Roll No. 2020319
PARMESH YADAV

SC-HW-1

Problem-1

$$\text{Absolute Condition No.} = \frac{\text{Absolute forward error}}{\text{Absolute backward error}} = \frac{|\cancel{x - x_0}|}{|a|.|\cancel{x - x_0}|} = \frac{1}{|a|}$$

$$\text{Relative Condition No.} = \frac{\text{Relative forward error}}{\text{Relative backward error}} = \frac{|\cancel{x - x_0}|}{\cancel{|x_0|}} \times \frac{\cancel{|a|}.\cancel{|x_0|}}{\cancel{|a|}.|\cancel{x - x_0}|} = 1$$

Problem-2

(a) $f(x) = (x-1)^{\alpha}$

$f'(x) = \alpha(x-1)^{\alpha-1}$

$$\text{Relative condition no.} = \frac{|x.f'(x)|}{|f(x)|} = \frac{|x.\alpha(x-1)^{\alpha-1}|}{|(x-1)^{\alpha}|}$$

$$= \left|\frac{x\alpha}{x-1}\right|$$

Let dx be the perturbation

$$\text{Absolute condition no.} = \frac{|\text{Absolute forward error}|}{|\text{Absolute backward error}|} = \frac{|dx.f'(x)|}{|x+dx-x|}$$

$$= \frac{|\cancel{dx}.\alpha(x-1)^{\alpha-1}|}{|\cancel{dx}|}$$

$$= |\alpha(x-1)^{\alpha-1}|$$

Therefore, as the value of x goes to 1, relative condition no. approaches infinity.

(b) $f(x) = \ln x$

$f'(x) = 1/x$

Relative condition no. $= \dfrac{|x \cdot f'(x)|}{|f(x)|} = \left| \dfrac{\cancel{x} \cdot \frac{1}{\cancel{x}}}{\ln x} \right| = \dfrac{1}{\ln x}$

Let dx be the perturbation

Absolute condition no. $= \dfrac{|dx \cdot 1/x|}{|x+dx-x|} = \dfrac{|dx \cdot 1/x|}{|dx|}$

$= \left| \dfrac{1}{x} \right|$

Therefore, as the value of x goes to 1, relative condition no. approaches infinity.

(c) $f(x) = x^{-1} \cdot e^{x}$

$f'(x) = e^{x} \cdot x^{-1} - e^{x} \cdot x^{-2}$

Relative condition no. $= \dfrac{|x \cdot f'(x)|}{|f(x)|} = \left| \dfrac{x[\cancel{e^{x}} \cdot x^{-1} - \cancel{e^{x}} \cdot x^{-2}]}{x^{-1} \cdot \cancel{e^{x}}} \right|$

$= \left| [1 - \frac{1}{x}] x \right|$

$= |x-1|$

Let dx be the perturbation

Absolute condition no. $= \dfrac{|dx \cdot e^{x} \cdot x^{-1}|}{|x+dx-x|} = \dfrac{|dx \cdot e^{x} \cdot x^{-1}|}{|dx|}$

$= |e^{x} \cdot x^{-1}|$

Therefore, as the value of x goes to infinity, the relative condition no. goes to infinity.

(d) $f(x) = \frac{1}{1+x^{-1}} = \frac{x}{x+1}$

$f'(x) = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$

Relative condition no. $= \frac{|x \cdot f'(x)|}{|f(x)|} = \left| \frac{\cancel{x}}{(x+1)^2} \times \frac{\cancel{(x+1)}}{\cancel{x}} \right|$

$= \left| \frac{1}{x+1} \right|$

Let dx be the perturbation

Absolute condition no. $= \frac{|dx \cdot 1/(x+1)^2|}{|x+dx-x|} = \frac{|dx \cdot 1/(x+1)^2|}{|dx|}$

$= \left| \frac{1}{(x+1)^2} \right|$

Therefore, as the value of x goes to -1, the relative condition no. goes to infinity.

