. Deuterium occurs naturally in water and, while 0.015% is not much of an abundance, water is clearly plentiful and so the supply is potentially enormous. Also, unlike uranium, deuterium is relatively easy to separate from hydrogen as the masses differ by a factor of two, rather than 1%. The most obvious reaction to make helium would be using two deuterons:

$$d + d \to {}^4_2\mathrm{He} + \gamma\,, \qquad Q = 23.8 \text{ MeV}$$

but although the energy release is large, this is an EM reaction with a correspondingly low cross section. The more common reactions are:

$$d + d \to {}^3_2\mathrm{He} + n\,, \qquad Q = 3.3 \text{ MeV}$$

and

$$d + d \to {}^3_1\mathrm{H} + p\,, \qquad Q = 4.0 \text{ MeV}.$$

This final reaction produces tritium, which is the third isotope of hydrogen. Tritium beta decays to ${}^3_2$He with a half-life of 12.3 years so none is found naturally and it has to be manufactured. This can be done from the above or by using the lithium reactions

$$ {}^{6}_{3}\mathrm{Li} + n \to {}^{4}_{2}\mathrm{He} + t\,, \qquad {}^{7}_{3}\mathrm{Li} + n \to {}^{4}_{2}\mathrm{He} + t + n\,. $$

As previously mentioned, tritium also reacts with a deuteron (the D-T reaction) to produce helium through a strong interaction:

$$ d + t \to {}^{4}_{2}\mathrm{He} + n\,, \qquad Q = 17.6 \text{ MeV}. $$

This releases a lot of energy as the helium is strongly bound and happens to have a large cross section, making it good for practical applications. The main disadvantage is that tritium is needed, which firstly must be manufactured and secondly must be replenished as it decays. Also, the neutron produced takes more than half the energy and extracting that for power uses is not straightforward.

Appendix I – non-examinable: The Fission Atomic Bomb

U-235 is the only naturally-occurring isotope which is thermally fissile, i.e. fission can occur from the capture of neutrons down to thermal energies. However, a chain reaction is not viable in natural uranium, which is mostly ${}^{238}_{92}\mathrm{U}$, as this isotope does not fission easily and it would instead absorb most neutrons and deexcite through gamma-rays. The conceptually easiest way around this problem is to separate the uranium isotopes and use only ${}^{235}_{92}\mathrm{U}$. This is the method used for atomic fission bombs, the so-called “A bomb”. However, uranium enrichment is a very difficult thing to do in practice. By definition, the uranium isotopes have the same nuclear charge and hence extremely similar chemistry, so they cannot be separated by forming different compounds. They differ by 3 nucleons in around 240, so their masses differ by only around 1%. The main method used is a centrifuge, where a suspension of a uranium compound is spun at very high angular velocity. The centrifugal force is proportional to mass and so the heavier isotope atoms tend to migrate to the outermost parts of the container.

Assuming some approximately pure ${}^{235}_{92}\mathrm{U}$ can be acquired, then enough has to be assembled to go supercritical. As stated above, each fission results in around 2.5 neutrons, each of around 2 MeV. From the cross sections above, a 2 MeV neutron has an 18% chance of causing a fission, which by itself would mean $m \sim 0.45$, which is subcritical. However, the other 82% of the total cross section is mainly scattering, not absorption, where the neutron simply loses energy to the nucleus and continues. The loss of energy actually makes the neutron more likely to induce fission, as the fission proportion of the total cross section increases as the neutron energy drops. After several such scatters, the neutron has a high probability of causing a fission event. The actual average number of scatters before fission is around six.

We can make an order of magnitude estimate of how long the neutron takes to fission and the average distance gone in this time quite simply. The mean free path is generally given by

$$\langle d \rangle = \frac{1}{\rho\sigma}\,,$$

where ρ is the number density of targets and σ is the cross section. For a neutron in ${}^{235}_{92}$U, it usually scatters down through energies from a few MeV to a few keV before being captured and causing fission. Take the average kinetic energy order of magnitude to be around 0.1 MeV in this range, which gives a cross section for scattering of around 10 barns. The number density of uranium is 5×10^{28} m^{-3}, so the mean free path is $\langle d \rangle \sim 2$ cm. As these neutrons are not relativistic, the kinetic energy is approximately $m_n v^2/2 = m_n c^2 \beta^2/2$ and with $m_n \sim 1{,}000$ MeV/c^2, then $\beta \sim 0.01$. Hence, the velocity is $\sim 3 \times 10^6$ m/s and so the time between scatters is around 10^{-8} s. The average of six scatters will take of order $\tau_f \sim 10^{-7}$ s and with a random walk the neutron will travel of order 10 cm.

Hence, the fraction of neutrons which actually cause fission is pretty high and the main factor limiting m in pure ${}^{235}_{92}$U is the loss of neutrons from the surface. Clearly, the bigger the volume, the smaller this effect. Hence, there is a critical sphere radius at which the loss is such as to make $m = 1$; this is clearly around the size of the distance a neutron goes and is in fact 8.7 cm. The corresponding quantity of uranium is known famously as the 'critical mass' and is 52 kg. This mass can be significantly reduced by the use of a neutron reflector (a 'tamper') around the uranium core.

A mass larger than this critical value will have a smaller surface loss, so that $m > 1$ and it will be supercritical. The fission rate will increase exponentially with a time constant of $\tau_f \sim 10^{-7}$ s and, as each fission releases around 200 MeV, there will be an enormous energy release in an extremely short time. The first atomic bombs had energy yields of around 10^{14} J; even then, only around 10% of the uranium actually fissioned before the rest was blown apart by the explosion.

Ignoring the difficulties of getting pure ${}^{235}_{92}$U, which are fortunately very significant, the main issue to overcome in an atomic bomb is how to assemble the critical mass at precisely the required time. If not done all at once, the subcritical masses can still produce enough heat to explode or melt any mechanical structure and so prevent the completion of the critical mass.

