Introductory notes to the Standard Model of Particle Physics

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Abstract: The aim of this course is to introduce the Higgs Mechanism, responsible for the mass generation of the fundamental particles in the Standard Model, as well as understanding the implications of this being a theory with gauge invariance, such as the emergence of electric charge in the fundamental particles. Without formal details, we will discuss how to draw Feynaman diagrams just by direct inspection of the interactions present in the Lagrangian. While Particle Physics is a relativistic Quantum Field Theory (QFT), in this introductory notes we will simply study its classical filed theory limit leaving field quantization for a more advanced course. In essence, we will first introduce the Lagrangian formulation for relativistic classical fields and then, in a second part of the course, use this knowledge to stepwise construct the Standard Model.

Keywords: transformation, invariance, gauge invariance, symmetry, gauge symmetry, spontaneous symmetry breaking, charge, mass

Contents

1 Special Relativity

The Standard Model of Particle Physics is a relativistic theory, that is, a theory that describes very energetic particles. It is also commonly denoted as High Energy Physics (HEP). We will start our journey by stepwise introducing a sequence of concepts until we have all ingredients to study the SM itself.

1.1 The principle of relativity

The first concept to introduce is that of an **inertial frame**. A given reference system is said to be inertial when a freely moving body in it proceeds with constant velocity. Clearly, if two frames move uniformly relative to each other, they are both inertial reference systems.

The **principle of relativity** states that *all laws in nature are identical in all inertial systems of reference*. This means that the equations that describe the laws of nature are **invariant** under transformations of coordinates. In other words, the equation that describes a given law, $\mathcal{L}\left(t,x,y,z\right)$, has one and the same form in any inertial frame, that is

$$\mathcal{L}\left(t,x,y,z\right)=\mathcal{L}'\left(t',x',y',z'\right). \tag{1.1}$$

This **invariance** principle will have outstanding implications throughout this course.

In Classical Mechanics, the interaction between the particles of a given material is described by a **potential energy of interaction**, which is a function of the spacial coordinates $U\left(x,y,z\right)$. This way of describing interactions contains an assumption of instantaneous propagation of information. This implies that any perturbation in the position of a certain particle of the material **instantaneously** changes the positions of all other particles. However, nature does not behave like this. In fact interactions are observed to propagate at a finite velocity. Such interactions that propagate from one particle to another are typically denoted as **signals**, while the propagation velocity is called **signal velocity**.

It follows from the principle of relativity that the laws that govern electromagnetic radiation (or light) phenomena are the same in all inertial frames. In turn, the signal velocity is the speed of light, $c=2.998\times10^{8}ms^{-1}$, **which is the same for every reference system.**

The classical mechanics limit is approached for small velocities $v \lll c$. Therefore, the limiting transition from relativistic mechanics to classical mechanics takes place when the signal velocity is $c\to\infty$, that is, instantaneous transmission of the interaction.

1.2 Space-time intervals

We start this subsection introducing the concept of an **event**. In general terms, an event $\mathcal{P}$ is defined by *the place where it occurred and the time when it occured*, and can be described by the coordinates

$$x^\mu = (t, x, y, z) \tag{1.2}$$

with $\mu = 0, 1, 2, 3$. The coordinates x^μ define the position of the event $\mathcal{P}$ in a four-dimensional space-time and is nothing more than a point. In such a space each particle is described by a certain line whose points determine the coordinates of the particle at all moments of time, denoted as **world line**.

Let us now consider two reference frames O and O' moving relative to each other with velocity v as depicted in Fig. 1. Consider now two events $\mathcal{P}_1$ and $\mathcal{P}_2$ defined by

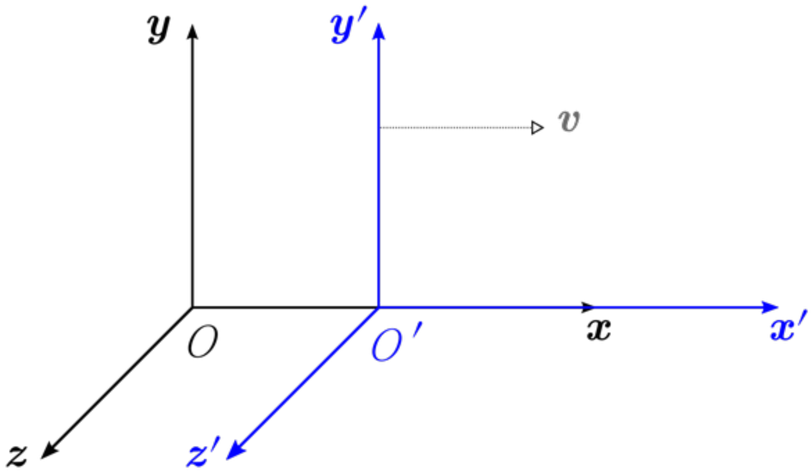


Figure 1: *Moving frame O' relative to a rest frame O.*

$\mathcal{P}_1$: At t_1, a signal is sent out from a point (t_1, x_1, y_1, z_1) in the frame O travelling at the speed of light c.

$\mathcal{P}_2$: At a time t_2 the signal is received at a point (t_2, x_2, y_2, z_2) as seen from the frame O.

Observing the propagation of the signal in the frame O, the distance covered by it is

$$\Delta r = c\,(t_2 - t_1)\,. \tag{1.3}$$

On the other hand, this same distance is equal to (norm of the position vector)

$$\Delta r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{1.4}$$

From (1.3) and (1.4) we can derive a relation between the coordinates of both events in the O frame as

$$\Delta s^2 \equiv c^2\,(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 = 0\,, \tag{1.5}$$

which is denoted as **interval**.

We can now do the same exercise for the moving coordinate system O'. Recalling that the speed of light is an invariant in nature we get a similar relation to (1.5) that reads

$$\Delta s^2 \equiv c^2 \left(t_2' - t_1'\right)^2 - \left(x_2' - x_1'\right)^2 - \left(y_2' - y_1'\right)^2 - \left(z_2' - z_1'\right)^2 = 0\,. \tag{1.6}$$

From here we conclude the following:

- **From the principle of invariance of the speed of light, if an interval is zero in one reference system, $\Delta s = 0$, then it is zero in any other inertial frame.**

If two events are infinitesimally close, the line element is written as

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \equiv c^2 dt^2 - dr^2. \tag{1.7}$$

With our choice of sign convention in ds^2, we can recast (1.7) as

$$ds^2 = \begin{pmatrix} c\,dt & dx & dy & dz \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} c\,dt \\ dx \\ dy \\ dz \end{pmatrix} \tag{1.8}$$

or written in component notation

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.9}$$

where Einstein sum convention is implicit. Note that $g_{\mu\nu} = \mathrm{diag}\,(+,-,-,-)$ is the Minkowski metric and it characterizes a four-dimensional flat space-time.

The metric is also used to raise or to lower space-time indices as

$$x_\alpha = g_{\alpha\mu} x^\mu, \tag{1.10}$$

and can be shown to obey the property

$$g_{\mu\nu} g^{\mu\alpha} = \delta_\nu^\alpha. \tag{1.11}$$

Note that if we have

$$x^\mu = (ct, x, y, z) \tag{1.12}$$

then through eq. (1.10) we find that

$$x_\mu = (ct, -x, -y, -z)\,. \tag{1.13}$$

We denote x^μ as a **covariant vector** and x_μ **contravariant vector**.

So far we have understood that due to the invariance of the speed of light, for a null world line $ds = 0$ in one inertial frame, then $ds' = 0$ in any other inertial reference system.

The question now is what happens if $ds \neq 0$?

Noting that for the null world-lines we have $ds = ds'$ and that ds and ds' are infinitesimals of the same order, then in general ds and ds' must be proportional to each other

$$ds^2 = a\, ds'^2\,, \tag{1.14}$$

where the coefficient a can only depend on the magnitude of the relative velocities between any two inertial frames. To determine the proportionality factor we will consider three inertial frames O, O_1 and O_2. Defining v_1 the velocity of O_1 relative to O, v_2 the velocity of O_2 relative to O and v_{12} the velocity of O_2 relative to O_1 we have the following:

$$ds^2 = a\,(v_1)\, ds_1^2 \qquad ds^2 = a\,(v_2)\, ds_2^2 \qquad ds_1^2 = a\,(v_{12})\, ds_2^2 \tag{1.15}$$

from where we obtain

$$a\,(v_{12}) = \frac{a\,(v_2)}{a\,(v_1)}. \tag{1.16}$$

The key point here is that v_{12} depends not only on the magnitudes v_1 and v_2, but also on the relative angle between the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Therefore, since this angular dependence is **only** present on the left hand side of eq. (1.16), the only way for this expression to be valid is if a is a constant equal to

$$a = \frac{a}{a} \Rightarrow a = 1. \tag{1.17}$$

With this we conclude that for two inertial frames and any ds

$$ds^2 = ds'^2. \tag{1.18}$$

With this result we arrive at a major conclusion:

- **The interval between any two events is the same in all inertial frames. In other words, ds is an invariant under coordinates transformations and is the mathematical description of the constancy of the speed of light.**

To finish this section we comment on the types of trajectories that a given particle's world line can undertake:

1. If the particle travels at the speed of light it exhibits a null trajectory $ds^2 = 0$. Its world line is along the $\overline{ab}$ or $\overline{cd}$ oblique lines in Fig. 2 and it is typically classified as light-like trajectory.

2. If the particle's velocity is smaller than the speed of light then $c^2\, dt^2 > dr^2$ which translates into $ds^2 > 0$. Its trajectory will be contained within the cone of Fig. 2. This particle's world line is classified as time-like.

3. Finally, if the particle's velocity is larger that the speed of light then $c^2\, dt^2 < dr^2$ which translates into $ds^2 < 0$. Its trajectory takes place outside the light cone and its world line is classified as space-like. This purely theoretical particles are often denoted as tachyons and their existence would violate some important physics laws such as causality.

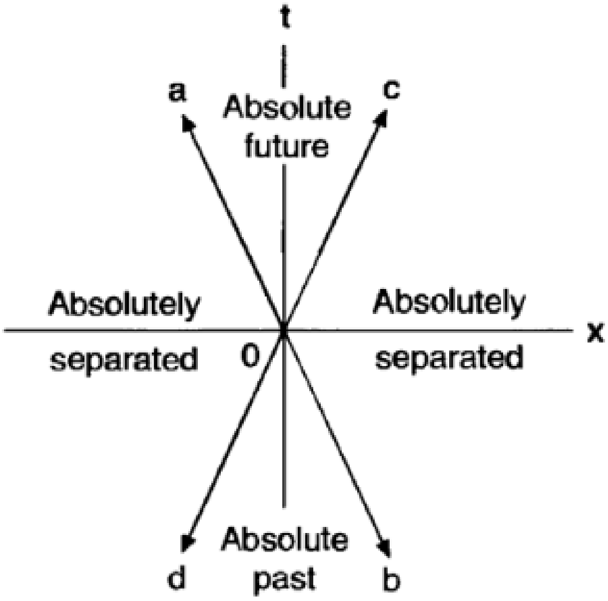


Figure 2: *Light cone(Extracted from the Landau's Classical Theory of Fields book).*

1.3 Lorentz transformations

In classical mechanics, due to the absolute nature of time, that is, given two inertial frames, O and O', we have $t = t'$. It is then very simple to determine the transformation of coordinates and for the case of Fig. 1 we have

$$t = t' \qquad x = x' + vt \qquad y = y' \qquad z = z'. \tag{1.19}$$

These are de so called Galileo transformations. The problem of these transformations is that they do not conserve the invariance of intervals in different frames. To see this consider the simple two dimensional case

$$ds^2 = c^2 dt^2 - dx^2. \tag{1.20}$$

Applying Galileo transformations we see that

$$ds'^2 = c^2 dt'^2 - d\left(x' + vt\right)^2 \neq c^2 dt'^2 - dx'^2 \Rightarrow ds'^2 \neq ds^2\,, \tag{1.21}$$

which confirms our statement.

The questions is then **how to leave the interval between events invariant?** It is this invariance requirement that will lead to relativist transformations.

To see this let us consider the boost along the x-direction represented in Fig. 1. Such a transformation must leave the squared distance $\Delta s = c^2t^2 - x^2$, the square of the distance of (ct, x) from the origin, invariant. Remember that in relativistic mechanics time is not absolute and hence the transformation of coordinates that leaves Δs invariant takes place in the tx plane. In other words, this transformation is nothing more than a **rotation** in the tx plane.

However, the Minkowski space with metric signature $(+,-,-,-)$ is not Euclidean but pseudo-Euclidean. The geometry in pseudo-Euclidean spaces is hyperbolic and rotations are described via hyperbolic functions and not standard trigonometric ones. It follows from this that the most general form for a transformation of coordinates between the O and the O' coordinate system is

$$x = x' \cosh\theta + ct' \sinh\theta \qquad ct = x' \sinh\theta + ct' \cosh\theta \tag{1.22}$$

with θ a rotation angle in the hyperbolic space. We can now verify that these transformations do in fact preserve the distance Δs^2:

$$\begin{aligned}\Delta s^2 &= c^2 t^2 - x^2 \\ &= \left(x' \sinh\theta + ct' \cosh\theta\right)^2 - \left(x' \cosh\theta + ct' \sinh\theta\right)^2 \\ &= \cosh^2\theta \left(c^2 t'^2 - x'^2\right) + \sinh^2\theta \left(x'^2 - c^2 t'^2\right) \\ &= \left(\cosh^2\theta - \sinh^2\theta\right)\left(c^2 t'^2 - x'^2\right) \\ &= \left(c^2 t'^2 - x'^2\right) = \Delta s'^2 \,.\end{aligned} \tag{1.23}$$

We have then shown that the transformations (1.22) do preserve the distance in the Minkowski space. For completeness, the reader can trivially verify that for an Euclidean space with metric signature $(+,+,+,+)$, the distance $\Delta s^2 = c^2t^2 + x^2$ is preserved if we replace $\cosh\theta$ and $\sinh\theta$ by $\cos\theta$ and $\sin\theta$ respectively.

So far we have only determined mathematical formulas that warrant the invariance of Δs^2 in different frames. What is then the physical meaning of the transformations (1.22)? To address this question let us consider the motion of the origin of O' in O, that is, setting $x' = 0$. In this case the transformations in (1.22) simplify to

$$x = ct' \sinh\theta \qquad ct = ct' \cosh\theta \,. \tag{1.24}$$

Dividing x/ct one obtains

$$\frac{x}{ct} = \tanh\theta, \tag{1.25}$$

where x/t is the velocity v of the frame O' relative to O, thus

$$\frac{v}{c} = \tanh\theta. \tag{1.26}$$

Typically one defines

$$\beta = \frac{v}{c} \tag{1.27}$$

which represents the velocity of a given particle as a fraction of the speed of light. From (1.26) and (1.27) we can write

$$\sinh\theta = \gamma\,\beta \qquad \cosh\theta = \gamma \tag{1.28}$$

with γ an arbitrary constant. Using the identity $\cosh\theta - \sinh\theta = 1$ we can determine an expression for γ

$$\gamma^2 - \beta^2\gamma^2 = 1 \Rightarrow \gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{1.29}$$

Finally, substituting this into the transformations (1.22) we get

$$ct = \gamma\left(\beta x' + ct'\right) \qquad x = \gamma\left(x' + \beta ct'\right) \qquad y = y' \qquad z = z'\,, \tag{1.30}$$

which represent the so called **Lorentz transformations** for a boost along the x-direction. Note that in the limit $c \to \infty$, thus $\gamma \to 1$, we recover the Galileo transformations.

It is often useful to use the Lorentz transformations in matrix notation. We define $\Lambda^\mu{}_\nu$, the Lorentz transformations, as:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.31}$$

and the transformation of coordinates can be recast in components as

$$x^\mu = \Lambda^\mu{}_\nu x'^\nu. \tag{1.32}$$

Note also that we can write the Lorentz transformations for coordinates as

$$\Lambda^\mu{}_\nu = \frac{\partial x^\mu}{\partial x'^\nu}. \tag{1.33}$$

To see this we can apply these derivatives to eq. (1.30), e.g.

$$\begin{aligned}
\frac{\partial x^0}{\partial x'^0} &= \frac{\partial(ct)}{\partial(ct')} = \gamma \\
\frac{\partial x^0}{\partial x'^1} &= \frac{\partial(ct)}{\partial(x')} = \gamma\beta \\
\frac{\partial x^0}{\partial x'^2} &= \frac{\partial(ct)}{\partial(y')} = 0 \\
\frac{\partial x^1}{\partial x'^0} &= \frac{\partial x}{\partial(ct')} = \gamma \\
\frac{\partial x^1}{\partial x'^1} &= \frac{\partial x}{\partial(x')} = \gamma\beta \\
&\cdots
\end{aligned} \tag{1.34}$$

Lorentz transformations **preserve the metric** as we can see from the following:

$$\begin{aligned}
ds^2 &= g_{\mu\nu}dx^\mu dx^\nu = g_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta dx'^\alpha dx'^\beta \\
ds'^2 &= g_{\alpha\beta}dx'^\alpha dx'^\beta \\
ds'^2 &= ds^2
\end{aligned} \tag{1.35}$$

from these conditions we conclude that

$$g_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = g_{\alpha\beta} \tag{1.36}$$

which proves our statement.

1.4 Proper time

In general terms, the proper time of a particle is the time as seen from its own reference frame O'. In this case we observe the following:

$$\begin{aligned} & ds^2 = c^2 dt'^2 \\ \Leftrightarrow\ & c^2 dt^2 - dx^2 = c^2 dt'^2 \\ \Leftrightarrow\ & c\, dt' = \sqrt{c^2 dt^2 - dx^2} \\ \Leftrightarrow\ & c\, dt' = c\, dt \sqrt{1 - \frac{1}{c^2}\frac{dx^2}{dt^2}} \\ \Leftrightarrow\ & dt' = dt\sqrt{1-\beta^2} \\ \Leftrightarrow\ & dt' = \frac{dt}{\gamma}\,. \end{aligned} \tag{1.37}$$

Since $1 < \gamma < \infty$ we see that the particle in its moving inertial frame, O', perceives time slower than an observer at rest in O. For example, in the limit of $v \to c$, that is, $\beta \to 1$, times tends to stop for the particle and its world-line approaches a null or light-like trajectory. This is the case of photons that proceed at the speed of light and do not perceive time, i.e. $ds = dt' = 0$. In fact photons do not decay and still today we are receiving information from the very early stages of the Universe carried by primordial photons.

Another relevant example is that of cosmic muons. These are created in the atmosphere approximately at 15 km altitude, when very energetic cosmic rays, typically high energy protons, collide with the molecules of the atmosphere inducing a shower of particles of which muons are one of the sub-products.

These muons are also very energetic and proceed at a velocity which is approximately

$$\beta_{\text{muon}} \approx 0.9998.$$

Knowing that the mean-life of a muon is

$$\Delta t \approx 2.197 \times\ 10^{-6} s\,,$$

if we calculate the distance covered by the muons from their production until they decay using classical mechanics we get

$$\Delta x = \beta_{\text{muon}}\, c\, \Delta t \approx 655\,\text{m} \ll 15\,\text{km}.$$

However we know that cosmic muons not only cross the entire atmosphere but can also penetrate through the surface and reach CERN Large Hadron Collider (LHC) detectors at 100 m deep. This is indeed an example where relativistic mechanics is critical for reliable results. Using $\Delta t = \gamma \Delta t'$ then we need to correct Δx with

$$\gamma = \frac{1}{\sqrt{1-\beta^2_{\text{muon}}}} \approx 50$$

which means that

$$\Delta x \approx 33 \text{ km}$$

in agreement with experimental observations.

1.5 The four-vectors velocity and energy-momentum

A given set of four quantities $A^\mu = \left(A^0, \boldsymbol{A}\right)$ is a four-vector if and only if it Lorentz transforms as the the space-time coordinates x^μ, i.e.

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu. \tag{1.38}$$

Consider a time-like particle, that is, a particle for which $ds^2 > 0$. If x^μ and $x^\mu + dx^\mu$ correspond to two events on its world-line, then in its moving frame (or proper frame) O' we have:

$$dx' = dy' = dz' = 0 \Rightarrow ds^2 = dt'^2. \tag{1.39}$$

We define the **four-vector velocity, or four-velocity** as

$$u^\mu = \lim_{\Delta S \to 0} \frac{\Delta x^\mu}{\Delta S} \equiv \frac{dx^\mu}{ds} = \frac{dx^\mu}{c\, dt'} \tag{1.40}$$

Let us now determine u^μ in the O and O' frames:

- **Frame $\boldsymbol{O}$**

$$\begin{aligned} dx^\mu = (c\, dt, d\boldsymbol{r}) \Rightarrow u^\mu = \frac{dx^\mu}{ds} &= \frac{1}{c}\left(\frac{c\, dt}{dt'}, \frac{d\boldsymbol{r}}{dt'}\right) = \frac{1}{c}\left(\frac{c\, dt}{\gamma^{-1} dt}, \frac{d\boldsymbol{r}}{\gamma^{-1} dt}\right) \\ &= \frac{\gamma}{c}\,(c, \boldsymbol{v}) = \gamma\,(1, \boldsymbol{\beta}) \end{aligned} \tag{1.41}$$

- **Frame $\boldsymbol{O'}$**

$$dx^\mu = \left(c\, dt', \boldsymbol{0}\right) \Rightarrow u^\mu = \frac{dx^\mu}{ds} = \frac{1}{c}\left(\frac{c\, dt'}{dt'}, \boldsymbol{0}\right) = (1, \boldsymbol{0}) \tag{1.42}$$

Note that the four-vector velocity is a dimensionless quantity and its inner product $u^\mu u_\mu$ in the Minkwoski space is:

$$u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu = \left(u^0\right)^2 - \sum_{i=1}^{3} \left(u^i\right)^2 = \gamma^2\left(1 - \boldsymbol{\beta}^{\,2}\right) = \gamma^2 \gamma^{-2} = 1 \tag{1.43}$$

By analogy with classical mechanics we can also define the **four-vector Energy-momentum** of a particle with mass m in the following way:

$$p^\mu = m\, c\, u^\mu, \tag{1.44}$$

such that in the spacial component we recover the standard formula for the linear momentum $\boldsymbol{p} = m\, c\, \boldsymbol{\beta} = m\, \boldsymbol{v}$. Form (1.44) we immediately see that

$$p^\mu p_\mu = \left(p^0\right)^2 - \boldsymbol{p}^{\,2} = m^2 c^2\,, \tag{1.45}$$

which is also a Lorentz Invariant quantity. In fact, the mass is the same as seen from any reference frame.

