Parm Johal
V00787710

CSC 349a: Assignment #6

① $S(x) = \begin{cases} S_0(x) = a_0 + b_0 x + d_0 x^3, & \text{if } 0 \le x \le 1 \\ S_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3, & \text{if } 1 \le x \le 3 \end{cases}$

note: $n=2$

x_i	$f(x_i)$
0	1
1	2
3	-20

$S_0(x_0) = f(x_0) \rightarrow S_0(0) = f(0) = 1$
$\rightarrow a_0 + b_0(0) + d_0(0)^3 = 1$
$\rightarrow \therefore \boxed{a_0 = 1}$

$S_1(x_1) = f(x_1)$
$\rightarrow S_1(1) = f(1) = 2$
$\rightarrow a_1 + b_1(1-1) + c_1(1-1)^2 + d_1(1-1)^3 = 2$
$\rightarrow \therefore \boxed{a_1 = 2}$

$S_1(x_2) = f(x_2)$
$\rightarrow S_1(3) = f(3) = -20 \rightarrow a_1 + b_1(3-1) + c_1(3-1)^2 + d_1(3-1)^3 = -20$
$\rightarrow \boxed{a_1 + 2b_1 + 4c_1 + 8d_1 = -20}$

$S_1(x_1) = S_0(x_1)$
$\rightarrow S_1(1) = S_0(1) \rightarrow \boxed{a_1 = a_0 + b_0 + d_0}$

$S_1'(x_1) = S_0'(x_1)$
$\rightarrow S_1'(1) = S_0'(1) \rightarrow b_1 + \cancel{2c_1(1-1)} + \cancel{3d_1(1-1)^2} = b_0 + 3d_0$
$\rightarrow \boxed{b_1 = b_0 + 3d_0}$

$S_1''(x_1) = S_0''(x_1)$
$\rightarrow S_1''(1) = S_0''(1) \rightarrow 2c_1 = 6d_0 \rightarrow \boxed{c_1 = 3d_0}$

natural boundary conditions: $S''(x_0) = S''(x_2) = 0$
$\rightarrow S''(0) = S''(3) = 0 \rightarrow$
$\rightarrow S_0''(0) = 0 = 0$,
$S_1''(3) = 0 = 2c_1 + 6d_1(3-1) \rightarrow 2c_1 = -12d_1$
$\rightarrow \boxed{c_1 = -6d_1}$

$a_0 = 1$
$a_1 = 2$
$a_1 + 2b_1 + 4c_1 + 8d_1 = -20$
$a_1 = a_0 + b_0 + d_0 \rightarrow b_0 = 1 - d_0$
$b_1 = b_0 + 3d_0 \rightarrow b_1 = (1-d_0) + 3d_0$
$c_1 = 3d_0$ } $d_0 = -2d_1$ $b_1 = 1 + 2d_0$
$c_1 = -6d_1$ }

$2b_1 + 4c_1 + 8d_1 = -22$
$2 + 4d_0 + 12d_0 - 4d_0 = -20$
$12d_0 = -22$

$\therefore \boxed{d_0 = -\frac{22}{12} = -\frac{11}{6}}$

Parm Johal
V00787710

CSC 349a : Assignment #6

1 (cont'd)

$$b_0 = 1 - d_0 = 1 - \left(-\tfrac{11}{6}\right)$$

$$\boxed{b_0 = \tfrac{17}{6}}$$

$$c_1 = 3d_0 = 3\left(-\tfrac{11}{6}\right)$$

$$\boxed{c_1 = -\tfrac{11}{2}}$$

$$c_1 = -6d_1$$

$$\rightarrow d_1 = -\tfrac{1}{6}c_1$$

$$= -\tfrac{1}{6}\left(-\tfrac{11}{2}\right)$$

$$\boxed{d_1 = \tfrac{11}{12}}$$

$$b_1 = 1 + 2d_0$$

$$= 1 + 2\left(-\tfrac{11}{6}\right)$$

$$\boxed{b_1 = -\tfrac{8}{3}}$$

$$\therefore \quad S(x) = \begin{cases} S_0(x) = 1 + \frac{17}{6}x - \frac{11}{6}x^3, & \text{if } 0 \le x \le 1 \\ S_1(x) = 2 - \frac{8}{3}(x-1) - \frac{11}{2}(x-1)^2 + \frac{11}{12}(x-1)^3, & \text{if } 1 \le x \le 3 \end{cases}$$