One way to achieve this is to remove a plug of uranium from the centre of a ${}^{235}_{92}$U sphere, and to 'shoot' this plug back into the sphere at the right moment, at which point the assembly goes supercritical. The bomb dropped on Hiroshima in 1945 was of this gun-type design. A second type is the implosion device, whereby a subcritical amount of material is compressed by conventional explosives and the inward momentum is then sufficient to allow supercriticality for a long enough period before the heat generated blows the mass apart. A neutron-emitting initiator is usually placed at the centre to produce neutrons to start the chain reaction going. The diagram in Figure 14.5 shows a similar design but for a plutonium bomb. The atomic bomb dropped on Nagasaki was of this type.

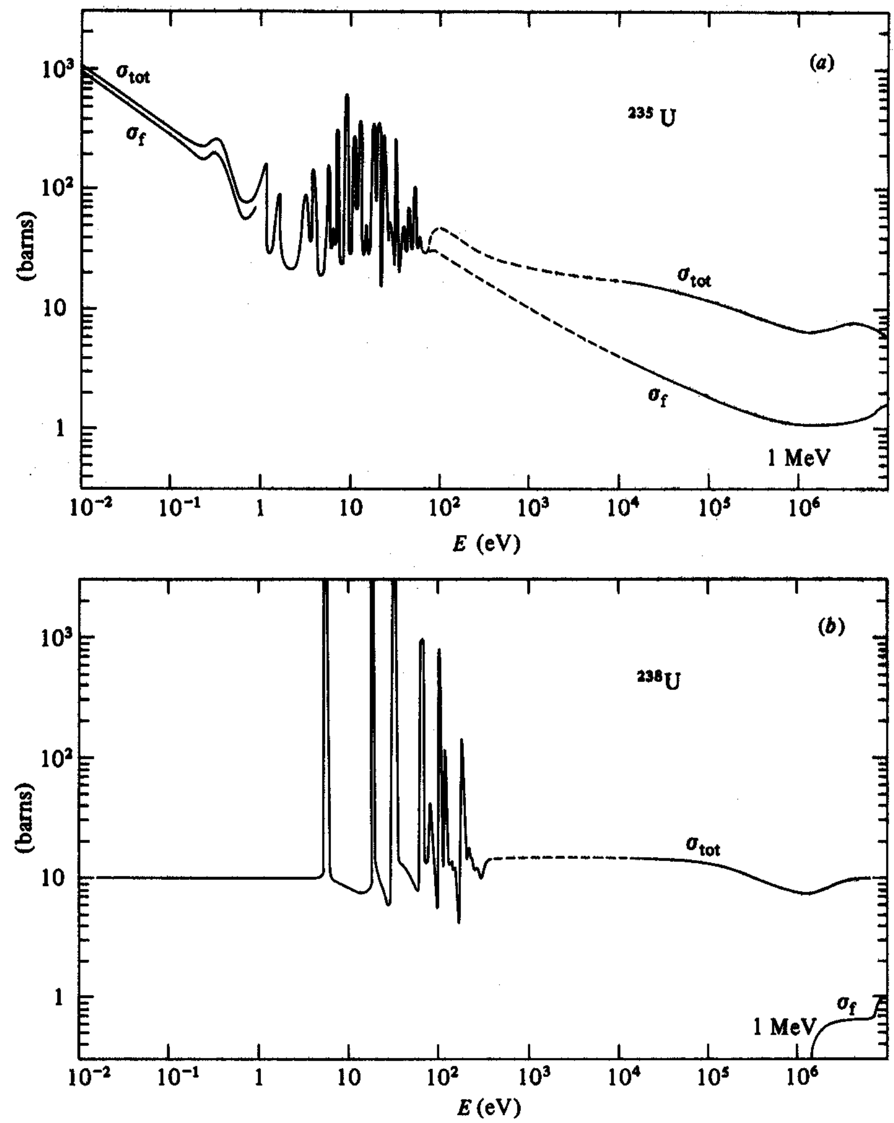


Figure 14.2: Neutron cross sections for U-235 and U-238, showing neutron-induced fission and the total cross section (which includes neutron capture resulting in gamma-ray emission).

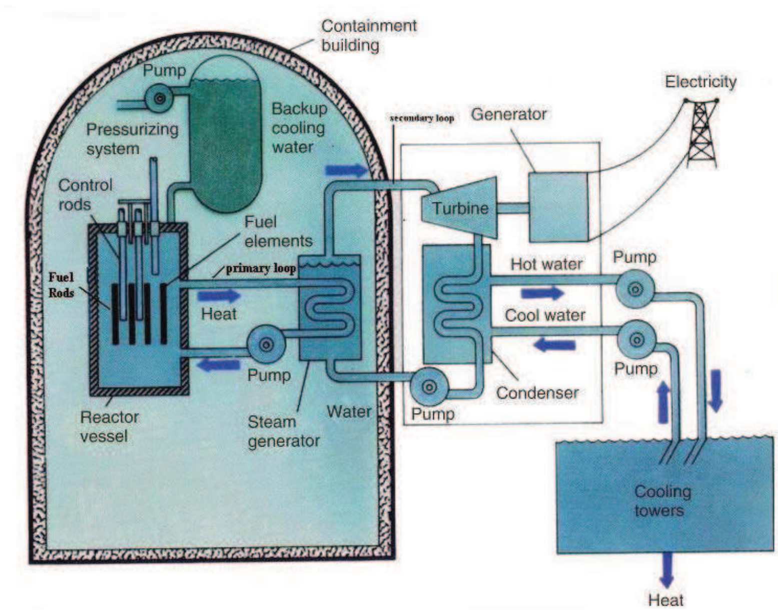


Figure 14.3: Nuclear power reactor.

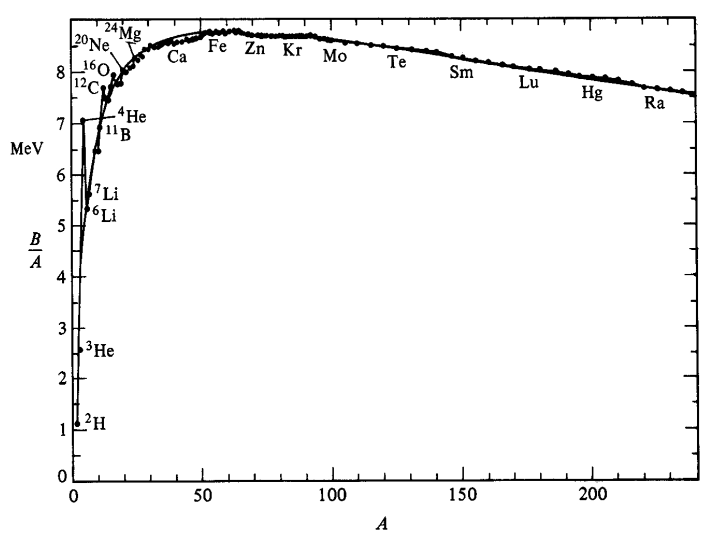


Figure 14.4: Nuclear binding energy per nucleon.

