Acknowledgements

Acknowledge ALL the people!

Resumo

Este tese é sobre alguma coisa

Palavras-chave: física (keywords em português)

Abstract

This thesis is about something, I guess.

Keywords: physics

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1. Introduction

1.1.

1.2. Structure outline

The first chapter of this work, this introduction, aims to establish the scene for the following content and to give an overview of the subject.

In the second chapter we first study the $3+1$ covariant formalism. This formalism creates the framework through which the spacetime manifold can be decomposed into a slicing of spacelike hypersurfaces, each perpendicular to a chosen timelike vector. Secondly, we check the conformal counterparts of the dynamical quantities of the formalism.

The third chapter is to present a general covariant and gauge invariant averaging procedure applied to the past light-cone of an observer and to introduce the correspoding Buchert-Ehlers commutation rules.

The fourth chapter is dedicated to the study and discussion of modified theories of gravity, mainly Brans-Dicke theories, where we introduce a new scalar field, the Brans-dicke fluid.

The fifth chapter is devoted initially to present the general Buchert equations, where we underline the various types of backreaction, followed by applying the previously obtained formalisms and averaging procedure to the Brans-Dicke theory, in order to determine the equivalent Friedmann equations.

1.3. Notation and conventions

- Spacetime indices are represented by Greek letters and adquire values from 0 to 3, i.e. $\mu, \nu, \lambda, \ldots - 0, 1, 2, 3$.
- Spatial indices are indicated by Latin letters starting from i onwards and run from 1 to 3, i.e. $i, j, k, \ldots - 1, 2, 3$.
- Einstein summation convenction is used: repeated index implies summation over all the possible values of the index.
- Quantities in bold denote tensor and vector fields without writing indeces, e.g. $\mathbf{u}, \mathbf{h}, \mathbf{K}, {}^3\mathbf{R}$.
- The signature of the metric is $(-,+,+,+)$.
- Natural units are employed, to the extent that $c = 1$.

- Equality by definition is indicated by $:=$.
- Equality by algebraic identity is represented by $\equiv$.
- Symmetrization is represented by round brackets, for example: $T_{(\mu\nu)} := \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$.
- Antisymmetrization is represented by square brackets, for example: $T_{[\mu\nu]} := \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$.
- Traceless tensors are represented by angle brackets, for example: $T_{\langle\mu\nu\rangle} := T_{(\mu\nu)} - \frac{1}{3}g_{\mu\nu}(g^{\lambda\rho}T_{\lambda\rho})$.
- The partial derivative w.r.t x^μ is indicated by ∂_μ or with a comma, for example: $u_{\mu,\nu} = \partial_\nu u_\mu$.
- The covariant derivative w.r.t x^μ is indicated by ∇_μ or with a semicolon, for example: $u_{\mu;\nu} = \nabla_\nu u_\mu$.

2. 3+1 Formalism

The 3+1 formalism is a method in general relativity, where we slice the spacetime manifold into a foliation of Cauchy hypersurfaces (three-dimensional surfaces). Due to the metric induced by the Lorentzian spacetime metric needing to be Riemannian, the hypersurfaces are required to be spacelike. From a naïve point of view, this can be seen as a decomposition from spacetime into ”space” + ”time”. It should be noted this spliting requires a well established and concise choice of time coordinate.

This formalism is also useful for solving the initial value problem, with this decomposition, solving the Einstein equation is equivalent to solving a Cauchy problem.

Studies with the 3+1 formalism started in the 1920s by Georges Darmois and continued until the 1950s, by direct affiliations. In 1952, Yvonne Choquet-Bruhat verified that the Cauchy problem, arising from the 3+1 decomposition has locally a unique solution []. Later in the decade, Dirac motivated a similar formalism, based on an Hamiltonian approach, which later was called ADM, by Arnowitt, Deser and Misner[]. Around the same time, Wheeler introduced the concept of geometrodynamics and coined the terms, soon mentioned, ”lapse” and ”shift”. In the 1970s, the 3+1 formalism became an essential tool, to numerical relativity.

In this chapter, we will be only considering a single general fluid.

2.1. Description of the geometry

Let us consider a four-dimensional manifold, $\mathcal{M}$, with a Lorentzian metric tensor $\mathbf{g}$ and described under a local coordinate system $x^\mu = (y, x^i)$, where $\mathbf{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu$. In this spacetime, matter is described by a matter content fluid, where the fluid flow is described by a timelike congruence with a unit, future-oriented, timelike tangent four-vector field $\mathbf{u}$, describing its four-velocity. Meanwhile we will not make any exact physical interpretation on this four-velocity[1].

The foliation is performed by slicing $\mathcal{M}$ in a family $y =$ const hypersurfaces which we denote Σ_y. Locally these hypersurfaces will share the same topology $\Sigma_y \simeq \Sigma$. Therefore locally we have $\mathcal{M} \simeq \mathbb{R} \times \Sigma$. And we denote by $\mathbf{n}$ their unit, future-oriented and timelike

[1] The four-velocity could be a energy frame of the fluid, a barycentric velocity, or the unit vector associated to another conserved current.

normal vector field, which generally is tilted with respect to the four-velocity $\mathbf{u}$. And we denote by $\boldsymbol{\partial_y}$ the vector parallel to the motion along y, when the x^i are held fixed. We can characterize this foliation by a scalar function y restrictively increasing along each flow line, and defined in such a way that each hypersurface is a level set of y. For the time being we choose the time coordinate t, as this stricly increasing function, implying that $y \equiv y(t)$.

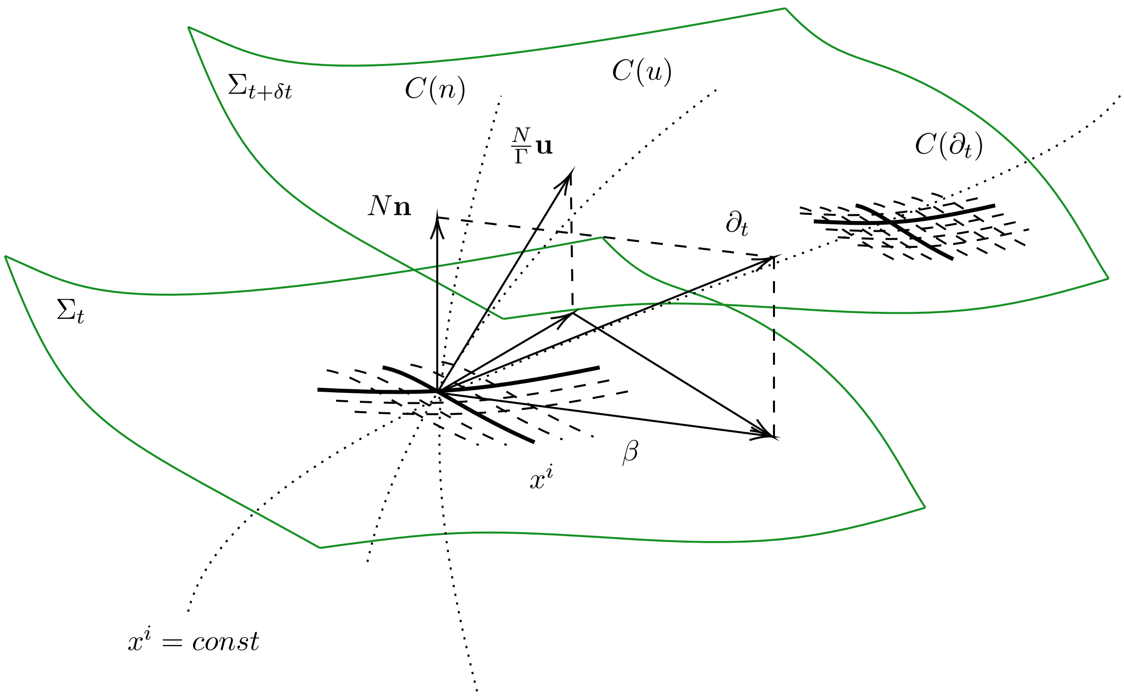


FIGURE 2.1: Scheme entailing the context of the geometrical quantities: $\mathbf{n}$ normal vector field to the hypersurface; $\mathbf{u}$ the 4-velocity vector field (fluid flow lines) with congruence $C(u)$; ∂_t is the tangent vector to the constant coordinates lines ($x^i = const$) with congruence $C(\partial_t)$ (time-vector of the coordinate basis).

Based on the Fig.2.1 we can write $\mathbf{n}$ as,

$$\mathbf{n} = \frac{1}{N}(\partial_t - \beta^i \partial_i) \Leftrightarrow n^\mu = \frac{1}{N}(1, -\beta^i) \tag{2.1}$$

where N, β^i are respectively known as the ”lapse function” and ”shift vector”. The components of the corresponding covector n_μ, is given by,

$$n_\mu = -N(1, 0), \tag{2.2}$$

where $n^\mu n_\mu = -1$.

The lapse function is a positive valued function and determines how far consecutive slices are from each other, in the time direction of each point in the slice. While the shift vector representes the spatial displacement between consecutive slices at the same spatial coordinate point.

In the 3+1 formalism, spacetime tensor are projected onto the hypersurfaces by applying a operator $\mathbf{h} = h_{\mu\nu} dx^{\mu} \otimes dx^{\nu}$, some important propeties are,

$$h_{\mu\nu} := g_{\mu\nu} + n_{\mu} n_{\nu}, \qquad h_{\mu\nu} n^{\nu} = 0, \qquad h^{\mu}_{\lambda} h^{\lambda}_{\nu} = h^{\mu}_{\nu}, \qquad h^{\mu\nu} h_{\mu\nu} = 3, \tag{2.3}$$

where the spatial component of the projector, h_{ij} is the induced Riemannian metric on the slices. The line element is then decomposed into,

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = -N^2 dt^2 + h_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \tag{2.4}$$

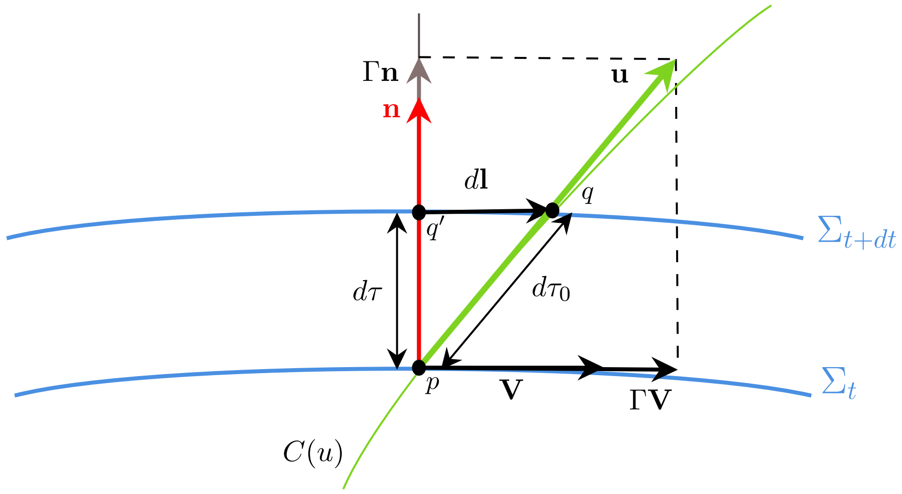
2.2. Description of the fluid

2.2.1 Decomposition of the fluid velocity

In the perfect fluid model, matter is represented as a vector $\mathbf{u}$, which is timelike and unitary, $u^{\mu} u_{\mu} = -1$. Usually we see it in the energy-momentum tensor,

$$T_{\mu\nu} = (\rho + P) u_{\mu} u_{\nu} + P g_{\mu\nu} \tag{2.5}$$

where ρ and P, are the matter density energy and the pressure, respectively, and both measured by an observer comoving with the fluid.



To determine this vector, let us first consider a fluid element at a point $p \in \Sigma_t$. Let τ be the Eulerian observer's [1] proper time at p. After a coordinate time $t + dt$, the fluid element as moved to the point $q \in \Sigma_{t+dt}$. Locally the event q', which for the observer happens at a time $\tau + d\tau$, is given by the orthogonal projection of q onto the observer's

[1]The Eulerian observer is the observer where it's velocity is the unit timelike vector $\mathbf{n}$ in Fig.2.1

worldline[1]. Defining the infinitesimal distance between q and q' as $d\mathbf{l}$. Then $d\tau_0$ be the proper time increment of the fluid, this time is related with the proper time of the observer by the Lorentz factor, Γ,

$$d\tau := \Gamma d\tau_0 \tag{2.6}$$

We also get the triangle identity,

$$d\tau_0 \mathbf{u} = d\tau \mathbf{n} + d\mathbf{l} \tag{2.7}$$

by taking the scalar product with $\mathbf{n}$,

$$d\tau_0 \mathbf{u} \cdot \mathbf{n} = d\tau \underbrace{\mathbf{n} \cdot \mathbf{n}}_{-1} + \underbrace{d\mathbf{l} \cdot \mathbf{n}}_{0} \tag{2.8}$$

hence with the relation in Eq.(2.6),

$$\Gamma = -\mathbf{u} \cdot \mathbf{n} \tag{2.9}$$

with respect to the coordinates (t, x^i),

$$\Gamma = N u^0 \tag{2.10}$$

The fluid velocity relative to the Eulerian observer is defined as,

$$\mathbf{V} = \frac{d\mathbf{l}}{d\tau}. \tag{2.11}$$

By dividing the identity 2.7 by $d\tau$ and using Eq.2.6,

$$\mathbf{u} = \Gamma(\mathbf{n} + \mathbf{V}). \tag{2.12}$$

We see that the fluid 4-velocity, $\mathbf{u}$, is now decomposed into a timelike and a spacelike part. The normalization of the fluid 4-velocity, $\mathbf{u} \cdot \mathbf{u} = -1$, gives us,

$$\Gamma = (1 - \mathbf{V} \cdot \mathbf{V})^{-1/2}. \tag{2.13}$$

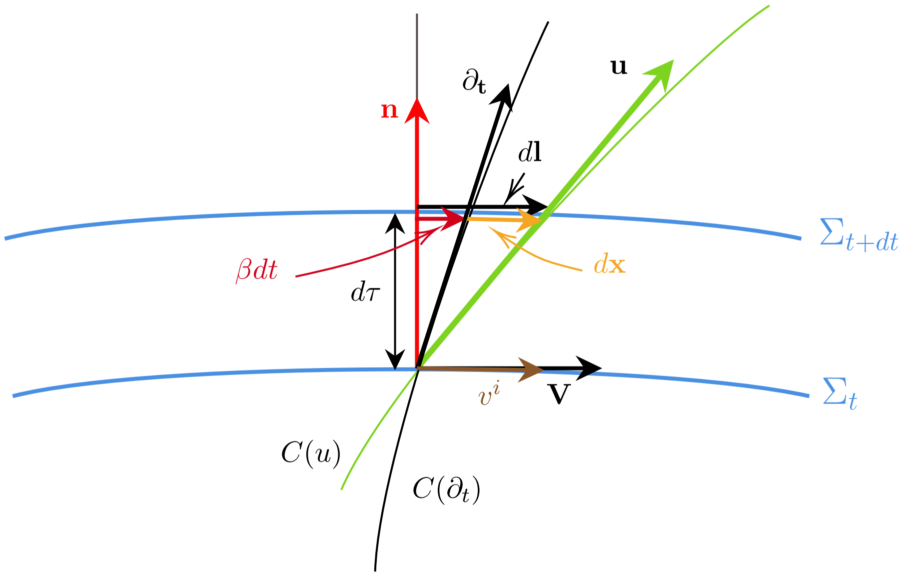
Notice this expression is similar to the Lorentz factor in special relativity, except the scalar product here is taken with the curved metric, instead of a flat metric.