Problem-3

(a) $f(x(\varepsilon)) + \varepsilon \cdot p(x(\varepsilon)) = 0$

differentiate w.r.t ε

$$\frac{d}{d\varepsilon}\left[f(x(\varepsilon)) + \varepsilon \cdot p(x(\varepsilon))\right] = 0$$

$$f'(x(\varepsilon)) \cdot x'(\varepsilon) + p(x(\varepsilon)) + \varepsilon \cdot p'(x(\varepsilon)) = 0$$

Now, acc. to the ques, $\varepsilon = 0$

$$f'(x(0)) \cdot x'(0) + p(x(0)) + 0 \cdot p'(x(0)) = 0$$

and, we know that $x(0) = x^*$

$$f'(x^*) \cdot \left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} + p(x^*) = 0$$

$$\Rightarrow \left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} = -\frac{p(x^*)}{f'(x^*)}$$

(b) $f(x) = (x-1)(x-2)(x-3)\dots(x-20)$

$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{20} x^{20}$

$p(x) = x^{19}$

Now, $\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} = -\frac{p(x^*)}{f'(x^*)}$

$$\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0,\, x^*=j} = \frac{-p(j)}{f'(j)} = \frac{-j^{19}}{f'(j)}$$

We need $f'(x)$,

$$f'(x) = (x-2)\dots(x-20) + (x-1)(x-3)\dots(x-20) + \dots + (x-1)(x-2)\dots(x-19)$$

$\Rightarrow f'(j) = \prod (x-\ell)$, where $\ell \neq j$

$\therefore \left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0,\, x^*=j} = \frac{-j^{19}}{\prod (j-\ell)} = -\prod \frac{j}{j-\ell}$, where ℓ takes all values except j.

(c) At $x^* = 1 = j$.

$$\Rightarrow \left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0,\, x^*=j} = -\prod \frac{(1)^{19}}{1-\ell} \;;\; 1 \neq \ell$$

$$= \frac{1}{19!}$$

At $x^* = 20 = j$

$$\Rightarrow \left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0,\, x^*=j} = -\prod \frac{(20)^{19}}{20-\ell} \;;\; 20 \neq \ell$$

$$= \frac{(20)^{19}}{19!}$$

$\therefore \frac{1}{19!} < \frac{20^{19}}{19!}$, hence, $x^* = 1$ is more stable to this perturbation.

Problem- 4

(a) When we run the given code snippet, as the loop goes on the value of x keeps on decreasing by half until it reaches 0. Last value printed just before reaching 0 is 5e-324 after which the value under flows and is rounded to 0.

(b) When we run the given code snippet, as the loop goes on the value of epsilon keeps on decreasing by half until the value of b equals a and that happens when the value of epsilon goes below epsilon machine value at which point value of b equals a.
Value of eps printed : 1.1102230246251565e-16

(c) When we run the given code snippet, as the loop goes on the value of x keeps on increasing by double until it reaches infinity. Last value printed just before reaching inf is 8.98846567431158e+307 after which the value over flows and is rounded to inf.

Problem- 5

Output of problem_5.py

```
[Running] python -u "d:\college\sem 5\Scientific Computing\HomeWork\HomeWork 1\Codes\problem._5.py"
(x, tan(x)) = (7.0685834705770345, 0.9999999999999994)
(x, tan(x)) = (63.6172512351933079, 0.9999999999999897)
(x, tan(x)) = (629.1039288813560688, 0.9999999999999456)
(x, tan(x)) = (6283.9707053429829102, 0.9999999999979700)
(x, tan(x)) = (62832.6384699592599645, 0.9999999999954939)
(x, tan(x)) = (628319.3161161219468340, 0.9999999998033864)
(x, tan(x)) = (6283186.0925777498632669, 0.9999999999777867)
(x, tan(x)) = (62831853.8571940287947655, 1.0000000012561285)
(x, tan(x)) = (628318531.5033566951751709, 0.9999997681704147)
(x, tan(x)) = (6283185307.9649848937988281, 1.0000005069523146)
(x, tan(x)) = (62831853072.5812606811523438, 0.9999954970141940)
(x, tan(x)) = (628318530718.7440185546875000, 0.9999453990134111)
(x, tan(x)) = (6283185307180.3710937500000000, 0.9984385423410914)
(x, tan(x)) = (62831853071796.6484375000000000, 0.9965461389148867)
(x, tan(x)) = (628318530717959.3750000000000000, 0.8900802593986616)
(x, tan(x)) = (6283185307179587.0000000000000000, 0.5766517306609554)
(x, tan(x)) = (62831853071795864.0000000000000000, -0.9682197486526081)
(x, tan(x)) = (628318530717958656.0000000000000000, -2.0518446488895292)
(x, tan(x)) = (6283185307179586560.0000000000000000, 5.5747653335805616)
(x, tan(x)) = (62831853071795863552.0000000000000000, -8.7118803024535953)
(x, tan(x)) = (628318530717958668288.0000000000000000, -1.0506941034214516)
```

As the value of x increases, the relative forward error also increases because of increasing round off error in the floating point system