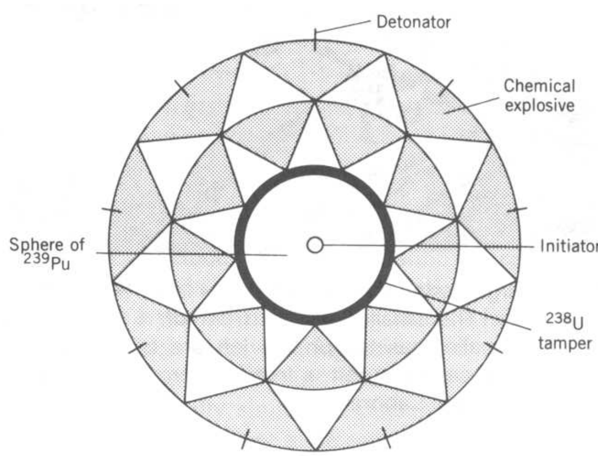


Figure 14.5: Schematic of the plutonium-239 implosion bomb.

NPP Lecture 15 – Nuclear Fusion – Part II

15.1 Fusion Reactors

As with fission, there is a peaceful application of fusion as a power source for energy generation. However, unlike for fission, this has never been achieved in a sustainable, power-neutral, efficient way. The benefits would be major, as the deuterium fuel is plentiful and there are no radioactive daughter products, unlike for fission, which result in long-lasting radioactive waste (at least in their current form). The main challenges are not in understanding the nuclear physics involved, but in the technology of how to produce the very high temperatures and pressures needed. This is obviously a very active area of research. One approach is to use lasers to rapidly implode capsules of D-T mixture using the photon pressure from the intense laser beams, as illustrated in Figure 15.1. The lasers have to be extremely fast and powerful, themselves requiring significant power input, so achieving a net power output is difficult.

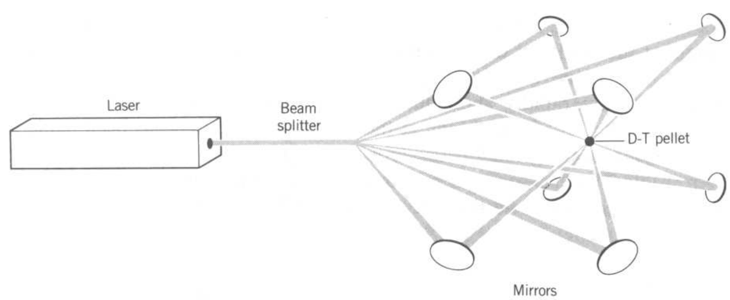


Figure 15.1: Concept of laser-induced nuclear fusion.

An alternative approach is to try to store the plasma at the temperature and pressure required while the reaction proceeds. Since the plasma would vaporise any solid material it touched, then a magnetic containment system is used, where magnetic fields are shaped such that the charged plasma particles spiral round in the field and cannot emerge and touch the vessel walls — see Figure 15.2.

It is assumed that any future reactor will be likely to use tritium and hence it needs to be able to manufacture this in such a way as to replenish its fuel supply. By surrounding the reactor with $^7_3$Li, the neutron capture reaction mentioned previously can be used to produce tritium from the emitted neutrons of the D-T

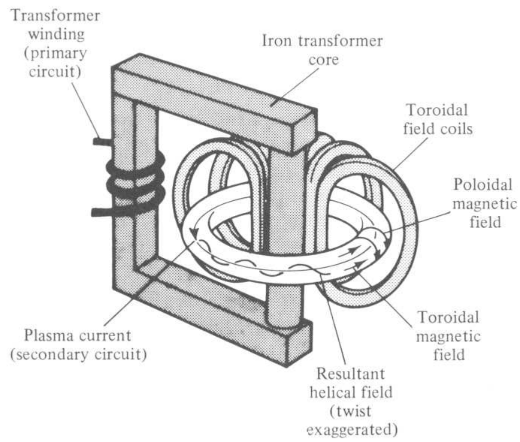


Figure 15.2: Concept of the Tokamak fusion reactor.

reaction. Since a further neutron is re-emitted in the lithium reaction, then it is feasible to believe enough neutrons are available to make the tritium at a rate at least equal to the rate it is used as fuel.

15.2 Nucleosynthesis

We now turn our attention to another practical application of fusion: as the power source of the stars, including our own Sun. The Sun works by combining protons into heavier nuclei, so giving out fusion energy. In fact, almost all nuclei heavier than helium were made in stars. However, firstly, we have to understand how the Sun is powered by hydrogen in the first place.

15.2.1 Big Bang nucleosynthesis

In the very early Universe, specifically within the first 4 minutes after the Big Bang, matter was too hot to form nuclei: the collisions were just too energetic for any nuclear binding to survive. The matter was all separate particles, namely electrons, neutrinos, protons and neutrons; the latter have a lifetime of around 15 minutes and so had not decayed significantly yet. There were also a lot of photons up to high energies. However, as the Universe expanded and hence cooled, it reached a temperature of around 10^9 K which corresponds to 0.1 MeV, when larger nuclei could form with a good probability of survival.

The first reaction which could occur was:

$$p + n \to d + \gamma\,,$$

which has $Q = 2.2$ MeV. This relatively low value indicates that the deuteron is not strongly bound and it is the ease of disintegrating this initial bound state which prevented nuclei from forming earlier. However, after 4 minutes, enough of the deuterons can live for a significant time that further reactions with the nucleons could occur:

$$p + d \to {}^3_2\text{He} + \gamma\,, \qquad n + d \to {}^3_1\text{H} + \gamma\,,$$

followed by

$$p + {}^3_1\text{H} \to {}^4_2\text{He} + \gamma \qquad n + {}^3_2\text{He} \to {}^4_2\text{He} + \gamma\,.$$

The helium nuclei are strongly bound and so are not easily knocked apart again; hence, once past the deuteron bottleneck, these reactions proceed quickly and give mainly helium.