Noting that p^0 has dimensions of **energy over velocity** we can recast eq. (1.45) as

$$\frac{E^2}{c^2} - \boldsymbol{p}^{\,2} = m^2c^2 \Rightarrow E^2 = m^2c^4 + c^2\boldsymbol{p}^2. \tag{1.46}$$

This means that for an observer in the inertial frame O the energy of the moving particle is

$$E = \sqrt{m^2c^4 + c^2\boldsymbol{p}^2} \qquad \text{observer in } O \tag{1.47}$$

whereas the **rest energy** of the particle ($\boldsymbol{p} = 0$) is

$$E = m\,c^2 \qquad \text{observer in } O'. \tag{1.48}$$

This popular equation, and its more general form (1.47), have profound implications. First, it highlights the equivalence between mass and energy. Second, if we read it from left to right, it can represent the materialization of particles as e.g. a photon being converted into an electron-positron pair, or if we read it from right to left, the annihilation of a particle-antiparticle pair into radiation (photons). This is also the principle behind the operation of collision experiments such as the LHC. Protons are accelerated to almost the speed of light becoming highly energetic, until they collide at the centre of detectors converting all that energy into mass and radiation.

For the case of photons, whose mass is zero, their energy is

$$E_{\text{photons}} = c\,|\boldsymbol{p}| \tag{1.49}$$

irrespective of the reference frame. In fact we cannot slow down photons (do not confuse it with absorbing them in a certain material) and, as we have discussed in the beginning of this section, their velocity is the same in every reference frame. As an analogy, we cannot "*store light in a bottle for later use*", or any other massless, light-like, particle.

If we consider the limit of small velocities we can Taylor expand (1.47) for small $|\boldsymbol{p}|\,/m$ ratio as

$$E = mc^2\sqrt{1 + \frac{\boldsymbol{p}^2}{m^2c^2}} \approx mc^2\left[1 + \frac{1}{2}\frac{\boldsymbol{p}^2}{m^2c^2} + \mathcal{O}\left(\frac{\boldsymbol{p}^4}{m^4}\right)\right] \approx mc^2 + \frac{\boldsymbol{p}^2}{2m} + \cdots \tag{1.50}$$

which, except for the first term that represents the rest energy, is the classical expression for the kinetic energy of a particle.

To conclude, and substituting (1.44) in eqs. (1.41) and (1.42), we write the final form of the four-vector energy-momentum in both O and O' frames:

$$\begin{aligned} p^\mu &= \gamma m\,c\,(1, \boldsymbol{\beta}) \\ p'^\mu &= m\,c\,(1, \boldsymbol{0})\,, \end{aligned} \tag{1.51}$$

from where we can obtain the relativistic expressions for the energy

$$\begin{cases} p^0 = \gamma mc \\ p'^0 = mc \end{cases} \Rightarrow \begin{cases} E = \gamma mc^2 \\ E' = mc^2 \end{cases} \tag{1.52}$$

and for the momentum

$$\begin{cases} \boldsymbol{p} = \gamma \boldsymbol{\beta} mc \\ \boldsymbol{p'} = \boldsymbol{0} \end{cases} \Rightarrow \begin{cases} \boldsymbol{p} = \gamma m \boldsymbol{v} \\ \boldsymbol{p'} = \boldsymbol{0}. \end{cases} \tag{1.53}$$

In particle physics we typically use **natural units of measurement** to express our physical quantities. For example, c is the natural unit of speed, e is the natural unit of electric charge and $\hbar$ the natural Plank unit. In practice what we do is to set $c = e = \hbar = 1$ in all equations. In natural units all formulas in this section should be recast with $c = 1$. For example, the rest energy of a particle reads

$$E = m \tag{1.54}$$

while the energy as seen in the laboratory frame is

$$E = \sqrt{m^2 + \boldsymbol{p}^2}. \tag{1.55}$$

From here onwards we will star using natural units and eq. (1.55) is the expression from where we will later derive the Klein-Gordon equation.

2 Relativistic equations

In this course we will study the Standard Model form a classical field theory point of view. It is then instructive to start introducing the notion of field with a brief reference to classical electrodynamics.

2.1 Covariant formulation of Maxwell theory

2.1.1 The electromagnetic field

A field is a physical entity that at every point in space can be represented e.g. either by a number, if it is a scalar field, or by a vector, if it is a vector field, or even by a tensor in general. For example, a temperature map is a scalar field as for every point in space there is a specific number that represents the temperature, at that point. For the case of a vector field, for every point in space it is characterized by a set of numbers containing directional information. A vary familiar example are the electric field lines created by point charges and can easily be seen in laboratory.

We now introduce the four-vector field $A^\mu(x^\alpha)$, which is a function of the space-time coordinates x^α

$$A^\mu(x^\alpha) = \left(A^0(x^\alpha), \boldsymbol{A}(x^\alpha)\right). \tag{2.1}$$

This vector field represents the **electromagnetic field** and in the context of Particle Physics it can describe not only photons but also gluons, the strong force mediators, or the weak-force mediators, the W and Z bosons. The time component $A^0\left(x^\alpha\right)$ is typically denoted as **scalar potential** and the spacial component $\boldsymbol{A}\left(x^\alpha\right)$ is as **vector potential**. In what follows, for ease of notation, we will drop the space-time argument in the electromagnetic field, which should be implicitly assumed unless otherwise stated.

From basic electromagnetism we also know that the electric, $\boldsymbol{E}$, and the magnetic, $\boldsymbol{B}$, fields written in terms of the A^0 and $\boldsymbol{A}$ potentials read

$$\begin{aligned} \boldsymbol{E} &= -\boldsymbol{\nabla} A^0 - \frac{\partial \boldsymbol{A}}{\partial t}\,, \\ \boldsymbol{B} &= \boldsymbol{\nabla} \wedge \boldsymbol{A}\,. \end{aligned} \tag{2.2}$$

In four-vector notation a spacial derivative can be written as

$$\frac{\partial}{\partial x^\mu} \equiv \partial_\mu = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) \qquad \frac{\partial}{\partial x_\mu} \equiv \partial^\mu = \left(\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right) \tag{2.3}$$

with

$$\partial^\mu = g^{\mu\nu}\partial_\nu \qquad \text{and} \qquad \partial^\mu\partial_\mu = \Box \tag{2.4}$$

Note that the derivative with respect to a contravariant vector is covariant vector, while de derivative with respect to a co-vector is a contra-vector.

Let us now introduce the tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\,, \tag{2.5}$$

denoted as **electromagnetic tensor**. We see readily from this expression that $F^{\mu\nu}$ is anti-symmetric with respect to $\mu \leftrightarrow \nu$ interchange, that is

$$F^{\mu\nu} = -F^{\nu\mu}\,. \tag{2.6}$$

If we expand $F^{\mu\nu}$, and taking into account the **sign of the gradient** in the derivatives defined in Eq. (2.3), we observe the following:

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = \frac{\partial}{\partial t}A^i - \frac{\partial}{\partial x_i}A^0 = \left(\frac{\partial}{\partial t}\boldsymbol{A} + \boldsymbol{\nabla} A^0\right)^i = -\boldsymbol{E}^i \tag{2.7}$$

and

$$\begin{aligned} F^{12} &= \partial^1 A^2 - \partial^2 A^1 = -\frac{\partial}{\partial x}A_y + \frac{\partial}{\partial y}A_x = -\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right) = -B_z \\ F^{13} &= \partial^1 A^3 - \partial^3 A^1 = -\frac{\partial}{\partial x}A_z + \frac{\partial}{\partial z}A_x = \left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right) = B_y \\ F^{23} &= \partial^2 A^3 - \partial^3 A^2 = -\frac{\partial}{\partial y}A_z + \frac{\partial}{\partial z}A_y = -\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right) = -B_x\,. \end{aligned} \tag{2.8}$$

Therefore, and using the anti-symmetry of the electromagnetic field tensor (2.6) we can explicitly write

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{2.9}$$

We conclude that the electromagnetic field tensor defines the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ respectively as partial derivatives of the electromagnetic field vector A^μ.

We have explicitly written the form of the electromagnetic tensor $F^{\mu\nu}$, which in a **condensed notation** we may recast as

$$F^{\mu\nu} = (-\boldsymbol{E}, \boldsymbol{B}) \ . \tag{2.10}$$

The question now is how does it look like if we lower both Lorentz indices? We have learned in the previous section that to lower one index of a given object we apply the metric $g_{\mu\nu}$. To lower both of the indices of the electromagnetic field tensor we will **then apply the metric twice** as follows:

$$F^{\mu\nu} g_{\mu\alpha}\, g_{\nu\beta} = F_{\alpha\beta} \, . \tag{2.11}$$

Specializing to the F^{0i} and F^{ij} blocks we have:

$$F_{0i} = F^{0j} g_{00}\, g_{ji} = F^{0j} g_{ji} = F^{0j} g_{ii} = -F^{0i} \tag{2.12}$$

and

$$F_{ij} = F^{kl} g_{ki}\, g_{lj} = F^{ij} g_{ii}\, g_{jj} = F^{ij} \tag{2.13}$$

thus

$$F_{\mu\nu} = (\boldsymbol{E}, \boldsymbol{B}) \, . \tag{2.14}$$

2.1.2 Gauge invariance

The concept of **gauge invariance** is of uttermost importance in Particle Physics. This concept has remarkable consequences such as

- It is a consequence of gauge invariance that the electromagnetic, weak and strong forces exist. This is why they are called **gauge interactions**;
- It is due to gauge invariance that photons and gluons are massless;
- It is also due to gauge protection that the W and Z **gauge bosons** only acquire small masses when compared to the Plank scale.

These consequences will become clearer further ahead, in particular after we introduce the Lagrangian formalism and start playing with symmetries.

What is then the concept of gauge invariance? To answer this let us consider the electromagnetic field vector A^μ. We note that A^μ **is not unique** as we have the freedom to redefine it as

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \alpha \tag{2.15}$$

with α a continuous function of the space-time coordinates with continuous first derivative. With (2.15) we see that A'^μ and A^μ are not the same with the former resulting from a **continuous local shift** on A^μ. Now, the question at hand is if the theory is left invariant upon this local redefinition of the electromagnetic field? The answer can be seen from the following:

$$
\begin{aligned}
F^{\mu\nu} \to F'^{\mu\nu} &= \partial^\mu A'^\nu - \partial^\nu A'^\mu = \partial^\mu A^\nu + \partial^\mu\partial^\nu\alpha - \partial^\nu A^\mu - \partial^\nu\partial^\mu\alpha \\
&= \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu} \,.
\end{aligned}
\tag{2.16}
$$

We conclude from here that the **gauge transformation** that we have operated on A^μ leaves the theory invariant. In fact, through $F^{\mu\nu} = F'^{\mu\nu}$, we see that the $\partial^\mu\alpha$ local transformation leaves the electric and magnetic fields unaltered. This is what we denote as **gauge invariance**.

2.1.3 Maxwell equations - covariant formulation

According to Maxwell's theory, the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ are related to each other through the scalar potential A^0 and the vector potential $\boldsymbol{A}$, that we have already **unified** in what we called the electromagnetic field A^μ.

In the standard formulation we can identify two pairs for the Maxwell's equations. The **first pair** reads

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 \,, \\
\boldsymbol{\nabla} \wedge \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} &= 0 \,,
\end{aligned}
\tag{2.17}
$$

representing the **Gauss Law** for magnetic fields and **Faraday's Law** respectively.

The **second pair** has the form

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{E} &= 4\pi\rho \,, \\
\boldsymbol{\nabla} \wedge \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} &= 4\pi\boldsymbol{j} \,,
\end{aligned}
\tag{2.18}
$$

where, in the presence of a source, ρ and $\boldsymbol{j}$ are non-vanishing and represent the charge and current densities respectively. The top equation is **Gauss' Law** for electric fields and the bottom one **Ampere's Law**.

What Maxwell had not noticed in his theory is that these equations encode a relativistic character. For example, from Ampere's Law, a varying electric field generates a magnetic field. For varying one can think of a moving charge, however, whether that charge is moving or not depends on the choice of reference frame. This observation is strongly calling for a covariant formulation, or in other words, **a formulation manifestly invariant under Lorentz Transformations**.

We first define the **dual** electromagnetic tensor as

$$
F^{*\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}, \tag{2.19}
$$

where $\varepsilon^{\mu\nu\alpha\beta}$ is the totally anti-symmetric 4D Levi-Civita symbol defined as:

$$\varepsilon^{\mu\nu\alpha\beta} = \begin{cases} 1 & \text{if } \mu = 0,\ \nu = 1,\ \alpha = 2,\ \beta = 3 \text{ or even permutations of indices} \\ -1 & \text{if odd permutations if indices} \\ 0 & \text{if any two repeated indices}\,. \end{cases} \tag{2.20}$$

Looking into the components of $F^{*\mu\nu}$ we see the following:

$$\begin{aligned} F^{*0i} &= \frac{1}{2}\left(\varepsilon^{0ijk}F_{jk} + \varepsilon^{0ikj}F_{kj}\right) \\ F^{*ij} &= \frac{1}{2}\left(\varepsilon^{ij0k}F_{0k} + \varepsilon^{ijk0}F_{k0}\right) \end{aligned} \tag{2.21}$$

from where we identify, e.g.

$$\begin{aligned} F^{*01} &= \frac{1}{2}\left(\varepsilon^{0123}F_{23} + \varepsilon^{0132}F_{32}\right) = -B_x \\ F^{*12} &= \frac{1}{2}\left(\varepsilon^{1203}F_{03} + \varepsilon^{1230}F_{30}\right) = E_z \\ &\dots \end{aligned} \tag{2.22}$$

such that

$$F^{*\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}. \tag{2.23}$$

Knowing the form of the dual tensor we can now calculate its spacial derivatives $\partial_\mu F^{*\mu\nu}$ which gives four equations corresponding to $\mu = 0, 1, 2, 3$:

$$\begin{aligned} &\partial_\mu F^{*\mu 0} = \partial_i F^{*i0} = \boldsymbol{\nabla}\cdot\boldsymbol{B} \\ &\partial_\mu F^{*\mu 1} = \partial_0 F^{*01} + \partial_2 F^{*21} + \partial_3 F^{*31} = -\partial_t B_x - \partial_y E_z + \partial_z E_y = -\left(\partial_t \boldsymbol{B} + \boldsymbol{\nabla}\wedge\boldsymbol{E}\right)_x \\ &\partial_\mu F^{*\mu 2} = -\left(\partial_t \boldsymbol{B} + \boldsymbol{\nabla}\wedge\boldsymbol{E}\right)_y \\ &\partial_\mu F^{*\mu 3} = -\left(\partial_t \boldsymbol{B} + \boldsymbol{\nabla}\wedge\boldsymbol{E}\right)_z \end{aligned} \tag{2.24}$$

from where we conclude that the covariant form for the first pair of Maxwell equations corresponds to

$$\partial_\mu F^{*\mu\nu} = 0 \tag{2.25}$$

which, in a unified way, we can call **Farady-Gauss Law**. Doing the same for the electromagnetic tensor,

$$\begin{aligned} &\partial_\mu F^{\mu 0} = \partial_i F^{i0} = \boldsymbol{\nabla}\cdot\boldsymbol{E} \\ &\partial_\mu F^{\mu 1} = \partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} = -\partial_t E_x + \partial_y B_z - \partial_z B_y = \left(-\partial_t \boldsymbol{E} + \boldsymbol{\nabla}\wedge\boldsymbol{B}\right)_x \\ &\partial_\mu F^{\mu 2} = \left(-\partial_t \boldsymbol{E} + \boldsymbol{\nabla}\wedge\boldsymbol{B}\right)_y \\ &\partial_\mu F^{\mu 3} = \left(-\partial_t \boldsymbol{E} + \boldsymbol{\nabla}\wedge\boldsymbol{B}\right)_z \end{aligned} \tag{2.26}$$

and defining the four-current vector

$$j^\mu = (\rho, \boldsymbol{j}) \tag{2.27}$$

we conclude that the covariant form for the second pair of Maxwell equations can be written as

$$\partial_\mu F^{\mu\nu} = 4\pi j^\nu \,, \tag{2.28}$$

which is also denoted as **Ampere-Gauss Law**.

To prove that the Maxwell equations, Eqs. (2.25) and (2.28), are in fact covariant we need to verify that they are preserved under Lorentz Transformations. To see this we consider the transformations

$$\begin{aligned} F^{\mu\nu} &= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F'^{\alpha\beta} \,, \\ F^{*\mu\nu} &= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F'^{*\alpha\beta} \,, \\ j^\nu &= \Lambda^\nu{}_\beta j'^\beta \\ \partial_\mu &= \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x'^\alpha} \frac{\partial x'^\alpha}{\partial x^\mu} = \partial'_\alpha \left[\Lambda^{-1}\right]^\alpha{}_\mu \,. \end{aligned} \tag{2.29}$$

For the Ampere-Gauss Law we have

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= j^\nu \\ \Leftrightarrow \partial'_\alpha \left[\Lambda^{-1}\right]^\alpha{}_\mu \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F'^{\alpha\beta} &= \Lambda^\nu{}_\beta j'^\beta \\ \Leftrightarrow \Lambda^\nu{}_\beta \partial'_\alpha F'^{\alpha\beta} &= \Lambda^\nu{}_\beta j'^\beta \\ \Rightarrow \partial'_\alpha F'^{\alpha\beta} &= j'^\beta \end{aligned} \tag{2.30}$$

which is manifestly covariant. The same steps also prove the covariance of Faraday-Gauss Law. With the covariant formalism it is also trivial to determine the Lorentz invariants of Maxwell theory, which read

$$\begin{aligned} F^{\mu\nu} F_{\mu\nu} &= 2\left(|\boldsymbol{B}|^2 - |\boldsymbol{E}|^2\right) \\ F^{*\mu\nu} F_{*\mu\nu} &= 2\left(|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2\right) \\ F^{*\mu\nu} F_{\mu\nu} &= -4\boldsymbol{E}\cdot\boldsymbol{B} \end{aligned} \tag{2.31}$$

and are the same in every inertial reference frame. In essence, from these invariants we conclude the following:

- If $\boldsymbol{E}$ and $\boldsymbol{B}$ are perpendicular in a given reference frame, $\boldsymbol{E}\cdot\boldsymbol{B} = 0$, they will be so in any other reference frame.
- If the angle between $\boldsymbol{E}$ and $\boldsymbol{B}$ is acute (obtuse) in a given reference frame, it will be so in any other inertial frame.
- If the magnitude of the electric field is larger (smaller) than that of the magnetic field in a given inertial frame, it will be so in any other reference frame.

To conclude this section we give an explicit example that shows how a magnetic field emerges from a moving electric charge.

Consider a point charge at rest in its proper frame, O', and the electric field generated by it $\boldsymbol{E'} = \left(E'_x, E'_y, E'_z\right)$. The electromagnetic tensor in the O' frame reads

$$F'^{\mu\nu} = \begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & 0 & 0 \\ E'_y & 0 & 0 & 0 \\ E'_z & 0 & 0 & 0 \end{pmatrix} . \tag{2.32}$$

If we now consider that the inertial frame O' is moving with velocity β along the positive x-direction relative to the laboratory frame O, how does the electromagnetic field tensor looks like from the perspective of an observer at rest in O? To answer this all we have to do is to apply a Lorentz boost along the x-axis with the form (1.31). In components we have $F^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} F'^{\alpha\beta} \Lambda_{\beta}{}^{\nu}$ which can be translated into matrix representation as

$$[F] = [\Lambda] \left[F'\right] [\Lambda]^{\top} . \tag{2.33}$$

which results in

$$\begin{aligned} [F] &= \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & 0 & 0 \\ E'_y & 0 & 0 & 0 \\ E'_z & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -E'_x & -\gamma E'_y & -\gamma E'_z \\ E'_x & 0 & -\gamma\beta E'_y & -\gamma\beta E'_z \\ \gamma E'_y & \gamma\beta E'_y & 0 & 0 \\ \gamma E'_z & \gamma\beta E'_z & 0 & 0 \end{pmatrix} . \end{aligned} \tag{2.34}$$

from where we see the emergence of a magnetic field as a **relativistic effect of a moving electric field** with components

$$\begin{aligned} B_x &= 0 \\ B_y &= \gamma\beta E'_z \\ B_z &= \gamma\beta E'_y \end{aligned} \tag{2.35}$$

while the electric field in the laboratory frame reads

$$\begin{aligned} E_x &= E'_x \\ E_y &= \gamma E'_y \\ E_z &= \gamma E'_z . \end{aligned} \tag{2.36}$$

2.2 The Klein-Gordon equation

In non-relativistic Quantum Mechanics, the quantum state ψ of a physical system is described by the Schrödinger equation. If we consider a free particle of mass m with energy

$$E = \frac{\boldsymbol{p}^2}{2m} \tag{2.37}$$

one of the postulates of Quantum Mechanics says that physical observables are represented by linear hermitian operators, that is

$$\begin{aligned} \hat{E} &\to i\frac{\partial}{\partial t} \\ \hat{\boldsymbol{p}} &\to -i\boldsymbol{\nabla} \end{aligned} \tag{2.38}$$

the Schrödinger equation written in units of the Plank constant follows directly from substituting these operators into $\hat{E}\psi = \frac{\hat{\boldsymbol{p}}^2}{2m}\psi$,

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2m}\boldsymbol{\nabla}^2\psi\,. \tag{2.39}$$

While the Schrödinger equation offers a good description of non-relativistic systems, it is no longer reliable for relativistic scenarios.

