Misalignment mechanism (before Inflation)

- Following: [ArXiv:1510.07633] and [The Early Universe, (Kolb)].

The soft-breaking term in the broken phase of ϕ is

$$V_{soft} \simeq -\frac{\mu_s^2 \, \vartheta_\sigma^2}{N^2} \cos\left(\frac{\theta(x)}{\vartheta_\sigma/N}\right)$$

where in our case $N=2$.

Near to the minimum, with $\frac{\theta}{\vartheta_s/N} \ll 1$, we have

$$V_{soft} \simeq \frac{m_\theta^2 \, \vartheta_s^2}{4 \quad N^2} \Theta^2 ,$$

$\Theta \equiv \frac{\theta}{\vartheta_s/N}$ → miselignment angle

$m_\theta^2 \cong -2\mu_s^2$

and lagrangian density can be written as

$$\mathcal{L} = \left(\frac{\vartheta_\sigma}{N}\right)^2 \left[\frac{\dot{\Theta}^2}{2} - \frac{m_\theta^2}{4} \Theta^2 \right]$$

The equation of motion can be obtained varying the action

$$S = \int d^4x \sqrt{-g} \, \mathcal{L}$$

where g is the determinant of the metric tensor

$$g = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & R^2 & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & R^2 \end{vmatrix} = -(R^3)^2$$

$R = R(t)$ is the scale factor of FRW metric

then

$$S = \int d^4x \, R^3 \mathcal{L}$$

finally the eq. of motion is given by

• from now ~~on~~ on we will assume that $\Theta(x)$ is spatially homogeneous.

$$\Box \Theta - \frac{\partial V}{\partial \Theta} = 0$$

where $\Box$ is the D'Alembertian

$$\Box = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\sigma} \partial_\sigma \right)$$

$$g^{\mu\sigma} = \mathrm{diag}(-1, R^2, R^2, R^2)$$

$$\frac{(\vartheta_\sigma / N)}{(\vartheta_\sigma / N)} \left(\Box \Theta - \frac{\partial V}{\partial \Theta} \right) = 0$$

$$\left(\frac{\vartheta_\sigma}{N}\right) \Box \Theta - \frac{N}{\vartheta_\sigma} \frac{\partial V}{\partial \Theta} = 0$$

$$\left(\frac{\vartheta_\sigma}{N}\right)^2 \Box \Theta - \frac{\partial V}{\partial \Theta} = 0,$$

where

$$V \simeq \frac{m_\Theta^2\, \vartheta_s^2}{4\, N^2} \cdot \Theta^2 \Rightarrow \frac{\partial V}{\partial \Theta} = \frac{m_\Theta^2}{2} \left(\frac{\vartheta_\sigma}{N}\right)^2 \Theta,$$

Then

$$\Box \Theta = \frac{1}{R^3} \partial_t \left(R^3 \partial_t \Theta \right)$$

$$= \frac{1}{R^3} \partial_t \left(R^3 \dot{\Theta} \right) = \frac{1}{R^3} \left(3R^2 \dot{R} \dot{\Theta} + R^3 \ddot{\Theta} \right)$$

$$= 3 H \dot{\Theta} + \ddot{\Theta}$$

then the eq. of motion is given by

$$\ddot{\Theta} + 3H\dot{\Theta} + \frac{m_\Theta^2}{2} \Theta = 0$$

As de relic density of Θ is defined as

$$\Omega_\Theta h^2 = \frac{\rho_\Theta}{\rho_{crit}/h^2},$$

where ρ_θ can be obtaned through the energy momentum tensor

$$\rho_\theta = \frac{\dot{\theta}^2}{2} + V$$

$$= \frac{(\vartheta_\sigma/N)^2}{(\vartheta_\sigma/N)^2}\left[\frac{\dot{\theta}^2}{2} + V\right]$$

$$= \left(\frac{\vartheta_\sigma}{N}\right)^2\left[\frac{\dot{\Theta}^2}{2} + \left(\frac{\vartheta_\sigma}{2}\right)^{-2} V\right]$$

$$\rho_\theta = \left(\frac{\vartheta_\sigma}{N}\right)^2\left[\frac{\dot{\Theta}^2}{2} + \frac{m_\theta^2}{4}\Theta^2\right]$$

Coming back to eq. of motion

$$\ddot{\Theta} + 3H\dot{\Theta} + \frac{m_\theta^2}{2}\Theta = 0$$

we observe that there is a period in the early universe where

$$H \gg m_\theta .$$

$H \simeq 1.66\, g_*^{1/2} \frac{T^2}{M_{Pl}}$ (radiation)

then the initial condition is defined as

$$\Theta(t_i) = \Theta_i \quad \text{and} \quad \dot{\Theta}(t) = 0$$

where we assume $H \gg \dot{\Theta}$.

And when $m_\theta \simeq 3H(T_{osc})$ the field Θ begins to oscillate, like a damped harmonic oscillator, where

$$T_{osc} \simeq 1.6\ \text{KeV}\left(\frac{m_\theta}{10^{-20}\,eV}\right)^{1/2}\left(\frac{3.5}{g_*(T_{osc})}\right)^{1/4}$$

Then, coming back to eq. of energy density

$$\rho_\Theta = \left(\frac{v_\sigma}{N}\right)^2 \left[\frac{\dot{\Theta}^2}{2} + \frac{m_\theta^2}{4} \Theta^2 \right]$$

we can claim that the initial energy density of θ is

$$\rho_\Theta(t) = \left(\frac{v_\sigma}{N}\right)^2 \frac{m_\theta^2}{4} \Theta_i^2 , \quad t_i \leq t \leq t_{osc}$$

By definition, the relic abundance of θ today is given by

$$\Omega_\theta h^2 = \frac{\rho_\theta(t_0)\, h^2}{\rho_{crit}} ,$$

where $\rho_\theta(t_0)$ represents the energy density of θ today.

Considering the conservation of entropy, we can obtain $\rho_\theta(t_0)$ from $\rho_\theta(t_{osc})$. Because, $\rho_\Theta(t_0)$ is the redshift of initial energy density

$$\rho_\theta(R) = \left(\frac{v_\sigma}{N}\right)^2 \frac{m_\theta^2}{4} \Theta_i^2 \left(\frac{R_{osc}}{R}\right)^3 , \quad R_{osc} > R$$

where R is the scale factor.

The entropy density is given by

$$s = \frac{S}{R^3} = \frac{2\pi^2}{45} g_s(T)\, T^3$$

Then, considering the conservation of entropy we obtain

$$\left(\frac{R_{osc}}{R_0}\right)^3 = \frac{g_s(T_0)}{g_s(T_{osc})}\left(\frac{T_0}{T_{osc}}\right)^3$$

$T_0 = 2.3\times 10^{-4}\,eV$

$g_s(T_0) = 3.91$

As $\rho_{crit} = 8.15\times 10^{-47}\,h^2\,GeV^4$, we obtain

$$\Omega_\theta h^2 = 0.11\left(\frac{m_\theta}{10^{-20}eV}\right)^{1/2}\left(\frac{v_\sigma}{10^{17}GeV}\right)^2\left(\frac{\Theta_i}{2.25\times10^{-1}}\right)^2\left(\frac{3.91}{g_s(T_{osc})}\right)\left(\frac{g_*(T_{osc})}{3.4}\right)^{3/4}$$

where we take $N=2$.