Problem-7

Output of problem_7.1.py

```
[Running] python -u "d:\college\se
n = 100, e = 0.5822073316515288
n = 200, e = 0.5797135815734098
n = 300, e = 0.5788814056433012
n = 400, e = 0.5784651440685238
n = 500, e = 0.5782153315683285
n = 600, e = 0.5780487667534508
n = 700, e = 0.5779297805478292
n = 800, e = 0.5778405346932214
n = 900, e = 0.5777711175764439
n = 1000, e = 0.5777155815682065
n = 1100, e = 0.5776701414855578
n = 1200, e = 0.5776322736978301
n = 1300, e = 0.5776002309764809
n = 1400, e = 0.5775727652416682
n = 1500, e = 0.5775489611978291
n = 1600, e = 0.5775281323494514
n = 1700, e = 0.5775097537135299
n = 1800, e = 0.5774934169591495
n = 1900, e = 0.5774787997122512
n = 2000, e = 0.5774656440682016
n = 2100, e = 0.5774537412431791
n = 2200, e = 0.5774429204111771
n = 2300, e = 0.5774330404528918
n = 2400, e = 0.5774239837672805
n = 2500, e = 0.5774156515681996
n = 2600, e = 0.5774079602664202
n = 2700, e = 0.5774008386555254
n = 2800, e = 0.5773942257008384
n = 2900, e = 0.5773880687857824
n = 3000, e = 0.5773823223089227
n = 3100, e = 0.5773769465525795
n = 3200, e = 0.5773719067634975
n = 3300, e = 0.5773671724007521
n = 3400, e = 0.5773627165162711
n = 3500, e = 0.5773585152416505
n = 3600, e = 0.5773545473603523
n = 3700, e = 0.5773507939494777
n = 3800, e = 0.5773472380778735
n = 3900, e = 0.5773438645508655
n = 4000, e = 0.5773406596931707
n = 4100, e = 0.5773376111636459
n = 4200, e = 0.5773347077964406
n = 4300, e = 0.5773319394643330
n = 4400, e = 0.5773292969607322
n = 4500, e = 0.5773267718973916
n = 4600, e = 0.5773243566154296
n = 4700, e = 0.5773220441077846
n = 4800, e = 0.5773198279512748
n = 4900, e = 0.5773177022470506
n = 5000, e = 0.5773156615681660
```

Output of problem_7.2.py

```
[Running] python -u "d:\college\se
n = 100, e = 0.5772197901404903
n = 200, e = 0.5772167013748222
n = 300, e = 0.5772161263242399
n = 400, e = 0.5772159246680912
n = 500, e = 0.5772158312352449
n = 600, e = 0.5772157804495590
n = 700, e = 0.5772157498141715
n = 800, e = 0.5772157299243794
n = 900, e = 0.5772157162847442
n = 1000, e = 0.5772157065265553
n = 1100, e = 0.5772156993055031
n = 1200, e = 0.5772156938126143
n = 1300, e = 0.5772156895374030
n = 1400, e = 0.5772156861448545
n = 1500, e = 0.5772156834077089
n = 1600, e = 0.5772156811674058
n = 1700, e = 0.5772156793105871
n = 1800, e = 0.5772156777544755
n = 1900, e = 0.5772156764374801
n = 2000, e = 0.5772156753129938
n = 2100, e = 0.5772156743452568
n = 2200, e = 0.5772156735064380
n = 2300, e = 0.5772156727746101
n = 2400, e = 0.5772156721323229
n = 2500, e = 0.5772156715655328
n = 2600, e = 0.5772156710628664
n = 2700, e = 0.5772156706149998
n = 2800, e = 0.5772156702142466
n = 2900, e = 0.5772156698542288
n = 3000, e = 0.5772156695296005
n = 3100, e = 0.5772156692358834
n = 3200, e = 0.5772156689692576
n = 3300, e = 0.5772156687264989
n = 3400, e = 0.5772156685048309
n = 3500, e = 0.5772156683019034
n = 3600, e = 0.5772156681156311
n = 3700, e = 0.5772156679442730
n = 3800, e = 0.5772156677862554
n = 3900, e = 0.5772156676402354
n = 4000, e = 0.5772156675050191
n = 4100, e = 0.5772156673795781
n = 4200, e = 0.5772156672629993
n = 4300, e = 0.5772156671544533
n = 4400, e = 0.5772156670532187
n = 4500, e = 0.5772156669586632
n = 4600, e = 0.5772156668702006
n = 4700, e = 0.5772156667873283
n = 4800, e = 0.5772156667095789
n = 4900, e = 0.5772156666365351
n = 5000, e = 0.5772156665678327
```