In 1926, Oskar Klein and Walter Gordon proposed an equation to describe relativistic electrons. To derive such an equation they have considered the total energy of a relativist particle ϕ of mass m

$$E = \sqrt{m^2 + \boldsymbol{p}^2} \Leftrightarrow E^2 - \boldsymbol{p}^2 = m^2 \tag{2.40}$$

and applied the postulate (2.38) to obtain

$$\begin{aligned} \left[(i\frac{\partial}{\partial t})^2 - (-i\boldsymbol{\nabla})^2\right]\phi &= m^2\phi \\ \Leftrightarrow -\left[\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2\right]\phi &= m^2\phi \\ \Leftrightarrow \left(\Box + m^2\right)\phi &= 0 \end{aligned} \tag{2.41}$$

which is the **Klein-Gordon equation**.

2.2.1 Covariance of the Klein-Gordon equation

Similarly to what we have demonstrated for the Maxwell equations, the Klein-Gordon (KG) equation is also manifestly Lorentz invariant. The covariance of the Klein-Gordon equation has important consequences for the type of field it describes as we will see in what follows. Let us then consider the following transformations:

$$\begin{aligned} \Box = g^{\mu\nu}\partial_\mu\partial_\nu &= \left[\Lambda^{-1}\right]^{\alpha}{}_{\mu}\left[\Lambda^{-1}\right]^{\beta}{}_{\nu}\partial'_\alpha\partial'_\beta \\ \phi\left(x\right) &= S\phi'\left(x'\right) \end{aligned} \tag{2.42}$$

where, for ease of notation, we write the four-vector space-time as $x \equiv x^\mu$, and where S determines the transformation of the field $\phi\left(x\right)$ and obeys $S^{-1}S = \mathbb{1}$. We will also need to take into account the Lorentz transformation on the derivatives as in the last line of Eq. (2.29) as well as the invariance of the metric under Lorentz transformations as in

Eq. (1.36). Starting from the equation as perceived in the laboratory frame we have:

$$\begin{aligned}\left(g^{\mu\nu}\partial_\mu\partial_\nu + m^2\right)\phi\left(x\right) &= 0\\ \Leftrightarrow \left(g^{\mu\nu}\left[\Lambda^{-1}\right]^{\alpha}{}_{\mu}\left[\Lambda^{-1}\right]^{\beta}{}_{\nu}\partial'_\alpha\partial'_\beta + m^2\right)S\phi'\left(x'\right) &= 0\\ \Leftrightarrow \left(g^{\alpha\beta}\partial'_\alpha\partial'_\beta + m^2\right)S\phi'\left(x'\right) &= 0\\ \Leftrightarrow \left(\Box' + m^2\right)S\phi'\left(x'\right) &= 0\,.\end{aligned} \tag{2.43}$$

The only way for the KG equation becoming manifestly invariant under Lorentz transformations is if

$$S\phi'\left(x'\right) = \phi'\left(x'\right) = \phi\left(x\right) \Rightarrow S = S^{-1} = \mathbb{1}. \tag{2.44}$$

We conclude form this that the KG-field $\phi\left(x\right)$ transforms as a **number** (like a temperature filed does) and therefore is a **scalar field**. At variance to what Oskar Klein and Walter Gordon proposed in the beginning, this equation cannot describe relativistic electrons as it does not account for spin. Instead, we can describe particles with spin 0, which are denoted as scalar bosons[1]. The only fundamental scalar so far discovered is the Higgs boson, however, there are several theories that go beyond the Standard Model rich in scalar particles. As examples we have multi-Higgs [1] models or supersymmetry [2].

2.2.2 Solutions of the Klein-Gordon equation

To study the solutions of the KG equation we start with the ansatz

$$\phi\left(x\right) = e^{-ip_\mu x^\mu} = e^{-iEt+i\boldsymbol{p}\cdot\boldsymbol{x}}\,. \tag{2.45}$$

Applying it to the KG equation we obtain

$$\begin{aligned}g^{\mu\nu}\partial_\mu\partial_\nu e^{-ip_\alpha x^\alpha} + m^2 e^{-ip_\mu x^\mu} &= 0\\ g^{\mu\nu}\left(-ip_{\alpha_1}\frac{\partial x^{\alpha_1}}{\partial x^\mu}\right)\left(-ip_{\alpha_2}\frac{\partial x^{\alpha_2}}{\partial x^\nu}\right)e^{-ip_\alpha x^\alpha} + m^2 e^{-ip_\mu x^\mu} &= 0\\ g^{\mu\nu}\left(-ip_{\alpha_1}\delta^{\alpha_1}_\mu\right)\left(-ip_{\alpha_2}\delta^{\alpha_2}_\nu\right)e^{-ip_\alpha x^\alpha} + m^2 e^{-ip_\mu x^\mu} &= 0\\ g^{\mu\nu}\left(-ip_\mu\right)\left(-ip_\nu\right)e^{-ip_\alpha x^\alpha} + m^2 e^{-ip_\mu x^\mu} &= 0\\ \left(-p_\mu p^\mu + m^2\right)e^{-ip_\mu x^\mu} &= 0\\ \left(-E^2 + \boldsymbol{p}^2 + m^2\right)\phi\left(x\right) &= 0\\ \Rightarrow E^2 &= \boldsymbol{p}^2 + m^2\,,\end{aligned} \tag{2.46}$$

from where we conclude that the energy of a scalar particle is

$$E = \pm\sqrt{\boldsymbol{p}^2 + m^2}\,, \tag{2.47}$$

that is, the KG equation admits two solutions for a free particle of momentum $\boldsymbol{p}$,

$$\phi_\pm\left(x\right) = N_\pm e^{\mp iEt\pm i\boldsymbol{p}\cdot\boldsymbol{x}}\,. \tag{2.48}$$

[1] Remember that from statistical physics systems of integer spin particles are described by the Bose-Einstein distribution while systems of half-integer spin are described by the Fermi-Dirac distribution. Therefore, we denote integer spin particles as bosons, e.g. Higgs and gauge bosons, while half integer ones are called fermions, e.g. quarks and leptons

The solution $\phi_+ (x)$ has positive energy while the solution $\phi_- (x)$ describes a state with **negative energy**. The emergence of negative energy solutions raised an interpretation issue. Furthermore, it is not possible to simply remove $\phi_- (x)$ as $\{\phi_+ (x), \phi_- (x)\}$ form a complete[2] basis.

Let us know interpret the meaning of the negative energy states. For a clearer physical meaning we will introduce here a minimal coupling of the Klein-Gordon field with the electromagnetic field. In other words, we will assume that the positive energy solution ϕ_+ has electric charge Q and therefore is our electric/scalar potential A^0, i.e.

$$A^\mu = (\phi, \boldsymbol{A}) \,. \tag{2.49}$$

A minimal coupling with the electromagnetic field can be described by

$$\partial_\mu \to \partial_\mu - iQA_\mu \tag{2.50}$$

and the KG equation for ϕ_+ assumes the form

$$\left[g^{\mu\nu} (\partial_\mu - iQA_\mu) (\partial_\nu - iQA_\nu) + m^2\right] \phi_+ = 0\,. \tag{2.51}$$

Noting that the negative energy solutions are the complex conjugation of the positive ones, $\phi_- = \phi_+^*$, we can write the complex conjugated KG equation that reads

$$\left[g^{\mu\nu} (\partial_\mu + iQA_\mu) (\partial_\nu + iQA_\nu) + m^2\right] \phi_- = 0\,, \tag{2.52}$$

from where we see that the negative energy solutions have the **opposite charge** of the positive energy solutions. The complex conjugation of a wave function corresponds to an operation called **charge conjugation**. This operation transforms **particles** in its respective **antiparticles**, which carry the same mass and spin but opposite charges. In fact, a scalar theory where ϕ and ϕ^* are both solutions of the KG equation but with opposite sign electric charges **is a theory that predicts the existence of antiparticles**. This property is common to all equations of relativistic Quantum Mechanics and **implies the existence of an antiparticle for each particle species.** Another way of interpreting the negative energy solutions is to consider that they describe positive energy particles travelling backwards in time.

If the solution is a real function, that is $\phi(x) = \phi^*(x)$, then the only way to make eqs. (2.51) and (2.52) consistent is if the charge $Q = 0$. Therefore, we conclude that while charged scalar fields are described with complex functions, a real scalar is described with a real function. For this type of scenarios the ansatz (2.45) would have to be modified and a possible solution for a free particle could be, e.g. $\varphi(x) = \cos(p_\mu x^\mu)$.

[2] Recall that we say that a given basis of functions is complete if we can span any continuous function $\Phi(x)$ as $\Phi(x) = c_- \phi_- (x) + c_+ \phi_+ (x)$.

2.3 The Dirac equation

In 1928, Paul Dirac proposed a relativistic equation linear in $\frac{d}{dt}$ with the same form as the Schrödinger equation

$$i\hbar\frac{d}{dt}\left|\psi\right\rangle = \hat{H}\left|\psi\right\rangle \, . \tag{2.53}$$

This Schrödinger-like form was motivated by one of the postulates of Quantum Mechanics according to which the time evolution of a given quantum state, $|\psi\rangle$, is described by (2.53).

Dirac's novel idea was to think that such an equation should also be linear in $\boldsymbol{p}$ in order to comply with the expression of the four-vector energy-momentum and the need for covariance of the theory (as we will see). This hypothesis lead Dirac to propose an Hamiltonian of the form

$$\hat{H} = \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m \tag{2.54}$$

with $\boldsymbol{\alpha}$ and β hermitian operators in order to warranty that $\hat{H}$ is also hermitian. Acting Dirac's Hamiltonian twice on the state $|\psi\rangle$ we obtain the squared value of the energy

$$\left(\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m\right)^2\left|\psi\right\rangle = E^2\left|\psi\right\rangle \, , \tag{2.55}$$

and from the relativistic nature of this physical system it follows that

$$\begin{aligned} \left(\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m\right)^2 &= \boldsymbol{p}^2 + m^2 \\ \Rightarrow \left(\boldsymbol{\alpha}\cdot\boldsymbol{p}\right)^2 + m\left(\boldsymbol{\alpha}\beta + \beta\boldsymbol{\alpha}\right)\cdot\boldsymbol{p} + \beta^2 m^2 &= \boldsymbol{p}^2 + m^2 \end{aligned} \tag{2.56}$$

from where we conclude that

$$\begin{aligned} \left(\boldsymbol{\alpha}\cdot\boldsymbol{p}\right)^2 &= \boldsymbol{p}^2 \, , \\ \boldsymbol{\alpha}\beta + \beta\boldsymbol{\alpha} &= \boldsymbol{0} \, , \\ \beta^2 &= 1 \, . \end{aligned} \tag{2.57}$$

These equations are verified if the α_i and β operators obey the following relations:

$$\begin{aligned} \alpha^2 = \beta^2 &= 1 \, , \\ \alpha_i\beta + \beta\alpha_i &= 0 \, , \\ \alpha_i\alpha_j + \alpha_j\alpha_i &= 0 \qquad \text{for} \quad i \neq j \quad \text{and} \quad i,j = 1,2,3 \, . \end{aligned} \tag{2.58}$$

Using the identity $\beta^2 = 1$ we can now rewrite Dirac's equation in the following way:

$$\begin{aligned} i\partial_t\psi &= \left(\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m\right)\psi \\ i\beta\partial_t\psi &= \left(\beta\boldsymbol{\alpha}\cdot\boldsymbol{p} + m\right)\psi \\ i\beta\partial_t\psi &= \left(-i\beta\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + m\right)\psi \end{aligned} \tag{2.59}$$

and if we define

$$\beta \equiv \gamma^0 \qquad \beta\boldsymbol{\alpha} \equiv \boldsymbol{\gamma} \tag{2.60}$$

it follows that

$$\begin{aligned} i\gamma^0\partial_t\psi &= \left(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m\right) \\ \left(i\gamma^0\partial_t + i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} - m\right)\psi &= 0 \\ \left(i\gamma^\mu\partial_\mu - m\right)\psi &= 0 \end{aligned} \tag{2.61}$$

which is the canonical form for the **Dirac equation**. Note that we have defined a new four-vector given by

$$\gamma^\mu = \left(\gamma^0, \boldsymbol{\gamma}\right) \,. \tag{2.62}$$

Using the relations in (2.58) and the definitions (2.60) we can determine the algebra that the γ^μ factors describe. The first relation follows from multiplying the second line in (2.58) by β on the left as

$$\begin{aligned} \alpha_i\beta + \beta\alpha_i &= 0 \\ \Rightarrow \beta\alpha_i\beta + \beta\beta\alpha_i &= 0 \\ \gamma^i\gamma^0 + \gamma^0\gamma^i &= 0 \,. \end{aligned} \tag{2.63}$$

The second relation follows from multiplying the third line in (2.58) by β both on the left and on the right

$$\begin{aligned} \alpha_i\alpha_j + \alpha_j\alpha_i &= 0 \\ \beta\alpha_i\alpha_j\beta + \beta\alpha_j\alpha_i\beta &= 0 \\ \gamma^i\alpha_j\beta + \gamma^j\alpha_i\beta &= 0 \\ -\gamma^i\beta\alpha_j - \gamma^j\beta\alpha_i &= 0 \\ -\gamma^i\gamma^j - \gamma^j\gamma^i &= 0 \\ \gamma^i\gamma^j + \gamma^j\gamma^i &= 0 \,, \end{aligned} \tag{2.64}$$

which is verified whenever $i \neq j$. Finally, we should also determine the relations verified for $i = j$:

$$\begin{aligned} \alpha_i\alpha_i + \alpha_i\alpha_i &= 2 \\ \beta\alpha_i\alpha_i\beta + \beta\alpha_i\alpha_i\beta &= 2\beta^2 \\ \gamma^i\alpha_i\beta + \gamma^i\alpha_i\beta &= 2 \\ -\gamma^i\beta\alpha_i - \gamma^i\beta\alpha_i &= 2 \\ \gamma^i\gamma^i + \gamma^i\gamma^i &= -2 \,, \end{aligned} \tag{2.65}$$

from where we conclude that in general we have

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \,, \tag{2.66}$$

which defines what is denoted as **Clifford Algebra**. If we try to construct new operators out of the γ^μ factors the only independent combinations are the following

$$\begin{aligned} \left(\gamma^0\right)^2 =& 1 \qquad \text{one component} \\ i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv& \gamma^5 \qquad \text{one component} \\ & \gamma^\mu \qquad \text{four components} \\ & \gamma^\mu\gamma^5 \qquad \text{four components} \\ \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \equiv& \sigma^{\mu\nu} \qquad \text{six components (anti-symmetric)} \end{aligned} \tag{2.67}$$

which contains a maximum of 16 independent elements. This means that the γ^μ factors can be represented by 4×4 matrices and one of the possible parametrizations is the so called

Pauli-Dirac representation that reads

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{2.68}$$

with $\mathbb{1}$ the 2×2 identity matrix and σ^i the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.69}$$

Note that we can write **any** 4×4 matrix as a linear combination of

$$\{\mathbb{1}, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\} \tag{2.70}$$

which is a dimension-16 basis for the space of all 4×4 matrices.

2.3.1 Solutions of the Dirac equation

In the previous subsection we have derived the Dirac equation which reads

$$(i\gamma^\mu\partial_\mu - m)\,\psi = 0\,, \tag{2.71}$$

where the γ^μ factors are 4×4 matrices. To find solutions for the Dirac equation we start by introducing the ansatz

$$\psi^\alpha = u^\alpha(p)e^{-ip_\mu x^\mu} \tag{2.72}$$

with $u^\alpha(p)$ a four component column matrix ($\alpha = 1, 2, 3, 4$) and where p denotes the four momentum vector p^μ. Noting that

$$\partial_\mu\psi^\alpha = -ip_\mu\psi^\alpha \tag{2.73}$$

we can rewrite Dirac's equation as

$$(\gamma^\mu p_\mu - m)\,u^\alpha(p) = 0 \tag{2.74}$$

Multiplying now on the left by $(\gamma^\nu p_\nu + m)$ we get

$$\begin{aligned} (\gamma^\nu p_\nu + m)\,(\gamma^\mu p_\mu - m)\,u^\alpha(p) &= 0 \\ \Rightarrow \left(\gamma^\nu\gamma^\mu p_\nu p_\mu - m^2\right)u^\alpha(p) &= 0 \\ \Rightarrow \gamma^\nu\gamma^\mu p_\nu p_\mu u^\alpha(p) &= m^2 u^\alpha(p)\,. \end{aligned} \tag{2.75}$$

Using the anti-commutator (2.66) and knowing that the momentum operators commute, $p_\nu p_\mu = p_\mu p_\nu$, we note the following

$$\begin{aligned} \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}p_\nu p_\mu &= g^{\mu\nu}p_\nu p_\mu \\ \Leftrightarrow \frac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)p_\nu p_\mu &= p^\mu p_\mu \\ \Leftrightarrow \frac{1}{2}\left(\gamma^\mu\gamma^\nu p_\nu p_\mu + \gamma^\nu\gamma^\mu p_\nu p_\mu\right) &= E^2 - \boldsymbol{p}^2 \\ \Leftrightarrow \frac{1}{2}\left(\gamma^\mu\gamma^\nu p_\mu p_\nu + \gamma^\nu\gamma^\mu p_\nu p_\mu\right) &= E^2 - \boldsymbol{p}^2 \\ \Leftrightarrow \gamma^\nu\gamma^\mu p_\nu p_\mu &= E^2 - \boldsymbol{p}^2\,, \end{aligned} \tag{2.76}$$

and replacing this in eq. (2.75) we obtain

$$\left(E^2 - \boldsymbol{p}^2\right) u^\alpha(p) = m^2 u^\alpha(p)\,. \tag{2.77}$$

The first case to consider is that we are in the proper frame of the particle which means that it is at rest thus $\boldsymbol{p} = 0$. Therefore the energy spectrum of Dirac's equation yields

$$E^2 u^\alpha = m^2 u^\alpha \Rightarrow E = \pm m \qquad \text{with} \qquad m > 0\,. \tag{2.78}$$

Similarly to what we have seen for the Klein-Gordon equation, the Dirac equation also predicts negative energy solutions that we identify once more e **positive energy anti-particles**.

Using Dirac's Hamiltonian (2.54) it is immediate to determine the eigenvectors of Dirac's equation for the stationary states with energy $\pm m$:

$$\begin{gathered} H u^\alpha = \gamma^0 m u^\alpha \\ \begin{pmatrix} E\mathbb{1} & 0 \\ 0 & E\mathbb{1} \end{pmatrix} u^\alpha = \begin{pmatrix} m\mathbb{1} & 0 \\ 0 & -m\mathbb{1} \end{pmatrix} u^\alpha \end{gathered} \tag{2.79}$$

which read

$$u^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad u^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad u^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.80}$$

So far we have determined the energy eigenstates and respective eigenvectors but we have not yet realized one of the most remarkable predictions encoded in this formalism. In fact, there is a double degeneracy embodied in eq. (2.79) which indicates that there must be another observable that commutes with the Hamiltonian and whose eigenvalues can be taken to discriminate the degenerate states. To see what this means let us first look to the result of the vector product $\boldsymbol{\gamma} \wedge \boldsymbol{\gamma}$:

$$\begin{aligned} \boldsymbol{\gamma} \wedge \boldsymbol{\gamma} &= \left(\gamma^2\gamma^3 - \gamma^3\gamma^2\right) \boldsymbol{e}_x + \left(\gamma^3\gamma^1 - \gamma^1\gamma^3\right) \boldsymbol{e}_y + \left(\gamma^1\gamma^2 - \gamma^2\gamma^1\right) \boldsymbol{e}_z \\ &= 2\left(\gamma^2\gamma^3 \boldsymbol{e}_x + \gamma^3\gamma^1 \boldsymbol{e}_y + \gamma^1\gamma^2 \boldsymbol{e}_z\right) \\ &= -2i \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} \boldsymbol{e}_x - 2i \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \boldsymbol{e}_y - 2i \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \boldsymbol{e}_z \end{aligned} \tag{2.81}$$

from where we conclude that the spin operator $\boldsymbol{S}$ can be defined as

$$\begin{aligned} \boldsymbol{S} = \frac{i}{4} \boldsymbol{\gamma} \wedge \boldsymbol{\gamma} &= \frac{1}{2} \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} \boldsymbol{e}_x + \frac{1}{2} \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \boldsymbol{e}_y + \frac{1}{2} \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \boldsymbol{e}_z \\ &= S_x \boldsymbol{e}_x + S_y \boldsymbol{e}_y + S_z \boldsymbol{e}_z\,. \end{aligned} \tag{2.82}$$

The spin operator clearly commutes with the Hamiltonian for the case $\boldsymbol{p} = 0$ which means that the basis $\{u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}\}$, with $u^{(i)}$ the eigenvectors (2.80), is also an eigenbasis of the spin operator. We conclude from this that the Dirac equation naturally contains the spin of a particle and contrary to the Klein-Gordon equation it can describe relativistic electrons. Note that the u^α column matrices transform in the spin space and not in the usual four-dimensional space-time, and for that reason are denoted as **Dirac spinors**.