[1]The space of simultaneous events for the Eulerian observer is the space orthogonal to his 4-velocity $\mathbf{n}$.

Now it is worth to see how the Eulerian vector $\mathbf{V}$, is expressed in the base of coordinates (t, x^i). The fluid coordinate velocity is defined by,

$$\mathbf{v} := \frac{d\mathbf{x}}{dt} \tag{2.14}$$

where $d\mathbf{x}$ is the displacement between the fluid worline and the line of constant spatial coordinates.



From the definition of the shift vector, and with the help of the Fig.**??**,

$$d\mathbf{l} = \boldsymbol{\beta} dt + d\mathbf{x} \tag{2.15}$$

dividing this relation with $d\tau$, using Eqs.(2.11) and (2.14),

$$\mathbf{V} = \frac{1}{N}(\boldsymbol{\beta} + \mathbf{v}). \tag{2.16}$$

Where the components of u_μ and u^μ are [1],

$$u_\mu = \frac{\Gamma}{N}(-N^2 + \beta^k(\beta_k + v_k), \beta_i + v_i), \qquad u^\mu = \frac{\Gamma}{N}(1, v^i) \tag{2.17}$$

check the footnote for more detail on how to obtain u_μ[1].

[1]Let $\boldsymbol{\chi}$ be a spatial vector, $\chi^0 = 0$ (however $\chi_0 \neq 0$) and it must satisfy, $n^\alpha \chi_\alpha = 0$, which in turn gives us that $\chi_0 = \beta^i \chi_i$.

2.2.2 Decomposition of the covariant derivative of the four-velocity

The operator that projects tensors onto the local rest frames of the fluid is, $\mathbf{b} = b_{\mu\nu} dx^{\mu} \otimes dx^{\nu}$,

$$b_{\mu\nu} := g_{\mu\nu} + u_{\mu} u_{\nu}, \qquad b_{\alpha\mu} u^{\alpha} = 0, \qquad b^{\mu}_{\alpha} b^{\alpha}_{\nu} = b^{\mu}_{\nu}, \qquad b^{\alpha\beta} b_{\alpha\beta} = 3. \tag{2.18}$$

The projectors $\mathbf{b}$ and $\mathbf{h}$ mainly differ due to the tilt between $\mathbf{u}$ and $\mathbf{n}$. Projected quantities using $\mathbf{b}$ or $\mathbf{h}$, represent what observers moving with $\mathbf{u}$ or $\mathbf{n}$ measure, respectively.

2.2.3 Decomposition of the energy-momentum tensor

The most important factor behind the usage of these two vectors, namely the Eulerian's and the fluid's velocities, $\mathbf{n}$ and $\mathbf{u}$, respectively, is their role in the interpretation of the energy-momentum tensor.

The decomposition with respect to the rest fluid is,

$$T_{\mu\nu} = \rho u_{\mu} u_{\nu} + 2 q_{(\mu} u_{\nu)} + \pi_{\mu\nu} + P b_{\mu\nu}, \tag{2.19}$$

where ρ is the energy density, q_{μ} is the spatial heat vector, $\pi_{\mu\nu}$ is the spatial and traceless anisotropic pressure, and P is the isotropic pressure, all w.r.t the fluid's rest frames.

The decomposition with respect to the Normal frames,

$$T_{\mu\nu} = E n_{\mu} n_{\nu} + 2 J_{(\mu} n_{\nu)} + S_{\mu\nu}, \tag{2.20}$$

where E is the energy density of the fluid, J_{μ} is the momentum density, and $S_{\mu\nu}$ is the stress density, all w.r.t the normal vector.

Now we want to see the relation between E and ρ, so that later we can substitute it in the Friedmann equation.

The energy density,

$$E = \Gamma^2 \rho + (\Gamma^2 - 1) P + 2\Gamma v^{\alpha} q_{\alpha} + V^{\alpha} V^{\beta} \pi_{\alpha\beta} \tag{2.21}$$

The pressure,

$$S = (\Gamma^2 - 1)\rho + (\Gamma^2 + 2) P + 2\Gamma V^{\alpha} q_{\alpha} + V^{\alpha} V^{\beta} \pi_{\alpha\beta} \tag{2.22}$$

2.3. Time derivatives

We have presented in Fig.(2.1), three different congruences and therefore multiple time derivatives can arise [1]:

- Comoving derivative along the normal flow and according to the proper time τ, i.e. $d/d\tau$.
- Comoving derivative along the fluid flow and according to the proper time τ_0, i.e. $d/d\tau_0$.
- Comoving derivative along the normal flow and according to the coordinate time, i.e. d/dt.
- Comoving derivative along the fluid flow and according to the coordinate time, i.e. D/Dt.
- Partial coordinate time derivative along the curves of $x^i=$ const., i.e. $\partial/\partial t := \partial_t|_{x^i}$.

Let $\mathbf{F}$ be a tensor of arbitrary rank. We can relate the d/dt and $\partial_t|_{x^i}$ by,

$$\frac{d\mathbf{F}}{dt} = \left.\frac{\partial \mathbf{F}}{\partial t}\right|_{X^i} = \left.\frac{\partial \mathbf{F}}{\partial t}\right|_{x^i} + \left.\frac{\partial x^i}{\partial t}\right|_{X^i}\frac{\partial \mathbf{F}}{\partial x^i} = \left.\frac{\partial \mathbf{F}}{\partial t}\right|_{x^i} - \beta^i\frac{\partial \mathbf{F}}{\partial x^i}, \tag{2.23}$$

where X^i are the comoving spatial coordinates w.r.t the normal flow. From the eq. (2.1) we obtain.

$$n^\mu\partial_\mu\mathbf{F} = \frac{d\mathbf{F}}{d\tau} = \frac{1}{N}\frac{d\mathbf{F}}{dt}. \tag{2.24}$$

The fluid's time derivative, can relate the D/Dt and $\partial_t|_{x^i}$ by,

$$\frac{D\mathbf{F}}{Dt} = \left.\frac{\partial \mathbf{F}}{\partial t}\right|_{X^i} = \left.\frac{\partial \mathbf{F}}{\partial t}\right|_{x^i} + \left.\frac{\partial x^i}{\partial t}\right|_{X^i}\frac{\partial \mathbf{F}}{\partial x^i} = \left.\frac{\partial \mathbf{F}}{\partial t}\right|_{x^i} - \beta^i\frac{\partial \mathbf{F}}{\partial x^i}, \tag{2.25}$$

where now X^i are the comoving spatial coordinates w.r.t the fluid flow. From the eq. (2.17) we obtain.

$$u^\mu\partial_\mu\mathbf{F} = \frac{d\mathbf{F}}{d\tau_0} = \frac{\Gamma}{N}\frac{D\mathbf{F}}{Dt}. \tag{2.26}$$

2.4. Kinematical quantities

In the 3+1 formalism, it is useful to introduce the spatial connection $\mathbf{D}$, i.e. the three-covariant derivative associated with the metric $\mathbf{h}$, is defined as

$$D_\mu T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q} = h^{\alpha_1}_{\gamma_1}\cdots h^{\alpha_p}_{\gamma_p}h^{\delta_1}_{\beta_1}\cdots h^{\delta_q}_{\beta_q}h^\nu_\mu\nabla_\nu T^{\gamma_1\cdots\gamma_p}{}_{\delta_1\cdots\delta_q}, \tag{2.27}$$

where ∇ is the four-covariant derivative. Here the spatial connection satisfies the metric compatability,

$$D_\mu h_{\alpha\beta} = 0. \tag{2.28}$$

The Frobenius theorem in **??** states that a foliation orthogonal to the fluid is only possible when there is no vorticity, and furthermore we will see that the three-torsion is proportional to vorticity (in the observers frame). Therefore this connection has to be torsion-free, and a Levi-Civita connection. Notice that here we construct the connection around the observer, by using $\mathbf{h}$, if we were to use the fluid we would obtain a connection with metric compatability but under the presence of torsion.

The acceleration vector is given by

$$A_\alpha := n^\mu \nabla_\mu n_\alpha = D_\alpha \ln N, \tag{2.29}$$

where in the last equality we used eqs. (2.1) and (2.2). Here it is noticable that if $N = 1$ we describe a geodesic observer.

One can decompose the four-velocity of the normal vector using eq. (2.27)

$$\nabla_\beta n_\alpha = g^\mu_\beta \nabla_\mu n_\alpha = (h^\mu_\beta - n_\beta n^\mu)\nabla_\mu n_\alpha = -n_\beta A_\alpha + D_\beta n_\alpha, \tag{2.30}$$

where the last term is the extrinsic curvature, given by

$$K_{\alpha\beta} := D_\beta n_\alpha. \tag{2.31}$$

The extrinsic curvature is a type of curvature in hypersurfaces, that defines the bending of the hypersurface domain Σ in the spacetime manifold $\mathcal{M}$. It can be decomposed into a traceful and traceless part by

$$K_{\alpha\beta} = \frac{1}{3} K h_{\alpha\beta} + A_{\alpha\beta}, \tag{2.32}$$

where K and $A_{\alpha\beta}$ are, respectively, the rate of expansion and shear viewed from the observer's perspective. If a tensor field is perpendicular to the normal vector, e.g. $n^\nu K_{\mu\nu} = 0$, then all its independent information lies in K_{ij}, which is the usual form we encounter in literature.

In the fluid's perspective the extrinsic curvature is

$$\Theta_{\alpha\beta} := D^{(\mathbf{u})}_\beta u_\alpha = \frac{1}{3}\Theta b_{\alpha\beta} + \sigma_{\alpha\beta} + w_{\alpha\beta}, \tag{2.33}$$

where Θ, $\sigma_{\alpha\beta}$ and $w_{\alpha\beta}$ are the rate of expansion, shear and vorticity of the fluid. This extrinsic curvature will now measure a tilted bending relative to $K_{\alpha\beta}$, because Σ is not the fluid's hypersurface.

From the spatial connection introduced, we also obtain a new type of curvature, the intrinsic curvature ${}^3\mathbf{R}$. The intrinsic curvature measures the non-commutativity between two consecutive three-covariant derivates $\mathbf{D}$, by the Ricci identity:

$$[D_i, D_j]\, v^k = {}^3R^k{}_{lij} v^l, \tag{2.34}$$

where $\mathbf{v}$ is a spatial vector and the prescript 3 indicates explicitely the quantity is spatial. When building the formalism using the fluid flow, it would yield an additional term proportional to a non-zero torsion ${}^3\mathbf{T}$.

2.5. Gauss-Codazzi-Ricci Relations

In this section we will introduce the decompositions of the spacetime Riemann tensor derived in **??????**. These decompositions are in terms of the quantities in the previous section, namely the Riemann tensor associated with the metric $\mathbf{h}$, ${}^3\mathbf{R}$, the extrinsic curvature $\mathbf{K}$, and the acceleration vector $\mathbf{A}$.

Throughout these relations we will not consider the extrinsic curvature symmetric. This choice is made so we can easily relate to the relations obtained using the fluid flow, where the vorticity is generally non-zero.