However, to get further fusion after ${}^4_2\text{He}$ is hard. Hence, most of the nuclear matter in the Universe was still single nucleons, with some helium in addition. In fact, after about 30 minutes the temperature had dropped by around a factor of three and most of the neutrons had decayed. The latter shut off the deuteron creation reaction, and the lower temperature made the other fusions much less likely due to the Coulomb barrier. Hence, the nuclei were "frozen" after 30 minutes and remained in that state for millions of years until stars formed. The composition of the Universe during this time was (by mass) 76% protons, 24% helium and traces of deuterons, ${}^3_2\text{He}$ and ${}^7_3\text{Li}$ — as all the other species, including the neutrons and tritium, had decayed. Even since their creation, the stars have only fused a small percentage of the matter in the Universe, so the overall composition of the Universe today is close to how it was 30 minutes after the Big Bang.

15.2.2 Stellar nucleosynthesis

When the Universe was around 10^6 years old, the temperature had dropped low enough, $\sim$ 2000 K, that neutral atoms could form without immediate reionisation. This turned off electromagnetism as the dominant force between matter particles and allowed gravity to take over. Being purely attractive, this meant that any fluctuations in the density of matter would grow, attracting more matter and eventually coalescing into stars. As the matter fell together (including dark matter!), the gravitational potential energy became kinetic energy which through collisions caused the temperature to rise. Given enough mass, very high temperatures could be reached, allowing thermonuclear fusion reactions to start.

The critical difference from nucleosynthesis in the early Universe was the absence of neutrons. The early neutrons had decayed soon after the Big Bang and so the stars which formed were effectively made from hydrogen with some helium. At the temperatures which were reached, these gases form a plasma. As before, the first step was to make deuterons but without neutrons this is very difficult. It was only possible by changing a proton to a neutron, which required a weak interaction:

$$p + p \to d + e^+ + \nu_e\,.$$

This is much slower than the proton-neutron reaction of the early Universe for two reasons. Firstly, there is now a Coulomb barrier to be overcome and, secondly, this is a weak, not electromagnetic, interaction. A very rough estimate of the cross section σ_d for this critical first process is to assume the reaction occurs through the two protons interacting through the strong force and then requiring a beta decay simultaneously. The strong interaction cross section for protons σ_{pp} is of order 10 barn at these energies. The strong part of the reaction will last for the typical strong interaction time of order $\tau_S \sim 10^{-23}$ s. An estimate of the typical time needed for a beta decay at these energies is given by the neutron lifetimes, which is of order $\tau_\beta \sim 10^3$ s. Hence, an order of magnitude estimate for the cross section for deuteron production is:

$$\sigma_d \sim \sigma_{pp}\frac{\tau_S}{\tau_\beta} \sim \frac{10 \times 10^{-23}}{10^3} \sim 10^{-25}\ \mathrm{b} \sim 10^{-53}\ \mathrm{m}^2\,.$$

This is extremely small and leads to a slow reaction rate. The density of matter within the Sun is around $n \sim 10^{32}$ protons/m^3 so the mean free path of protons before they undergo this reaction is $1/n\sigma_d \sim 10^{21}$ m, which takes them $\sim 10^9$ years on average. This is why the Sun burns up its hydrogen so slowly, and hence why we are able to exist. If the deuterons could be formed more quickly, the Sun would have exhausted all its fuel many billions of years ago. Hence, what took 30 minutes in the early Universe has taken 10^9 years in stars and is not complete yet.

After the deuteron is formed then, as for the early Universe, further reactions can proceed quickly. Here, due to the lack of neutrons, the dominant ones are:

$$p + d \to {}^3_2\mathrm{He} + \gamma\,,$$

followed by

$$2 \times {}^3_2\mathrm{He} \to {}^4_2\mathrm{He} + 2p\,.$$

This is called the PPI cycle. (There are other, less important, chains of reactions which result in helium, called the PPII and PPIII cycles.) The combined result of these reactions is to turn four protons into a helium nucleus:

$$4p \to {}^4_2\mathrm{He} + 2e^+ + 2\nu_e\,.$$

However, the positrons annihilate very quickly with the electrons in the plasma, giving off high energy photons that contribute to the Sun's power output. Hence, in terms of the Sun's power, the reaction can be thought of as

$$4p + 2e^- \to {}^4_2\mathrm{He} + 2\nu_e\,,$$

which has $Q = 26.8$ MeV. The neutrinos are created along with the positrons and deuterons in the initial proton-proton reaction which has $Q = 0.42$ MeV. Effectively, all this available energy is taken by the two leptons and, of this, the neutrino takes a little more than half on average (being lighter than the positrons), i.e. ~ 0.3 MeV. The neutrinos usually escape from the Sun so they do not contribute to its luminous power output, and so the observable power is actually around 26.2 MeV per helium nucleus formed. Comparing this to the measured power output of the Sun allows us to calculate the rate of this reaction and hence predict the number of solar neutrinos we would expect. It was the discrepancy between these numbers which lead to the solar neutrino problem and its interpretation in terms of neutrino mixing.

15.2.3 Heavier nuclei

The Sun is currently burning its hydrogen to helium but then will contract as this fuel starts to run out. However, the contraction will release enough gravitational energy to heat it up to higher temperatures and start helium burning, for which the overall reaction is:

$$3 \times \ {}^{4}_{2}\mathrm{He} \rightarrow \ {}^{12}_{6}\mathrm{C}\,,$$

which has to proceed through the fusion of two helium nuclei into ${}^{8}_{4}\mathrm{Be}$, which spontaneously fissions back to two ${}^{4}_{2}\mathrm{He}$ nuclei with a lifetime of order 10^{-16} s. Hence, the third helium nucleus has to react with the ${}^{8}_{4}\mathrm{Be}$ within this very short timescale to get this reaction to proceed. This reaction therefore goes relatively slowly, although it is speeded up to some extent by a resonance of excited carbon. The higher temperatures are needed as the Coulomb barriers are larger for doubly-charged helium than for hydrogen. This reaction therefore occurs in the hotter core, while hydrogen burning continues as before in the 'cooler' outer layers.