Acting with S_z on the Dirac spinors we see that

$$S_z u^{(1,3)} = \frac{1}{2} u^{(1,3)} \,, \qquad S_z u^{(2,4)} = -\frac{1}{2} u^{(2,4)} \,, \tag{2.83}$$

from where we conclude that

$$\begin{aligned}
u^{(1)} &\longrightarrow \text{Particle solution with } E = m \,, S_z = \frac{\hbar}{2} \\
u^{(2)} &\longrightarrow \text{Particle solution with } E = m \,, S_z = -\frac{\hbar}{2} \\
u^{(3)} &\longrightarrow \text{Antiparticle solution with } E = -m \,, S_z = \frac{\hbar}{2} \\
u^{(4)} &\longrightarrow \text{Antiparticle solution with } E = -m \,, S_z = -\frac{\hbar}{2}
\end{aligned} \tag{2.84}$$

If we consider the laboratory frame, that is $\boldsymbol{p} \neq 0$, and noting that $\gamma^0 H = \boldsymbol{\gamma} \cdot \boldsymbol{p} + m$ eq. (2.79) should be recast to

$$\begin{gathered}
\gamma^0 H u^\alpha (p) = (\boldsymbol{\gamma} \cdot \boldsymbol{p} + m\mathbb{1}) u^\alpha (p) \Rightarrow \\
\begin{pmatrix} E\mathbb{1} & 0 \\ 0 & -E\mathbb{1} \end{pmatrix} \begin{pmatrix} u_{\mathrm{A}} \\ u_{\mathrm{B}} \end{pmatrix} = \begin{pmatrix} m\mathbb{1} & \boldsymbol{\sigma} \cdot \boldsymbol{p} \\ -\boldsymbol{\sigma} \cdot \boldsymbol{p} & m\mathbb{1} \end{pmatrix} \begin{pmatrix} u_{\mathrm{A}} \\ u_{\mathrm{B}} \end{pmatrix}
\end{gathered} \tag{2.85}$$

where we have also conveniently redefined the four component u^α spinor as $u^\alpha = (u_{\mathrm{A}} \quad u_{\mathrm{B}})^\top$ with u_{A} and u_{B} arbitrary two-component spinors. It follows from (2.85) that

$$\begin{aligned}
(E - m)\, u_{\mathrm{A}} &= \boldsymbol{\sigma} \cdot \boldsymbol{p}\, u_{\mathrm{B}} \Rightarrow u_{\mathrm{A}} = \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E - m} u_{\mathrm{B}} \\
(E + m)\, u_{\mathrm{B}} &= \boldsymbol{\sigma} \cdot \boldsymbol{p}\, u_{\mathrm{A}} \Rightarrow u_{\mathrm{B}} = \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E + m} u_{\mathrm{A}}
\end{aligned} \tag{2.86}$$

from where replacing the second equation in the first one gives

$$u_{\mathrm{A}} = \frac{(\boldsymbol{\sigma} \cdot \boldsymbol{p})^2}{(E^2 - m^2)} u_{\mathrm{A}} \,. \tag{2.87}$$

Noting that

$$\begin{aligned}
\boldsymbol{\sigma} \cdot \boldsymbol{p} &= p_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + p_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}
\end{aligned} \tag{2.88}$$

thus

$$\begin{aligned}
(\boldsymbol{\sigma} \cdot \boldsymbol{p})^2 &= \begin{pmatrix} p_z^2 + (p_x - ip_y)\,(p_x + ip_y) & p_z\,(p_x - ip_y) - p_z\,(p_x - ip_y) \\ p_z\,(p_x + ip_y) - p_z\,(p_x + ip_y) & (p_x + ip_y)\,(p_x - ip_y) + p_z^2 \end{pmatrix} \\
&= \begin{pmatrix} p_z^2 + p_x^2 + p_y^2 & 0 \\ 0 & p_x^2 + p_y^2 + p_z^2 \end{pmatrix} = \boldsymbol{p}^2 \mathbb{1} \,.
\end{aligned} \tag{2.89}$$

Equation (2.87) is then rewritten as

$$u_{\rm A} = \frac{\boldsymbol{p}^2}{E^2 - m^2} u_{\rm A} \,, \tag{2.90}$$

from where we conclude that $\boldsymbol{p}^2 = E^2 - m^2$ which implies that

$$E = \pm\sqrt{\boldsymbol{p}^2 + m^2} \,. \tag{2.91}$$

Once again we see that Dirac's equation admits both positive and negative energy solutions thus predicting both particle and antiparticle states.

In general we can write the spinors that describe free spin-1/2 particles as

$$\begin{aligned} u^{(+)}(p) &= N \begin{pmatrix} u_{\rm A} \\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m} u_{\rm A} \end{pmatrix}, \qquad E > 0 \\ u^{(-)}(p) &= N \begin{pmatrix} \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E-m} u_{\rm B} \\ u_{\rm B} \end{pmatrix}, \qquad E < 0 \end{aligned} \tag{2.92}$$

with N a normalization constant. It is convenient to attribute to each of the 2-spinors $u_{\rm A}$ and $u_{\rm B}$ the eigenvectors of the Pauli matrix σ^3,

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{2.93}$$

With this basis, replacing the Pauli spinors in the Dirac ones we obtain **two spinors for particle solutions** with $E > 0$

$$u^{(1)}(p) = N \begin{pmatrix} 1 \\ 0 \\ \dfrac{p_z}{E+m} \\ \dfrac{p_x + ip_y}{E+m} \end{pmatrix} \qquad u^{(2)}(p) = N \begin{pmatrix} 0 \\ 1 \\ \dfrac{p_x - ip_y}{E+m} \\ -\dfrac{p_z}{E+m} \end{pmatrix} \tag{2.94}$$

and **two spinors for particle solutions** with $E < 0$

$$u^{(3)}(p) = N \begin{pmatrix} \dfrac{p_z}{E-m} \\ \dfrac{p_x + ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \qquad u^{(4)}(p) = N \begin{pmatrix} \dfrac{p_x - ip_y}{E-m} \\ -\dfrac{p_z}{E-m} \\ 0 \\ 1 \end{pmatrix} . \tag{2.95}$$

It is also usual to redefine the negative energy solutions as **positive energy antiparticle** states where, using the *Feynman-Stuckelberg convention*, *i.e.* $v^{(1)}(p) = u^{(4)}(-p)$ and $v^{(2)}(p) = u^{(3)}(-p)$, one has

$$v^{(2)}(p) = N \begin{pmatrix} \dfrac{p_z}{E+m} \\ \dfrac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix} \qquad v^{(1)}(p) = N \begin{pmatrix} \dfrac{p_x - ip_y}{E+m} \\ -\dfrac{p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} . \tag{2.96}$$

In essence, we have the freedom to regard both particles and antiparticles as positive energy states and their wave functions can be written as

$$
\begin{aligned}
\psi &= u^{(1)}(p)e^{-ip\cdot x}\\
\psi &= u^{(2)}(p)e^{-ip\cdot x}
\end{aligned}
\tag{2.97}
$$

for particles, and

$$
\begin{aligned}
\psi &= v^{(1)}(p)e^{ip\cdot x}\\
\psi &= v^{(2)}(p)e^{ip\cdot x}
\end{aligned}
\tag{2.98}
$$

for anti-particles. To completely determine the solutions of Dirac's equation we need to find the normalization constant N. It is usual to define the convention

$$
u^{(i)\dagger}u^{(j)} = v^{(i)\dagger}v^{(j)} = 2E\delta_{ij}\,, \tag{2.99}
$$

and using the particle solution $u^{(1)}\,(p)$,

$$
\begin{aligned}
u^{(1)}(p)u^{(1)}(p) &= |N|^2 \begin{pmatrix} 1 & 0 & \dfrac{p_z}{E+m} & \dfrac{p_x+ip_y}{E+m}\end{pmatrix}\begin{pmatrix} 1\\ 0\\ \dfrac{p_z}{E+m}\\ \dfrac{p_x+ip_y}{E+m}\end{pmatrix}\\
&= |N|^2\left(1+\frac{p_z^2+p_x^2+p_y^2}{(E+m)^2}\right)\\
&= |N|^2\left(1+\frac{\boldsymbol{p}^2}{(E+m)^2}\right)\\
&= |N|^2\left(1+\frac{E^2-m^2}{(E+m)^2}\right)\\
&= |N|^2\left(1+\frac{E-m}{E+m}\right)\\
&= |N|^2\left(\frac{2E}{E+m}\right) = 2E\\
&\Rightarrow N = \sqrt{E+m}
\end{aligned}
\tag{2.100}
$$

up to an arbitrary phase. The same result can be obtained using the antiparticle states.

2.3.2 Helicity

We have verified that the solutions of Dirac equations in the proper frame of the particle are also solutions of the spin operator $\boldsymbol{S}$. However it is no longer true in the case of $\boldsymbol{p} \neq 0$ as the commutator

$$
[\boldsymbol{S}, H] \neq 0\,. \tag{2.101}
$$

To see this take for example the Dirac Hamiltonian and the spin operator in in the matrix form:

$$\begin{aligned}\boldsymbol{S}H &= \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}\begin{pmatrix} m\mathbb{1} & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & m\mathbb{1} \end{pmatrix} = \begin{pmatrix} m\boldsymbol{\sigma} & \boldsymbol{\sigma}\,(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \\ \boldsymbol{\sigma}\,(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & m\boldsymbol{\sigma} \end{pmatrix} \\ H\boldsymbol{S} &= \begin{pmatrix} m\mathbb{1} & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & m\mathbb{1} \end{pmatrix}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} = \begin{pmatrix} m\boldsymbol{\sigma} & (\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\boldsymbol{\sigma} \\ (\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\boldsymbol{\sigma} & m\boldsymbol{\sigma} \end{pmatrix}.\end{aligned} \tag{2.102}$$

Since the Pauli matrices $\boldsymbol{\sigma}$ do not commute we conclude that $\boldsymbol{S}H \neq H\boldsymbol{S}$ and contrary to the case of $\boldsymbol{p} = 0$ the general solutions of the Dirac equation are not spin eigenstates.

Alternatively, we define a new operator compatible with the Hamiltonian

$$h = 2\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{|\boldsymbol{p}|}\,, \tag{2.103}$$

which represents the **spin projection along the direction of motion of the particle** and is typically denoted as **helicity**. Let us verify that the helicity is in fact compatible with the Hamiltonian:

$$\begin{aligned}hH &= \begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} \end{pmatrix}\begin{pmatrix} m\mathbb{1} & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & m\mathbb{1} \end{pmatrix} = \begin{pmatrix} m\boldsymbol{\sigma}\cdot\boldsymbol{p} & \boldsymbol{p}^2 \\ \boldsymbol{p}^2 & m\boldsymbol{\sigma}\cdot\boldsymbol{p} \end{pmatrix} \\ Hh &= \begin{pmatrix} m\mathbb{1} & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & m\mathbb{1} \end{pmatrix}\begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} \end{pmatrix} = \begin{pmatrix} m\boldsymbol{\sigma}\cdot\boldsymbol{p} & \boldsymbol{p}^2 \\ \boldsymbol{p}^2 & m\boldsymbol{\sigma}\cdot\boldsymbol{p} \end{pmatrix} \\ &\Rightarrow [h, H] = 0\,,\end{aligned} \tag{2.104}$$

from where we conclude that the helicity is a good quantum number. We can further note that the square of the helicity operator is

$$h^2 = \frac{4}{|\boldsymbol{p}|^2}\frac{1}{4}\begin{pmatrix} (\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 & 0 \\ 0 & (\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.105}$$

which means that the eigenvalues of the helicity operator are $h_\pm = \pm 1$, where the label $h_+ = 1$ stands for **plus helicity spinor** and indicates that the spin projection is parallel to the particle's momentum, whereas $h_- = -1$ stands for **minus helicity spinor** and implies that the spin projection is anti-parallel to the particle's momentum. Note that, in general, helicity **is not** the same as chirality, whose discussion is left as an exercise.

We can finally characterize the spinors $u^{(i)}(p)$ and $v^{(i)}(p)$ with respect to their helicity. For simplicity and clarity we will consider that the momentum aligned with the z-directions, that is $\boldsymbol{p} = p_z \boldsymbol{e}_z$.

$$\begin{aligned}
hu^{(1)} &= \frac{1}{p_z}\begin{pmatrix} p_z & 0 & 0 & 0 \\ 0 & -p_z & 0 & 0 \\ 0 & 0 & p_z & 0 \\ 0 & 0 & 0 & -p_z \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \dfrac{p_z}{E+m} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \dfrac{p_z}{E+m} \\ 0 \end{pmatrix} = u^{(1)} \\
hu^{(2)} &= \frac{1}{p_z}\begin{pmatrix} p_z & 0 & 0 & 0 \\ 0 & -p_z & 0 & 0 \\ 0 & 0 & p_z & 0 \\ 0 & 0 & 0 & -p_z \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\dfrac{p_z}{E+m} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ \dfrac{p_z}{E+m} \end{pmatrix} = -u^{(2)} \\
hv^{(1)} &= \frac{1}{p_z}\begin{pmatrix} p_z & 0 & 0 & 0 \\ 0 & -p_z & 0 & 0 \\ 0 & 0 & p_z & 0 \\ 0 & 0 & 0 & -p_z \end{pmatrix}\begin{pmatrix} 0 \\ -\dfrac{p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{p_z}{E+m} \\ 0 \\ -1 \end{pmatrix} = -v^{(1)} \\
hv^{(2)} &= \frac{1}{p_z}\begin{pmatrix} p_z & 0 & 0 & 0 \\ 0 & -p_z & 0 & 0 \\ 0 & 0 & p_z & 0 \\ 0 & 0 & 0 & -p_z \end{pmatrix}\begin{pmatrix} \dfrac{p_z}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{p_z}{E+m} \\ 0 \\ -1 \\ 0 \end{pmatrix} = v^{(2)}
\end{aligned} \tag{2.106}$$

which means that, like the spin for the case of a particle at rest, the helicity lifts the double degeneracy of the stationary states. In particular

$$\begin{aligned}
u^{(1)}(p) &\longrightarrow \quad \text{fermion with energy E and + helicity,} \\
u^{(2)}(p) &\longrightarrow \quad \text{fermion with energy E and } - \text{ helicity,} \\
v^{(1)}(p) &\longrightarrow \quad \text{antifermion with energy E and } - \text{ helicity,} \\
v^{(2)}(p) &\longrightarrow \quad \text{antifermion with energy E and + helicity.}
\end{aligned} \tag{2.107}$$

To understand chirality it is convenient to follow the same steps in this section but using the Weyl representation of the gamma matrices instead (this is left as an exercise).

2.3.3 Covariance of the Dirac equation

To study the covariance of the Dirac equation let us admit that the spinors $\psi(x)$ transform as

$$\psi(x) = S\psi'(x') \tag{2.108}$$

with $SS^{-1} = \mathbb{1}$. Following the same steps as for the Klein-Gordon equation

$$\begin{aligned}
\left(i\gamma^\mu \partial_\mu - m\right)\psi(x) &= 0 \\
\Rightarrow \left(i\gamma^\mu \left[\Lambda^{-1}\right]^\nu{}_\mu \partial'_\nu - m\right) S\psi'(x') &= 0 \\
\Rightarrow \left(i\gamma^\mu S \left[\Lambda^{-1}\right]^\nu{}_\mu \partial'_\nu - mS\right) \psi'(x') &= 0 \\
\Rightarrow S^{-1}\left(i\gamma^\mu S \left[\Lambda^{-1}\right]^\nu{}_\mu \partial'_\nu - mS\right) \psi'(x') &= 0 \\
\Rightarrow \left(iS^{-1}\gamma^\mu S \left[\Lambda^{-1}\right]^\nu{}_\mu \partial'_\nu - mS^{-1}S\right) \psi'(x') &= 0 \\
\Rightarrow \left(iS^{-1}\gamma^\mu S \left[\Lambda^{-1}\right]^\nu{}_\mu \partial'_\nu - m\right) \psi'(x') &= 0 \\
\left(i\gamma'^\nu \partial'_\nu - m\right) \psi'(x') &= 0
\end{aligned} \tag{2.109}$$

provided that

$$\gamma'^\nu = S^{-1}\gamma^\mu S \left[\Lambda^{-1}\right]^\nu{}_\mu \,. \tag{2.110}$$

Specifying all indices explicitly

$$\left[\gamma'^\nu\right]_{\alpha\beta} = \left[S^{-1}\right]_\alpha{}^\rho \left[\gamma^\mu\right]_{\rho\eta} \left[S\right]^\eta{}_\beta \left[\Lambda^{-1}\right]^\nu{}_\mu \tag{2.111}$$

where μ and ν are Lorentz indices denoting vector space transformations and where α, β, ρ and η are Lorentz indices that denote transformations in the spinor space.

3 Lagrangian formulation in field theory

3.1 The Euler-Lagrange equations in classical mechanics

The fundamental quantity in classical mechanics is the action, $\mathcal{S}$. We can think of it as measure of the path taken between any two points $q_1(t_1)$ and $q_2(t_2)$ as shown in Fig. 3. However, out of all possible trajectories nature has chosen to pick that one that **extremizes** $\mathcal{S}$, which reads[3]

$$\mathcal{S}[q] = \int dt L(q, \dot{q}, t) \,, \tag{3.1}$$

with the **Lagrangian** $L(q, \dot{q}, t)$ defined by the difference between the kinetic and potential energies

$$L(q, \dot{q}, t) = T(\dot{q}) - V(q, t) \tag{3.2}$$

and where q are the generalized coordinates and $\dot{q} = \frac{dq}{dt}$. A trajectory that extremizes the

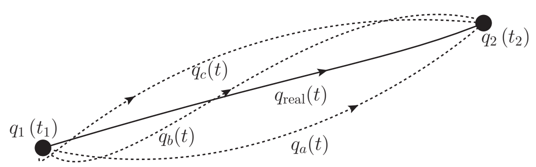


Figure 3: *Possible trajectories between two points $q_1(t_1)$ and $q_2(t_2)$. The solid line corresponds to a realistic trajectory $q_{\text{real}}(t)$.*

[3] Note that the action is not a standard function but instead a functional. While the former maps numbers into numbers, e.g. $f(x) : \mathbb{R} \to \mathbb{R}$, the latter transform functions into numbers, e.g. $\mathcal{S}[f] : \mathcal{F} \to \mathbb{R}$.

action, which we denote as $q_{\mathrm{real}}(t)$ in Fig: 3, **is a trajectory that on average has the least difference of kinetic and potential energy**. For example, if one throws an object up in a constant gravitational field it undergoes a parabolic trajectory starting with a certain kinetic energy and then slowing down as the potential energy grows. If the trajectory was completely different randomly going up and down, the thrown object would need to be at times animated with a lot more of kinetic energy in order to reach the endpoint $q_2\left(t_2\right)$ in the same amount of available time $t_2 - t_1$. It is possible to understand from this example that a non-parabolic trajectory would on average have a larger $T\left(\dot{q}\right) - V\left(q, t\right)$ difference and the action would by no means be extremized.

Let us then consider an infinitesimal shift to a certain trajectory $q_i(t)$ as

$$q_i'(t) = q_i(t) + \delta q_i(t)\,. \tag{3.3}$$

Under this conditions, the variation[4] of the action reads

$$\begin{aligned}
\delta\mathcal{S} &= \int_{t_1}^{t_2} dt\, L\left(q_i', \dot{q}_i', t\right) - \int_{t_1}^{t_2} dt\, L\left(q_i, \dot{q}_i, t\right) \\
&= \int_{t_1}^{t_2} dt\, \left[L\left(q_i + \delta q_i, \dot{q}_i + \delta\dot{q}_i, t\right) - L\left(q_i, \dot{q}_i, t\right)\right] \\
&= \int_{t_1}^{t_2} dt\, \delta L\left(q_i, \dot{q}_i, t\right) \\
&= \int_{t_1}^{t_2} dt\, \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i\right) \\
&= \int_{t_1}^{t_2} dt\, \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\delta q_i\right) \\
&= \int_{t_1}^{t_2} dt\, \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) \\
&= \left[\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} dt\, \left(\frac{\partial L}{\partial q_i}\delta q_i - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)\,.
\end{aligned} \tag{3.4}$$

Provided that all trajectories converge both in t_1 and in t_2, then $\delta q_i(t_1) = \delta q_i(t_2) = 0$ which implies that

$$\left[\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right]_{t_1}^{t_2} = 0\,. \tag{3.5}$$

Therefore, the variation on the action becomes

$$\delta\mathcal{S} = \int_{t_1}^{t_2} dt\, \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i\,, \tag{3.6}$$

and applying the **principle of the stationary of the action** we obtain the **Euler-Lagrange** equations for a classical particle:

$$\delta\mathcal{S} = 0 \Rightarrow \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\,. \tag{3.7}$$

[4]In functional analysis, the analogous of a derivative of a function is called variation and denoted with δ. In other words, the differential calculus is generalized to the the calculus of variations and the stationarity conditions are determined by requiring that the variation of a given functional $\mathcal{F}$ is zero, that is $\delta\mathcal{F} = 0$.

Let us study a simple example. Consider the classical one-particle Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - V(x)\,, \tag{3.8}$$

take the following derivatives

$$\begin{aligned}\frac{\partial L}{\partial x} &= -\frac{\partial V}{\partial x}\\ \frac{\partial L}{\partial \dot{x}} &= m\dot{x} = p_x\end{aligned} \tag{3.9}$$

and apply the Euler-Lagrange equations in order to extract the equations of motion, that is

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \Rightarrow -\frac{\partial V}{\partial x} - \frac{d}{dt}p_x = 0 \Rightarrow -\frac{\partial V}{\partial x} = m\ddot{x} \Leftrightarrow F = m\,a\,. \tag{3.10}$$

In conclusion, we see that from the principle of the least action we have elegantly derived Newton's second Law.