From the Codazzi relation in **??**,

$$\perp_u (R^\alpha{}_{\beta\gamma\delta} n^\beta) = h^\alpha_\epsilon h^\phi_\gamma h^\mu_\delta R^\epsilon{}_{\beta\phi\mu} u^\beta = 4K_{[\alpha\delta]} A^\alpha + D_\gamma K^\alpha{}_\delta - D_\delta K^\alpha{}_\gamma \Leftrightarrow \times\delta^\alpha_\gamma \tag{2.35}$$

$$\Leftrightarrow h^\phi_\epsilon h^\mu_\delta R^\epsilon{}_{\beta\phi\mu} n^\beta = 4K_{[\alpha\delta]} A^\alpha + D_\alpha K^\alpha{}_\delta - D_\delta K^\alpha{}_\alpha \Leftrightarrow \tag{2.36}$$

$$\Leftrightarrow (\delta^\phi_\epsilon + n^\phi n_\epsilon) h^\mu_\delta R^\epsilon{}_{\beta\phi\mu} n^\beta = 4K_{[\alpha\delta]} A^\alpha + D_\alpha K^\alpha{}_\delta - D_\delta K \Leftrightarrow \tag{2.37}$$

$$\Leftrightarrow \delta^\phi_\epsilon h^\mu_\delta R^\epsilon{}_{\beta\phi\mu} n^\beta + h^\mu_\delta R^\epsilon{}_{\beta\phi\mu} n^\beta n^\phi n_\epsilon = 4K_{[\alpha\delta]} A^\alpha + D_\alpha K^\alpha{}_\delta - D_\delta K \Leftrightarrow \tag{2.38}$$

$$\Leftrightarrow h^\mu_\delta R^\epsilon{}_{\beta\epsilon\mu} n^\beta + h^\mu_\delta \underbrace{R_{\epsilon\beta\phi\mu}}_{\beta\leftrightarrow\epsilon \text{ Anti-symmetric}} n^\phi \underbrace{n^\beta n^\epsilon}_{\text{Symmetric}} = 4\Theta_{[\alpha\delta]} A^\alpha + D_\alpha K^\alpha{}_\delta - D_\delta K \Leftrightarrow \tag{2.39}$$

$$\Leftrightarrow h^\mu_\delta R_{\beta\mu} n^\beta = 4K_{[\alpha\delta]} A^\alpha + D_\alpha K^\alpha{}_\delta - D_\delta K \tag{2.40}$$

In **??** we obtained

$$\perp_{\mathbf{n}} R^\alpha{}_{\beta\gamma\delta} = {}^3R^\alpha{}_{\beta\gamma\delta} + K^\alpha{}_\gamma K_{\beta\delta} - K^\alpha{}_\delta K_{\beta\gamma}. \tag{2.41}$$

Contracting the α and γ indices in the Gauss relation, we obtain

$$\perp_{\mathbf{n}} R_{\alpha\delta} = {}^3R_{\alpha\delta} + KK_{\alpha\delta} - K^{\gamma}{}_{\delta}K_{\alpha\gamma} - h^{\beta}_{\alpha}h^{\mu}_{\delta}R_{\epsilon\beta\phi\mu}n^{\phi}n^{\epsilon}, \tag{2.42}$$

where the last term, we want the ϕ and ϵ to be the second and last indices of the Riemann tensor and using the Ricci relation in **??**

$$h^{\beta}_{\alpha}h^{\mu}_{\delta}R_{\epsilon\beta\phi\mu}n^{\phi}n^{\epsilon} = h_{\beta\alpha}h^{\mu}_{\delta}R^{\beta}{}_{\epsilon\mu\phi}n^{\phi}n^{\epsilon} = \perp_u (R_{\alpha\epsilon\delta\phi}n^{\phi}n^{\epsilon}) = D_{\delta}A_{\alpha} + A_{\alpha}A_{\delta} + K^{\beta}{}_{\delta}K_{\beta\alpha} - \dot{K}_{\alpha\delta}, \tag{2.43}$$

where we define the overdot as $\dot{\mathbf{T}} = \mathcal{L}_{\mathbf{n}}\mathbf{T}$.

Inserting eq. (2.43) in eq. (2.42)

$$\perp_{\mathbf{n}} R_{\alpha\delta} = {}^3R_{\alpha\delta} + KK_{\alpha\delta} - D_{\delta}A_{\alpha} - A_{\alpha}A_{\delta} + \dot{K}_{\alpha\delta} - K^{\beta}{}_{\delta}(K_{\alpha\beta} + K_{\beta\alpha}). \tag{2.44}$$

From the Ricci relation in **??**,

$$\perp_{\mathbf{n}} (R^{\alpha}{}_{\beta\gamma\delta}n^{\beta}n^{\delta}) = D_{\gamma}A^{\alpha} + A^{\alpha}A_{\gamma} + K^{\delta}{}_{\gamma}K_{\delta}{}^{\alpha} - \dot{K}^{\alpha}{}_{\gamma} \Leftrightarrow \times\delta^{\gamma}_{\alpha} \tag{2.45}$$

$$\Leftrightarrow R_{\beta\delta}n^{\beta}n^{\delta} = D_{\alpha}A^{\alpha} + A^{\alpha}A_{\alpha} - K^{\delta}{}_{\alpha}K_{\alpha}{}^{\delta} - \dot{K}. \tag{2.46}$$

Contracting the indices in eq. (2.44) and using eq. (2.46), we get the Ricci scalar decomposition:

$$R = {}^3R + K^2 - 2(D^{\alpha}A_{\alpha} + A_{\alpha}A^{\alpha}) + 2\dot{K} - K_{\alpha\beta}\dot{h}^{\alpha\beta} - K^{\alpha\beta}K_{\alpha\beta}. \tag{2.47}$$

2.6. Dynamical Equations

From another definition of the extrinsic curvature with no vorticity:

$$K_{ij} := \frac{1}{2}\mathcal{L}_{\mathbf{n}}h_{ij}, \tag{2.48}$$

i.e. the symmetric part of the extrinsic curvature. It is worth to note that $\mathcal{L}_{\mathbf{n}}h^{ij} \neq K^{ij}$ due to the Lie derivative's properties, in the **??**. From this we can get the evolution equation for the metric $\mathbf{h}$:

$$\frac{dh_{ij}}{dt} = 2NK_{ij} + (NV^k)\partial_k h_{ij} + h_{ik}\partial_j\beta^k + h_{kj}\partial_i\beta^k \tag{2.49}$$

Considering a spacetime that obeys the Einstein equation:

$$\mathbf{R} - \frac{1}{2}R\mathbf{g} = \kappa^2\mathbf{T}, \tag{2.50}$$

where $\kappa^2 = 8\pi G$. It can also be written as

$$\mathbf{R} = \kappa^2 \left(\mathbf{T} - \frac{1}{2} T \mathbf{g} \right), \tag{2.51}$$

where T is trace of the EM tensor.

We can project these equations in three ways: a full projection along the normal vector (and perpendicular to Σ); a mixed projection along the normal vector and the hypersurface; and a full projection onto Σ.

Projecting the eq. (2.50) along the normal vector in

$$n^\mu n^\nu R_{\mu\nu} + \frac{1}{2} R = \kappa^2 T_{\mu\nu} n^\mu n^\nu, \tag{2.52}$$

by using eqs. (2.20), (2.46) and (2.47), we obtain the energy equation or called Hamiltonian constraint [2]:

$$\frac{1}{2} \left({}^3R + K^2 - K_{\mu\nu} K^{\mu\nu} \right) = \kappa^2 E. \tag{2.53}$$

Projecting eq. (2.50) along the normal vector and the hypersurface:

$$h^\mu_\alpha n^\nu R_{\mu\nu} = \kappa^2 T_{\mu\nu} h^\mu_\alpha n^\nu \tag{2.54}$$

by using eqs. (2.20) and (2.40) we get the momentum constraint:

$$D_i K - D_j K^j{}_i = \kappa^2 J_i. \tag{2.55}$$

Projecting eq. (2.51) onto the hypersurface

$$\perp_{\mathbf{n}} R_{\mu\nu} = \kappa^2 \perp_{\mathbf{n}} \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right), \tag{2.56}$$

by using eqs. (2.20) and (2.44) we get

$$\mathcal{L}_{\mathbf{n}} K_{\mu\nu} = \kappa^2 \left(S_{\mu\nu} - \frac{1}{2}(S - E) h_{\mu\nu} \right) + N^{-1} D_\nu D_\mu N + K^\alpha{}_\nu (K_{\mu\alpha} + K_{\alpha\mu}) - K K_{\mu\nu} - {}^3R_{\mu\nu},$$

notice again the above equation only contains spatial tensor fields, and therefore we can write with spatial indices:

$$\mathcal{L}_{\mathbf{n}} K_{ij} = \kappa^2 \left(S_{ij} - \frac{1}{2}(S - E) h_{ij} \right) + N^{-1} D_j D_i N + K^k{}_j (K_{ik} + K_{ki}) - K K_{ij} - {}^3R_{ij}, \tag{2.57}$$

where we use eq. (2.29) to simplify $D_j A_i + A_j A_i = N^{-1} D_j D_i N$.

Instead of projecting the equivalent form of the Einstein equation, let us project the Einstein equation onto the hypersurface:

$$\perp_{\mathbf{n}} R_{\mu\nu} - \frac{3}{2} R = \kappa^2 \perp_{\mathbf{n}} T_{\mu\nu}, \tag{2.58}$$

where taking the trace and using eqs. (2.20) and (2.44) yields

$$\begin{aligned} &{}^3R + K^2 - N^{-1} D^i D_i N + h^{ij} \dot{K}_{ij} - K^{ij} K_{ji} - K^{ij} K_{ij} && (2.59)\\ &\quad - \frac{3}{2} \left({}^3R + K^2 - 2N^{-1} D^i D_i N + 2\dot{K} - K_{ij} \dot{h}^{ij} - K^{ij} K_{ij} \right) = \kappa^2 S \Leftrightarrow && (2.60)\\ &\Leftrightarrow \frac{1}{2} \left(-{}^3R - K^2 + 4N^{-1} D^i D_i N - 4\dot{K} - 3K_{ij} K^{ji} \right) = \kappa^2 S. && (2.61) \end{aligned}$$

From the conservation law $\nabla_\alpha T^{\alpha\beta} = 0$, projecting along the fluid flow:

$$\begin{aligned} u_\beta \nabla_\alpha T^{\alpha\beta} = 0 \Leftrightarrow u_\beta \nabla_\alpha \left(\rho u^\alpha u^\beta + 2q^{(\alpha} u^{\beta)} + \pi_{\alpha\beta} + P b_{\alpha\beta} \right) = 0 \Leftrightarrow \\ \Leftrightarrow \frac{\Gamma}{N} \frac{D\rho}{Dt} + \Theta(\rho + P) = -A^{(\mathbf{u})}_\alpha q^\alpha - \nabla_\alpha q^\alpha - \pi^{\alpha\beta} \Sigma_{\alpha\beta}, \end{aligned} \tag{2.62}$$

where we use the total time derivative according to the fluid flow in eq. (2.26). And the energy conservation law according to the observer is

$$\begin{aligned} n_\beta \nabla_\alpha T^{\alpha\beta} = 0 \Leftrightarrow n_\beta \nabla_\alpha \left(E n^\alpha n^\beta + 2J^{(\alpha} n^{\beta)} + S_{\alpha\beta} \right) = 0 \Leftrightarrow \\ \Leftrightarrow \frac{1}{N} \frac{dE}{dt} + K \left(E + \frac{S}{3} \right) = -2A_\alpha J^\alpha - D_\alpha J^\alpha - S_{\alpha\beta} A^{\alpha\beta}. \end{aligned} \tag{2.63}$$

2.7. Conformal 3+1 Formalism

A conformal transformation is a mathematical operation that alters the spacetime metric while preserving angles but not distances, it is a very useful tool in modified theories as we will establish later, however for the moment let us focus in its role in the 3+1 formalism.

2.7.1 Introduction

A conformal transformation is given by

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu}, \qquad g^{\mu\nu} \to \tilde{g}^{\mu\nu} = \Omega^2 g^{\mu\nu} \tag{2.64}$$

where Ω is some strictly positive scalar field. It is important to overstate that this transformation does not alter the basis of coordinates.

Since the conformal metric is still a valid metric, there exists a unique Levi-Civita connection $\tilde{\nabla}$ associated with it. Therefore, given a tensor field $\mathbf{T}$ with rank $\begin{pmatrix} p \\ q \end{pmatrix}$, the covariant derivatives are related by

$$\nabla_\mu T^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \cdots \beta_q} = \tilde{\nabla}_\mu T^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \cdots \beta_q} - \sum_{r=1}^{p} C^{\alpha_r}{}_{\mu\lambda} T^{\alpha_1 \cdots \lambda \cdots \alpha_p}{}_{\beta_1 \cdots \beta_q} + \sum_{r=1}^{q} C^{\lambda}{}_{\mu\alpha_r} T^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \cdots \lambda \cdots \beta_q}, \tag{2.65}$$

where we have that

$$C^\lambda{}_{\mu\nu} := \tilde{\Gamma}^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\mu\nu}, \tag{2.66}$$

where $\tilde{\Gamma}^\lambda{}_{\mu\nu}$ is the Christoffel symbol associated with $\tilde{\nabla}$. We can further simplify $C^\lambda{}_{\mu\nu}$, by converting the Christoffel symbol of the connection ∇ with the conformal transformation, after some computations we obtain

$$C^\lambda{}_{\mu\nu} = -\left(\delta^\lambda_\mu \tilde{\nabla}_\nu + \delta^\lambda_\nu \tilde{\nabla}_\mu - \tilde{g}_{\mu\nu} \tilde{\nabla}^\lambda\right) \ln \Omega. \tag{2.67}$$

Having determined the relation between the covariant derivatives, it is time to check the relation between the Ricci scalars. We can start with the Ricci identity:

$$\left[\nabla_\mu, \nabla_\nu\right] X^\lambda = R^\lambda{}_{\rho\mu\nu} X^\rho. \tag{2.68}$$

2.7.2 Kinematical quantities

In the 3+1 formalism, we saw we could decompose the spacetime metric $g_{\mu\nu}$ into an induced hypersurface metric $h_{\mu\nu}$. With the conformal transformation we obtain

$$\tilde{g}_{\mu\nu} = \Omega^{-2} h_{\mu\nu} - \Omega^{-1} n_\mu \Omega^{-1} n_\nu = \tilde{h}_{\mu\nu} - \tilde{n}_\mu \tilde{n}_\nu, \tag{2.69}$$

where the conformal induced hypersurface metric and the conformal normal vector are, respectively

$$\tilde{h}_{\mu\nu} = \Omega^{-2} h_{\mu\nu}, \qquad \tilde{n}_\mu = \Omega^{-1} n_\mu. \tag{2.70}$$

From Eq.(2.70) and Eq.(2.64), the covector of the normal vector can be obtain by

$$n^\mu = g^{\mu\nu} n_\nu = \Omega^{-2} \tilde{g}^{\mu\nu} \Omega \tilde{n}_\nu = \Omega^{-1} \tilde{n}^\mu, \tag{2.71}$$

here one should remember the presence of a metric in upper raised indices, and therefore a difference to its lower raised counterpart.