Even heavier stars than the Sun can contract to reach high enough temperatures to overcome larger Coulomb barriers and hence allow carbon burning to produce oxygen and furthermore oxygen burning to produce silicon. For the biggest stars, which have the mass to produce the required temperatures, silicon burning produces iron, at which point there is no further energy release from fusion. In these cases, there are many different fusion reactions occurring and these steps are complex. They actually result in a whole range of nuclei up to iron, not just the carbon, oxygen, silicon and iron mentioned — although these are the most common.

However, this does not mean that no heavier nuclei can ever be made. These heavier nuclei require energy to be formed but they can be produced from neutron absorption on the existing nuclei, working their way up to higher and higher A. The kinetic energy of the neutrons supplies the energy needed to create the heavier nuclei. Under normal conditions, the neutrons are created in the complex reactions of oxygen burning and above. The neutron absorption is relatively slow and the nuclei formed usually beta decay back to the beta-stability curve before absorbing the next neutron. This is called the "s-process" ("s" for slow) of heavy nucleus creation. However, if the neutron flux is very high, then the nuclei are pumped full of neutrons too fast for beta decay to occur and they are driven to extremely neutron-rich states with very high A. This happens only during a *supernova* explosion and this rapid creation is called the "r-process". It results in nuclei much higher up in A than would be expected in the s-process. Specifically, any nucleus with $A > 209$ had to be made by the r-process as they would have alpha-decayed too rapidly during an s-process. These heavy nuclei beyond $A > 209$ are indeed seen in the Solar System, as shown in Figure 15.3. It is also seen that the abundance of nuclei above iron is much lower; this reflects the fact that they require energy to be formed.

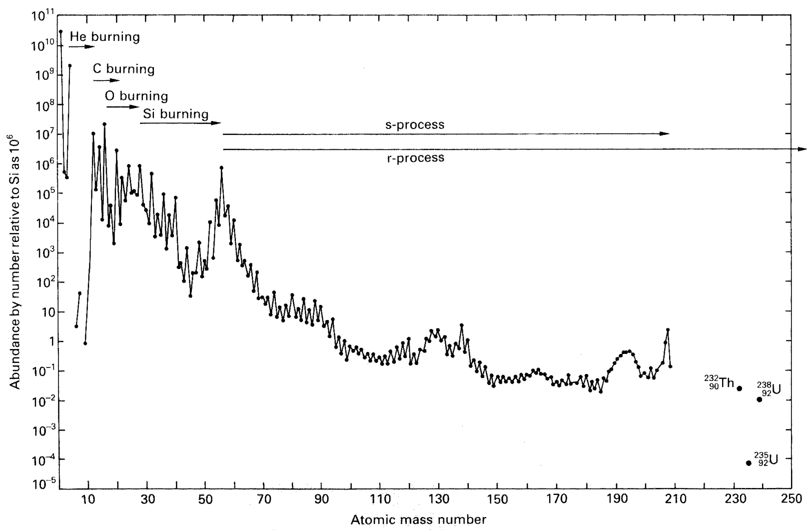


Figure 15.3: Abundances of isotopes in the Solar System as a function of atomic mass (normalised such that silicon abundance is 10^6).

Spectroscopy of the Sun shows it contains small amounts of carbon, oxygen and higher nuclei, which could not have been manufactured by the Sun itself, given its temperature. In addition, the Earth contains both thorium and uranium heavy nuclei which have $A > 209$. The only conclusion is that the Earth (and also the Sun) must be the product of the remnants of a supernova which exploded many billions of years ago. All the matter we see on Earth, including ourselves, was formed in the centre of at least one star before our Sun even existed. This material was blasted out into space when the star exploded at the end of its lifetime. We owe our existence to the recycling of matter through the stars.

Appendix I – non-examinable: The Hydrogen Bomb

The simplest conceptual use of fusion reactions is in the hydrogen bomb, called the "H bomb" to distinguish it from the fission "A bomb". A diagram of a D-T reacting H bomb is shown in Figure 15.4.

To start the reaction, the temperature must be made high enough. This is done using a fission A bomb (which itself is started from conventional explosives). This generates enough heat and pressure (from the vaporised material surrounding the fusion material) for the fusion reaction to proceed. The fusion reaction clearly generates heat itself and so is self-sustaining for the short period until the material is blown apart. The diagram shows that the main part of the fusion fuel in this case is lithium deuteride, $^6_3$LiD, where D is deuterium, which is chemically

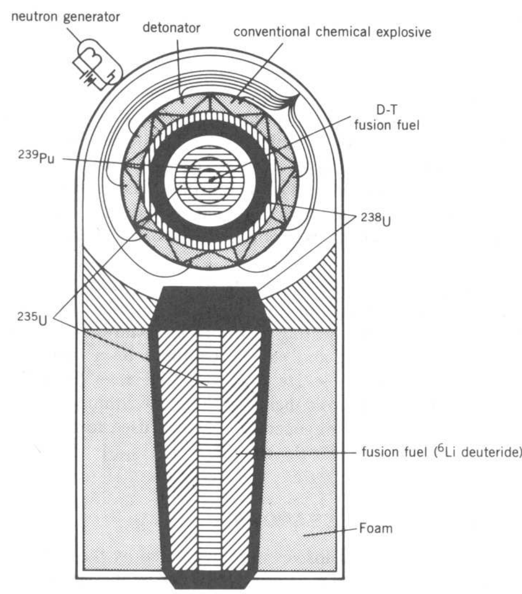


Figure 15.4: The thermonuclear weapon.

similar to the salt lithium hydride, but with deuterium rather than hydrogen. The neutrons emitted from the fission reaction actually convert the lithium to tritium as shown above as part of the explosion, hence providing the fuel for the fusion reaction and reducing the need for tritium manufacture beforehand. H bombs can be made much more powerful than fission bombs due to the more efficient use of the fuel. Yields of around 1000 times those obtained from fission bombs can be produced from a bomb which is small and light enough to easily be carried in an aeroplane. The first such thermonuclear weapon was tested in 1952. Most of the nuclear arsenal of the world's superpowers consists of this type of device.

