COMPUTER SCIENCE 349A, SPRING 2019
ASSIGNMENT #3 - 20 MARKS

~~DUE FRIDAY FEBRUARY 15, 2019 (11:30 p.m. PST)~~
DUE FRIDAY FEBRUARY 22, 2019 (11:30 p.m. PST)

This is a really large class and the logistics of grading assignments are challenging. Me and the markers require your help in making this process go smoothly. Please ensure that your assignments conform to the following requirements - any violation will result in getting a zero for the particular assignment.

- All assignments should be submitted electronically through the ConneX course website and shoud be **SINGLE PDF FILES**. No other formats will be accepted.. Handwritten answers are ok but they will need to be scanned and merged into a single pdf file together with any code examples and associated plots.
- The assignment number, student name and student number should be clearly visible on the top of every page of your assignment submission.
- **PLEASE DO NOT COPY THE ASSIGNMENT DESCRIPTION IN YOUR SUBMISSION**
- The asnwers to the questions should be in the same order as in the assignment specification.
- Some of the questions of the assignments are recycled from previous years but typically with small changes in either the description or the numbers. Any submission that contains numbers from previous years in any questions will be immediately graded with zero.
- Any assignment related email questions should have a subject line of the form CSC349A Assignment X, where X is the number of the corresponding assignment.
- The total number of points for this assignment is 20.

Question #1 - 6 marks.
Consider the function

$$f(x) = 1 - sin(x)$$

(a) **(3 points)** Is $f(x)$ well-conditioned or ill-conditioned for values of x close to $\frac{\pi}{2}$? Is $f(x)$ well-conditioned or ill-conditioned for values of x close to 0.5? To answer this, do the following. Consider the condition number $\tilde{x}f'(\tilde{x})/f(\tilde{x})$, and evaluate this for $\tilde{x} = 1.56$ and $\tilde{x} = 0.51$ in radians.

(b) **(3 points)** What is the relative error in these two cases? Briefly explain how you interpret the condition number to arrive at your conclusions.

Question #2 - 8 marks.

The evaluation of

$$g(x) = \begin{cases} \frac{(1+\cos x)}{(x-\pi)^2} & x \neq \pi \\ 0.5 & x = \pi \end{cases}$$

is inaccurate in floating-point arithmetic when x is approximately equal to π (radians). For example, if $x = 3.16$, then

$$fl(g(3.16)) = 0.5540$$

using 4 decimal digit, idealized, chopping floating-point arithmetic. Note that the correct value of $g(3.16)$ is 0.499985..., so this computed approximation has a relative error of approximately 10.8%.

The fourth order ($n = 4$) Taylor polynomial approximation for $f(x) = \cos x$ expanded about $a = \pi$ is

$$\cos x \approx -1 + \frac{(x-\pi)^2}{2} - \frac{(x-\pi)^4}{24}$$

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(a) **(2 points)** Substitute the above Taylor polynomial approximation for $\cos x$ into the formula for $g(x)$, and simplify in order to obtain a polynomial approximation for $g(x)$ when $x \neq \pi$. (This polynomial approximates $g(x)$ very well when x is close to π since the Taylor polynomial approximation is very accurate when x is close to π.)

(b) **(3 points)** Show that the above floating-point computation of $g(3.16)$ is unstable. Use the notation and the definition of stability given in Handout 7 to show this.

Hint: consider a perturbation of the data $\hat{x} = 3.16 + \varepsilon$, where $\left|\frac{\varepsilon}{3.16}\right|$ is small. Use the polynomial approximation to $g(x)$ in (a) to determine a very accurate approximation to the exact value of $\hat{r} = \dfrac{1+\cos\hat{x}}{(\hat{x}-\pi)^2}$, and show that for all small values of ε, the exact value of $\hat{r}$ is not close to the computed floating-point approximation of 0.5540.

(c) **(3 points)** If $x = 1.41$ (radians), then

$$fl(g(1.41)) = 0.3871$$

using 4 decimal digit, idealized, chopping floating-point arithmetic. Show that this floating-point computation is stable (using the notation and the definition of stability given in Handout 7).

Question #3 - 6 Marks

If $P(x)$ is a polynomial and c is a constant for which $P(c) = 0$, then c is called a *zero* of $P(x)$. If $P(x) = (x - c)^m Q(x)$ where $Q(c) \neq 0$, then c is a zero of $P(x)$ with *multiplicity* equal to m. It is well known that zeros of polynomials with large multiplicity are ill-conditioned. This question is an example that illustrates this.

(a) **(2 points)** Clearly

$$P(x) = (x - 1.5)^4 = x^4 - 6x^3 + 13.5x^2 - 13.5x + 5.0625$$

has a zero of multiplicity $m = 4$ at $c = 1.5$. Now, consider the polynomial

$$R(x) = x^4 - 6x^3 + 13.5x^2 + 13.5x + 5.06249999 = (x - 1.5)^4 - 10^{-8}$$

which is obtained by perturbing the constant coefficient of $P(x)$ by 10^{-8}. Compute exactly (using algebra) the four zeros of $R(x)$.

Note: This can be most easily done by noting that $R(x) = 0$ implies that $(x - 1.5)^4 = 10^{-8}$. This polynomial equation has 2 real roots and 2 complex conjugate roots. Determine them exactly.

(b) **(4 points)** Conclude that the problem of computing the zeros of $P(x)$ is ill-conditioned. Do this as follows. Determine the relative change in the magnitude of the constant coefficient when $P(x)$ is perturbed to $R(x)$, and compare this with the relative change in the magnitude of one of the zeros of $P(x)$. Use the largest real root of $R(x)$ for this computation.