The fluid vector u_μ, transforms in the same way as the normal vector

$$\tilde{u}_\mu = \Omega^{-1} u_\mu, \qquad \tilde{u}^\mu = \Omega u^\mu, \tag{2.72}$$

and therefore

$$\tilde{u}^\mu = \Omega u^\mu = \Omega \Gamma \left(n^\mu + \frac{1}{N}(\beta^\mu + v^\mu) \right) = \Gamma \left(\tilde{n}^\mu + \frac{1}{\tilde{N}}(\beta^\mu + v^\mu) \right) \tag{2.73}$$

where the lapse function is transformed as $N \to \tilde{N} = \Omega^{-1} N$, and the shift covector remains unchanged as well as the fluid coordinate velocity v^μ. However, as mentioned, their contravectors are dependent on the transformation as

$$\tilde{v}_\mu = \Omega^2 v_\mu, \qquad \tilde{\beta}_\mu = \Omega^2 \beta_\mu. \tag{2.74}$$

The acceleration of the normal vector is equal to

$$\begin{aligned}
& A_\mu = n^\nu \nabla_\nu n_\mu = \\
\text{Eq.(2.65)} \to \quad & = n^\nu \left[\tilde{\nabla}_\nu + C^\lambda{}_{\nu\mu} n_\lambda \right] = \\
\text{Eq.(2.70)} \to \quad & = \Omega^{-1} \tilde{n}^\nu \tilde{\nabla}_\nu \left(\Omega \tilde{n}_\mu \right) + \tilde{n}^\nu \tilde{n}_\lambda C^\lambda{}_{\nu\mu} = \\
\text{Eq.(2.67)} \to \quad & = \tilde{A}_\mu + \tilde{\nabla}_\mu \ln \Omega + \tilde{n}_\mu \tilde{n}^\nu \tilde{\nabla}_\nu \ln \Omega = \\
\text{Eq.(2.69)} \to \quad & = \tilde{A}_\mu + \tilde{D}_\mu \ln \Omega,
\end{aligned} \tag{2.75}$$

where the conformal acceleration is

$$\tilde{A}_\mu = \tilde{n}^\nu \tilde{\nabla}_\nu \tilde{n}_\mu, \tag{2.76}$$

and where $\tilde{D}$ is the conformal spatial connection, and is defined as

$$\tilde{D}_\mu T^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \cdots \beta_q} := \tilde{h}^{\alpha_1}_{\gamma_1} \cdots \tilde{h}^{\alpha_p}_{\gamma_p} \tilde{h}^{\delta_1}_{\beta_1} \cdots \tilde{h}^{\delta_q}_{\beta_q} \tilde{h}^\lambda_\mu \tilde{\nabla}_\lambda T^{\gamma_1 \cdots \gamma_p}{}_{\delta_1 \cdots \delta_q}, \tag{2.77}$$

Like before, according to the Frobenius Theorem, $\tilde{D}$ is only a proper three-dimensional covariant derivative if the vector $\tilde{n}_\mu$ has no ”vorticity”, i.e. $\tilde{D}_{[\nu} \tilde{n}_{\mu]} = 0$. Therefore, now it is a good opportunity to determine the conformal extrinsic curvature.

As mentioned before, when ”vorticity” is not present in the hypersurface, a useful definition for the conformal extrinsic curvature is

$$\tilde{K}_{\mu\nu} := \frac{1}{2} \mathcal{L}_{\tilde{n}} \tilde{h}_{\mu\nu}, \tag{2.78}$$

with this we can compute the relation between the usual and conformal extrinsic curvature as

$$K_{\mu\nu} = \Omega \tilde{K}_{\mu\nu} + \tilde{h}_{\mu\nu} \mathcal{L}_{\tilde{n}} \Omega, \qquad \tilde{K}_{\mu\nu} = \Omega^{-1} K_{\mu\nu} - \Omega^{-2} h_{\mu\nu} \mathcal{L}_{n} \Omega. \tag{2.79}$$

Decomposing the conformal extrinsic curvature in the following way

$$\tilde{K}_{\mu\nu} = \frac{1}{3} \tilde{K} \tilde{h}_{\mu\nu} + \tilde{A}_{\mu\nu} \tag{2.80}$$

where $\tilde{A}_{\mu\nu}$ is the traceless part and $\tilde{K}$ is the trace of the conformal extrinsic curvature, we can see that they relate to their usual counterparts as

$$\tilde{K} = \Omega K - 3 \mathcal{L}_{n} \Omega, \qquad \tilde{A}_{\mu\nu} = \Omega^{-1} A_{\mu\nu}. \tag{2.81}$$

3. Averaging procedure

3.1. Buchert's approach

This is a bare-bones approach to the averaging problem, doing a direct averaging of scalar measurable quantities, under a spatial hypersurface. The first papers of Buchert started by assuming a "dust" like spacetime[3][4], he recently adapted this approach with an extensive generalised spacetime[1] and foliation, which is fully described in Figure 2.1.

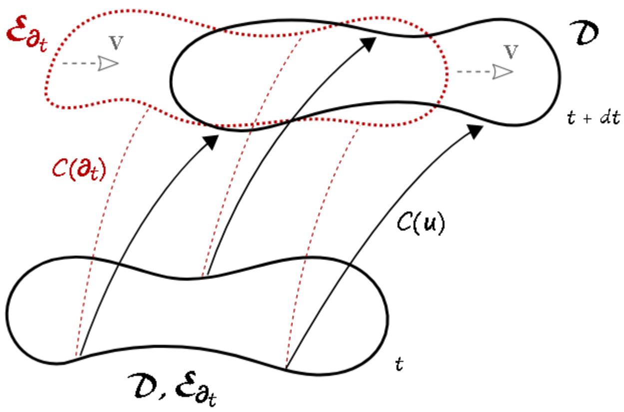
In a synchronous frame, the inhomogeneous and anisotropic metric is $ds^2 = -dt^2 + h_{ij}dx^i dx^j$. The average of any scalar quantity over a spatial hypersurface of constant proper time, Σ, is,

$$\langle S(t,x)\rangle_\Sigma \equiv \frac{1}{V_\Sigma(t)} \int_\Sigma d^3x \sqrt{h} S(t,x), \tag{3.1}$$

The commutative relation between the time derivative and the averaging operator is given by,

$$[\partial_t \cdot, \langle\cdot\rangle_\Sigma] S = \langle \Theta S\rangle_\Sigma - \langle\Theta\rangle_\Sigma \langle S\rangle_\Sigma, \tag{3.2}$$

where Θ is the expansion rate of the dust, assuming the dust is comoving with the observer. The scale factor is defined as $a_\Sigma \propto V_\Sigma^{1/3}$, and so $\langle\Theta\rangle_\Sigma = 3\partial_t \ln a_\Sigma$.



Aqui seguimos buchert, mas aplicamos para a congruencia de n^μ

This figure represents how the hypersurface Σ is transported along the congruence $C(n)$, with $X^i =$ const.. And is transported along the congruence $C(\partial_t)$, with $x^i =$ const., which coincides with Σ at the time t. The domain undergoes a spatial motion, with velocity β, in the coordinate basis (t, x^i). Therefore d/dt and $\int_\Sigma d^3x$ do not commute.

Consider a family of maps, $\mathbf{\Phi}_t = \mathbf{f}(t, \cdot)$, in order to change the coordinates x^i to X^i,

$$x^i = f^i(t, \mathbf{X}), \qquad d^3x = \text{det}\left(\frac{\partial \mathbf{f}(t, \mathbf{X})}{\partial \mathbf{X}}\right) d^3X = J(t, \mathbf{X}) d^3X, \tag{3.3}$$

while the domain transforms as, $\Sigma_x \to \Sigma_X = \mathbf{\Phi}_t^{-1}(\Sigma_x)$, the volume of the domain is then given by,

$$V_\Sigma(t) = \int_{\Sigma_x} d^3x \sqrt{h(t, x^i)} \to V_\Sigma(t) = \int_{\Sigma_X} d^3X J(t, \mathbf{X}) \sqrt{h(t, f^i(t, \mathbf{X}))}. \tag{3.4}$$

The total derivative of coordinate time of the volume of domain,

$$\frac{dV_\Sigma}{dt} = \int_{\Sigma_X} d^3X \frac{d}{dt}\left(J(t, \mathbf{X})\sqrt{h(t, f^i(t, \mathbf{X}))}\right) = \int_{\Sigma_X} d^3x J^{-1} \frac{d}{dt}\left(J\sqrt{h}\right) = \tag{3.5}$$

$$= \int_{\Sigma_X} d^3x \left(\frac{d}{dt}\sqrt{h} - J\sqrt{h}\frac{d}{dt}\left(J^{-1}\right)\right) = \tag{3.6}$$

$$= \int_{\Sigma_X} d^3x \left(\partial_t\sqrt{h} - \beta^k\partial_k\sqrt{h} + J^{-1}\sqrt{h}\frac{d}{dt}J\right) \tag{3.7}$$

something useful here is,

$$\beta^i = -\left.\frac{dx^i}{dt}\right|_{\mathbf{X}} = -\frac{df^i(t, \mathbf{X})}{dt} = -\partial_t|_{\mathbf{X}} f^i(t, X) \Leftrightarrow \times \frac{d}{dX^i} \tag{3.8}$$

$$\Leftrightarrow J\partial_i\beta^i = -\partial_{X^i}\partial_t|_{\mathbf{X}} f^i(t, X) \Leftrightarrow \tag{3.9}$$

$$\Leftrightarrow J\partial_i\beta^i = -\partial_t|_{\mathbf{X}}\partial_{X^i} f^i(t, X) \Leftrightarrow \tag{3.10}$$

$$\Leftrightarrow J\partial_i\beta^i = -\partial_t|_{\mathbf{X}} J \tag{3.11}$$

$$\frac{dV_\Sigma}{dt} = \int_{\Sigma_X} d^3x \left(\partial_t\sqrt{h} - \beta^k\partial_k\sqrt{h} - \sqrt{h}\partial_k\beta^k\right) \Leftrightarrow \tag{3.12}$$

$$= \int_{\Sigma_X} d^3x \left(\frac{1}{2}\sqrt{h}h^{ij}\partial_t h_{ij} - \frac{1}{2}h^{ij}\sqrt{h}\beta^k\partial_k h_{ij} - \sqrt{h}\partial_k\beta^k\right) = \tag{3.13}$$

$$= \int_{\Sigma_X} d^3x\sqrt{h}\left(\frac{1}{2}h^{ij}\partial_t h_{ij} - \frac{1}{2}h^{ij}\beta^k\partial_k h_{ij} - \partial_k\beta^k\right) = \tag{3.14}$$

$$= \int_{\Sigma_X} d^3x\sqrt{h}\left(\frac{1}{2}h^{ij}\partial_t h_{ij} - D_k\beta^k\right) \tag{3.15}$$

By the trace of Eq.(**??**) we see that,

$$\frac{dV_\Sigma}{dt} = \int_{\Sigma_X} d^3x\sqrt{h}\,(NK) \tag{3.16}$$

where D_k is the three-covariant derivative. Now diving by the volume we obtain,

$$\frac{1}{V_\Sigma}\frac{dV_\Sigma}{dt} = \langle NK\rangle_\Sigma \tag{3.17}$$

The commutative relation between the time derivative and the averaging operator is given by,

$$\left[\frac{d}{dt}, \langle\cdot\rangle_\Sigma\right] S = \langle NKS\rangle_\Sigma - \langle NK\rangle_\Sigma \langle S\rangle_\Sigma, \tag{3.18}$$

3.2. Lightcone Averaging Procedure

The averaging formalism made by Buchert in which we average over hypersurfaces at fixed proper times is not sufficient for observations dealing with photon detection, as photons travel along the null light-cone, a light-cone averaging procedure would be more observationally meaningful[5].

In this procedure we still obtain Buchert's equations, by utilizing the Buchert-Ehlers commutation rules[6]. Another advantage in this procedure is that the fluid flow vector is not necessarily orthogonal to the hypersurface.

A scalar field, $S(x)$, under a 4-dimensional domain Ω in $\mathcal{M}_4$ with coordinates x^μ, is given by,

$$F(S,\Omega) = \int_\Omega d^4x\sqrt{-g}S(x), \tag{3.19}$$

this expression is not generally gauge invariant, because the region Ω depends on the choice of coordinates. Gauge invariance can be achieved by introducing a window function $\mathcal{W}_\Omega(x)$ which acts as a filtering of the main manifold $\mathcal{M}_4$ to the domain in interest Ω, where,

$$F(S,\Omega) = \int_\Omega d^4x\sqrt{-g}S(x) \equiv \int_{\mathcal{M}_4} d^4x\sqrt{-g}S(x)\mathcal{W}_\Omega(x), \tag{3.20}$$

where the window function will require a region, inside the past light-cone of the observer, bounded by the hypersurface of the past $A(x) = A_0$,

$$F(S; \underbrace{-}_{\text{Dirac}}; \underbrace{A_0, V_0}_{\text{Heavyside}}) = \int_{\mathcal{M}_4} d^4x\sqrt{-g}\Theta(A(x) - A_0)\Theta(V_0 - V(x))S(x), \tag{3.21}$$

where $V(x)$ is a scalar satisfying $\partial_\mu V\partial^\mu V = 0$; where V_0 specifies the past light-cone of the observer; and where "$-$" symbolises there are no Dirac delta functions in the window function.