NPP Lecture 16 – Revision of Fundamental Concepts

16.1 Introduction

In this revision lecture we will concentrate on the particle physics part of the course, as this contains more unfamiliar, often counter-intuitive concepts. We will, nonetheless, summarise some key nuclear physics aspects too.

There are several fundamental concepts in the particle physics part of the course; five of the most important are:

1. Matter and fermions
2. Forces and bosons
3. Feynman diagrams
4. Conservation laws
5. Bound state masses, decays and reactions

We discuss these in turn below.

16.2 Matter and Fermions

In the Standard Model what we consider as 'matter' is composed of spin-1/2 fermions. Atoms are made of electrons orbiting a nucleus. Nuclei are made of protons and neutrons, generically known as nucleons. Nucleons in turn are made of u and d quarks; there are three quarks in every nucleon with the proton being uud and the neutron being udd. These are all spin-1/2 fermions.

The first generation of fundamental fermions consists of the electron and the u and d quarks and, in addition, a fourth fermion: the electron neutrino. The electron and its neutrino are known together as a lepton doublet. This pattern of quarks and leptons is then replicated in second and third generations to give particles which have higher mass but in all other ways are identical to these first generation particles. The fact that they have identical charges, spins, and other quantum numbers is called "universality". The second and third generations 'behave' like heavier copies of the first.

All charged fermions have a corresponding, separate antifermion, as predicted by the Dirac equation. These antiparticles have identical mass but opposite charge (and opposite other quantum numbers also), i.e. $Q_{\overline{f}} = -Q_f$, etc.

The quarks are never seen individually but only within two types of composite particles, which are generically known as hadrons. The nucleons are examples of one of these types, baryons, which all have three quarks and a baryon number $B = +1$ (or three antiquarks in the case of antibaryons, with $B = -1$); hence they are fermions. The proton is the only absolutely stable baryon; a free neutron can decay, although its lifetime is quite long (~ 900 s), while protons cannot as there are no lighter baryons. However, note that neutrons can be stable when bound into nuclei and, in contrast, protons can be unstable: this depends on the relative nuclear masses of the parent and daughter nuclei involved.

The other type of hadrons, the mesons, are made of a quark and an antiquark and so are bosons. The lightest mesons are the pions, made of u and d quarks, and then the kaons, made of an s quark together with a u or d quark. No mesons are absolutely stable and so they are not found within normal matter. As they carry $B = 0$ they can decay to leptons.

16.3 Forces and Bosons

In contrast, the Standard Model forces occur through the exchange of fundamental particles which are bosons. In fact, for the three forces considered in this course, the strong, electromagnetic and weak, all the force bosons are spin 1. The strong force bosons are called gluons and there are eight of these. The electromagnetic boson is called the photon, and there is only one. The weak bosons are the $W^\pm$ and Z, being three particles.

All forces are propagated by exchange of these bosons. The range of the force is inversely proportional to the mass:

$$\text{range} = \frac{\hbar}{mc}\,,$$

so that heavy bosons are more limited in range. The gluons and photon have no mass and so are said to have infinite range (although the strong force is 'masked' by the feature of colour confinement which restricts its range significantly, to $\sim 10^{-15}$ m). In contrast, the $W^\pm$ and Z have masses of 80 GeV/c^2 and 91 GeV/c^2, respectively, and so have a very short range of $\sim 2 \times 10^{-18}$ m.

The strength of the force depends on the size of the charge carried by the matter fermions. However, while in electromagnetism the different particles are allowed to carry different charges, they only differ by a factor of three, $Q_d = -e/3$, $Q_u = 2e/3$, $Q_e = -e$, and so have similar strengths reflected in the value of the fine structure constant:

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \sim \frac{1}{137}\,.$$

For the strong and weak forces, all fermions have to have the same charge for each force (or zero charge) called g_S and g_W, respectively. Equivalents of the fine structure constant can then be formed:

$$\alpha_S = \frac{g_S^2}{4\pi\hbar c} \sim \frac{1}{10}\,, \qquad \alpha_W = \frac{g_W^2}{4\pi\hbar c} \sim \frac{1}{29}\,.$$

Hence, the weak force is in fact somewhat stronger than the electromagnetic force; it appears weaker at low energies because of the very short range of the force, due in turn to the high masses of the $W^\pm$ and Z bosons.

The only fundamental fermions with non-zero strong charge (colour) are the quarks. The strong force exhibits the features of asymptotic freedom and confinement. The former means that we can draw 'asymptotically' free quarks in a Feynman diagram (i.e. quarks can behave as free for very short times or at very high energies) and the latter means that only bound states of quarks (the hadrons) are allowed: the baryons and mesons mentioned above, which must be colour neutral.

All particles with electromagnetic charge feel the electromagnetic force. All the fundamental particles are charged except the neutrinos, so the charged leptons and all the quarks can interact electromagnetically.

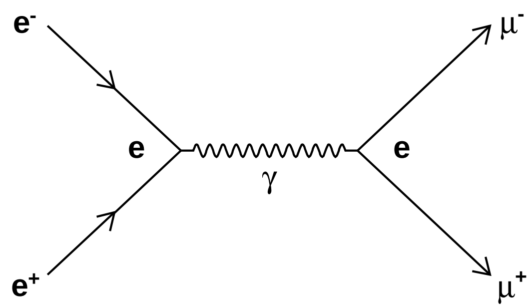
Finally, all fundamental fermions carry weak charge, so all can interact weakly. Specifically, neutrinos only have weak charge and so any interaction which includes a neutrino must involve the weak force. Contrary to the other two, the weak force can change the type of particle, and so it is responsible for many types of decay, both of fundamental particles and of nuclei.

A fourth force which was mentioned in the course was the nuclear force. This is the residual Van der Waals-like strong force between the nucleons, even though they are colour neutral. This is not a fundamental force as such, but its properties are clearly important in understanding how nuclei are bound together.