3.2 The Euler-Lagrange equations in field theory

So far we have derived the Euler-Lagrange equations for a classical particle. If we go one step forward and consider a classical system consisting of n-particles the Lagrangian (3.8) is generalized to

$$L = \frac{1}{2}\sum_{k=1}^{n} m_k \dot{x}_k^2 - V(x_1, \ldots, x_n)\,, \tag{3.11}$$

where the space coordinates become indexed by the particle k. We can now generalize it to continuum variables replacing the $x_k(t)$ coordinates by a quantity in each space point $\phi\,(t, \boldsymbol{x})$. This quantity is a classical field. The coordinates $\boldsymbol{x}$ replace the k-index in the discrete case and instead of a sum over all particles, $\sum_k$, we integrate over all space. The Lagrangian then becomes

$$L = \int d^3x\,\mathcal{L}\,(\phi, \partial_\mu \phi) \tag{3.12}$$

thus the action

$$\mathcal{S} = \int d^4x\,\mathcal{L}\,(\phi, \partial_\mu \phi) \tag{3.13}$$

with $\mathcal{L}\,(\phi, \partial_\mu \phi)$ a **Lagrangian density**. As it is evident from this construction, in field theory the generalized coordinates are no longer the standard space coordinates and their time derivatives but the fields itself and their four-derivatives.

In particle physics the fundamental quantity used to define a certain model is the Lagrangian density. Choosing a set of initial symmetries it will contain all needed information to completely characterize e.g. the nature of particle interactions, their charges and masses as well as how symmetries can be broken. Note, particle physicists tend to simply call the Lagrangian density as Lagrangian.

By inspection of eqs. (3.12) and (3.13), knowing that the Lagrangian L has units of energy and that from the De Broglie relation

$$E = \frac{hc}{\lambda} \tag{3.14}$$

we have

$$[L] = [E] = \mathrm{L}^{-1} \tag{3.15}$$

with $h = c = 1$ and where L stands for length units in the LMT system, we conclude that

$$[L] = \mathrm{L}^3 [\mathcal{L}] \Rightarrow \mathrm{L}^{-1} = \mathrm{L}^3 [\mathcal{L}] \Rightarrow [\mathcal{L}] = \mathrm{L}^{-4} = \left[E^4\right] = \left[m^4\right] \,, \tag{3.16}$$

that is, the Lagrangian density has units of energy or mass to the forth power. This information is extremely relevant in the construction of particle physics models.

To derive the Euler-Lagrange equations in field theory let us consider a four-sphere with volume V_4 whose surface is a three-dimensional space S_3 with normal four-vectors n^μ as pictured in Fig. 4. The field value $\phi\,(t, \boldsymbol{x})$ are arbitrary everywhere except on the boundary

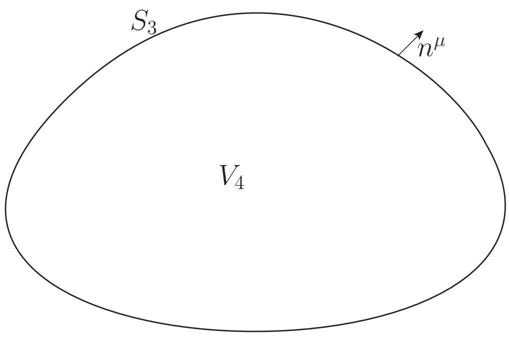


Figure 4: *Four dimensional space of integration of the action.*

where they converge to a common value. This is a generalization of the classical trajectories $q^i\,(t)$ converging to a common point at the boundaries t_1 and t_2. As such, the variation of the fields at the surface S_3 vanishes, that is

$$\phi\,(t, \boldsymbol{x})\Big|_{S_3} = \phi'\,(t, \boldsymbol{x})\Big|_{S_3} \Rightarrow \delta\phi\,(t, \boldsymbol{x})\Big|_{S_3} = 0\,. \tag{3.17}$$

Similarly to the variation of the classical trajectories (3.3) one can define the field variation as

$$\phi'\,(t, \boldsymbol{x}) = \phi\,(t, \boldsymbol{x}) + \delta\phi\,(t, \boldsymbol{x}) \,, \tag{3.18}$$

as well as

$$\partial_\mu \phi'\,(t, \boldsymbol{x}) = \partial_\mu \phi\,(t, \boldsymbol{x}) + \delta\left[\partial_\mu \phi\,(t, \boldsymbol{x})\right] \,, \tag{3.19}$$

where we have

$$\delta\left[\partial_\mu \phi\,(t, \boldsymbol{x})\right] = \partial_\mu \phi'\,(t, \boldsymbol{x}) - \partial_\mu \phi\,(t, \boldsymbol{x}) = \partial_\mu \left(\phi'\,(t, \boldsymbol{x}) - \phi\,(t, \boldsymbol{x})\right) = \partial_\mu \delta\phi\,(t, \boldsymbol{x}) \,. \tag{3.20}$$

With this information we can calculate the variation of the action as follows:

$$
\begin{aligned}
\delta\mathcal{S} &= \int_{V_4} d^4x \, \left[\mathcal{L}\left(\phi', \partial_\mu \phi'\right) - \mathcal{L}\left(\phi, \partial_\mu \phi\right)\right] \\
&= \int_{V_4} d^4x \, \left[\mathcal{L}\left(\phi + \delta\phi, \partial_\mu \phi + \delta\left(\partial_\mu \phi\right)\right) - \mathcal{L}\left(\phi, \partial_\mu \phi\right)\right] \\
&= \int_{V_4} d^4x \, \delta\mathcal{L} \\
&= \int_{V_4} d^4x \, \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\delta\left(\partial_\mu \phi\right)\right] \\
&= \int_{V_4} d^4x \, \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\partial_\mu \delta\phi\right] \\
&= \int_{V_4} d^4x \, \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\delta\phi\right) - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\right)\delta\phi\right] \\
&= \int_{V_4} d^4x \, \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\right)\right]\delta\phi + \int_{V_4} d^4x \, \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\delta\phi\right) .
\end{aligned} \tag{3.21}
$$

Using a generalization of Gauss-Ostrogradsky's theorem, which generically states that the integral of the divergence of a vector field $\boldsymbol{F}$ over a volume V is identical to the surface integration of that same vector field over the boundary of V, i.e.

$$
\int_V dV \, \boldsymbol{\nabla} \cdot \boldsymbol{F} = \int_S dS \, \boldsymbol{F} \cdot \boldsymbol{n} \, , \tag{3.22}
$$

the last term in the last line of eq. (3.21) can be converted in an integral over S_3

$$
\int_{V_4} d^4x \, \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\delta\phi\right) = \int_{S_3} d^3x \, n_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\delta\phi = 0 \, , \tag{3.23}
$$

which vanishes since the variation of the field ϕ at the surface S_3 is zero, that is $\delta\phi\Big|_{S_3} = 0$. Therefore, the stationarity of the action implies that

$$
\delta\mathcal{S} = \int_{V_4} d^4x \, \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\right]\delta\phi = 0 \tag{3.24}
$$

from where we obtain the Euler-Lagrange equations in field theory

$$
\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} = 0 \, . \tag{3.25}
$$

3.2.1 The Klein-Gordon Lagrangian

Consider the Klein-Gordon equation for a real scalar field φ

$$
\left(g^{\mu\nu}\partial_\mu \partial_\nu + m^2\right)\varphi = 0 \tag{3.26}
$$

which can be recast as

$$
-m^2\varphi - \partial_\mu\left(g^{\mu\nu}\partial_\nu \varphi\right) = 0 \tag{3.27}
$$

from where we directly read that

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \varphi} &= -m^2 \varphi \\ \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \varphi\right)} &= g^{\mu\nu} \partial_\nu \varphi \,.\end{aligned} \tag{3.28}$$

We can then write the Klein-Gordon Lagrangian for a **free real scalar field** as

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} m^2 \varphi^2 \,, \tag{3.29}$$

where the $\frac{1}{2}$ factors are combinatorial factors which take into account the interchange $\mu \leftrightarrow \nu$. Recalling that the Lagrangian density has dimensions of $\left[m^4\right]$, we conclude from (3.29) that the mass dimensions of a scalar field are

$$[\varphi] = [m] \,. \tag{3.30}$$

If we now define a complex scalar field as

$$\phi = \frac{1}{\sqrt{2}} \left(\varphi_1 + i\varphi_2\right) \,, \tag{3.31}$$

each of the real scalars φ_1 and φ_2 obey a Klein-Gordon equation

$$\begin{aligned}\left(\Box + m^2\right) \varphi_1 &= 0 \\ \left(\Box + m^2\right) \varphi_2 &= 0\end{aligned} \tag{3.32}$$

which means that the Lagrangian can be written as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi_1 \partial_\nu \varphi_1 + \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi_2 \partial_\nu \varphi_1 - \frac{1}{2} m^2 \varphi_1^2 - \frac{1}{2} m^2 \varphi_2^2 \\ &= g^{\mu\nu} \partial_\mu \left[\frac{1}{\sqrt{2}} \left(\varphi_1 - i\varphi_2\right)\right] \partial_\nu \left[\frac{1}{\sqrt{2}} \left(\varphi_1 + i\varphi_2\right)\right] - m^2 \left[\frac{1}{\sqrt{2}} \left(\varphi_1 - i\varphi_2\right)\right] \left[\frac{1}{\sqrt{2}} \left(\varphi_1 + i\varphi_2\right)\right] \\ &= g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi - m^2 \phi^* \phi \,.\end{aligned} \tag{3.33}$$

If we now apply the Euler-Lagrange equations to each of the fields, ϕ^* and ϕ,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi^*\right)} &= 0 \Longrightarrow \left(g^{\mu\nu} \partial_\mu \partial_\nu + m^2\right) \phi = 0 \\ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} &= 0 \Longrightarrow \left(g^{\mu\nu} \partial_\mu \partial_\nu + m^2\right) \phi^* = 0\end{aligned} \tag{3.34}$$

from where we see that there is one Klein-Gordon equation for each degree of freedom, which we identify with **the particle and antiparticle solutions** previously studied. We conclude from this that the Lagrangian (3.33) fully describes a non-interacting particle-antiparticle system whose scalar potential contains only a single mass term

$$V\left(\phi, \phi^*\right) = m^2 \phi^* \phi \,. \tag{3.35}$$

If we add a quartic self interactions to the theory the potential for a complex scalar with field operators up to mass-dimension four becomes

$$V(\phi, \phi^*) = m^2 \phi^* \phi + \frac{1}{4} \lambda (\phi^* \phi)^2 \tag{3.36}$$

and the Klein-Gordon equations for both the particle and the antiparticle are modified to

$$\begin{aligned} \left(g^{\mu\nu} \partial_\mu \partial_\nu + m^2 + \frac{\lambda}{2} \phi^* \phi \right) \phi = 0 \Leftrightarrow \left(\Box + V'' \right) \phi = 0 \\ \left(g^{\mu\nu} \partial_\mu \partial_\nu + m^2 + \frac{\lambda}{2} \phi^* \phi \right) \phi^* = 0 \Leftrightarrow \left(\Box + V'' \right) \phi^* = 0 \end{aligned} \tag{3.37}$$

with V'' the second derivative of the scalar potential in order to ϕ and ϕ^*. This example clearly shows that in an interacting theory the **physical mass** of the Klein-Gordon field is no longer given by the m^2 parameter but instead by the second derivatives of the potential. For the theory described by the potential (3.36) the squared mass of the Klein-Gordon field is modified to

$$V'' = m^2 + \frac{\lambda}{2} \phi^* \phi \,. \tag{3.38}$$

We will come back to this potential later on as a starting point to learn the Higgs mechanism.

3.2.2 The Dirac Lagrangian

The Dirac equation, that we derived in Sec. 2.3, reads

$$(i\gamma^\mu \partial_\mu - m) \psi = 0 \Leftrightarrow i\gamma^0 \partial_0 \psi + i\gamma^k \partial_k \psi - m\psi = 0 \,. \tag{3.39}$$

Let us derive the adjoint Dirac equation by hermitian conjugation:

$$\begin{aligned} \left(i\gamma^0 \partial_0 \psi \right)^\dagger + \left(i\gamma^k \partial_k \psi \right)^\dagger - m\psi^\dagger = 0 \\ -i\partial_0 \psi^\dagger \left(\gamma^0 \right)^\dagger - i\partial_k \psi^\dagger \left(\gamma^k \right)^\dagger - m\psi^\dagger = 0 \\ -i\partial_0 \psi^\dagger \gamma^0 + i\partial_k \psi^\dagger \gamma^k - m\psi^\dagger = 0 \,. \end{aligned} \tag{3.40}$$

The opposite sign in the first two terms follows from the hermeticity of γ^0 and anti-hermeticity of γ^k. If we multiply on the right by γ^0 and using the anti-commutation relations of the γ^μ matrices we have

$$\begin{aligned} -i\partial_0 \psi^\dagger \gamma^0 \gamma^0 + i\partial_k \psi^\dagger \gamma^k \gamma^0 - m\psi^\dagger \gamma^0 = 0 \\ -i\partial_0 \psi^\dagger \gamma^0 \gamma^0 - i\partial_k \psi^\dagger \gamma^0 \gamma^k - m\psi^\dagger \gamma^0 = 0 \\ i\partial_0 \overline{\psi} \gamma^0 + i\partial_k \overline{\psi} \gamma^k + m\overline{\psi} = 0 \\ \overline{\psi} \left(i \overleftarrow{\partial}_\mu \gamma^\mu + m \right) = 0 \,, \end{aligned} \tag{3.41}$$

where we have defined the adjoint spinor

$$\overline{\psi} = \psi^\dagger \gamma^0 \,. \tag{3.42}$$

In fact, the quantity $\overline{\psi}\psi$ is a Lorentz invariant while $\psi^\dagger\psi$ is not. A demonstration of the Lorentz invariance of $\overline{\psi}\psi$ is beyond the scope of this course.

The Dirac Lagrangian for a free fermion can be written as

$$\mathcal{L} = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi\,. \tag{3.43}$$

The first relevant comment is to note the mass-dimension of the fermion fields. Since the Lagrangian has mass dimension 4, if we look to the mass term we realize that

$$[m]^4 = [m]\left[\overline{\psi}\right][\psi] \Rightarrow \left[\overline{\psi}\right][\psi] = [m]^3 \Rightarrow \left[\overline{\psi}\right] = [\psi] = [m]^{\frac{3}{2}}\,. \tag{3.44}$$

Taking the derivatives with respect to the adjoint fermion $\overline{\psi}$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\overline{\psi}} &= i\gamma^\mu\partial_\mu\psi - m\psi\\ \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\overline{\psi}\right)} &= 0\end{aligned} \tag{3.45}$$

and replacing in the Euler-Lagrange equations we obtain the Dirac equation,

$$\frac{\partial\mathcal{L}}{\partial\overline{\psi}} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\overline{\psi}\right)} = 0 \Longrightarrow \left(i\gamma^\mu\partial_\mu - m\right)\psi = 0\,. \tag{3.46}$$

On the other hand, if we take the derivatives with respect to ψ

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\psi} &= -m\overline{\psi}\\ \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)} &= i\overline{\psi}\gamma^\mu\end{aligned} \tag{3.47}$$

which leads to the adjoint Dirac equation

$$\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)} = 0 \Longrightarrow \overline{\psi}\left(i\overleftarrow{\partial}_\mu\gamma^\mu + m\right) = 0\,. \tag{3.48}$$

So far we have only studied a free Dirac field. Let us now consider that our theory also contains a complex Klein-Gordon field ϕ that couples to the fermion ψ. Know that the scalars have mass dimension-1 and the fermions 3/2, we can construct a dimension-4 scalar-fermion interaction Lagrangian that of the form

$$\mathcal{L}_{\text{int}} = -y\phi\overline{\psi}\psi\,. \tag{3.49}$$

These terms are typically denoted as Yukawa terms and y is a Yukawa coupling. Once again, taking the field derivatives

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\overline{\psi}} &= i\gamma^\mu\partial_\mu\psi - m\psi - y\phi\psi\\ \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\overline{\psi}\right)} &= 0\end{aligned} \tag{3.50}$$

which yields the modified Dirac equation

$$(i\gamma^\mu\partial_\mu - m - y\phi)\,\psi = 0 \Leftrightarrow (i\gamma^\mu\partial_\mu - M)\,\psi = 0 \tag{3.51}$$

with the redefined mass

$$M = m + y\phi\,. \tag{3.52}$$

Similarly for the adjoint equation

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\psi} &= -m\overline{\psi} - y\phi\overline{\psi}\\ \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)} &= i\overline{\psi}\gamma^\mu\end{aligned} \tag{3.53}$$

thus

$$\overline{\psi}\left(i\overleftarrow{\partial}_\mu\gamma^\mu + m + y\phi\right) = 0 \Leftrightarrow \overline{\psi}\left(i\overleftarrow{\partial}_\mu\gamma^\mu + M\right) = 0\,. \tag{3.54}$$

This example highlights a crucial feature about Yukawa interactions. If the mass parameter vanishes, $m = 0$, Yukawa terms can regenerate a mass to the Dirac filed $M = y\phi$. The mechanism of fermion mass generation in the SM is in fact similar to this simplified example where all fermions become massive due to Yukawa interactions with the Higgs. However, the complete story is a bit more involved and will be revisited in detail in secs.??

3.2.3 The Maxwell Lagrangian

So far we have introduced Lagrangians for

1. a free massive scalar,
2. a massive scalar with a quartic self-interactions,
3. a free massive fermion,
4. a fermion-scalar interacting theory.

We will now introduce the Lagrangian for vector fields focusing on the example of the electromagnetic theory.

The Lagrangian is a scalar quantity neutral under all symmetries of the theory. For instance, both the Klein-Gordon and the Dirac Lagrangians are invariants under the Lorentz symmetry. Note that a scalar field ϕ is a Lorentz invariant on its own (recall eq. (2.44)), thus any power of ϕ is also Lorentz invariant, but the Dirac field is not (see (2.109)) and needs to be contracted with an adjoint spinor, $\overline{\psi}\psi$, in order to warrant such an invariance.

For the case of the Maxwell theory we have already seen that an invariant of the theory is the product $F_{\mu\nu}F^{\mu\nu}$, therefore, the most generic **non-interacting** Maxwell Lagrangian reads

$$\mathcal{L}_{\text{MAX}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,. \tag{3.55}$$

Let us derive the equations of motion using, as usual, the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)} = 0\,. \tag{3.56}$$

Since the derivatives are taken with respect to contravariant indices it is convenient to lower all indices in (3.55) by recasting it as

$$\begin{aligned}\mathcal{L}_{\rm MAX} &= -\frac{1}{4} g^{\alpha\delta} g^{\beta\gamma} F_{\alpha\beta} F_{\delta\gamma} \\ &= -\frac{1}{4} g^{\alpha\delta} g^{\beta\gamma} \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)\left(\partial_\delta A_\gamma - \partial_\gamma A_\delta\right)\,.\end{aligned} \tag{3.57}$$

This Lagrangian is only a function of $\partial_\mu A_\nu$ and not A_ν, thus

$$\frac{\partial \mathcal{L}_{\rm MAX}}{\partial A_\nu} = 0\,. \tag{3.58}$$

On the other hand the derivatives with respect to $\partial_\mu A_\nu$ read

$$\frac{\partial \mathcal{L}_{\rm MAX}}{\partial\left(\partial_\mu A_\nu\right)} = -\frac{1}{4} g^{\alpha\delta} g^{\beta\gamma}\left[\frac{\partial F_{\alpha\beta}}{\partial\left(\partial_\mu A_\nu\right)} F_{\delta\gamma} + F_{\alpha\beta}\frac{\partial F_{\delta\gamma}}{\partial\left(\partial_\mu A_\nu\right)}\right]\,. \tag{3.59}$$

Since contracted indices are dummy in each term, we can interchange $\alpha \leftrightarrow \delta$ and $\beta \leftrightarrow \gamma$ on the second term on the right-hand-side of the above equation, which yields

$$\begin{aligned}\frac{\partial \mathcal{L}_{\rm MAX}}{\partial\left(\partial_\mu A_\nu\right)} &= -\frac{1}{4}\left[g^{\alpha\delta} g^{\beta\gamma}\frac{\partial F_{\alpha\beta}}{\partial\left(\partial_\mu A_\nu\right)} F_{\delta\gamma} + g^{\delta\alpha} g^{\gamma\beta} F_{\delta\gamma}\frac{\partial F_{\alpha\beta}}{\partial\left(\partial_\mu A_\nu\right)}\right] \\ &= -\frac{1}{2} g^{\alpha\delta} g^{\beta\gamma}\frac{\partial F_{\alpha\beta}}{\partial\left(\partial_\mu A_\nu\right)} F_{\delta\gamma} \\ &= -\frac{1}{2} g^{\alpha\delta} g^{\beta\gamma}\left[\frac{\partial}{\partial\left(\partial_\mu A_\nu\right)}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)\right] F_{\delta\gamma} \\ &= -\frac{1}{2} g^{\alpha\delta} g^{\beta\gamma}\left[\left(\frac{\partial\left(\partial_\alpha A_\beta\right)}{\partial\left(\partial_\mu A_\nu\right)} - \frac{\partial\left(\partial_\beta A_\alpha\right)}{\partial\left(\partial_\mu A_\nu\right)}\right)\right] F_{\delta\gamma} \\ &= -\frac{1}{2} g^{\alpha\delta} g^{\beta\gamma}\left[\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right] F_{\delta\gamma} \\ &= -\frac{1}{2}\left[g^{\mu\delta} g^{\nu\gamma} - g^{\nu\delta} g^{\mu\gamma}\right] F_{\delta\gamma} \\ &= -\frac{1}{2}\left[F^{\mu\nu} - F^{\nu\mu}\right] \\ &= -F^{\mu\nu}\,.\end{aligned} \tag{3.60}$$

From the Euler-Lagrange equations (3.56) the free Maxwell theory yields

$$\begin{aligned}\frac{\partial \mathcal{L}_{\rm MAX}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}_{\rm MAX}}{\partial\left(\partial_\mu A_\nu\right)} &= 0 \\ \Rightarrow \partial_\mu F^{\mu\nu} &= 0\end{aligned} \tag{3.61}$$

which corresponds to the second pair of Maxwell equations, the Ampere-Gauss Law, in the absence of sources. The other pair of Maxwell equations can be obtained just by a redefinition of the fields. In fact, we have previously defined the dual tensor

$$F^{*\mu\nu} = \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{3.62}$$

from where we readily obtain the Faraday-Gauss Law

$$
\begin{aligned}
\partial_\mu F^{*\mu\nu} &= \varepsilon^{\mu\nu\alpha\beta}\partial_\mu\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) \\
&= \underbrace{\varepsilon^{\mu\nu\alpha\beta}\partial_\mu\partial_\alpha A_\beta}_{0} - \underbrace{\varepsilon^{\mu\nu\alpha\beta}\partial_\mu\partial_\beta A_\alpha}_{0} \\
&= 0
\end{aligned}
\tag{3.63}
$$

where we have used that the derivatives are symmetric under $\mu \leftrightarrow \alpha$ and $\mu \leftrightarrow \beta$ interchange while the Levi-Civita symbol is antisymmetric for such permutations.