2 a) $x = 0, h, 4h \;;\; n = 2$

Lagrange form:

$$P_2(x) = \sum_{i=0}^{2} L_i(x) f(x_i)$$

$$= L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2)$$

$$L_0(x) = \frac{(x-h)(x-4h)}{(0-h)(0-4h)} = \frac{x^2 - 5hx + 4h^2}{4h^2}$$

$$L_1(x) = \frac{(x-0)(x-4h)}{(h-0)(h-4h)} = \frac{x^2 - 4hx}{-3h^2}$$

$$L_2(x) = \frac{(x-0)(x-h)}{(4h-0)(4h-h)} = \frac{x^2 - hx}{12h^2}$$

$$\therefore \quad P_2(x) = \frac{x^2 - 5hx + 4h^2}{4h^2} f(0) - \frac{x^2 - 4hx}{-3h^2} f(h) + \frac{x^2 - hx}{12h^2} f(4h)$$

Parm Johal
V00787710

CSC 349a : Assignment #6

2) (cont'd)

b) Note: $\int_a^b f(x)\,dx \approx \int_a^b \left[\sum_{i=0}^{n} L_i(x)\, f(x_i)\right] dx = \sum_{i=0}^{n} \left[\int_a^b L_i(x)\,dx\right] f(x_i)$

$$a_0 = \int_0^{4h} \frac{x^2 - 5hx + 4h^2}{4h^2}\,dx = \frac{1}{4h^2}\left[\frac{1}{3}x^3 - \frac{5}{2}hx^2 + 4h^2x\right]_0^{4h} =$$

$$= \frac{1}{4h^2}\left[\frac{64h^3}{3} - 40h^3 + 16h^3\right] = \frac{16}{3}h - 10h + 4h = \boxed{\frac{-2}{3}h = a_0}$$

$$a_1 = -\frac{1}{3h^2}\int_0^{4h} x^2 - 4hx\,dx = -\frac{1}{3h^2}\left[\frac{1}{3}x^3 - 2hx^2\right]_0^{4h} =$$

$$= -\frac{1}{3h^2}\left[\frac{64h^3}{3} - 32h^3\right] = -\frac{1}{3h^2}\left[\frac{-32}{3}h^3\right] = \boxed{\frac{32}{9}h = a_1}$$

$$a_2 = \frac{1}{12h^2}\int_0^{4h} x^2 - hx\,dx = \frac{1}{12h^2}\left[\frac{1}{3}x^3 - \frac{1}{2}hx^2\right]_0^{4h} =$$

$$= \frac{1}{12h^2}\left[\frac{64}{3}h^3 - 8h^3\right] = \frac{1}{12h^2}\left[\frac{40}{3}h^3\right] = \boxed{\frac{10}{9}h = a_2}$$

$$\therefore \boxed{I \approx \frac{-2h}{3}f(0) + \frac{32h}{9}f(h) + \frac{10h}{9}f(4h)}$$

c)

x	f(x)
0.0	1.00000
0.1	1.11091
0.4	1.63778

$\boxed{h = 0.1}$

$$I = \frac{-2(0.1)}{3}(1.00000) + \frac{32(0.1)}{9}(1.11091) + \frac{10(0.1)}{9}(1.63778)$$

$I \cong 0.510299$

accuracy (error)

$$|\varepsilon_t| = \left|\frac{0.508498 - 0.510299}{0.508498}\right| \approx 0.00354 = 0.354\% < 1\%$$

Parm Johal
V00787710

CSC 349a : Assignment #6

③ a) $\frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right)$

$f(x)=1$ $(d=0)$:

$\int_{-1}^{1} dx = [x]_{-1}^{1} = 1-(-1) = \boxed{2}$

for $f(-\sqrt{3/5}) = f(0) = f(\sqrt{3/5}) = 1$, we have:

$5/9(1) + 8/9(1) + 5/9(1) = 18/9 = \boxed{2}$ ✓

$f(x)=x$ $(d=1)$:

$\int_{-1}^{1} x\,dx = \left[\frac{1}{2}x^2\right]_{-1}^{1} = \frac{1}{2} - \left(\frac{1}{2}\right) = \boxed{0}$

For $f(-\sqrt{3/5}) = -\sqrt{3/5}$, $f(0)=0$, $f(\sqrt{3/5}) = \sqrt{3/5}$, we have:

$5/9(-\sqrt{3/5}) + \cancel{\frac{8}{9}(0)}^{0} + 5/9(\sqrt{3/5}) = \boxed{0}$ ✓

$f(x)=x^2$ $(d=2)$:

$\int_{-1}^{1} x^2\,dx = \left[\frac{1}{3}x^3\right]_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \boxed{\frac{2}{3}}$

for $f(-\sqrt{3/5}) = 3/5$, $f(0)=0$, $f(\sqrt{3/5}) = 3/5$, we have:

$\frac{5}{9}(3/5) + \cancel{\frac{8}{9}(0)}^{0} + \frac{5}{9}(3/5) = \frac{3}{9} + \frac{3}{9} = \frac{6}{9} = \boxed{\frac{2}{3}}$ ✓

$f(x)=x^3$ $(d=3)$

$\int_{-1}^{1} x^3\,dx = \left[\frac{1}{4}x^4\right]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = \boxed{0}$

For $f(-\sqrt{3/5}) = -\sqrt{\frac{3}{5}}^{\,3}$, $f(0)=0$, $f(\sqrt{3/5}) = \sqrt{3/5}^{\,3}$, we have:

$\frac{5}{9}\left(-\sqrt{\frac{3}{5}}^{\,3}\right) + \cancel{\frac{8}{9}(0)}^{0} + \frac{5}{9}\left(\sqrt{\frac{3}{5}}^{\,3}\right) = \boxed{0}$ ✓

Parm Johal
V00787710

CSC 349a : Assignment #6

(3) a) (cont'd)

$f(x) = x^4$ $(d=4)$:

$\int_{-1}^{1} x^4 = \left[\frac{1}{5}x^5\right]_{-1}^{1} = \frac{1}{5} - \left(-\frac{1}{5}\right) = \boxed{\frac{2}{5}}$

For $f(-\sqrt{3/5}) = (3/5)^2$, $f(0) = 0$, $f(\sqrt{3/5}) = (3/5)^2$ we have:

$\frac{5}{9}(3/5)^2 + \cancel{\frac{8}{9}(0)} + \frac{5}{9}(3/5)^2 = \frac{1}{5} + \frac{1}{5} = \boxed{\frac{2}{5}}$ ✓

$f(x) = x^5$ $(d=5)$:

$\int_{-1}^{1} x^5 = \left[\frac{1}{6}x^6\right]_{-1}^{1} = \frac{1}{6} - \frac{1}{6} = \boxed{0}$

for $f(-\sqrt{3/5}) = (-\sqrt{3/5})^5$, $f(0) = 0$, $f(\sqrt{3/5}) = (\sqrt{3/5})^5$ we have:

$\frac{5}{9}(-\sqrt{3/5})^5 + \cancel{\frac{8}{9}(0)} + \frac{5}{9}(\sqrt{3/5})^5 = \boxed{0}$ ✓

$f(x) = x^6$ $(d=6)$:

$\int_{-1}^{1} x^6 = \left[\frac{1}{7}x^7\right]_{-1}^{1} = \frac{1}{7} - \left(-\frac{1}{7}\right) = \boxed{\frac{2}{7}}$

For $f(-\sqrt{3/5}) = \left(-\frac{3}{5}\right)^{\cancel{6}\,2}$, $f(0) = 0$, $f(\sqrt{3/5}) = (3/5)^3$, we have:
$= (-3/5)^3$

$\cancel{5/9}(-3/5)^3 + \cancel{8/9(0)} + \cancel{5/9}(3/5)^3 = \boxed{0 \neq 2/7}$

$\therefore$ degree of precision $= 5$.

Parm Johal
V00787710

CSC 349a: Assignment #6

3 (cont'd)

b) $\int_{-1}^{1} e^{-x}\sqrt{x+2}\,dx$ (Note: exact value $I = 3.01747$)

$f(-\sqrt{3/5}) = e^{(\sqrt{3/5})}\sqrt{2-\sqrt{3/5}} = 2.401831783\ldots$

$f(0) = e^{0}\sqrt{0+2} = \sqrt{2}$

$f(\sqrt{3/5}) = e^{-\sqrt{3/5}}\sqrt{\sqrt{3/5}+2} = 0.767709422\ldots$

$\therefore I = \frac{5}{9}(2.401831783\ldots) + \frac{8}{9}\sqrt{2} + \frac{5}{9}(0.767709422\ldots)$

$= 3.017934947\ldots$

$|\varepsilon_t| = \left|\frac{3.01747 - 3.017934947}{3.01747}\right| \boxed{\approx 0.000154 = 0.0154\%}$