16.4 Feynman Diagrams

The exchange of bosons between fermions can be pictorially represented by Feynman diagrams. These are much more than just diagrams: they allow the calculation of perturbative mathematical expressions in Quantum Field Theory which allow interaction rates to be calculated. Every interaction can be drawn in terms of a Feynman diagram. The allowed forms of the Feynman diagrams are limited by the properties of the forces. For all three forces, the only vertices allowed, where fermions and bosons meet, have an ingoing and an outgoing fermion together with a boson which can be going either in or out. Because antiparticles can be considered as particles going backwards in time, reversing the fermion directions gives vertices involving antiparticles also. If an interaction cannot be drawn in terms of these vertices, then it cannot happen.

A reaction with only one vertex in the diagram cannot occur as it cannot conserve energy and momentum, hence an actual reaction needs more than one vertex. For example for $e^+e^- \to \mu^+\mu^-$ the lowest order diagram, which has two vertices, is:



and this gives a cross section:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2\hbar^2 c^2}{4s}(1+\cos^2\theta)\,.$$

The above diagram has two vertices and so gives an amplitude which is proportional to e^2. Hence, the probability for this reaction is proportional to e^4 or α^2, as seen from the formula above.

16.5 Conservation Laws

There is a fundamental connection between symmetries of nature and conservation laws. Each symmetry has a conserved quantity associated with it. If the symmetry is broken, then the quantity will no longer be conserved.

The various forces respect (or not) various symmetries and so conserve (or not) various quantities. All forces conserve total energy, momentum and angular momentum. The conservation of these quantities is due to the symmetries arising from the homogeneity and isotropy of space-time and this is thought to hold absolutely. All forces must also conserve the total electric charge.

In addition, no forces change the total number of quarks or of leptons, so the number of each is conserved. The conservation of number of quarks results in the conservation of the number of baryons, as they contain an odd number of quarks, so one property leads to the other. This baryon number conservation (B), as well as lepton number conservation (L), is not thought to hold absolutely, as there is a matter-antimatter asymmetry in the Universe (many baryons, i.e. nucleons, but no antibaryons, and also many electrons but no positrons). This is thought to have arisen from baryon- and lepton-number violating processes in the early Universe.

The other quantities which can be conserved are parity, charge conjugation and time reversal. We studied the first two, and we add the last one here just for completeness. The first, $\hat{P}$, is the operation of reflecting polar vectors through the origin $\boldsymbol{r} \to -\boldsymbol{r}$. The second, $\hat{C}$, is the operation of replacing all particles with their antiparticles. A symmetry of any of these would say the rate of an interaction is the same before and after the relevant operation. The strong and electromagnetic forces are thought to respect all three symmetries. However, the weak force violates P and C directly. It has also been seen to violate the combination $CP = PC$ but only at a lower level, and only in quark interactions (so far...). [$\hat{T}$ is the equivalent operation for time reversal $t \to -t$. The weak force is also thought to violate T in a similar way because the total combination CPT (in any order) must also be conserved absolutely due to the fundamental properties of space-time.]

The strong and electromagnetic forces do not differ in any quantities which are conserved or not, so any reaction which is allowed for the electromagnetic force in terms of conservation laws is also allowed for the strong force. The main difference is the lack of strong charge on the charged leptons, e.g. the e^-, which therefore only interact electromagnetically (or weakly). In contrast, the weak force violates several other quantities, so any reaction which does not conserve them must be weak.

16.6 Bound State Masses, Decays and Reactions

At an experimental level, as well as via conserved quantities, the properties of forces are measured in three main ways, namely through mass/energy levels of bound states, decays of unstable states, and reactions between particles.

The mass levels of bounds states give information on the strong and electromagnetic forces; weak force bound states are too weakly bound to be observable. The levels are really levels of different masses, although the mass differences can be so small as to only be observable through the energies of the emitted particles as the states decay. This is the case for atomic transitions, where the mass differences between the levels in atoms is a very small fraction of the total mass. The differences are, however, easily observed by the photon energies emitted as the atoms made transitions between the states. In principle, if we know about quantum mechanics, then an observation of the spectrum of hydrogen would tell us about the electromagnetic force, although historically this was done the other way round. For the strong force, the mass levels of mesons and baryons do provide real information on the strong force. In both cases, the main binding energy term is seen to be due to a quark spin-spin coupling, and this gives a good description of the hadron masses.

Decays of unstable particles give information on all three forces. In almost all cases, the decay rates are given by

$$\frac{dN}{dt} = -\lambda t\,,$$

resulting in an exponential law, where the probability of surviving to at least time t is given by:

$$S(t) = e^{-t/\tau} = e^{-\lambda t}\,,$$

where τ is the lifetime, i.e. the mean time the particle lives for, and $\lambda = 1/\tau$ is the decay constant, giving the probability per unit time for a decay. Any decaying particle must also have a spread in its mass, due to the Heisenberg uncertainty principle, as they only survive for a finite time. This spread is expressed as an energy, the width Γ, and is given by $\Gamma = \hbar/\tau = \hbar\lambda$. For the strong force, hadron decays from the excited to ground states can occur and these tend to have lifetimes around 10^{-23} s, which are characteristic of the strong force. Nuclear alpha and fission decays are also strong (or nuclear) force decays, but have a large suppression due to quantum tunnelling and so can have much longer lifetimes. Electromagnetic decays include excited atom decays, some meson decays such as $\pi^0 \to \gamma\gamma$, and nuclear gamma decays. In the absence of suppression effects, these decays tend to have lifetimes of $\sim 10^{-16}$ s, which are characteristic of the electromagnetic force.

Weak decays include muon decay, ground-state meson decays and nuclear beta decays. These tend to be much slower than the previous cases, as they tend to be lower energy and so have a suppressed weak decay rate due to the short range of the weak force. Characteristic lifetimes are from about 10^{-13} s for tau decay, 10^{-8} s for charged pion decay, all the way up to 10^3 s for a free neutron decay. A very nice weak decay, illustrating the 'left-handed' nature of the weak force, is the charged pion decay, as discussed in lectures.

Reactions usually only ever involve two incoming particles as the probability for three or more reacting at once is too small. Reactions are usually quantified in terms of a cross section σ which is the effective target area presented to one of the incoming particles for the reaction to occur. In terms of the cross section, the reaction rate per incident particle (i.e. the probability of an interaction) in a thin material is $\sigma n d$, where n is the density of targets per unit volume and d the material thickness. In a thick slab of material, it makes more sense to talk of mean free path, which is given by $1/n\sigma$.