3.3. Conformal LCA

In this section, we will look at how the lightcone averaging procedure changes under a conformal transformation. Immediately, the integral over a scalar function S, changes in the following manner,

$$\begin{aligned}I(S;A_0;V_0) &= \int_{\mathcal{M}} d^4x\sqrt{-g}\delta(A-A_0)\Theta(V_0-V)\sqrt{-\partial_\mu A\partial^\mu A}S(x) = \\ &= \int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\Omega^4\delta(A-A_0)\Theta(V_0-V)\sqrt{-g^{\mu\nu}\partial_\mu A\partial_\nu A}S(x) = \\ &= \int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\Omega^3\delta(A-A_0)\Theta(V_0-V)\sqrt{-\tilde{g}^{\mu\nu}\partial_\mu A\partial_\nu A}S(x) = \\ &= \tilde{I}(S\Omega^3;A_0;V_0)\end{aligned} \tag{3.22}$$

where we take into consideration that $A(x)$ and $V(x)$ are true scalars, and not scalars through the contraction of indices, and therefore remain unchanged, however $\sqrt{-\partial_\mu A\partial^\mu A}$ is a scalar by contraction of indices and therefore a metric tensor is present. Here we also defined the conformal integral as,

$$\tilde{I}(S;A_0;V_0) := \int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\delta(A-A_0)\Theta(V_0-V)\sqrt{-\tilde{\nabla}_\mu A\tilde{\nabla}^\mu A}S(x), \tag{3.23}$$

where the new window function will filter the manifold into a conformal hypersurface $\tilde{\Sigma}_{A_0}$, in the context of conformal LCA I shall omit the A_0 from the hypersurface, for brevity. Here the manifold remains the same, it is not changed by the conformal transformation, this is another advantage of using the window function to filter the manifold.

Defining the conformal average as,

$$\langle S\rangle_{\tilde{\Sigma}} := \frac{\tilde{I}(S)}{\tilde{I}(1)}, \tag{3.24}$$

from Eq.(3.22) we can obtain the following relations,

$$\langle S\rangle_{\tilde{\Sigma}} = \frac{\left\langle S\Omega^{-3}\right\rangle_{\Sigma}}{\left\langle \Omega^{-3}\right\rangle_{\Sigma}}, \qquad \langle S\rangle_{\Sigma} = \frac{\left\langle S\Omega^{3}\right\rangle_{\tilde{\Sigma}}}{\left\langle \Omega^{3}\right\rangle_{\tilde{\Sigma}}}. \tag{3.25}$$

Assuming A is homogeneous and V does not depend on the time coordinate, a convenient way to write the conformal integral is,

$$\begin{aligned}\tilde{I}(S;A_0;V_0) &= \int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\delta(A-A_0)\Theta(V_0-V)\sqrt{-\tilde{\nabla}_\mu A\tilde{\nabla}^\mu A}S(x) \\ &= \int_{\mathcal{M}} d^3x\left(\frac{\partial t}{\partial A}dA\right)\sqrt{-\tilde{g}}\delta(A-A_0)\Theta(V_0-V)\sqrt{-\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu A\tilde{\nabla}_\nu A}S(x) \\ &= \int_{\mathcal{M}} d^3x dA\frac{1}{\partial_0 A}\sqrt{-\tilde{g}}\delta(A-A_0)\Theta(V_0-V)\sqrt{-\tilde{g}^{00}\tilde{\nabla}_0 A\tilde{\nabla}_0 A}S(x) \\ &= \int_{\mathcal{M}} d^3x dA\sqrt{-\tilde{g}}\delta(A-A_0)\Theta(V_0-V)\frac{1}{\tilde{N}}\frac{\sqrt{\tilde{\nabla}_0 A\tilde{\nabla}_0 A}}{\partial_0 A}S(x) \\ &= \int_{\tilde{\Sigma}_{A_0}} d^3x\sqrt{\tilde{h}(t_0,\vec{x})}\Theta(V_0-V)S(t_0,\vec{x}),\end{aligned} \tag{3.26}$$

where $\sqrt{-\tilde{g}} = \tilde{N}\sqrt{\tilde{h}}$ and reminder that A_0 is the hypersurface level chosen at a time t_0.

Defining the conformal volume as,

$$\tilde{V} := \int_{\tilde{\Sigma}} d^3\sqrt{\tilde{h}}\Theta(V_0 - V), \tag{3.27}$$

and using eq. (2.64), allows us to see that,

$$\tilde{V} = V\langle\Omega^{-3}\rangle_{\Sigma}, \tag{3.28}$$

where if we derive w.r.t d/dt and divide by $\tilde{V}$, we obtain,

$$\frac{1}{\tilde{V}}\frac{d\tilde{V}}{dt} = \frac{1}{V}\frac{dV}{dt} + \frac{1}{\langle\Omega^{-3}\rangle_{\Sigma}}\frac{d}{dt}\langle\Omega^{-3}\rangle_{\Sigma}. \tag{3.29}$$

The conformal volume evolution is given by,

$$\frac{1}{\tilde{V}}\frac{d\tilde{V}}{dt} = \left\langle\tilde{N}\tilde{K}\right\rangle_{\tilde{\Sigma}} \tag{3.30}$$

where the term inside the averaging operators, relates to its usual form as,

$$\tilde{N}\tilde{K} = NK - 3\frac{d}{dt}\ln\Omega. \tag{3.31}$$

The commutation rule is,

$$\left[\frac{d}{dt}, \langle\cdot\rangle_{\tilde{\Sigma}}\right] S = \left\langle\tilde{N}\tilde{K}S\right\rangle_{\tilde{\Sigma}} - \left\langle\tilde{N}\tilde{K}\right\rangle_{\tilde{\Sigma}}\langle S\rangle_{\tilde{\Sigma}}, \tag{3.32}$$

where $NV^i = \tilde{N}\tilde{V}^i$ due to v^i and β^i being conformally invariant.

4. Scalar-tensor theories

Among modified theories of gravity, scalar-tensor theories have gained more attention than others. The main characteristic of these theories is the introduction of an hypothetical scalar field, such fields are also present in the standard model of particle physics and unified field theories.

This scalar field can be inserted in Dirac's Large number hypothesis, where the possibility for the Gravitational "constant" G to vary in time was raised. In fact, Robert Dicke and Carl Brans, exploited this possibility with a time dependent scalar field ϕ [7].

4.1. Jordan-Brans-Dicke theory

The simplest way of introducing a time-dependent gravitational constant is represented in the so-called Jordan frame by the following action:

$$S_{\text{JBD}} = \int_{\mathcal{M}} d^4x \left[\sqrt{-g}\frac{1}{2k'}\left[R\phi - \omega\frac{\phi_{,\rho}\phi^{,\rho}}{\phi}\right] + \sqrt{-g}\mathcal{L}_m\right] \tag{4.1}$$

where ω is the dimensionless Brans-Dicke coupling constant, $k' = 8\pi$, and $\mathcal{L}_m$ is the matter Lagrangian describing ordinary matter, i.e. any form of matter different from the scalar field ϕ. The second term is introduced in order to make the scalar field dynamical, where the factor ϕ in the denominator is responsible for making ω dimensionless.

In this action, matter is not directly coupled to ϕ, in the sense that the matter Lagrangian $\mathcal{L}_m$ does not depend on ϕ, and is minimally coupled to ϕ. However, in the first term we see that ϕ is directly coupled to the metric, via the Ricci scalar. Here the metric and scalar field describe the gravitational field, contrary to other theories where the metric alone describes it, the gravitational field together with the matter will describe the dynamics.

4.1.1 Jordan frame

The field equations of the Brans-Dicke theory:

$$G_{\mu\nu} = k'\left\{\frac{T^{(m)}_{\mu\nu}}{\phi} + \frac{\omega}{k'\phi^2}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\right] + \frac{1}{k'\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\right)\right\}, \tag{4.2}$$

$$\Box\phi = \frac{k'T^{(m)}}{2\omega+3}, \tag{4.3}$$

where

$$T^{(m)}_{\mu\nu} := \frac{-2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}^{(m)}\right)}{\delta g^{\mu\nu}} \tag{4.4}$$

is the energy-momentum tensor for ordinary matter. A full derivation of the field equations is shown in appendix A.

In eq. (4.3), it is clear the scalar field is sourced by matter with non zero trace, i.e. $T^{(m)} \neq 0$ (this type of matter is usually called non-conformal matter for reasons that will apparent in the next section). From the relation between matter and the scalar field in the field equation of ϕ, one would be lead to believe there is a coupling between the two, however even in the derivation this relation only happens via the field equation of the metric, hence a minimal coupling between matter and ϕ.

By defining an effective energy-momentum tensor as

$$T^{\text{eff}}_{\mu\nu} = \frac{T^{(m)}_{\mu\nu}}{\phi} + \frac{\omega}{k'\phi^2}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\right] + \frac{1}{k'\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\right), \tag{4.5}$$

we see that its components w.r.t the observer's four-velocity n_μ, are as follows. The energy density:

$$\begin{aligned}
E = T_{\mu\nu}n^\mu n^\nu &= \left[\frac{T^{(M)}_{\mu\nu}}{\phi} + \frac{1}{k'}\left(\frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\right) + \frac{\omega}{\phi^2}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2\right]\right)\right]n^\mu n^\nu \\
&= \frac{E_m}{\phi} + \frac{\omega}{k'\phi^2}\left((n^\mu\partial_\mu\phi)(n^\nu\partial_\nu\phi) - \frac{1}{2}g_{\mu\nu}n^\mu n^\nu(\nabla\phi)^2\right) + \frac{1}{k'\phi}\left(n^\mu n^\nu\nabla_\mu\nabla_\nu - g_{\mu\nu}n^\mu n^\nu\Box\right)\phi \\
&= \frac{E_m}{\phi} + \frac{\omega}{k'\phi^2}\left(\left(\frac{1}{N}\frac{d\phi}{dt}\right)^2 + \frac{1}{2}(\nabla\phi)^2\right) + \frac{1}{k'\phi}\left[\frac{1}{N}\frac{d}{dt}\left(\frac{1}{N}\frac{d\phi}{dt}\right) - A^\mu\nabla_\mu\phi + \Box\phi\right] = \\
&= \frac{E_m}{\phi} + \frac{\omega}{k'\phi^2}\left(\frac{1}{2}\left(\frac{1}{N}\frac{d\phi}{dt}\right)^2 + \frac{1}{2}h^{\mu\nu}D_\mu\phi D_\nu\phi\right) + \frac{1}{k'\phi}\left[D^\alpha D_\alpha\phi - \frac{K}{N}\frac{d\phi}{dt}\right]
\end{aligned} \tag{4.6}$$

where we use the eq. (2.24). And the pressure is given by

$$\begin{aligned}
S = T_{\mu\nu}h^{\mu\nu} = \frac{S_m}{\phi} &+ \frac{\omega}{k'\phi^2}\left(\frac{3}{2}\left(\frac{1}{N}\frac{d\phi}{dt}\right)^2 - \frac{1}{2}h^{\alpha\beta}D_\alpha\phi D_\beta\phi\right) + \\
&+ \frac{1}{k'\phi}\left[3N\frac{d^2\phi}{dt^2} - 3\frac{d\phi}{dt}\frac{dN}{dt} - 3A^\mu\nabla_\mu\phi - 2D^\alpha D_\alpha\phi + 2\frac{K}{N}\frac{d\phi}{dt}\right],
\end{aligned} \tag{4.7}$$

where we use $D_\mu\phi = h^\alpha_\mu\nabla_\alpha\phi \Leftrightarrow h^\mu_\beta D_\mu\phi = h^\alpha_\beta\nabla_\alpha\phi$.

It is useful to determine

$$E + S = \frac{E_m + S_m}{\phi} + \frac{\omega}{k'\phi^2}\left(\frac{1}{N}\frac{d\ln\phi}{dt}\right)^2 + \frac{1}{k'\phi}\left[\frac{3}{N}\frac{d}{dt}\left(\frac{1}{N}\frac{d\phi}{dt}\right) - 3A^\mu\nabla_\mu\phi - D^\alpha D_\alpha\phi + \frac{K}{N}\frac{d\phi}{dt}\right]. \tag{4.8}$$

4.1.2 Einstein frame

In 1961, Brans and Dicke showed that under a conformal transformation, in eq. (2.64), the Brans-Dicke action can recover a form similar to the Hilbert-Einstein action [7], we call this the Einstein frame. This is a very common technique used in gravitational theories and cosmology, to go from the Jordan frame to the Einstein frame.