If one introduces in out theory a source of electromagnetic field, an interaction term should be introduced in the Lagrangian as e.g.

$$
\mathcal{L}_{\text{int}} = -j^\nu A_\nu \,, \tag{3.64}
$$

with j^ν an electromagnetic current. The main difference now is that the derivative of the Lagrangian with respect to the fields is no longer zero

$$
\frac{\partial \mathcal{L}_{\text{int}}}{\partial A_\nu} = -j^\nu \tag{3.65}
$$

and applying the Euler-Lagrange equations to $\mathcal{L} = \mathcal{L}_{\text{MAX}} + \mathcal{L}_{\text{int}}$ we recover the Ampere-Gauss Law in the presence of sources, that is

$$
\partial_\mu F^{\mu\nu} = j^\nu \,, \tag{3.66}
$$

where j^μ can involve both interactions with fermions or scalars as we will see in the next subsection.

3.2.4 Continuous symmetries - Noëther's Theorem

Using the antisymmetry of the electromagnetic tensor it is trivial to see that the current density j^ν is a conserved quantity. This can be seen by differentiating (3.66) by ∂_ν on both sides

$$
\partial_\nu j^\nu = \partial_\nu \partial_\mu F^{\mu\nu} = 0 \,. \tag{3.67}
$$

This result is a consequence **Noëther Theorem** which states the following:

Noëther Theorem
The invariance of the Lagrangian $\mathcal{L}$ under a continuous symmetry implies the existence of a conserved quantity.

In field theory such continuous symmetries and corresponding conserved quantities are the following

1. **Space-time symmetries** such as Lorentz boosts and translations. These are described by the Poincaré group and lead to energy-momentum conservation, i.e. $\partial_\mu T^{\mu\nu} = 0$ with $T^{\mu\nu}$ the stress energy-momentum tensor. The conserved quantities, or charges, emerging from this law are the energy, linear momentum and total angular momentum.

2. **Internal symmetries.** These correspond to transformations in the field space and can either be global or local. The latter ones are also denoted as **gauge symmetries**. This is the principle behind the emergence of internal particle charges such as, e.g. the electric charge, hypercharge, weak isospin or colour charge. In general, the conserved quantity is the current-density j^μ and the conservation law reads $\partial_\mu j^\mu = 0$.

In this course we will briefly refer to space time transformations but will mostly focus on internal symmetries starting with the case of **global** ones.

Global internal symmetries - the case of U(1) invariance

Let us then consider the free Klein-Gordon Lagrangian

$$\mathcal{L}_{\mathrm{KG}} = g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - m^2\phi^*\phi\,. \tag{3.68}$$

It is straightforward to verify that this Lagrangian is invariant under a **global phase transformation** of the form

$$\begin{aligned} \phi &\to \phi' = e^{-iq_1\alpha}\phi \\ \phi^* &\to \phi'^* = e^{-iq_2\alpha}\phi^* \end{aligned} \tag{3.69}$$

with α a **global** parameter, i.e. equal at every point in space, and $q_{1,2}$ the global charges of ϕ and ϕ^* respectively, provided that

$$\mathcal{L}_{\mathrm{KG}}\left(\phi, \partial_\mu\phi\right) = \mathcal{L}_{\mathrm{KG}}\left(\phi', \partial_\mu\phi'\right)\,. \tag{3.70}$$

This invariance condition fixes the global charges of each of the fields as follows:

$$\begin{aligned} \mathcal{L}_{\mathrm{KG}}\left(\phi', \partial_\mu\phi'\right) &= g^{\mu\nu}\partial_\mu\phi'^*\partial_\nu\phi' - m^2\phi'^*\phi' \\ &= e^{-i(q_1+q_2)\alpha}\left(g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - m^2\phi^*\phi\right) \\ &= \mathcal{L}_{\mathrm{KG}}\left(\phi, \partial_\mu\phi\right) \Rightarrow q_1 = -q_2 \end{aligned} \tag{3.71}$$

that is, the invariance implies that the global charges q_1 and q_2 are equal and of opposite sign if such a phase symmetry exists in the theory, consistently with what we expect for particle and antiparticle. The actual field phase transformation reads

$$\begin{aligned} \phi &\to \phi' = e^{-iq\alpha}\phi \\ \phi^* &\to \phi'^* = e^{iq\alpha}\phi^*\,, \end{aligned} \tag{3.72}$$

which is actually a **unitary transformation of dimension one**, that is, only one global parameter, α, such that defining $U = e^{-iq\alpha}$ we have $U^*U = U^{-1}U = 1$. This class of transformations are actually denoted as U(1) symmetries and play a crucial role in Particle Physics.

Note that introducing the quartic interaction

$$V_{\mathrm{int}}\left(\phi, \phi^*\right) = \frac{\lambda}{4}\left(\phi^*\phi\right)^2 \tag{3.73}$$

the U(1) global symmetry is preserved

$$V_{\text{int}}\left(\phi', \phi'^*\right) = \underbrace{e^{-2i\alpha(q-q)}}_{1} \frac{\lambda}{4} \left(\phi^*\phi\right)^2 = V_{\text{int}}\left(\phi, \phi^*\right) \,. \tag{3.74}$$

However, if one was to introduce a cubic self interactions of the form

$$V_{\text{int}}\left(\phi, \phi^*\right) = a\,\phi^*\phi^2 + \text{c.c.} \tag{3.75}$$

with a a mass dimension one real parameter, we see that

$$V_{\text{int}}\left(\phi', \phi'^*\right) = \underbrace{e^{-2iq\alpha}}_{\neq 1} a\,\phi^*\phi^2 + \text{c.c.} \neq V_{\text{int}}\left(\phi, \phi^*\right) \tag{3.76}$$

thus the theory would not be invariant under the global U(1) symmetry. For the case of the free Dirac theory

$$\mathcal{L}_{\text{D}} = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi \tag{3.77}$$

it is immediate to see, just by inspection of $\mathcal{L}_{\text{D}}$, that the global U(1) transformations

$$\begin{aligned} \psi \to \psi' &= e^{-iq\alpha}\psi\,, \\ \overline{\psi} \to \overline{\psi'} &= e^{iq\alpha}\overline{\psi}\,, \end{aligned} \tag{3.78}$$

leave $\mathcal{L}_{\text{D}}$ invariant.

We have now understood that both the Dirac and the Klein-Gordon theories contain a continuous global U(1) symmetry. Therefore, as a result of Noëther's theorem there should be a conserved current associated to it. A U(1) infinitesimal field transformation can be represented as

$$\delta\Psi^i = -iq_i\alpha\Psi^i\,. \tag{3.79}$$

and for the field derivatives

$$\delta\left(\partial_\mu\Psi^i\right) = \partial_\mu\delta\Psi^i = -iq_i\alpha\partial_\mu\Psi^i\,. \tag{3.80}$$

Since

$$\mathcal{L}\left(\Psi'^i, \partial_\mu\Psi'^i\right) - \mathcal{L}\left(\Psi^i, \partial_\mu\Psi^i\right) = 0 \Rightarrow \delta\mathcal{L}\left(\Psi^i, \partial_\mu\Psi^i\right) = 0 \tag{3.81}$$

then

$$\sum_i \left[\frac{\partial\mathcal{L}}{\partial\Psi^i}\delta\Psi^i + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Psi^i\right)}\delta\left(\partial_\mu\Psi^i\right)\right] = 0\,. \tag{3.82}$$

It follows from the Euler-Lagrange equations that

$$\frac{\partial\mathcal{L}}{\partial\Psi^i} = \partial_\mu\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Psi^i\right)} \tag{3.83}$$

thus eq. (3.82) becomes

$$\begin{aligned}
\sum_i \left[\partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Psi^i\right)} \delta \Psi^i + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Psi^i\right)} \partial_\mu \delta \Psi^i\right] &= 0 \\
\Rightarrow \partial_\mu \left(\sum_i \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Psi^i\right)} \delta \Psi^i\right) &= 0 \\
\Rightarrow \partial_\mu \left(\sum_i \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Psi^i\right)} \left(-i q_i \alpha \Psi^i\right)\right) &= 0
\end{aligned} \tag{3.84}$$

Since α is a universal constant we can eliminate it and the conservation law takes the form

$$\partial_\mu \left(-i \sum_i q_i \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Psi^i\right)} \Psi^i\right) = 0 \tag{3.85}$$

where the Noëther current density is given by

$$j^\mu = -i \sum_i q_i \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Psi^i\right)} \Psi^i \,. \tag{3.86}$$

Assuming that all fields vanish at spacial infinity, the total charge

$$Q = \int d^3x\, j^0 \Rightarrow \frac{dQ}{dt} = \int d^3x\, \partial_0 j^0 - \underbrace{\int d^3x \partial_i j^i}_{=0} = \int d^3 \partial_\mu j^\mu = 0 \tag{3.87}$$

is a constant in time, or in other words, a conserved quantity. Note that $\int d^3x \partial_i j^i$ is a total divergence, thus a vanishing surface integral on the space boundary. If we specialize eq. (3.86) to the case of the Klein-Gordon theory with a global U(1) symmetry the four-current reads

$$\begin{aligned}
j^\mu_{\mathrm{KG}} &= -i \left(q \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \phi + (-q) \phi^* \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^*\right)}\right) \\
&= -i\, q\, g^{\mu\nu} \left(\left(\partial_\nu \phi\right)^* \phi - \phi^* \partial_\nu \phi\right) \\
&= iq \left(\phi^* \partial^\mu \phi - \left(\partial^\mu \phi\right)^* \phi\right)
\end{aligned} \tag{3.88}$$

while for the U(1) Dirac theory we have

$$\begin{aligned}
j^\mu_{\mathrm{D}} &= -i \left(q \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi\right)} \psi + (-q) \overline{\psi} \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \overline{\psi}\right)}\right) \\
&= q \overline{\psi} \gamma^\mu \psi \,.
\end{aligned} \tag{3.89}$$

U(1) **gauge symmetry: local electromagnetic currents**

Electromagnetic currents j^μ_{EM} emerge from interaction of locally charged particles with the electromagnetic field. These are the currents that show up in eqs. (3.64), (3.65) and (3.66) as we will see.

Let us start with the free Klein-Gordon theory, which we rewrite here:

$$\mathcal{L}_{\mathrm{KG}}\left(\phi, \partial_\mu \phi\right) = g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi - m^2 \phi^* \phi \,. \tag{3.90}$$

Now, replace the global parameter α in eq. (3.72) by a local one $\alpha(x)$ dependent on the space-time coordinates. The local, or gauge, transformations become

$$\begin{aligned}\phi &\to \phi' = e^{-iq\alpha(x)}\phi \\ \phi^* &\to \phi'^* = e^{iq\alpha(x)}\phi^* .\end{aligned} \tag{3.91}$$

If we apply these transformations on the Klein-Gordon Lagrangian we need to take into account that the derivative also acts on the argument of the exponential, that is

$$\begin{aligned}\partial_\nu \phi' &= -iq\partial_\nu \alpha(x)\, e^{-iq\alpha(x)}\phi + e^{-iq\alpha(x)}\partial_\nu\phi \\ \partial_\mu \phi'^* &= iq\partial_\mu \alpha(x)\, e^{iq\alpha(x)}\phi^* + e^{iq\alpha(x)}\partial_\mu\phi^*\end{aligned} \tag{3.92}$$

and then

$$\begin{aligned}\mathcal{L}_{\text{KG}}\left(\phi', \partial_\mu\phi'\right) = \mathcal{L}_{\text{KG}}\left(\phi, \partial_\mu\phi\right) &+ g^{\mu\nu}q^2\partial_\mu\alpha(x)\,\partial_\nu\alpha(x)\,\phi^*\phi \\ &+ iqg^{\mu\nu}\partial_\mu\alpha(x)\,\phi^*\partial_\nu\phi - iqg^{\mu\nu}\partial_\mu\phi^*\partial_\nu\alpha(x)\,\phi ,\end{aligned} \tag{3.93}$$

from where we see that the free Klein-Gordon theory is by no means U(1) gauge invariant. To cure this problem we need to redefine the kinetic terms. Recalling that Maxwell's theory is invariant under local gauge transformations of the form

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\alpha(x) \ , \tag{3.94}$$

it is possible to cancel the extra terms in (3.93) if we redefine the derivatives as

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu , \tag{3.95}$$

denoted as **covariant derivative**. Note that a local gauge transformation is such that at every point in space we are performing a different rotation on the field space. We can think of it as a different field-axis orientation for each space-time point. Therefore, the standard derivative ∂_μ is not only measuring the field change but also the basis change. A proper covariant derivative ensures that we are always comparing the change in fields with respect to a fixed basis. In very general terms if at x we have a field $\psi(x)$, then at $x + dx$ the field becomes $\psi(x + dx) = \psi(x) + d\psi$. However, since the axis are changing, the variation $d\psi = \psi(x + dx) - \psi(x)$ is measuring fields with respect to different axis. To account for the basis change we need to *parallel transport* the vector ψ from the coordinate system at x to that at $x + dx$. In other words, the components of ψ become identical in each **local** coordinate system, which results in a different vector at $x + dx$ denoted as $\psi + \delta\psi$. Therefore, the variation $d\psi = \psi(x + dx) - \psi(x)$ should be changed to:

$$\begin{aligned}D\psi &= \psi(x + dx) - (\psi + \delta\psi) \\ &= (\psi + d\psi) - (\psi + \delta\psi) \\ &= d\psi - \delta\psi\end{aligned} \tag{3.96}$$

Assuming that $\delta\psi$ is proportional to ψ and to dx^μ we can write

$$\delta\psi = ig\boldsymbol{T}^a A^a_\mu dx^\mu \psi , \tag{3.97}$$

where $\boldsymbol{T}^a$ are generators of a continuous transformations group and A^a_μ fields carrying the information on how internal-space axis vary from point to point. Equation (3.96) then becomes

$$\begin{aligned} \frac{D\psi}{dx^\mu} &= \frac{d\psi}{dx^\mu} - ig\boldsymbol{T}^a A^a_\mu \psi \\ D_\mu \psi &= \partial_\mu \psi - ig\boldsymbol{T}^a A^a_\mu \psi \,. \end{aligned} \tag{3.98}$$

In essence, the A^a_μ play the role of affine connections $\Gamma^\alpha_{\beta\gamma}$ in general relativity.

It follows from the field transformations (3.91) and (3.94) that

$$\begin{aligned} (D_\nu \phi)' &= \left(\partial_\nu + iqA'_\nu\right) e^{-iq\alpha(x)}\phi \\ &= \left[\partial_\nu + iqA_\nu + iq\partial_\nu \alpha\left(x\right)\right] e^{-iq\alpha(x)}\phi \\ &= e^{-iq\alpha(x)}\left[-iq\partial_\mu \alpha\left(x\right) + \partial_\nu + iqA_\nu + iq\partial_\nu \alpha\left(x\right)\right]\phi \\ &= e^{-iq\alpha(x)}\left(\partial_\nu + iqA_\nu\right)\phi \\ &= e^{-iq\alpha(x)}\left(D_\nu \phi\right) \\ (D_\nu \phi)'^* &= e^{iq\alpha(x)}\left(D_\nu \phi\right)^* \end{aligned} \tag{3.99}$$

which shows that the $D_\mu \phi$ and $(D_\mu \phi)^*$ transform as the fields itself. This means that the **electromagnetic interacting** Klein-Gordon theory

$$\mathcal{L}_{\text{KG}} = g^{\mu\nu}\left(D_\mu \phi\right)^* \left(D_\nu \phi\right) - m^2 \phi^* \phi \tag{3.100}$$

is gauge invariant by making the replacements $\partial_\mu \to D_\mu$. Note that q can now be interpreted as the electric charge of the scalar field ϕ.

The Klein-Gordon electromagnetic current can be calculated from the modified kinetic terms that read

$$\begin{aligned} \mathcal{L}_{\text{kin}} &= g^{\mu\nu}\left(D_\mu \phi\right)^* D_\nu \phi \\ &= g^{\mu\nu}\left(\partial_\mu \phi^* \partial_\nu \phi + iq\partial_\mu \phi^* A_\nu \phi - iqA_\mu \phi^* \partial_\nu \phi + q^2 A_\mu A_\nu \phi^* \phi\right) \,. \end{aligned} \tag{3.101}$$

Then

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} &= g^{\mu\nu}\partial_\nu \phi^* - iqg^{\mu\nu}A_\nu \phi^* = g^{\mu\nu}\left(\partial_\nu - iqA_\nu\right)\phi^* = g^{\mu\nu}\left(D_\nu \phi\right)^* \\ \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^*\right)} &= g^{\mu\nu}\partial_\nu \phi + iqg^{\mu\nu}A_\nu \phi = g^{\mu\nu}\left(\partial_\nu + iqA_\nu\right)\phi = g^{\mu\nu}\left(D_\nu \phi\right) \end{aligned} \tag{3.102}$$

and using eq. (3.86) the electromagnetic Klein-Gordon current becomes

$$j^\mu_{\text{KG}} = i\, q\, g^{\mu\nu}\left[\phi^* D_\nu \phi - \left(D_\nu \phi\right)^* \phi\right] \,. \tag{3.103}$$

Interestingly, the derivatives of the kinetic terms (3.101) with respect to the electromagnetic field vector A_ν are

$$\begin{aligned}
\frac{\partial \mathcal{L}_{\rm kin}}{\partial A_\nu} &= i\,q\,g^{\mu\nu}\left(\partial_\mu\phi\right)^*\phi - iqg^{\nu\mu}\phi^*\left(\partial_\mu\phi\right) + q^2 g^{\mu\nu}A_\mu\phi^*\phi + q^2 g^{\nu\mu}A_\mu\phi^*\phi \\
&= iq\left[g^{\mu\nu}\left(\partial_\mu\phi\right)^*\phi - g^{\nu\mu}\phi^*\left(\partial_\mu\phi\right) - iqg^{\mu\nu}\left(A_\mu\phi\right)^*\phi - iqg^{\nu\mu}\phi^*\left(A_\mu\phi\right)\right] \\
&= iq\left[g^{\mu\nu}\left(\partial_\mu\phi\right)^*\phi - g^{\nu\mu}\phi^*\left(\partial_\mu\phi\right) + g^{\mu\nu}\left(iqA_\mu\phi\right)^*\phi - g^{\nu\mu}\phi^*\left(iqA_\mu\phi\right)\right] \\
&= iq\left[g^{\mu\nu}\left(\partial_\mu\phi + iqA_\mu\phi\right)^*\phi - g^{\nu\mu}\phi^*\left(\partial_\mu\phi + iqA_\mu\phi\right)\right] \\
&= i\,q\,g^{\mu\nu}\left[\left(D_\mu\phi\right)^*\phi - \phi^* D_\mu\phi\right] \\
&= -j^\nu_{\rm KG}\,.
\end{aligned} \tag{3.104}$$

This type of theory is typically denoted as **scalar electrodynamics** where the sources of electromagnetic field are the charged ϕ and ϕ^* scalars. In fact, the interaction terms that we mentioned in eq. (3.65) are the kinetic terms (3.101) and Ampere-Gauss law (3.66) can be recast as

$$\partial_\mu F^{\mu\nu} = j^\nu_{\rm KG}\,. \tag{3.105}$$

For the Dirac theory we can also promote the global transformations (3.78) to local ones,

$$\begin{aligned}
\psi \to \psi' &= e^{-iq\alpha(x)}\psi\,, \\
\overline{\psi} \to \overline{\psi'} &= e^{iq\alpha(x)}\overline{\psi}\,,
\end{aligned} \tag{3.106}$$

if we the covariant derivative is added to the Lagrangian

$$\mathcal{L}_{\rm D} = \overline{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi = \overline{\psi}\left(i\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu - m\right)\psi\,. \tag{3.107}$$

The Dirac electromagnetic current reads

$$\begin{cases}
\dfrac{\partial \mathcal{L}}{\partial\left(\partial_\mu\psi\right)} = q\overline{\psi}\gamma^\mu \\
\dfrac{\partial \mathcal{L}}{\partial\left(\partial_\mu\overline{\psi}\right)} = 0
\end{cases} \quad \Rightarrow j^\mu_{\rm D} = q\overline{\psi}\gamma^\mu\psi \tag{3.108}$$

which, unlike charged scalars, is identical the previous U(1)-global Dirac theory. We can also recast the Dirac Lagrangian as

$$\mathcal{L}_{\rm D} = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - j^\mu_{\rm D}A_\mu\,. \tag{3.109}$$

and if our theory consists solely on a charged fermion then the second pair of Maxwell equations takes the form

$$\partial_\mu F^{\mu\nu} = j^\nu_{\rm D}\,. \tag{3.110}$$

Finally, if we consider a $\mathrm{U}(1)_{\rm EM}$ **gauge theory** with a scalar and a fermion both coupled to the electromagnetic field, the Lagrangian containing **all allowed interactions** (including self scalar terms) reads

$$\mathcal{L} = g^{\mu\nu}\left(D_\mu\phi\right)^*\left(D_\nu\phi\right) - V\left(\phi^*\phi\right) + \overline{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{3.111}$$

with

$$V\left(\phi^{*}\phi\right)=m^{2}\phi^{*}\phi+\frac{\lambda}{4}\left(\phi^{*}\phi\right)^{2}\ . \tag{3.112}$$

Note that a mass term for the photon

$$m_{A}^{2}A_{\mu}A^{\mu} \tag{3.113}$$

is **forbidden by gauge invariance** from where we conclude that **gauge symmetries protect gauge bosons from developing mass terms**.