As well as the cross section itself (which just indicates how likely an interaction or scattering is) we often discussed differential cross sections, such as $d\sigma/d\Omega$ which is the cross section for a scattering to occur into a particular solid angle, where the solid angle element $d\Omega = \sin\theta\, d\theta\, d\phi$ (also written as $d(\cos\theta)\, d\phi$). To obtain the total cross section one integrates over, in this case, θ and ϕ.

The result of any decay or reactions can be several (N) particles in the final state. To describe each particle requires three variables for the three components of momentum $\boldsymbol{p}_i$ as the energy is then fixed by $E_i = \sqrt{p_i^2 c^2 + m_i^2 c^4}$. There are four constraints overall from total energy and momentum conservation so the total number of free variables is $3N - 4$. A two-body final state is particularly simple: for $N = 2$ there are two free variables which simply describe the orientation of the system in space, i.e. θ and ϕ. The momenta magnitudes, and hence energies, are all fixed. (In contrast, for $N = 3$ there are five free variables, which allows the momenta magnitudes to vary from one interaction to the next.) Recall that it can be useful to work in the centre-of-mass frame, defined to be the frame where the total momentum $\boldsymbol{p}_T = 0$; in this frame $m_T c^2 = E_{\text{cm}} = \sqrt{s}$. Hence, the total mass of the system is often called the centre-of-mass energy, and its square is usually denoted by s.

To describe reactions and decays we have also used the momentum 4-vector $P = (E/c, \boldsymbol{p})$. This 4-vector is conserved (i.e. the same before and after) in reactions and decays. And while its components are not Lorentz-invariant themselves, its magnitude is: $|P|^2 = m^2c^2$, i.e. the length of the momentum 4-vector is essentially the invariant mass of the system in all frames. This simplifies many relativistic kinematic calculations.

16.7 Nuclear Physics Concepts

Understanding what makes certain nuclei stable is very important, as we saw. The mass of a nucleus is given by

$$m_N = Zm_p + Nm_n - \frac{BE}{c^2}\,,$$

where BE is the binding energy, defined as the energy needed to break the nucleus into its constituent nucleons, i.e. into neutrons and protons. For a nucleus to be stable, then BE must be > 0. Consequently, we spent some time working out how to calculate the binding energy.

We used the nature of the nuclear force as a stepping stone to derive:

- Semi-empirical mass formula (SEMF) – The basic nuclear force model gave us the liquid-drop model, and adding to this three additional effects introduced dependence on the number of neutrons and protons and gave the very useful SEMF which can be used to calculate the energetics of many nuclear reactions.

- Shell model – We took a quantum approach by filling discrete energy levels in an 'independent particle approximation' model for each nucleon moving in an average potential provided by all other nucleons. We used the basic nuclear force model to suggest an appropriate radial potential (Woods-Saxon), but found that to satisfy observations we also needed to introduce a spin-orbit term to perturb the radial potential.

The SEMF does a good job of describing trends but not the non-smooth behaviour of the binding energy such as the magic numbers, for which the shell model is needed. In the shell model we saw that, for odd-A nuclei, the single unpaired nucleon determines the properties of the whole nucleus to good approximation.

For nuclei along the beta-stability curve, the binding energy per nucleon as a function of A peaks around $^{56}_{26}$Fe, which means that this is the most strongly bound nucleus. It is, therefore, energetically favourable both to break up heavier nuclei to bring them closer to Fe and also to combine lighter nuclei.

Alpha and beta radioactivity also help bring unstable nuclei closer to the stability curve, while gamma decay deexcites nuclei without changing any nucleons. We learned that beta decay is a weak force process mediated by virtual W bosons, and the emission of neutrinos in $\beta^{\pm}$ decay means that the emitted electron/positron are not monoenergetic (three-body decay). Electron Capture competes with β^+ decay and in this instance the neutrino is indeed monoenergetic as this is a two-body decay. In alpha decay we worked with a model whereby the alpha particle pre-exists inside a heavy nucleus and can be emitted from it by tunnelling through a Coulomb barrier. This explained the very large range of lifetimes associated with this decay. We saw that often alpha decays happen in longer decay chains, such as the U-238 series. Beta-minus decays are often needed to bring the nuclei closer to the stability curve, as the neutron fraction would otherwise increase too much for a particular A.

The final topics were nuclear fission and fusion. We saw that fission can again involve quantum tunnelling in an oscillating nucleus (spontaneous fission), but it can be induced much more easily by careful choice of material and consideration of the pairing term in the binding energy: we looked in particular at the neutron-induced fission of U-235. Central to this discussion was the possibility of chain reactions to be produced, fed by additional neutron emission following fission events. We saw how the number of free neutrons increases or decreases exponentially depending on a criticality condition being met, and we looked at applications in nuclear power generation, which operates at the critical threshold.

Fusion was illustrated via stellar nuclear processes and nucleosynthesis. For fusion to occur typically positive charges must overcome a Coulomb barrier and this is most effectively done at extremely high temperatures rather than through tunnelling (thermonuclear fusion). The making of ${}^4_2\mathrm{He}$, a well-bound nucleus, is a particularly important stepping stone in element formation, in that most of the energy available in fusing elements up to ${}^{56}_{26}\mathrm{Fe}$ is released in that first step.

Big Bang nucleosynthesis formed elements up to ${}^4_2\mathrm{He}$ in the first few minutes, but as the Universe cooled down this sequence stalled until the first stars were formed, producing high enough temperatures to start thermonuclear fusion once again — now without the presence of neutrons, which had decayed in the early universe.

In stellar nucleosynthesis the 'PP I' cycle starts with a weak-force reaction between two protons to produce deuterons (solving the neutron problem, albeit very slowly), and from there through several steps to ${}^4_2\mathrm{He}$. The total energy release is 26.8 MeV per helium nucleus created, with two neutrinos carrying a bit of this energy away from the star. Later stages of stellar evolution produce elements by fusion up to ${}^{56}_{26}\mathrm{Fe}$. The heavier elements beyond iron can be produced by neutron capture, either slowly (s-process), when unstable isotopes have time to beta-decay back to the stability valley, or rapidly, when they do not (r-process) — the latter happens in supernova explosions.