By reviewing the action in eq. (4.32), and (eq do ricci):

$$\begin{aligned} S_{\mathrm{JBD}} = &\int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\Omega^4 \frac{1}{2k'}\left[\Omega^{-2}\left(\tilde{R} - 6\tilde{\nabla}_\mu\tilde{\nabla}^\mu \ln\Omega - 6\tilde{\nabla}_\mu \ln\Omega\tilde{\nabla}^\mu \ln\Omega\right)\phi - \omega\frac{\Omega^{-2}\tilde{\nabla}_\mu\phi\tilde{\nabla}^\mu\phi}{\phi}\right] \\ &+ \int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\Omega^4\mathcal{L}_m(\Omega^{-2}\tilde{g}^{\mu\nu}, \Psi), \end{aligned} \tag{4.9}$$

where if we choose the conformal factor to be

$$\Omega^2 = (G\phi)^{-1}, \tag{4.10}$$

the corresponding action will be

$$\begin{aligned} S_{\mathrm{EF}} = &\int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\frac{1}{2k}\left[\tilde{R} + 3\tilde{\nabla}_\mu\tilde{\nabla}^\mu \ln\phi - \frac{3}{2}\tilde{\nabla}_\mu \ln\phi\tilde{\nabla}^\mu \ln\phi - \omega\frac{\tilde{\nabla}_\mu\phi\tilde{\nabla}^\mu\phi}{\phi^2}\right] \\ &+ \int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\Omega^4\mathcal{L}_m(\Omega^{-2}\tilde{g}^{\mu\nu}, \Psi), \end{aligned} \tag{4.11}$$

and hence, we obtain an action written in the Einstein frame. We can further simplify by noting the second term, inside the square brackets, is a surface term and therefore is removed from our action, turning our action into:

$$S_{\mathrm{EF}} = \int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{2k} - \frac{1}{2k}\left(\frac{3}{2} + \omega\right)\tilde{\nabla}_\mu \ln\phi\tilde{\nabla}^\mu \ln\phi + \Omega^4\mathcal{L}_m(\Omega^{-2}\tilde{g}^{\mu\nu}, \Psi)\right]. \tag{4.12}$$

We can define a new scalar field φ as

$$\varphi = \sqrt{\frac{3 + 2\omega}{2k}}\ln\phi. \tag{4.13}$$

In eq. (4.12), we still do not have the matter Lagrangian written in the Einstein frame. In fact, depending on the ordinary matter being treated, this will acquire vastly different situations. These differences are due to two types of matter:

Conformal matter, the Lagrangian scales with the conformal factor in the following way:

$$\mathcal{L}_m(g^{\mu\nu}\Omega^{-2}, \Psi) = \Omega^{-4}\mathcal{L}_m(\tilde{g}^{\mu\nu}, \Psi), \tag{4.14}$$

therefore making $\sqrt{-g}\mathcal{L}_m$ conformally invariant. This scaling can arise, for example, in radiation fields and scalar fields with a zero or quadratic potential. For example, the Maxwell field:

$$\mathcal{L}_m(g^{\mu\nu}, A_\mu) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}F_{\mu\nu} = -\frac{1}{4}\Omega^{-4}\tilde{g}^{\mu\alpha}\tilde{g}^{\nu\beta}F_{\alpha\beta}F_{\mu\nu} = \\ = \Omega^{-4}\mathcal{L}_m(\tilde{g}^{\mu\nu}, \tilde{A}_\mu), \tag{4.15}$$

where the Maxwell tensor is and has $\tilde{F}_{\alpha\beta} = F_{\alpha\beta}$.

From eq. (4.4), the energy-momentum tensor scales by

$$T_{\mu\nu} = \Omega^2\tilde{T}_{\mu\nu}, \tag{4.16}$$

and is traceless, i.e. $T = T^\mu_\mu = 0$. This choice is seen in Brans' paper of 1961 [7], and this recent paper [8], contains proofs for a radiative field, a fermionic field and a perfect fluid.

Non-conformal matter, where the Lagrangian does not scale with the conformal factor, and therefore, from eq. (4.4), the energy-momentum tensor scales as

$$T_{\mu\nu} = \Omega^{-2}\tilde{T}_{\mu\nu}, \tag{4.17}$$

and has a non zero trace. With this matter we obtain a coupling between the matter Lagrangian and the conformal factor.

Deixar claro qual o tipo de materia que se vai usar

Therefore, the action in eq. (4.12):

$$S_{\text{EF}} = \int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{2k} - \frac{1}{2}\tilde{\nabla}_\mu\varphi\tilde{\nabla}^\mu\varphi + \frac{1}{G^2\phi^2}\mathcal{L}_m(\tilde{g}^{\mu\nu}, \Psi)\right]. \tag{4.18}$$

With help from the appendix A, we can determine the variation w.r.t the metric of the Einstein frame's action to be

$$\delta_{\tilde{g}}S = \int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\delta\tilde{g}^{\mu\nu}\left\{\frac{\tilde{G}_{\mu\nu}}{2k} - \frac{1}{2}\left(\tilde{\nabla}_\mu\varphi\tilde{\nabla}_\nu\varphi - \frac{1}{2}g_{\mu\nu}\tilde{\nabla}_\lambda\varphi\tilde{\nabla}^\lambda\varphi\right) - \frac{1}{2}\alpha(\varphi)\tilde{T}^{(m)}_{\mu\nu}\right\} \tag{4.19}$$

where $\alpha(\varphi) = \exp\left(-2\sqrt{\frac{2k}{3+2\omega}}\varphi\right)/G^2$ refers to the coupling between matter and the field φ. And w.r.t the scalar field:

$$\delta_\varphi S = \int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\delta\varphi\tilde{\Box}\varphi + \delta_\varphi S_m(g^{\mu\nu}). \tag{4.20}$$

The extra term can be varied as

$$\delta_\varphi S_m(g^{\mu\nu}) = \delta_\varphi g^{\mu\nu}\frac{\delta S_m(g^{\mu\nu})}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{\frac{2\kappa^2}{2\omega+3}}(-\rho_m+3P_m). \tag{4.21}$$

By taking $\delta_{\tilde{g}}S=0$ and $\delta_\varphi S=0$, we obtain the following field equations:

$$\tilde{G}_{\mu\nu} = k\left\{\alpha(\varphi)\tilde{T}^{(m)}_{\mu\nu} + \left(\tilde{\nabla}_\mu\varphi\tilde{\nabla}_\nu\varphi - \frac{1}{2}g_{\mu\nu}\tilde{\nabla}_\lambda\varphi\tilde{\nabla}^\lambda\varphi\right)\right\}, \tag{4.22}$$

$$\tilde{\Box}\varphi = \frac{1}{2}\sqrt{\frac{2\kappa^2}{2\omega+3}}(-\rho_m+3P_m). \tag{4.23}$$

As seen in the Jordan frame, we can write the RHS of eq. (4.22), as an effective energy-momentum tensor (conformal) given by

$$\tilde{T}^{\text{eff}}_{\mu\nu} := \alpha(\varphi)\tilde{T}^{(m)}_{\mu\nu} + \tilde{\nabla}_\mu\varphi\tilde{\nabla}_\nu\varphi - \frac{1}{2}g_{\mu\nu}\tilde{\nabla}_\lambda\varphi\tilde{\nabla}^\lambda\varphi, \tag{4.24}$$

and its energy density according to the conformal observer is

$$\tilde{E} = \tilde{T}^{\text{eff}}_{\mu\nu}\tilde{n}^\mu\tilde{n}^\nu = \alpha(\varphi)\tilde{E}_m + \frac{1}{2}\left(\frac{1}{\tilde{N}}\frac{d\varphi}{dt}\right)^2 + \frac{1}{2}\tilde{D}_\lambda\varphi\tilde{D}^\lambda\varphi, \tag{4.25}$$

and the pressure is

$$\tilde{S} = \tilde{T}^{\text{eff}}_{\mu\nu}\tilde{h}^{\mu\nu} = 3\alpha(\varphi)\tilde{S}_m + \frac{3}{2}\left(\frac{1}{\tilde{N}}\frac{d\varphi}{dt}\right)^2 - \frac{1}{2}\tilde{D}_\lambda\varphi\tilde{D}^\lambda\varphi \tag{4.26}$$

In this frame, due to the direct coupling between matter and the scalar field, the conservation law of the matter EM tensor transforms as

$$\begin{aligned}
\tilde{\nabla}_\alpha\tilde{T}^{\alpha\beta}_{(m)} &= \tilde{\nabla}_\alpha\left(\Omega^6T^{\alpha\beta}_{(m)}\right) = \nabla_\alpha\left(\Omega^6T^{\alpha\beta}_{(m)}\right) + \Omega^6\left(C^\alpha_{\alpha\lambda}T^{\lambda\beta}_{(m)} + C^\beta_{\alpha\lambda}T^{\alpha\lambda}_{(m)}\right)\\
&= \Omega^6\nabla_\alpha T^{\alpha\beta}_{(m)} + 6\Omega^5T^{\alpha\beta}_{(m)}\nabla_\alpha\Omega - \Omega^5\left[4T^{\lambda\beta}_{(m)}\nabla_\lambda\Omega + T^{\alpha\lambda}_{(m)}\left(\delta^\beta_\alpha\nabla_\lambda + \delta^\beta_\lambda\nabla_\alpha - \tilde{g}_{\alpha\lambda}\nabla^\beta\right)\Omega\right]\\
&= \Omega^6\nabla_\alpha T^{\alpha\beta}_{(m)} + 6\Omega^5T^{\alpha\beta}_{(m)}\nabla_\alpha\Omega - \Omega^5\left[4T^{\lambda\beta}_{(m)}\nabla_\lambda\Omega + \left(T^{\beta\lambda}_{(m)}\nabla_\lambda + T^{\alpha\beta}_{(m)}\nabla_\alpha - T_{(m)}\Omega^{-2}\nabla^\beta\right)\Omega\right]\\
&= \Omega^6\nabla_\alpha T^{\alpha\beta}_{(m)} + \Omega^3T^{(m)}\nabla^\beta\Omega = \Omega^3T^{(m)}\nabla^\beta\Omega.
\end{aligned} \tag{4.27}$$

By noticing the trace of the EM tensor transforms as

$$\tilde{T}^{(m)} \equiv \tilde{g}^{\alpha\beta}\tilde{T}^{(m)}_{\alpha\beta} = \Omega^4T^{(m)}. \tag{4.28}$$

Therefore, $\tilde{T}^{(m)}$ vanishes iff. $T^{(m)}=0$. Then eq. (4.27) becomes

$$\tilde{\nabla}_\alpha\tilde{T}^{\alpha\beta}_{(m)} = \tilde{T}^{(m)}\tilde{\nabla}^\beta(\ln\Omega). \tag{4.29}$$

In the JBD theory the choice of transfomation is $\Omega = (G\phi)^{-1/2}$, then we obtain

$$\tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}_{(m)} = -\frac{1}{2}\tilde{T}^{(m)}\tilde{\nabla}^\beta \ln\phi, \tag{4.30}$$

or, in terms of the new scalar field:

$$\tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}_{(m)} = -\frac{1}{2}\sqrt{\frac{2\kappa^2}{2\omega+3}}\tilde{T}^{(m)}\tilde{\nabla}^\beta \varphi. \tag{4.31}$$

4.2. Bergmann-Wagoner theory

Bergmann [9] and Wagoner [10] proposed the most general scalar-tensor theory of gravitation, by introducing a cosmological function $\Lambda(\phi)$ and a ϕ-dependent coupling, $\omega := \omega(\phi)$, yield the following action:

$$S_{\text{BW}} = \int_{\mathcal{M}} d^4x \left[\sqrt{-g}\frac{1}{2k'}\left[R\phi - \omega\frac{\phi_{,\rho}\phi^{,\rho}}{\phi} - V(\phi)\right] + \sqrt{-g}\mathcal{L}_m\right]. \tag{4.32}$$

It can be shown that the JBD theory is a special case of BW theory, by

$$\omega(\phi) = \omega = C^{te}, \qquad \Lambda(\phi) = 0. \tag{4.33}$$

Under the variational principle, the variation w.r.t the metric, will be the same as in eq. (A.28) with an extra term:

$$\delta_g \int_{\mathcal{M}} d^4x\sqrt{-g}\frac{1}{2k'}\Lambda(\phi) = \frac{1}{2k'}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(\frac{1}{2}\Lambda(\phi)g_{\mu\nu}\right), \tag{4.34}$$

this will yield the following field equation:

$$G_{\mu\nu} = k'\left\{\frac{T^{(m)}_{\mu\nu}}{\phi} + \frac{\omega}{k'\phi^2}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\right] + \frac{1}{k'\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\right) - \frac{\Lambda(\phi)g_{\mu\nu}}{2k'\phi}\right\}. \tag{4.35}$$

The variation w.r.t the scalar field, will be

$$\delta_\phi S_{\text{BW}} = \frac{1}{2k'}\int_{\mathcal{M}} d^4x\sqrt{-g}\left\{R - \left(\frac{\omega}{\phi} + \frac{d\omega(\phi)}{d\phi}\right)\frac{\phi_{,\mu}\phi^{,\mu}}{\phi} + 2\omega\frac{\Box\phi}{\phi} - \frac{d\Lambda(\phi)}{d\phi}\right\}\delta\phi \tag{4.36}$$

which with $\delta_\phi S_{\rm BW} = 0$ and the trace of eq. (4.35):

$$2\omega\Box\phi = -\phi R + \left(\frac{\omega}{\phi} + \frac{d\omega(\phi)}{d\phi}\right)\phi_{,\mu}\phi^{,\mu} + \phi\frac{d\Lambda(\phi)}{d\phi} \Leftrightarrow$$
$$\Leftrightarrow 2\omega\Box\phi = \phi\left[\frac{k'T^{(m)}}{\phi} - \omega\frac{\phi_{,\mu}\phi^{,\mu}}{\phi^2} - 3\frac{\Box\phi}{\phi}\right] + \left(\frac{\omega}{\phi} + \frac{d\omega(\phi)}{d\phi}\right)\phi_{,\mu}\phi^{,\mu} + \phi\frac{d\Lambda(\phi)}{d\phi} \Leftrightarrow$$
$$\Leftrightarrow (2\omega+3)\Box\phi = k'T^{(m)} + \frac{d\omega(\phi)}{d\phi}\phi_{,\mu}\phi^{,\mu} + \phi\frac{d\Lambda(\phi)}{d\phi} \tag{4.37}$$

4.2.1 Einstien frame

$$S_{\text{EF}} = \int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{2k} - \frac{1}{2k}\left(\frac{3}{2}+\omega\right)\tilde{g}^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \frac{1}{2k}\frac{\Lambda(\phi)}{G\phi^2} + \alpha(\phi)\mathcal{L}_m(\tilde{g}^{\mu\nu},\Psi)\right] \tag{4.38}$$