A brief glance at Feynman diagrams and Feynman rules

Typically, particle interactions can be perturbatively treated provided that the couplings in the theory are not large. This is the case of the SM at high energies where the strength of all three forces is indeed small enough. Particle interactions will then emerge at various orders of the expansion of the generating functional

$$Z=\exp\left(i\int d^{4}x\,\mathcal{L}_{\rm int}\right) \tag{3.114}$$

Note, however, that at low energies the strong interaction coupling becomes too large and a pertrubative analysis is no longer valid.

Considering again the $\mathrm{U}(1)_{\mathrm{EM}}$ Lagrangian (3.111) we can separate the interaction terms from the free-theory ones as follows

$$\begin{aligned}\mathcal{L}=&g^{\mu\nu}\left(\partial_{\mu}\phi\right)^{*}\left(\partial_{\nu}\phi\right)-m^{2}\phi^{*}\phi+\overline{\psi}\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi+\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\\&+i\,q\,g^{\mu\nu}\left[\left(\partial_{\mu}\phi\right)^{*}A_{\nu}\phi-A_{\mu}\phi^{*}\partial_{\nu}\phi\right]+q^{2}g^{\mu\nu}A_{\mu}A_{\nu}\phi^{*}\phi-\frac{\lambda}{4}\left(\phi^{*}\phi\right)^{2}-q\overline{\psi}\gamma^{\mu}\psi A_{\mu}\\=&\mathcal{L}_{\rm KG}^{\rm free}+\mathcal{L}_{\rm D}^{\rm free}+\mathcal{L}_{\rm MAX}+\mathcal{L}_{\rm int}\,,\end{aligned} \tag{3.115}$$

where the second line corresponds to the interaction terms $\mathcal{L}_{\rm int}$. At first order in perturbation theory, which is commonly denoted **tree-level**, Feynman diagrams can be readily drawn from the Lagrangian where scalar fields are identified with dashed lines, fermions with continuous lines and photons with wavy lines. The corresponding Feynaman rules are the coefficients in front of the fields multiplied by symmetry factors (when there are identical fields) and by an i that comes from the expansion of the generating functional Z. The Feynman rules for three last terms in the second line of eq. (3.115) are direct to read, however, the first terms contains derivative interactions which means that the Feynman rule will depend on the particle's momentum. To see this let us consider the solution of the Klein-Gordon equation

$$\phi=Ne^{-ip\cdot x}\Rightarrow\partial_{\mu}\phi=-ip_{\mu}\phi\,. \tag{3.116}$$

Then

$$\begin{aligned}&i\,q\,g^{\mu\nu}\left[\left(\partial_{\mu}\phi\right)^{*}A_{\nu}\phi-A_{\nu}\phi^{*}\partial_{\mu}\phi\right]\\=&i\,q\,g^{\mu\nu}A_{\nu}\left[ip_{\mu}\phi^{*}\phi+ip'_{\mu}\phi^{*}\phi\right]\\=&i\,q\,\left[ip_{\mu}+ip'_{\mu}\right]A^{\mu}\phi^{*}\phi\\=&-q\left(p+p'\right)_{\mu}A^{\mu}\phi^{*}\phi\,.\end{aligned} \tag{3.117}$$

The Feynman diagrams and respective Feynamn rules are shown in Fig. 5, where the factor of 2 in the bottom-left diagram is a symmetry factor accounting for $A_\mu \leftrightarrow A_\nu$ interchange, whereas the factor of four in the bottom-right diagram results from the fact that both ϕ^* and ϕ appear twice, thus providing a symmetry factor of $2 \times 2 = 4$.

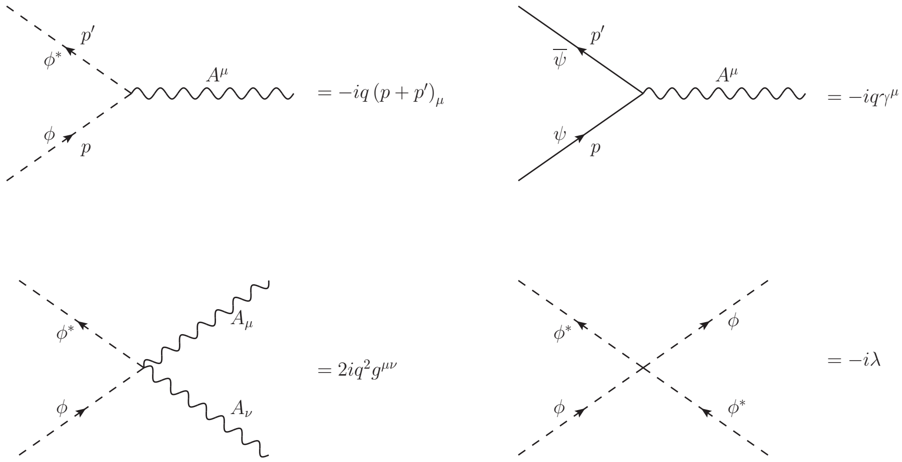


Figure 5: *Feynman diagrams and Feynman rules at tree-level for a* $\mathrm{U}(1)_{\mathrm{EM}}$*-gauge theory with one fermion and one scalar.*

Space-time transformations

We focus now on the case of Lorentz transformations, boosts and rotations, which are represented as

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \,, \tag{3.118}$$

as well as constant space-time translations

$$x'^\mu = x^\mu + a^\mu \,. \tag{3.119}$$

To sum this up, Lorentz transformations and space-time translations read

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{3.120}$$

and form a group denoted as **Poincaré group**. A thorough studty of the Poincaré group is beyond the scope of this course. For further details see e.g. the link in footnote[5].

Pure translations

The transformation on the fields due to a pure translation are defined as

$$\Psi'\left(x'\right) = S\Psi\left(x\right) \,, \tag{3.121}$$

where S can be written as an exponential

$$S = e^s = 1 + s + \mathcal{O}\left(s^2\right) \,. \tag{3.122}$$

[5] http://cftp.ist.utl.pt/~gernot.eichmann/2014-hadron-physics/hadron-app-2.pdf

Using the expansion above, the transformation on the field becomes

$$
\begin{aligned}
\Psi'\left(x'\right) &\simeq \Psi\left(x\right) + s\Psi\left(x\right) \\
&= \underbrace{\Psi\left(x'\right) - \Psi\left(x'\right)}_{=0} + \Psi\left(x\right) + s\Psi\left(x\right) \\
&= \Psi\left(x'\right) - \left[\Psi\left(x'\right) - \Psi\left(x\right)\right] + s\Psi\left(x\right) \\
&= \Psi\left(x'\right) - \delta x^{\nu}\partial_{\nu}\Psi\left(x\right) + s\Psi\left(x\right) \\
\Rightarrow \Psi'\left(x'\right) - \Psi\left(x'\right) &\simeq -\delta x^{\nu}\partial_{\nu}\Psi\left(x\right) + s\Psi\left(x\right) \\
\Rightarrow \delta\Psi &\simeq -\delta x^{\nu}\partial_{\nu}\Psi\left(x\right) + s\Psi\left(x\right)
\end{aligned}
\tag{3.123}
$$

To lowest order, and noting that $\delta x^{\nu} = a^{\nu}$, we can write the field transformation as

$$
\delta\Psi = -a^{\nu}\partial_{\nu}\Psi\,. \tag{3.124}
$$

At variance to what we have verified for the global internal symmetry case we also have a variation in the coordinates. Therefore, eq. (3.82) should recast to

$$
\begin{aligned}
\sum_i \left[\frac{\partial\mathcal{L}}{\partial\Psi^i}\delta\Psi^i + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Psi^i\right)}\delta\left(\partial_\mu\Psi^i\right) + \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu\right] &= 0 \\
\Rightarrow \partial_\mu\left(\sum_i \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Psi^i\right)}\delta\Psi^i + \mathcal{L}\delta x^\mu\right) - \mathcal{L}\partial_\mu\delta x^\mu &= 0 \\
\Rightarrow \partial_\mu\left(\sum_i \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Psi^i\right)}\delta\Psi^i + \mathcal{L}\delta x^\mu\right) &= 0
\end{aligned}
\tag{3.125}
$$

since $\delta x^{\mu} = a^{\mu}$ is a constant. Using (3.124) the above equation can be rewritten as

$$
\begin{aligned}
\partial_\mu\left(\sum_i \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Psi^i\right)}\left(-a^\alpha\partial_\alpha\Psi^i\right) + \mathcal{L}a^\mu\right) &= 0 \\
\Rightarrow \partial_\mu\left(\sum_i \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Psi^i\right)}a^\alpha\partial_\alpha\Psi^i - \mathcal{L}a^\alpha g^{\mu}{}_{\alpha}\right) &= 0 \\
\Rightarrow \partial_\mu\left(\sum_i \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Psi^i\right)}\partial_\alpha\Psi^i - \mathcal{L}g^{\mu}{}_{\alpha}\right) &= 0 \\
\Rightarrow \partial_\mu\left(\sum_i \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Psi^i\right)}\partial^\nu g_{\nu\alpha}\Psi^i - \mathcal{L}g^{\mu\nu}g_{\nu\alpha}\right) &= 0 \\
\Rightarrow \partial_\mu\left(\sum_i \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Psi^i\right)}\partial^\nu\Psi^i - \mathcal{L}g^{\mu\nu}\right) &= 0
\end{aligned}
\tag{3.126}
$$

where the Noëther conservation law reads

$$
\partial_\mu T^{\mu\nu} = 0 \tag{3.127}
$$

with

$$
T^{\mu\nu} = \sum_i \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Psi^i\right)}\partial^\nu\Psi^i - \mathcal{L}g^{\mu\nu} \tag{3.128}
$$

the stress energy-momentum tensor (or deformation tensor). The **conserved charge** associated to **time-translations** is the total energy and reads

$$
\begin{aligned}
H &= \int d^3x\, T^{00} \\
&= \int d^3x \sum_i \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \Psi^i\right)} \partial^0 \Psi^i - \mathcal{L} \\
&= \int d^3x \sum_i \Pi^i \dot{\Psi}^i - \mathcal{L} \\
&= \int d^3x\, \mathcal{H}
\end{aligned}
\tag{3.129}
$$

with $\mathcal{H} = T^{00}$ the Hamiltonian density, whereas the conserved charged associated to **space-translations** is the physical **linear momentum**

$$
\begin{aligned}
P^k &= \int d^3x\, T^{0k} \\
&= \int d^3x \sum_i \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \Psi^i\right)} \partial^k \Psi^i - \mathcal{L} g^{0k} \\
&= -i \int d^3x \sum_i \Pi^i \left(p^k \Psi^i\right) \\
&= \int d^3x\, \mathcal{P}^k
\end{aligned}
\tag{3.130}
$$

with $\mathcal{P}^k = T^{0k}$ the linear momentum density.

Pure Lorentz transformations

Lorentz transformations can be written in exponential form as

$$
\Lambda = e^{\omega} = \mathbb{1} + \omega + \mathcal{O}\left(\omega^2\right) , \tag{3.131}
$$

which, in components reads

$$
x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} = \left[\delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} + \mathcal{O}\left(\omega^{\mu}{}_{\alpha}\omega^{\alpha}{}_{\nu}\right)\right] x^{\nu} = x^{\mu} + \delta x^{\mu} \tag{3.132}
$$

with $\omega^{\mu}{}_{\nu}$ infinitesimal. Recalling the invariance of the metric as previously seen in (1.36)

$$
g_{\alpha\beta} = g_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} \,. \tag{3.133}
$$

and using Lorentz transformations expanded to lowest order, we observe that $\omega^{\mu}{}_{\nu}$ is anti-symmetric:

$$
\begin{aligned}
g_{\alpha\beta} &= g_{\mu\nu}\left(\delta^{\mu}{}_{\alpha} + \omega^{\mu}{}_{\alpha}\right)\left(\delta^{\nu}{}_{\beta} + \omega^{\nu}{}_{\beta}\right) \\
&= \left(g_{\alpha\nu} + \omega_{\nu\alpha}\right)\left(\delta^{\nu}{}_{\beta} + \omega^{\nu}{}_{\beta}\right) \\
&= g_{\alpha\beta} + \omega_{\beta\alpha} + \omega_{\alpha\beta} + \underbrace{\omega_{\nu\alpha}\omega^{\nu}{}_{\beta}}_{\mathcal{O}(\omega^2)\sim 0} \\
&= g_{\alpha\beta} + \underbrace{\omega_{\beta\alpha} + \omega_{\alpha\beta}}_{=0} \\
&\Rightarrow \omega_{\alpha\beta} = -\omega_{\beta\alpha}.
\end{aligned}
\tag{3.134}
$$

If the transformations are purely Lorentz, eq. (3.123) then becomes

$$\delta\Psi = -\omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\Psi\left(x\right) + s\Psi\left(x\right)\,, \tag{3.135}$$

with s typically given by

$$s = -\frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}\,, \tag{3.136}$$

which follows from the fact that spinors transform as

$$\psi\left(x\right) \to \psi'\left(x'\right) = \psi\left(x\right) - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\psi\left(x\right)\,, \tag{3.137}$$

where

$$S^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^{\mu},\gamma^{\nu}\right]\,. \tag{3.138}$$

The expression for $S^{\mu\nu}$ follows from the covariance of Dirac's equation and represents a spin angular momentum matrix. However, a thorough demonstration is beyond the scope of this course.

With field transformation rule (3.135), all we have to do is to replace $a_{\nu} \to \omega_{\nu\alpha}x^{\alpha}$ in (3.126) and add the spin term which yields

$$\partial_{\mu}\left(T^{\mu\nu}\omega_{\nu\alpha}x^{\alpha} - \frac{i}{2}\sum_{i}\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\Psi^{i}\right)}\omega_{\nu\alpha}S^{\nu\alpha}\Psi^{i}\right) = 0\,. \tag{3.139}$$

Note that due to the **antisymmetry** of $\omega_{\nu\alpha}$ the symmetry of $T^{\mu\nu}x^{\alpha}$ under exchange of $\alpha \leftrightarrow \mu$ one has

$$\begin{aligned} T^{\mu\nu}x^{\alpha}\omega_{\nu\alpha} + T^{\mu\alpha}x^{\nu}\omega_{\alpha\nu} &= 0 \\ \Rightarrow \left(T^{\mu\nu}x^{\alpha} - T^{\mu\alpha}x^{\nu}\right)\omega_{\nu\alpha} &= 0 \end{aligned} \tag{3.140}$$

therefore

$$\begin{aligned} \partial_{\mu}\left(T^{\mu\nu}x^{\alpha} - T^{\mu\alpha}x^{\nu} - \frac{i}{2}\sum_{i}\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\Psi^{i}\right)}S^{\nu\alpha}\Psi^{i}\right)\omega_{\nu\alpha} &= 0 \\ \Rightarrow \partial_{\mu}\left(T^{\mu\nu}x^{\alpha} - T^{\mu\alpha}x^{\nu} - \frac{i}{2}\sum_{i}\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\Psi^{i}\right)}S^{\nu\alpha}\Psi^{i}\right) &= 0\,. \end{aligned} \tag{3.141}$$

from where we find the conserved Noëther current

$$\partial_{\mu}\mathcal{M}^{\mu\nu\alpha} = 0 \tag{3.142}$$

with

$$\mathcal{M}^{\mu\alpha\beta} = \ell^{\mu\alpha\beta} + s^{\mu\alpha\beta} \tag{3.143}$$

and where

$$\ell^{\mu\alpha\beta} = T^{\mu\alpha}x^{\beta} - T^{\mu\beta}x^{\alpha}\,, \qquad s^{\mu\alpha\beta} = -\frac{i}{2}\sum_{i}\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\Psi^{i}\right)}S^{\alpha\beta}\Psi^{i}\,, \tag{3.144}$$

are the orbital angular momentum and spin currents respectively. The **conserved charge** is the total angular momentum and can be written as

$$M^{\alpha\beta} = \int d^3x\, \mathcal{M}^{0\alpha\beta} \tag{3.145}$$

where we have

$$\begin{aligned}
M^{12} &= \int d^3x\, \left(\ell^{012} + s^{012}\right) \\
&= \int d^3x\, \left(T^{01}x^2 - T^{02}x^1 - \frac{i}{2}\sum_i \Pi^i \left(S^{12}\Psi^i\right)\right) \\
&= \int d^3x\, \left(\mathcal{P}_x Y - \mathcal{P}_y X + \mathscr{S}_z\right) \\
&= \int d^3x\, \left(\mathscr{L}_z + \mathscr{S}_z\right) \\
&= L_z + S_z \\
&= J_z
\end{aligned} \tag{3.146}$$

with the **angular momentum and spin density charges** along the z axis denoted by $\mathscr{L}_z$ and $\mathscr{S}_z$ respectively. Similarly we have that $M^{23} = J_x$ and $M^{31} = J_y$.

4 Symmetry breaking and the Higgs mechanism

4.1 Spontaneous symmetry breaking

In the previous section we have introduced the notion of continuous symmetry and discussed its physical implications such as, e.g., the emergence of both global and local charges. This is particularly relevant as in the Standard Model the electric charges of the particles are a manifestation of a continuous local U(1) symmetry, or gauge symmetry, and reflect the interaction strength of of such particles with the electromagnetic field. On the other hand we have also shown that mass terms for gauge bosons are forbidden by gauge invariance, however, nature tells us that there are at least three massive ones, the $W^\pm$ and Z bosons. This means that the symmetries that we observe to be preserved in the current state of the Universe, such as quantum electrodynamics $\mathrm{U}(1)_{\mathrm{EM}}$, must be relics of a larger symmetry broken at an earlier stage of the Universe evolution.

4.1.1 Pure scalar theory

To understand the mechanism of symmetry breaking and its implications we start our analysis considering a global U(1) symmetry with Lagrangian

$$\mathcal{L} = \left[\partial^\mu \Phi\right]^* \partial_\mu \Phi - V(\Phi) \tag{4.1}$$

where Φ is a complex scalar field and $V(\Phi)$ the potential

$$V(\Phi) = \frac{\lambda}{4}\left(\Phi^*\Phi\right)^2 + \mu^2 \Phi^*\Phi\,. \tag{4.2}$$

Here λ, μ^2 are both real parameters i.e., μ^2 can also be negative. The square is just a conventional notation to indicate that the parameter has dimensions of squared mass.

It is trivial to verify that $\mathcal{L}$ is invariant under the following global U(1) transformation:

$$\Phi \to e^{i\alpha}\Phi \tag{4.3}$$

provided that $\mathcal{L}(\Phi) = \mathcal{L}\left(e^{i\alpha}\Phi\right)$. Since Φ is a complex field we can expand it in terms of its real and imaginary components

$$\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \tag{4.4}$$

and study the shape of the potential for all possible values of λ and μ^2. There are four distinct possibilities

a) $\lambda > 0$ and $\mu^2 > 0$,

b) $\lambda > 0$ and $\mu^2 < 0$,

c) $\lambda < 0$ and $\mu^2 > 0$,

d) $\lambda < 0$ and $\mu^2 < 0$,

which result in four distinct shapes as shown in Fig. 6. Note that irrespective of the $\left(\lambda, \mu^2\right)$ configuration the potential is always invariant under a phase rotation in the (ϕ_1, ϕ_2)-plane. This is in fact a reflection of the U(1) symmetry present in the theory.

