Statistical Inference

Worksheet 1:

1. When rolling a pair of fair dice, the sum of the numbers on the faces up can be one of 11 values $2, 3, \ldots, 12$. Can we assign probability $1/11$ to each of these outcomes? If not, what probability should be assigned to each outcome?

2. One box contains 4 white and 2 black balls and another box contains 3 white and 5 black balls. One ball is taken from each box. What is the probability of:

   (a) both being white?

   (b) both being black?

   (c) one being white and the other being black?

3. Compute the probability of obtaining the number 4 at least once in two rolls of an honest die.

4. A group of consumers was chosen to order three products, X, Y and Z, according to their moisturizing capacity. Consider the following events:

$$
\begin{aligned}
A &= \{\text{Product } X \text{ is preferable to } Y\} \\
B &= \{\text{Product } X \text{ is sorted first}\} \\
C &= \{\text{Product } X \text{ is sorted in second}\} \\
D &= \{\text{Product } X \text{ is sorted third}\}
\end{aligned}
$$

   where a better position corresponds to greater hydration. If the consumer group is not made up of specialists and, consequently, assigns orders to the products through a random method so that each possible ordering is equiprobable, is event A independent of events B, C and D?

5. Suppose that 5% of the Portuguese population suffers from hypertension and that, among these, 75% drink alcoholic beverages. Among those who are not hypertensive, 50% drink alcoholic beverages.

   (a) What percentage of people drink alcohol?

   (b) What percentage of people drinking alcohol suffer from hypertension.

6. In a given population of mice there are 60% females of which 6% weigh more than 200 g and there are 40% males of which 2% weigh more than 200 g. One mouse was selected from that population and found to weigh more than 200 g. What is the probability that this mouse is female?

7. The percentage of alcohol (100%X) in a certain compound is a random variable where X has a probability density function f defined by

$$
f(x) = \begin{cases} 20x^3(1-x), & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}
$$

   (a) Determine the probability distribution function $F(x)$.

   (b) Calculate $\mathbb{P}(X \leq 2/3)$

   (c) Assume that the selling price of the compound depends on the alcohol content. If $1/3 < x < 2/3$, the price is C1 euros per litre; otherwise, is C2 euros per litre. If the production cost is C3 euros per liter, calculate the probability function of the random variable Y, which represents the net profit per liter.

8. A random variable X is defined by

$$X = \begin{cases} -2 & \text{with probability } 1/3 \\ 3 & \text{with probability } 1/2 \\ 1 & \text{with probability } 1/6 \end{cases}$$

   (a) Determine and sketch the probability mass function and distribution function of X.
   (b) Calculate $\mathbb{E}(X)$, $\mathbb{E}(2X+5)$, $\mathbb{E}(X^2)$, $\mathbb{V}(X)$, $\mathbb{V}(2X)$, $\mathbb{V}(2X+5)$, the mode and the median.

9. The pH of water samples from a specific lake is a r.v., X, with probability density function

$$f(x) = \begin{cases} \frac{3}{8}(7-x)^2, & 5 \leq x \leq 7 \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Determine and sketch the graph of the distribution function F for the random variable X.
   (b) Determine the mean and variance of the random variable X.
   (c) How often would you expect the ph value was less than 5.5?

10. Compute the mean and variance of the number of points that can appear when one a rolls an unbiased die. Describe the probability distribution function of the random variable you considered.

11. Let X be a random variable with probability density function f given by

$$f(x) = \begin{cases} c, & -2 \leq x \leq 2 \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Determine the value of the constant c.
   (b) For the value of c determined in (a) calculate:
      i. the probability distribution function.
      ii. $\mathbb{E}(X)$, $\mathbb{V}(X)$, the standard deviation and the 0.95-quantile.
      iii. the mode (modal class) and the median of X.
      iv. $\mathbb{P}(X \leq 0)$, $\mathbb{P}(X > 1/2)$, $\mathbb{P}(-1 < X < 1)$, $\mathbb{P}(-1 \leq X < 1)$, $\mathbb{P}(-1 < X \leq 3)$.
      v. Determine $\mathbb{E}(2X+5)$ and $\mathbb{V}(3X+4)$.

12. Determine the mean and standard deviation of the sum of points obtained when rolling a pair of unbiased dice.

13. Consider the random variable X with the following probability distribution function

$$F(x) = \begin{cases} 0, & x < 0 \\ 1/6, & 0 \leq x < 2 \\ 1/4, & 2 \leq x < 4 \\ 1/2, & 4 \leq x < 6 \\ 1, & 6 \leq x \end{cases}$$

   (a) Find the probability function of X and sketch its graph.
   (b) Calculate $\mathbb{P}(X \leq 1)$, $\mathbb{P}(X > 5)$, $\mathbb{P}(X \geq 1)$, $\mathbb{P}(2 \leq X < 6)$, $p(2 < X < 6)$, $\mathbb{P}(0 < X \leq 2)$.
   (c) Find the mean, median, mode and variance of X.

(d) Calculate $\mathbb{E}(X)$, $\mathbb{E}(X^2)$, $\mathbb{E}(X-5)$, $\mathbb{V}(-X)$ and $\mathbb{V}(X-5)$.

14. A random variable takes the values $2, 4, 7$ and 8 with the following probabilities, respectively,

$$\frac{1+3x}{4}, \frac{1-x}{4}, \frac{1+2x}{4}, \frac{1-4x}{4}.$$

    (a) For which values of x do you have a probability function? Take $x = 1/8$ and plot the probability and distribution functions.

15. Suppose that the daily demand for blood pressure measuring devices in a particular pharmacy in Porto is a r.v. X with the following probability mass function:

$$F(x) = \begin{cases} \frac{k2^x}{x!}, & x = 1, 2, 3, 4 \\ 0, & \text{otherwise.} \end{cases}$$

    (a) Classify the random variable X.
    (b) Determine the value of k.
    (c) Determine $\mathbb{E}(X)$, $\mathbb{V}(X)$, and the mode of X.
    (d) Determine the probability distribution function and sketch its graph.
    (e) What should be the minimum 'stock' of blood pressure measuring devices at the beginning of each day so that daily demand is satisfied with a probability of at least 0.8?
    (f) What is the probability that, when considering two days, the demand for devices is greater, than two devices, in each of them? (Assuming that the demand on each day is independent of the demand on any other day.)