5. Backreaction

5.1. Jordan-Brans-Dicke theory

5.1.1 Jordan frame

The scale factor is given by

$$\frac{a_\Sigma}{a_{\Sigma_0}} := \left(\frac{V_\Sigma}{V_{\Sigma_0}}\right)^{1/3}. \tag{5.1}$$

The effective Hubble expansion w.r.t the normal frame is thus defined as

$$H_\Sigma := \frac{1}{a_\Sigma}\frac{da_\Sigma}{dt} \equiv \frac{1}{3V_\Sigma}\frac{dV_\Sigma}{dt} \equiv \frac{1}{3}\left\langle NK\right\rangle_\Sigma. \tag{5.2}$$

The objective now is to obtain a Friedmannian equation, we can do this by taking the energy equation in eq. (2.53), multiplying by N^2 and taking the average:

$$k\langle N^2 E\rangle = \frac{1}{2}\langle N^{2\,3}R\rangle + \frac{1}{2}\langle N^2(K^2 + K_{ij}K^{ij})\rangle, \tag{5.3}$$

from here by summing $3\left(\frac{1}{a}\frac{da}{dt}\right)^2$ on the RHS and $\frac{1}{3}\langle NK\rangle^2$ on the LHS, as they are the same, the equation still stands, and after rearranging we get the following equation:

$$3\left(\frac{1}{a}\frac{da}{dt}\right)^2 = k\langle N^2 E\rangle - \frac{1}{2}\langle N^{2\,3}R\rangle - \frac{1}{2}\left[\left\langle N^2(K^2 + K_{ij}K^{ij})\right\rangle - \frac{2}{3}\langle NK\rangle^2\right]. \tag{5.4}$$

Now for the Raychaudhuri equation, by summing the eqs. (2.53) and (2.61), multiplying by N^2 and taking the average:

$$\frac{k}{2}\langle N^2(E+S)\rangle_\Sigma = -\left\langle N\frac{dK}{dt}\right\rangle_\Sigma + \langle N^2 K_{ij}K^{ij}\rangle_\Sigma + \langle ND_iD^iN\rangle_\Sigma, \tag{5.5}$$

from here by determining

$$\begin{aligned}
\frac{1}{a}\frac{d^2a}{dt^2} = \frac{d}{dt}\left(\frac{1}{a}\frac{da}{dt}\right) + \left(\frac{1}{a}\frac{da}{dt}\right)^2 &= \frac{1}{3}\frac{d}{dt}\left\langle NK\right\rangle_\Sigma + \frac{1}{9}\left\langle NK\right\rangle_\Sigma^2 \\
&= \frac{1}{3}\left\langle\frac{d}{dt}(NK)\right\rangle_\Sigma + \frac{1}{3}\langle N^2K^2\rangle_\Sigma - \frac{1}{3}\langle NK\rangle_\Sigma^2 + \frac{1}{9}\left\langle NK\right\rangle_\Sigma^2 \\
&= \frac{1}{3}\left\langle\frac{d}{dt}(NK)\right\rangle_\Sigma + \frac{1}{3}\langle N^2K^2\rangle_\Sigma - \frac{2}{9}\langle NK\rangle_\Sigma^2,
\end{aligned} \tag{5.6}$$

in order to sum on the eq. (5.5), we obtain

$$3\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{k}{2}\langle N^2(E+S)\rangle_\Sigma + \left\langle N^2(K^2 + K_{ij}K^{ij})\right\rangle_\Sigma - \frac{2}{3}\langle NK\rangle_\Sigma^2 + \left\langle ND_iD^iN + K\frac{dN}{dt}\right\rangle.$$

The averaged Friedman and Raychaudhuri equations as

$$3\left(\frac{1}{a_\Sigma}\frac{da_\Sigma}{dt}\right)^2 = 8\pi\left\langle N^2 E\right\rangle_\Sigma - \frac{1}{2}\left(\mathcal{R}_\Sigma + \mathcal{Q}_\Sigma\right), \tag{5.7}$$

$$3\frac{1}{a_\Sigma}\frac{d^2 a_\Sigma}{dt^2} = -4\pi\left\langle N^2(E+S)\right\rangle_\Sigma + \mathcal{Q}_\Sigma + \mathcal{P}_\Sigma \tag{5.8}$$

where the kinematical and dynamical backreactions are defined as

$$\mathcal{R}_\Sigma := \left\langle N^{2\,(3)}R\right\rangle_\Sigma \tag{5.9}$$

$$\mathcal{Q}_\Sigma := \left\langle N^2\left(K^2 + K_{ij}K^{ij}\right)\right\rangle_\Sigma - \frac{2}{3}\left\langle NK\right\rangle_\Sigma^2, \tag{5.10}$$

$$\mathcal{P}_\Sigma := \left\langle N D_i D^i N + K\frac{dN}{dt}\right\rangle_\Sigma. \tag{5.11}$$

By writing the RHS of eqs. (5.7) and (5.8) in terms of an effective energy density and effective pressur, in the following way

$$E_{\text{eff}} = \left\langle N^2\rho\right\rangle_\Sigma - \frac{1}{16\pi}\left(\mathcal{R}_\Sigma + \mathcal{Q}_\Sigma + \sum_a \mathcal{T}_\Sigma^{(a)}\right), \tag{5.12}$$

$$S_{\text{eff}} := 3\left\langle N^2 P\right\rangle_\Sigma - \frac{1}{16\pi}\left[3\mathcal{Q}_\Sigma + 4\mathcal{P}_\Sigma - \mathcal{R}_\Sigma + \sum_a \mathcal{T}_\Sigma^{(a)}\right], \tag{5.13}$$

where we introduce a new kind of backreaction, the stress energy backreaction $\mathcal{T}_\Sigma^{(a)}$, in which the index a refers to the species (matter, radiation or scalar field), in order to express the energy density and pressure according to the multiple fluid flows, e.g. $\rho = \rho_m + \rho_\varphi$ and $P = P_m + P_\varphi$, where it is given by

$$\mathcal{T}_\Sigma^{(a)} = -16\pi\left\langle N^2\left(n^\mu n^\nu - u^\mu_{(a)}u^\nu_{(a)}\right)T^{(a)}_{\mu\nu}\right\rangle_\Sigma, \tag{5.14}$$

where $u^\mu_{(a)}$ is the four-velocity of the corresponding species (a).

Due to the possible presence of multiple fluid flows, as in the JBD theory, each with their own four-velocity $\mathbf{u}_{(a)}$, and subsequently time derivative.

With the new backreaction we get

$$3\left(\frac{1}{a_\Sigma}\frac{da_\Sigma}{dt}\right)^2 = 8\pi\left\langle N^2\rho\right\rangle_\Sigma - \frac{1}{2}\left(\mathcal{R}_\Sigma + \mathcal{Q}_\Sigma + \mathcal{T}_\Sigma\right), \tag{5.15}$$

$$3\frac{1}{a_\Sigma}\frac{d^2 a_\Sigma}{dt^2} = -4\pi\left\langle N^2(\rho + 3P)\right\rangle_\Sigma + \mathcal{Q}_\Sigma + \mathcal{P}_\Sigma + \frac{1}{2}\mathcal{T}_\Sigma, \tag{5.16}$$

where the sum on the species is implicit.

A necessary condition for eq. (5.15) to yield eq. (5.16) is the integrability condition given by [1],

$$\frac{1}{a_\Sigma^6}\left[\frac{d}{dt}\left(a_\Sigma^6 \mathcal{Q}_\Sigma\right) + a_\Sigma^4 \frac{d}{dt}\left(a_\Sigma^2 \mathcal{R}_\Sigma\right) + a_\Sigma^2 \frac{d}{dt}\left(a_\Sigma^4 \mathcal{T}_\Sigma\right) + 4a_\Sigma^5 \frac{da_\Sigma}{dt}\mathcal{P}_\Sigma\right] =$$
$$= 2k\left(\frac{d}{dt}\left\langle N^2\rho\right\rangle_\Sigma + \frac{3}{a_\Sigma}\frac{da_\Sigma}{dt}\left\langle N^2(\rho+P)\right\rangle_\Sigma\right),$$

where the LHS are the various backreactions and in the RHS is an "averaged" continuity equation. To prove this, take the continuity equation in **??**, by multiplying $\times N^3$ and taking its average we get

$$\left\langle N^2 \frac{dE}{dt}\right\rangle + \left\langle N\Theta N^2(E+S)\right\rangle = -\left\langle N^3\left(2A_\alpha J^\alpha + D_\alpha J^\alpha + S^{\alpha\beta}A_{\alpha\beta}\right)\right\rangle, \tag{5.17}$$

where via the commutation rule in eq. (3.18), the relation in , and the scale factor in **??**, we obtain

$$\frac{d}{dt}\left\langle N^2\rho\right\rangle + 3\frac{1}{a_\Sigma}\frac{da_\Sigma}{dt}\langle N^2(\rho+P)\rangle = 3\frac{1}{a_\Sigma}\frac{da_\Sigma}{dt}\langle N^2 P\rangle + \left\langle\left(2\frac{d\ln N}{dt} - \frac{d\ln\Gamma}{dt}\right)N^2\rho\right\rangle \tag{5.18}$$

$$-\left\langle\left(NK + \frac{d\ln\Gamma}{dt}\right)N^2 P\right\rangle - \left\langle\frac{N^3}{\Gamma}\left(2A_\alpha q^\alpha + D_\alpha^{(u)}q^\alpha\right)\right\rangle \tag{5.19}$$

5.1.2 Homogeneous scalar field

Let us consider a homogeneous field $\phi = \phi(t)$, in this scenario we have

$$(\nabla\phi)^2 = g^{00}(\partial_0\phi)^2 = -\frac{1}{N^2}\left(\frac{d\phi}{dt}\right)^2, \qquad A^\mu\nabla_\mu\phi = g^{0i}A_i\nabla_0\phi = -\frac{\beta^i A_i}{N^2}\frac{d\phi}{dt} \tag{5.20}$$

$$\Box\phi = g^{00}\nabla_0\nabla_0\phi = -\frac{1}{N^2}\frac{d^2\phi}{dt^2}, \qquad \frac{1}{N}\frac{d}{dt}\left(\frac{1}{N}\frac{d\phi}{dt}\right) = \frac{1}{N^2}\left[\frac{d\phi}{dt}\frac{d}{dt}\ln\left(\frac{1}{N}\right) + \frac{d^2\phi}{dt^2}\right] \tag{5.21}$$

5.1.3 Einstein frame

The effective conformal scale factor is given by

$$\frac{\tilde{a}_{\tilde{\Sigma}}}{\tilde{a}_{\tilde{\Sigma}_0}} := \left(\frac{\tilde{V}_{\tilde{\Sigma}}}{\tilde{V}_{\tilde{\Sigma}_0}}\right)^{1/3}. \tag{5.22}$$

The effective conformal Hubble expansion is thus defined as

$$\tilde{H} := \frac{1}{\tilde{a}}\frac{d\tilde{a}}{dt} \equiv \frac{1}{3\tilde{V}}\frac{d\tilde{V}}{dt} \equiv \frac{1}{3}\left\langle\tilde{N}\tilde{K}\right\rangle_{\tilde{\Sigma}}. \tag{5.23}$$

From the same procedure as in the Jordan frame, the averaged conformal Friedman equation are

$$3\left(\frac{1}{\tilde{a}}\frac{d\tilde{a}}{dt}\right)^2 = k\left\langle \tilde{N}^2\tilde{E}\right\rangle_{\tilde{\Sigma}} - \frac{1}{2}\tilde{\mathcal{R}}_{\tilde{\Sigma}} - \frac{1}{2}\tilde{\mathcal{Q}}_{\tilde{\Sigma}}, \tag{5.24}$$

$$3\frac{1}{\tilde{a}_{\tilde{\Sigma}}}\frac{d^2\tilde{a}_{\tilde{\Sigma}}}{dt^2} = -\frac{k}{2}\left\langle \tilde{N}^2\left(\tilde{E}+\tilde{S}\right)\right\rangle_{\tilde{\Sigma}} + \tilde{\mathcal{Q}}_{\tilde{\Sigma}} + \tilde{\mathcal{P}}_{\tilde{\Sigma}}, \tag{5.25}$$

where the average intrinsic curvature, the kinematical and dynamical backreactions are defined as

$$\tilde{\mathcal{Q}}_{\tilde{\Sigma}} := \left\langle \tilde{N}^2\left(\tilde{K}^2 + \tilde{K}_{ij}\tilde{K}^{ij}\right)\right\rangle_{\tilde{\Sigma}} - \frac{2}{3}\left\langle \tilde{N}\tilde{K}\right\rangle^2_{\tilde{\Sigma}}, \tag{5.26}$$

$$\tilde{\mathcal{P}}_{\tilde{\Sigma}} := \left\langle \tilde{N}\tilde{D}_i\tilde{D}^i\tilde{N} + \tilde{K}\frac{d\tilde{N}}{dt}\right\rangle_{\tilde{\Sigma}}. \tag{5.27}$$

and they relate to the usual backreactions by

$$\mathcal{Q}_{\Sigma} = \tilde{\mathcal{Q}}_{\tilde{\Sigma}} - 4\frac{d}{dt}\ln\Omega\left\langle \tilde{N}\tilde{K}\right\rangle_{\tilde{\Sigma}} + 6\left(\frac{d}{dt}\ln\Omega\right)^2 \tag{5.28}$$

$$\mathcal{P}_{\Sigma} = \tilde{\mathcal{P}}_{\tilde{\Sigma}} + \frac{d}{dt}\ln\Omega\left\langle \tilde{N}\tilde{K} + 3\frac{d}{dt}\ln\tilde{N}\right\rangle_{\tilde{\Sigma}} + 3\left(\frac{d}{dt}\ln\Omega\right)^2 \tag{5.29}$$

where we assume the conformal factor to be homogeneous.