As we can clearly see in both the outputs that the second formula converges more rapidly than the first

Problem - 8

Output of problem_8a.py by testing all the matrices.

```
[Running] python -u "d:\college\sem 5\Scientific Computing\HomeWork\HomeWor
Random Matrix Generated
============================================
Matrix of size n = 10
--------------------------------------------
Condition Number based on the input A = 142.0286722507285
Error from unpivoted gaussian elimination = 1.2092336963815043e-13
Residual from unpivoted gaussian elimination = 1.780792194218199e-14
The error from np.linalg.solve = 2.504777312823609e-15
The residual from np.linalg.solve = 2.5121479338940403e-15
--------------------------------------------
Matrix of size n = 20
--------------------------------------------
Condition Number based on the input A = 407.4441249057004
Error from unpivoted gaussian elimination = 3.30102987362851e-13
Residual from unpivoted gaussian elimination = 1.6259488158202205e-13
The error from np.linalg.solve = 2.2379739031219244e-14
The residual from np.linalg.solve = 6.705600017716362e-15
--------------------------------------------
Matrix of size n = 30
--------------------------------------------
Condition Number based on the input A = 484.99769790906726
Error from unpivoted gaussian elimination = 2.3707699452347816e-13
Residual from unpivoted gaussian elimination = 3.579259853767149e-14
The error from np.linalg.solve = 1.0666424013646061e-13
The residual from np.linalg.solve = 1.2178088937988185e-14
--------------------------------------------
Matrix of size n = 40
--------------------------------------------
Condition Number based on the input A = 1775.6893944897872
Error from unpivoted gaussian elimination = 1.0748104537647466e-12
Residual from unpivoted gaussian elimination = 3.5854256806523836e-13
The error from np.linalg.solve = 4.5626384373773446e-14
The residual from np.linalg.solve = 2.2817716744798718e-14
```

```
Hilbert Matrix Generated
============================================
Matrix of size n = 10
--------------------------------------------
Condition Number based on the input A = 16024413500363.82
Error from unpivoted gaussian elimination = 0.00014527351138774312
Residual from unpivoted gaussian elimination = 5.087681048627601e-16
The error from np.linalg.solve = 0.0002741818135171358
The residual from np.linalg.solve = 5.661048867003676e-16
--------------------------------------------
Matrix of size n = 20
--------------------------------------------
Condition Number based on the input A = 1.3193976166344822e+18
Error from unpivoted gaussian elimination = 14.087302749717958
Residual from unpivoted gaussian elimination = 1.1322097734007351e-15
The error from np.linalg.solve = 95.04955010093468
The residual from np.linalg.solve = 2.869448674962546e-15
--------------------------------------------
Matrix of size n = 30
--------------------------------------------
Condition Number based on the input A = 3.8719824664564173e+18
Error from unpivoted gaussian elimination = 302.8967470969635
Residual from unpivoted gaussian elimination = 3.708876114780783e-15
The error from np.linalg.solve = 350.69922165064173
The residual from np.linalg.solve = 9.356881666874292e-15
--------------------------------------------
Matrix of size n = 40
--------------------------------------------
Condition Number based on the input A = 6.581732387647914e+18
Error from unpivoted gaussian elimination = 306.2920198702936
Residual from unpivoted gaussian elimination = 7.621824591168512e-15
The error from np.linalg.solve = 84.79151266330496
The residual from np.linalg.solve = 2.1727492905721985e-15
```

```
Third Example Matrix Generated
============================================
Matrix of size n = 10
--------------------------------------------
Condition Number based on the input A = 6.313751514675046
Error from unpivoted gaussian elimination = 0.0
Residual from unpivoted gaussian elimination = 0.0
The error from np.linalg.solve = 0.0
The residual from np.linalg.solve = 0.0
--------------------------------------------
Matrix of size n = 20
--------------------------------------------
Condition Number based on the input A = 12.706204736174712
Error from unpivoted gaussian elimination = 0.0
Residual from unpivoted gaussian elimination = 0.0
The error from np.linalg.solve = 0.0
The residual from np.linalg.solve = 0.0
--------------------------------------------
Matrix of size n = 30
--------------------------------------------
Condition Number based on the input A = 19.08113668772823
Error from unpivoted gaussian elimination = 0.0
Residual from unpivoted gaussian elimination = 0.0
The error from np.linalg.solve = 0.0
The residual from np.linalg.solve = 0.0
--------------------------------------------
Matrix of size n = 40
--------------------------------------------
Condition Number based on the input A = 25.451699579357093
Error from unpivoted gaussian elimination = 0.0
Residual from unpivoted gaussian elimination = 0.0
The error from np.linalg.solve = 0.0
The residual from np.linalg.solve = 0.0

[Done] exited with code=0 in 0.56 seconds
```