Like before, the Friedmann equations can be simplified into a more familiar form:

$$3\left(\frac{1}{\tilde{a}}\frac{d\tilde{a}}{dt}\right)^2 = k\tilde{E}_{\text{eff}}, \tag{5.30}$$

$$3\frac{1}{\tilde{a}_{\tilde{\Sigma}}}\frac{d^2\tilde{a}_{\tilde{\Sigma}}}{dt^2} = -\frac{k}{2}(\tilde{E}_{\text{eff}} + \tilde{S}_{\text{eff}}), \tag{5.31}$$

where the effective energy density and pressure are given by the same form as in **????**, and by introducing the eqs. (4.25) and (4.26), we obtain

$$\tilde{E}_{\text{eff}} := \left\langle \tilde{N}^2\alpha(\varphi)\tilde{\rho}_m\right\rangle_{\tilde{\Sigma}} + \left\langle \frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 + \frac{1}{2}\tilde{N}^2\tilde{D}_\alpha\varphi\tilde{D}^\alpha\varphi\right\rangle_{\tilde{\Sigma}} \tag{5.32}$$

$$-\frac{1}{2k'}\left(\mathcal{R}_{\Sigma} + \mathcal{Q}_{\Sigma} + \mathcal{T}^{(m)}_{\Sigma}\right), \tag{5.33}$$

$$\tilde{S}_{\text{eff}} := 3\left\langle \tilde{N}^2\alpha(\varphi)\tilde{P}_m\right\rangle_{\tilde{\Sigma}} + \left\langle \frac{3}{2}\left(\frac{d\varphi}{dt}\right)^2 - \frac{1}{2}\tilde{N}^2\tilde{D}_\alpha\varphi\tilde{D}^\alpha\varphi\right\rangle_{\tilde{\Sigma}} \tag{5.34}$$

$$-\frac{1}{2k'}\left[3\mathcal{Q}_{\Sigma} + 4\mathcal{P}_{\Sigma} - \mathcal{R}_{\Sigma} + \mathcal{T}^{(m)}_{\Sigma}\right], \tag{5.35}$$

5.2. Bergmann-Wagoner theory

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A. Hamilton's principle

This appendix serves to explain in detail the field equations in Section obtain through the variational principle. For simplicity we start by splitting the BD action into a purely gravitational and matter part, respectively,

$$S_{\mathrm{BD}}^{\mathrm{grav}} = \int_{\mathcal{M}} d^4x\sqrt{-g}\frac{1}{2k'}\left[R\phi - \frac{\omega}{\phi}\phi_{,\mu}\phi^{,\mu}\right], \tag{A.1}$$

$$S_{\mathrm{BD}}^{(m)} = \int_{\mathcal{M}} d^4x\sqrt{-g}\mathcal{L}_m. \tag{A.2}$$

From the definition of the energy-momentum tensor in Eq.(**??**), the variation of the matter part w.r.t the $g^{\mu\nu}$ is,

$$\delta_g S_{\mathrm{BD}}^{(m)} = \int_{\mathcal{M}} d^4x\delta\left(\sqrt{-g}\mathcal{L}_m\right) = -\frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}\delta g^{\mu\nu}T_{\mu\nu}^{(m)}. \tag{A.3}$$

and since this Lagrangian does not depend on ϕ, $\delta_\phi S_{\mathrm{BD}}^{(m)} = 0$.

The variation of the purely gravitational part w.r.t the metric is,

$$\delta_g S_{\mathrm{BD}}^{\mathrm{grav}} = \int_{\mathcal{M}} d^4x\left\{\frac{\delta\sqrt{-g}}{2k'}\left[\phi R - \frac{\omega}{\phi}\phi_{,\mu}\phi^{,\mu}\right] + \frac{\sqrt{-g}}{2k'}\left[\phi\delta R - \frac{\omega}{\phi}\partial_\mu\phi\partial_\nu\phi\delta g^{\mu\nu}\right]\right\}, \tag{A.4}$$

where we have taken into account that,

$$\delta_g\left(\phi_{,\mu}\phi^{,\mu}\right) = \delta_g\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) = \partial_\mu\phi\partial_\nu\phi\delta g^{\mu\nu}. \tag{A.5}$$

The variation of the purely gravitational part w.r.t the scalar field is,

$$\delta_\phi S_{\mathrm{BD}}^{\mathrm{grav}} = \frac{1}{2k'}\int_{\mathcal{M}} d^4x\sqrt{-g}\left\{R - \omega\frac{\phi_{,\mu}\phi^{,\mu}}{\phi^2} + 2\omega\frac{\Box\phi}{\phi}\right\}\delta\phi \tag{A.6}$$

where we take into consideration that,

$$\delta\left(\frac{\phi_{,\mu}\phi^{,\mu}}{\phi}\right) = -\frac{\phi_{,\mu}\phi^{,\mu}}{\phi^2}\delta\phi + \frac{1}{\phi}\delta\left(\phi_{,\mu}\phi^{,\mu}\right) = -\frac{\phi_{,\mu}\phi^{,\mu}}{\phi^2}\delta\phi + \frac{2}{\phi}\nabla^\mu\phi\nabla_\mu\delta\phi = \tag{A.7}$$

$$= -\frac{\phi_{,\mu}\phi^{,\mu}}{\phi^2}\delta\phi + 2\nabla_\mu\left(\frac{\nabla^\mu\phi}{\phi}\delta\phi\right) - 2\nabla_\mu\left(\frac{\nabla^\mu\phi}{\phi}\right)\delta\phi = \tag{A.8}$$

$$= -\frac{\phi_{,\mu}\phi^{,\mu}}{\phi^2}\delta\phi - 2\frac{\Box\phi}{\phi}\delta\phi + 2\frac{\nabla^\mu\phi\nabla_\mu\phi}{\phi^2}\delta\phi = \tag{A.9}$$

$$= \frac{\phi_{,\mu}\phi^{,\mu}}{\phi^2}\delta\phi - 2\frac{\Box\phi}{\phi}\delta\phi. \tag{A.10}$$

The second term in Eq.(A.8) is a boundary term, given by,

$$\int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_\mu\left(\frac{\nabla^\mu\phi}{\phi}\delta\phi\right) \tag{A.11}$$

and therefore, due to the requirement set by the stationary action principle (i.e. Hamilton's principle) it must vanish[1].

The following property can be used to rewrite Eq.(A.4),

$$\delta\left(\sqrt{-g}\right) = -\frac{1}{2}\frac{1}{\sqrt{-g}}\delta\det(g_{\mu\nu}) = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}, \tag{A.12}$$

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}, \tag{A.13}$$

where in the first equation the Jacobi formula is used, i.e. $\delta(\det(A)) = \det(A)Tr\left(A^{-1}\delta A\right)$.

Then Eq.(A.4) is rewritten as,

$$\delta_g S_{\rm BD}^{\rm grav} = \int_{\mathcal{M}} d^4x\sqrt{-g}\frac{\delta g^{\mu\nu}}{2k'}\left\{\phi G_{\mu\nu} - \frac{\omega}{\phi}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\right]\right\} + \frac{1}{2k'}\delta_g\bar{S}, \tag{A.14}$$

where,

$$\delta_g\bar{S} = \int_{\mathcal{M}} d^4x\sqrt{-g}\phi g^{\mu\nu}\delta R_{\mu\nu}. \tag{A.15}$$

Let us focus on this variation for a moment, in standard Hilbert-Einstein action this term generally vanishes, but now due to the presence of the scalar field ϕ, it will not. By the Palatini identity,

$$\delta R_{\mu\nu} = \nabla_\lambda\left(\delta\Gamma^\lambda_{\mu\nu}\right) - \nabla_\nu\left(\delta\Gamma^\lambda_{\mu\lambda}\right), \tag{A.16}$$

we rewrite the Eq.(A.15) as,

$$\delta_g\bar{S} = \int_{\mathcal{M}} d^4x\sqrt{-g}\phi g^{\mu\nu}\left[\nabla_\lambda\left(\delta\Gamma^\lambda_{\mu\nu}\right) - \nabla_\nu\left(\delta\Gamma^\lambda_{\mu\lambda}\right)\right]. \tag{A.17}$$

By virtue of the Stokes-Gauss-Ostrogradski theorem (i.e. Gauss theorem),

$$\int_{\mathcal{M}} d^4x\sqrt{-g}\phi g^{\mu\nu}\nabla_\lambda\left(\delta\Gamma^\lambda_{\mu\nu}\right) = \int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_\lambda\left(\phi g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}\right) - \int_{\mathcal{M}} d^4x\sqrt{-g}g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}\nabla_\lambda\phi, \tag{A.18}$$

$$\int_{\mathcal{M}} d^4x\sqrt{-g}\phi g^{\mu\nu}\nabla_\nu\left(\delta\Gamma^\lambda_{\mu\lambda}\right) = \int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_\nu\left(\phi g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda}\right) - \int_{\mathcal{M}} d^4x\sqrt{-g}g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda}\nabla_\nu\phi, \tag{A.19}$$

[1]The stationary action principle requires that on the boundary of integration variations of the metric and its first derivatives must vanish

both the first terms in the RHS, are boundary terms which vanish. Therefore, putting these equations in Eq.(A.17),

$$\delta_g \bar{S} = \int_{\mathcal{M}} d^4x\sqrt{-g}\left[g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\lambda}\nabla_\nu\phi - g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu}\nabla_\lambda\phi\right] = \\ = \int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_\lambda\phi\left[g^{\mu\lambda}\delta\Gamma^{\nu}_{\mu\nu} - g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu}\right]. \tag{A.20}$$

The variation of the Christoffel symbol can be written as,

$$\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left(\nabla_\mu\delta g_{\nu\rho} + \nabla_\nu\delta g_{\mu\rho} - \nabla_\rho\delta g_{\mu\nu}\right) \tag{A.21}$$

from this we see that,

$$\delta\Gamma^{\nu}_{\mu\nu} = \frac{1}{2}g^{\nu\rho}\nabla_\mu\delta g_{\nu\rho}, \tag{A.22}$$

$$g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left(2\nabla^\nu\delta g_{\nu\rho} - g^{\mu\nu}\nabla_\rho\delta g_{\mu\nu}\right) \tag{A.23}$$

putting this in Eq.(A.20),

$$\delta_g \bar{S} = \int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_\lambda\phi\left[g^{\nu\rho}\nabla^\lambda\delta g_{\nu\rho} - g^{\lambda\rho}\nabla^\nu\delta g_{\nu\rho}\right] = \\ = \int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_\lambda\phi\left[\nabla_\nu\delta g^{\nu\lambda} - g_{\alpha\beta}\nabla^\lambda\delta g^{\alpha\beta}\right] \tag{A.24}$$

where in the last step the relation $\delta g_{\mu\nu} = -g_{\mu\alpha}g_{\nu\beta}\delta g^{\alpha\beta}$ was useful. From these new terms, the Gauss theorem is,

$$\int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_\lambda\phi\nabla_\nu\delta g^{\nu\lambda} = \int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_\nu\left(\delta g^{\nu\lambda}\nabla_\lambda\phi\right) - \int_{\mathcal{M}} d^4x\sqrt{-g}\delta g^{\nu\lambda}\nabla_\nu\nabla_\lambda\phi, \tag{A.25}$$

$$\int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_\lambda\phi g_{\alpha\beta}\nabla^\lambda\delta g^{\alpha\beta} = \int_{\mathcal{M}} d^4x\sqrt{-g}\nabla^\lambda\left(g_{\alpha\beta}\delta g^{\alpha\beta}\nabla_\lambda\phi\right) - \int_{\mathcal{M}} d^4x\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}\nabla^\lambda\nabla_\lambda\phi, \tag{A.26}$$

where again the boundary terms vanish, giving us finally,

$$\delta_g \bar{S} = \int_{\mathcal{M}} d^4x\sqrt{-g}\left(g_{\mu\nu}\Box\phi - \nabla_\mu\nabla_\nu\phi\right)\delta g^{\mu\nu}. \tag{A.27}$$

Hence, we obtain the action variation,

$$\delta_g S_{\mathrm{BD}} = \delta_g S^{\mathrm{grav}}_{\mathrm{BD}} + \delta_g S^{(m)}_{\mathrm{BD}} = \\ = \frac{1}{2k'}\int_{\mathcal{M}} d^4x\sqrt{-g}\delta g^{\mu\nu}\left\{\phi G_{\mu\nu} - \frac{\omega}{\phi}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\right]\right. \tag{A.28} \\ \left. + \left(g_{\mu\nu}\Box\phi - \nabla_\mu\nabla_\nu\phi\right) - k'T^{(m)}_{\mu\nu}\right\},$$

and now by requiring $\delta_g S_{\mathrm{BD}} = 0$, we obtain the Brans-Dicke field equations,

$$G_{\mu\nu} = k' \left\{ \frac{T^{(m)}_{\mu\nu}}{\phi} + \frac{\omega}{k'\phi^2} \left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2} g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda} \right] + \frac{1}{k'\phi} \left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi \right) \right\} \tag{A.29}$$

where all the terms inside the curved brackets can be viewed as an effective energy-momentum tensor $T^{\mathrm{eff}}_{\mu\nu}$. And by taking the trace of Eq.(A.29),

$$R = -k'\frac{T^{(m)}}{\phi} + \frac{\omega}{\phi^2}\phi_{,\lambda}\phi^{,\lambda} + 3\frac{\Box\phi}{\phi} \tag{A.30}$$

and employing it in Eq.(A.6) and by setting $\delta_\phi S_{\mathrm{BD}} = 0$, we obtain the field equation for the scalar field ϕ,

$$\Box\phi = \frac{kT^{(m)}}{2\omega + 3}. \tag{A.31}$$