Output of problem_8b.py by testing all the matrices.

```
[Running] python -u "d:\college\sem 5\Scientific Computing\HomeWor
Random Matrix Generated
=======================
Matrix of size n = 10
Condition Number based on the input A = 60.557730120440475
Error from Pivoted gaussian elimination = 1.5946296824571388e-14
Residual from Pivoted gaussian elimination = 16.08149759967513
The error from np.linalg.solve = 4.186586107782731e-15
The residual from np.linalg.solve = 1.9860273225978185e-15
--------------------------------------------
Matrix of size n = 20
Condition Number based on the input A = 100.72840087606055
Error from Pivoted gaussian elimination = 4.917924594380282e-14
Residual from Pivoted gaussian elimination = 34.004408654172
The error from np.linalg.solve = 1.028500447299271e-14
The residual from np.linalg.solve = 1.0048591735576161e-14
--------------------------------------------
Matrix of size n = 30
Condition Number based on the input A = 1797.3787222761448
Error from Pivoted gaussian elimination = 1.8789971868018506e-12
Residual from Pivoted gaussian elimination = 63.11393245458833
The error from np.linalg.solve = 2.9849194106370166e-13
The residual from np.linalg.solve = 1.109334490649587e-14
--------------------------------------------
Matrix of size n = 40
Condition Number based on the input A = 676.5539102586423
Error from Pivoted gaussian elimination = 3.1837665734196403e-13
Residual from Pivoted gaussian elimination = 102.93943903812328
The error from np.linalg.solve = 8.753276018315711e-14
The residual from np.linalg.solve = 2.1683198553528097e-14
```

```
Hilbert Matrix Generated
=======================
Matrix of size n = 10
Condition Number based on the input A = 16024413500363.82
Error from Pivoted gaussian elimination = 0.001243116759688825
Residual from Pivoted gaussian elimination = 9.265394397183494
The error from np.linalg.solve = 0.0002741818135171358
The residual from np.linalg.solve = 5.661048867003676e-16
-----------------------------------------
Matrix of size n = 20
Condition Number based on the input A = 1.3193976166344822e+18
Error from Pivoted gaussian elimination = 889.9291304471374
Residual from Pivoted gaussian elimination = 20.46127935051444
The error from np.linalg.solve = 95.04955010093468
The residual from np.linalg.solve = 2.869448674962546e-15
-----------------------------------------
Matrix of size n = 30
Condition Number based on the input A = 3.8719824664564173e+18
Error from Pivoted gaussian elimination = 1753.9647715159579
Residual from Pivoted gaussian elimination = 31.853250771747422
The error from np.linalg.solve = 350.69922165064173
The residual from np.linalg.solve = 9.356881666874292e-15
-----------------------------------------
Matrix of size n = 40
Condition Number based on the input A = 6.581732387647914e+18
Error from Pivoted gaussian elimination = 809.9848580741321
Residual from Pivoted gaussian elimination = 43.32910328573028
The error from np.linalg.solve = 84.79151266330496
The residual from np.linalg.solve = 2.1727492905721985e-15
```

```
Third Example Matrix Generated
=======================
Matrix of size n = 10
Condition Number based on the input A = 6.313751514675046
Error from Pivoted gaussian elimination = 0.0
Residual from Pivoted gaussian elimination = 81.24038404635961
The error from np.linalg.solve = 0.0
The residual from np.linalg.solve = 0.0
--------------------------------------------
Matrix of size n = 20
Condition Number based on the input A = 12.706204736174712
Error from Pivoted gaussian elimination = 0.0
Residual from Pivoted gaussian elimination = 326.1901286060018
The error from np.linalg.solve = 0.0
The residual from np.linalg.solve = 0.0
--------------------------------------------
Matrix of size n = 30
Condition Number based on the input A = 19.08113668772823
Error from Pivoted gaussian elimination = 0.0
Residual from Pivoted gaussian elimination = 734.4385610791416
The error from np.linalg.solve = 0.0
The residual from np.linalg.solve = 0.0
--------------------------------------------
Matrix of size n = 40
Condition Number based on the input A = 25.451699579357093
Error from Pivoted gaussian elimination = 0.0
Residual from Pivoted gaussian elimination = 1305.9862173851607
The error from np.linalg.solve = 0.0
The residual from np.linalg.solve = 0.0

[Done] exited with code=0 in 0.614 seconds
```