The stationarity equations for this potential can be calculated taking the first derivatives with respect to ϕ_1 and ϕ_2:

$$\begin{aligned} \frac{\partial V}{\partial \phi_1} &= \mu^2\phi_1 + \frac{1}{4}\lambda\phi_1^3 + \frac{1}{4}\lambda\phi_1\phi_2^2 = 0 \\ \frac{\partial V}{\partial \phi_2} &= \mu^2\phi_2 + \frac{1}{4}\lambda\phi_2^3 + \frac{1}{4}\lambda\phi_2\phi_2^2 = 0 \end{aligned} \tag{4.5}$$

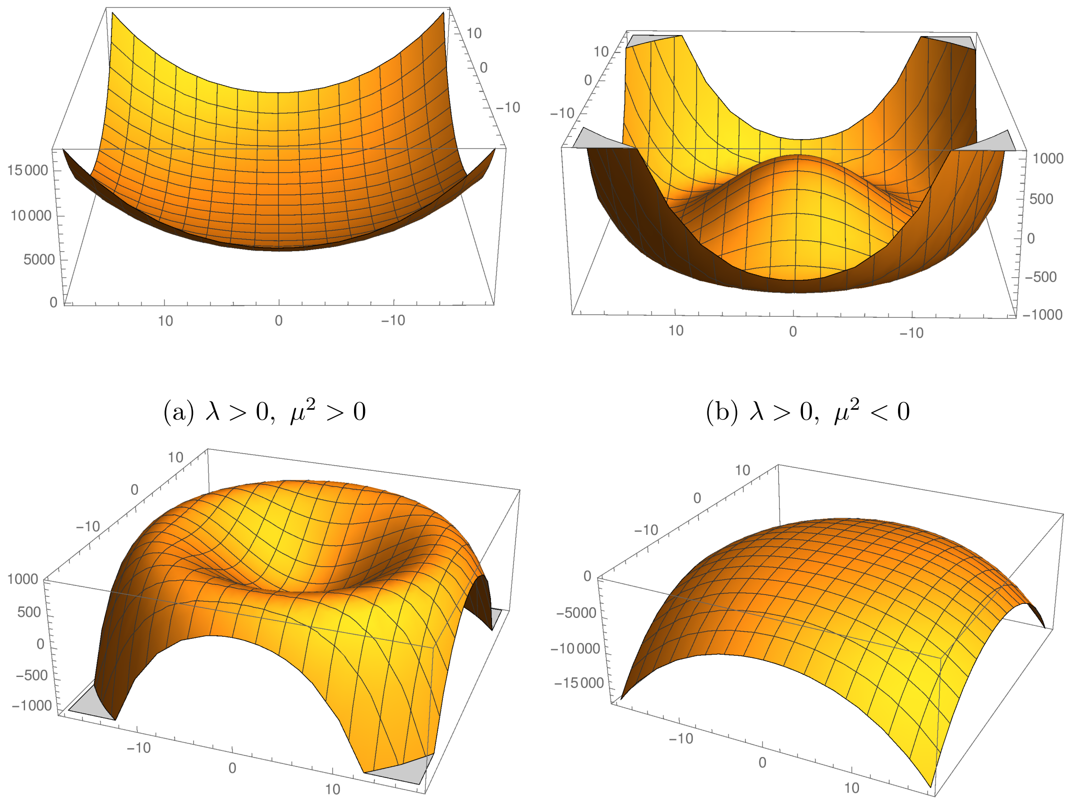
yielding

$$\phi_1 = \phi_2 = 0 \qquad \text{or} \qquad \mu^2 = -\frac{1}{4}\lambda\left(\phi_1^2 + \phi_2^2\right) . \tag{4.6}$$

If we plot the scalar potential for each of the four $\left(\lambda, \mu^2\right)$-configuration, see Fig. 6, we observe the following

a) There is a global **stable** minimum at $\phi_1 = \phi_2 = 0$ satisfying the first condition in (4.6),

b) There is a local **unstable** maximum at $\phi_1 = \phi_2 = 0$ and a global **stable** minimum at $\phi_1^2 + \phi_2^2 = v^2 > 0$ satisfying the first and second conditions in (4.6) respectively, and where v represents the distance to the origin,



(a) $\lambda > 0,\ \mu^2 > 0$

(b) $\lambda > 0,\ \mu^2 < 0$

(c) $\lambda < 0,\ \mu^2 > 0$

(d) $\lambda < 0,\ \mu^2 < 0$

Figure 6: Possible shapes of the scalar potential $V(\Phi)$

c) there is a local minimum, **unstable or metastable**, at $\phi_1 = \phi_2 = 0$ that satisfies the first condition in (4.6),

d) there is a global **unstable** maximum at $\phi_1 = \phi_2 = 0$ satisfying the first condition in (4.6).

The only stable solutions are a) and b). In fact, when $\lambda < 0$ as in c) and d), the potential becomes unbounded-from-below for large field values with potentially catastrophic consequences to the Universe. Remember that the preferred vacuum configuration is that of minimal energy, therefore, a scalar field sitting in the local minimum c) has a non-zero probability of transiting to $\phi_1 = \phi_2 \to -\infty$ upon quantum fluctuations or quantum tunnelling. If the tunnelling probability is tiny it means that the local minimum is long-lived (or metastable) and may also describe a realistic model if its life-time is larger than the age of the Universe.

Choosing the stable solutions, the scalar mass spectrum can be calculated taking the Hessian matrix of the potential and evaluating it at the minimum:

$$\partial^2_{ij} V \to \begin{pmatrix} \dfrac{\partial^2 V}{\partial \phi_1^2} & \dfrac{\partial^2 V}{\partial \phi_1 \partial \phi_2} \\ \dfrac{\partial^2 V}{\partial \phi_2 \partial \phi_1} & \dfrac{\partial^2 V}{\partial \phi_2^2} \end{pmatrix} \equiv \left[\mathbf{M^2}\right]_{ij} . \tag{4.7}$$

Taking the second derivatives of the potential the Hessian matrix becomes

$$\left[\mathbf{M^2}\right]_{ij} = \begin{pmatrix} \mu^2 + \frac{3}{4}\lambda\phi_1^2 + \frac{1}{4}\lambda\phi_2^2 & \frac{1}{2}\lambda\phi_1\phi_2 \\ \frac{1}{2}\lambda\phi_1\phi_2 & \mu^2 + \frac{1}{4}\lambda\phi_1^2 + \frac{3}{4}\lambda\phi_2^2 \end{pmatrix} . \tag{4.8}$$

For the potential shape a), the minimization condition (4.6) implies that $\phi_1 = \phi_2 = 0$, therefore, in the minimum of the potential, the mass matrix reads

$$\left[\mathbf{M^2}\right]_{ij}\Big|_{\phi_1=\phi_2=0} = \begin{pmatrix} \mu^2 & 0 \\ 0 & \mu^2 \end{pmatrix} , \tag{4.9}$$

with $\mu^2 > 0$. Note that the potential in the minimum a) preserves the U(1) symmetry ($V = 0$ is trivially invariant under U(1) transformations) yielding two physical scalar fields with equal mass

$$m_1^2 = m_2^2 = \mu^2 . \tag{4.10}$$

Note that this solution is also valid for the local minimum c).

Let us now verify what happens for case b). Here, the minimization equation (4.6) is verified for any field values $\phi_1^2 + \phi_2^2 = v^2 > 0$ where v is the distance to the origin of the potential in the field space. The mass matrix can then be cast as

$$\left[\mathbf{M^2}\right]_{ij}\Big|_{\mu^2=-\frac{1}{4}\lambda\left(\phi_1^2+\phi_2^2\right)} = \begin{pmatrix} \frac{1}{2}\lambda\phi_1^2 & \frac{1}{2}\lambda\phi_1\phi_2 \\ \frac{1}{2}\lambda\phi_1\phi_2 & \frac{1}{2}\lambda\phi_2^2 \end{pmatrix} . \tag{4.11}$$

To obtain the physical mass spectrum we diagonalize $\left[\mathbf{M^2}\right]_{ij}$ and, for ease of notation, take $\phi_1^2 + \phi_2^2 = v^2$:

$$\left[\mathbf{m^2}\right]_{ij} = \begin{pmatrix} \frac{1}{2}\lambda v^2 & 0 \\ 0 & 0 \end{pmatrix} . \tag{4.12}$$

The physical mass spectrum is then described by two particles with masses

$$m_1^2 = \frac{1}{2}\lambda v^2 \text{ and } m_2^2 = 0 , \tag{4.13}$$

valid everywhere in the valley of stable solutions. Note that for this scenario the U(1)-symmetry is no longer preserved in the ground state. In fact, if the particle is sitting in a given point of the minimum valley, the same solution only gets repeated upon a phase rotation of 2π. It is this broken U(1) invariance **only by the ground state** that we denote as **spontaneously broken symmetry**. Note that masses are the same everywhere in the minimum. This degeneracy is a consequence of the original symmetry and all solutions are physically identical.

Without loss of generality, we can choose our vacuum (minimum) performing a shift on the real component of Φ as

$$\begin{aligned} \phi_1 &= v + h \\ \phi_2 &= 0 + G\,. \end{aligned} \tag{4.14}$$

Then, inserting

$$\Phi = \frac{1}{\sqrt{2}}\,(v + h + iG) \tag{4.15}$$

in the potential and using the minimization condition $\mu^2 = \frac{1}{4}\lambda v^2$ for $\mu^2 < 0$ we get

$$V(h,G) = \frac{1}{16}\lambda G^4 + \frac{1}{8}\lambda G^2 h^2 + \frac{1}{16}\lambda h^4 + \frac{1}{4}\lambda v h G^2 + \frac{1}{4}\lambda v h^3 + \frac{1}{4}\lambda v^2 h^2 - \frac{1}{16}\lambda v^4\,. \tag{4.16}$$

Note that the process of spontaneous symmetry breaking generates cubic interactions of the form hG^2 and h^3 at the minimum which violate the U(1) symmetry. On the other hand the field G only comes in even powers meaning that the minimum of the potential contains a residual $\mathbb{Z}_2$ symmetry $G \to -G$. This is equivalent to say that the **Vacuum Expectation Value**, or VEV for short, $\langle\Phi\rangle = v$ induces the symmetry breaking pattern

$$\mathrm{U}(1) \to \mathbb{Z}_2\,. \tag{4.17}$$

We also see that only the field h acquires a quadratic term which means that we are left with one massive and one massless physical state in the minimum

$$m_h^2 = \frac{1}{2}\lambda v^2 \text{ and } m_G^2 = 0\,, \tag{4.18}$$

in agreement with (4.13). This is a particular case of **Goldstone's theorem** that says: *Whenever a continuous symmetry group of the scalar potential is broken to a smaller group (less symmetric), then the final scalar potential will contain one massless particle (a Goldstone) for each symmetry generator that is broken.* In this case there is only one continuous parameter (the α angle) to characterise the symmetry (one symmetry generator only), so there is only one symmetry to be broken. Later we will see in the Standard Model (SM) an example with more broken symmetry generators.

These are the principles behind the **Higgs Mechanism**. In the current model the massive state h is our **Higgs boson** whereas the massless one, G, is a **Goldstone boson** denoted after Goldstone's theorem.

It is also instructive to look at this problem from a geometrical approach. Since the field Φ is complex, we can also parametrize about a real VEV v as

$$\Phi = \frac{1}{\sqrt{2}}\,(h_r + v)\, e^{i\frac{h_\theta}{v}} \tag{4.19}$$

where h_r denote radial oscillations about v in the potential 6b and h_θ represents angular oscillations along the minimum valley of radius v. Expanding Φ for small h_θ to first order and keeping only linear terms we get

$$\Phi \approx \frac{1}{\sqrt{2}}\left(v + h_r + i h_\theta\right) \tag{4.20}$$

which can be replaced in the potential to give

$$V(h_r, h_\theta) = \frac{1}{16}\lambda h_\theta^4 + \frac{1}{8}\lambda h_\theta^2 h_r^2 + \frac{1}{16}\lambda h_r^4 + \frac{1}{4}\lambda v h_r h_\theta^2 + \frac{1}{4}\lambda v h_r^3 + \frac{1}{4}\lambda v^2 h_r^2 - \frac{1}{16}\lambda v^4\,, \tag{4.21}$$

from where we take that

$$m_{h_r}^2 = \frac{1}{2}\lambda v^2 \text{ and } m_{h_\theta}^2 = 0\,. \tag{4.22}$$

The Higgs boson is then associated with radial oscillations around the vacuum and its mass can be thought of as the energy cost of fluctuating in the radial direction. On the other hand, the Goldstone boson can be seen as a phase and it does not cost any energy to fluctuate in the angular direction.

4.1.2 Explicit symmetry breaking

We have studied above the effects of spontaneously breaking a continuous symmetry. However, although not as elegant, certain particle physics models contain explicitly broken symmetries. To see what it means let us modify our theory to

$$V(\Phi) \to V(\Phi) + \mu_1^2\left(\Phi^2 + (\Phi^*)^2\right)\,. \tag{4.23}$$

The new quadratic terms violate our U(1) symmetry because the transformation $\Phi \to \Phi e^{i\alpha}$ yields

$$\Phi^2 \to \Phi^2 e^{2i\alpha} \neq \Phi^2 \text{ and } \left(\Phi^*\right)^2 \to \left(\Phi^*\right)^2 e^{-2i\alpha} \neq \left(\Phi^*\right)^2 \tag{4.24}$$

which does not leave the theory invariant. Since these terms were **explicitly** introduced rather than **spontaneously** generated in the minimum, we say that the U(1) symmetry was **explicitly broken**.

On the other hand it is possible to verify that the new model contains a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry as the potential is invariant under the transformations

$$\begin{aligned} \phi_1 &\to -\phi_1 \qquad (\Leftrightarrow \Phi \to -\Phi^*)\,, \\ \phi_2 &\to -\phi_2 \qquad (\Leftrightarrow \Phi \to \Phi^*)\,. \end{aligned} \tag{4.25}$$

For example, for the first $\mathbb{Z}_2$ we have

$$\begin{aligned} \Phi^2 &\to \left(-\Phi^*\right)^2 = \left(\Phi^*\right)^2 \\ \left(\Phi^*\right)^2 &\to \left[\left(-\Phi^*\right)^*\right]^2 = \Phi^2 \\ \Phi^*\Phi &\to \left(-\Phi^*\right)^*\left(-\Phi^*\right) = \Phi\Phi^* \end{aligned} \tag{4.26}$$

which preserves the invariance of the potential (4.23). The stationarity conditions yield the following five possibilities

$$\begin{aligned} \phi_1 &= 0\,, & \phi_2 &= \pm\frac{2\sqrt{2\mu_1^2-\mu^2}}{\sqrt{\lambda}} \\ \phi_1 &= 0\,, & \phi_2 &= 0 \\ \phi_1 &= \pm\frac{2\sqrt{-2\mu_1^2-\mu^2}}{\sqrt{\lambda}}\,, & \phi_2 &= 0 \end{aligned} \tag{4.27}$$

and provided that both quadratic couplings are real, the stationarity conditions imply that

$$\mu^2 < 0 \text{ and } \mu^2 < 2\mu_1^2 < -\mu^2\,. \tag{4.28}$$

Requiring the potential to be bounded-from-below, that is, $\lambda > 0$, its shape becomes as in Fig. 7. It is clear that the potential no longer contains a phase-rotation invariance and a Goldstone direction linking all minima of equivalent masses is absent. Just by inspection of Fig. 7 we see that there is a local maximum, two saddle points and two stable global minima. Note that the two saddle points and the two minima be mapped into each other via the $\mathbb{Z}_2$ transformations $\phi_1 \to -\phi_1$ and $\phi_2 \to -\phi_2$.

Replacing the first and third conditions into the Hessian matrix we get the mass forms

$$\begin{pmatrix} 4\mu_1^2 & 0 \\ 0 & 4\mu_1^2 - 2\mu^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -4\mu_1^2 - 2\mu^2 & 0 \\ 0 & -4\mu_1^2 \end{pmatrix}\,. \tag{4.29}$$

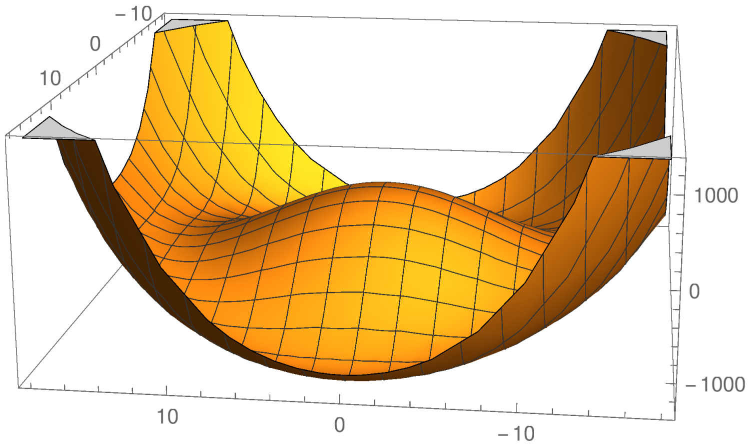
For example, if $0 < 2\mu_1^2 < -\mu^2$ then $\phi_1 = 0$ and $\phi_2 = \pm\frac{2\sqrt{2\mu_1^2-\mu^2}}{\sqrt{\lambda}}$ are two identical minima with physical masses given by the first matrix, whereas $\phi_1 = \pm\frac{2\sqrt{-2\mu_1^2-\mu^2}}{\sqrt{\lambda}}$, and $\phi_2 = 0$ are two identical saddle points. Note that the absence of a continuous symmetry implies that we no longer have massless Goldstone bosons in the particle spectrum. In fact, there is no continuous generator to break and Goldstone's theorem does not apply.

4.2 The abelian Higgs mechanism: vector boson mass generation

In section 4.1 we studied the behaviour of the scalar potential with an global U(1) symmetry. In what follows we localise this symmetry such that transformations on the scalar fields take the form (3.91), that is, the generator of the U(1) symmetry, $\alpha(x)$, becomes dependent on the space-time position. We have also shown in section 3.2.4 that gauge invariance requires the introduction of covariant derivatives $\partial^\mu \to \mathcal{D}^\mu$ from where interactions with **massless** gauge fields A^μ will be induced.

Let us then update the pure-scalar theory Lagrangian (4.1) to

$$\mathcal{L} = [\mathcal{D}^\mu \Phi]^* \, \mathcal{D}_\mu \Phi - V(\Phi) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{4.30}$$



(a) $\lambda > 0,\ \mu^2 < 0$

Figure 7: Scalar potential with explicitly broken U(1)-symmetry.

where the scalar potential is the same as in (4.2), thus both stationary points classification and vacuum stability conditions are exactly the same. Note that the theory in no longer purely scalar as it is now also describing interactions between scalars and vector fields A^μ. Note that the Maxwell terms has to be introduced provided that it is invariant under the gauge and Lorentz symmetries.

With the minimization conditions already known, $\mu^2 = \dfrac{1}{4}\lambda v^2$ in the stable minimum of potential shape b), and with the shift in Φ given by (4.15), we have obtained one massive Higgs boson with mass $m_h^2 = \dfrac{1}{2}\lambda v^2$ and a massless Goldstone boson $m_G^2 = 0$. The question to ask now is what happens in the case of a U(1) gauge theory, in particular, to the massless vector boson A^μ?

Let us expand the kinetic terms in (4.30):

$$\begin{aligned}
&[\mathcal{D}^\mu\Phi]^*\,\mathcal{D}_\mu\Phi = \\
=&\frac{1}{2}\left[\left(\partial^\mu - iqA^\mu\right)\left(v + h(x) - iG(x)\right)\right]\left[\left(\partial_\mu + iqA_\mu\right)\left(v + h(x) + iG(x)\right)\right] \\
=&\tfrac{1}{2}\partial^\mu h\partial_\mu h + \tfrac{1}{2}\partial^\mu G\partial_\mu G + \tfrac{1}{2}q^2A^\mu A_\mu h^2 - \tfrac{1}{2}q^2A^\mu A_\mu G^2 + \tfrac{1}{2}q^2v^2A^\mu A_\mu \\
&+ q\left(\partial^\mu G\right)A_\mu h - iq\partial^\mu G^2 A_\mu + q^2vA^\mu A_\mu h - iq^2A^\mu A_\mu hG - iq^2vA^\mu A_\mu G \\
&+ q\left(\partial^\mu G\right)A_\mu v\,.
\end{aligned} \tag{4.31}$$

The very first thing that we note is the generation of a mass term to the vector boson A^μ of the form $\frac{1}{2}m_A^2A^\mu A_\mu$. However, before we take any conclusion, we see that the last term in the Lagrangian (4.31) contains an off-diagonal bilinear interactions between the Goldstone G and the gauge boson A^μ. This means that the particle spectrum in (4.31) is not yet physical[6] In fact, after generating mass to the gauge boson we have raised its polarization scalar degrees of freedom from 2 to 3. While a massless gauge boson is only transversely polarized, a massive one also oscillates in the direction of motion, that is, contains a longitudinal polarization. However, simply performing the shift in Φ does not create new degrees of freedom as it appears in (4.31), 3 from A_μ plus 2 from h and G, while before the breaking we had 2 from A_μ and 2 from Φ. Something clearly needs to be fixed here!

As we have shown above, the shift $\Phi(x) = \frac{1}{2}(v + h(x) + iG(x))$ is actually the lowest order of the expansion

$$\Phi(x) = \tfrac{1}{2}\left(v + h(x) + iG(x)\right) \simeq \tfrac{1}{\sqrt{2}}(v + h(x))e^{iG(x)/v} \tag{4.32}$$

where the Goldstone direction is just a phase. Recalling that the U(1) gauge transformations acting on Φ and A_μ that leave the theory invariant are

$$\begin{aligned} \Phi &\to \Phi e^{i\alpha(x)} \\ A_\mu &\to A_\mu + \partial_\mu \alpha(x) \end{aligned} \tag{4.33}$$

where $\alpha(x)$ is left arbitrary, we can instead choose a gauge that eliminates unphysical fields from the Lagrangian. Using the expansion (4.32) we can fix our gauge as follows:

$$\Phi \to \tfrac{1}{\sqrt{2}}\left(v + h(x)\right) e^{i\frac{G(x)}{v}} \tag{4.34}$$

from there we take that the transformation parameter is

$$\alpha(x) = \frac{G(x)}{qv}\,. \tag{4.35}$$

This also sets the transformation on the gauge fields that reads

$$A_\mu \to A_\mu + \frac{1}{qv}\partial_\mu G(x)\,. \tag{4.36}$$

With this gauge choice the theory becomes independent of $G(x)$ and it can be simply removed from the Lagrangian (4.30) yielding

$$\left[\mathcal{D}^\mu \Phi\right]^* \mathcal{D}_\mu \Phi = \tfrac{1}{2}\partial^\mu h \partial_\mu h + \tfrac{1}{2}q^2 A^\mu A_\mu h^2 + \tfrac{1}{2}q^2 v^2 A^\mu A_\mu + q^2 v A^\mu A_\mu h\,, \tag{4.37}$$

and

$$V\left(\Phi\right) = V\left(h\right) = \frac{1}{16}\lambda h^4 + \frac{1}{4}\lambda v h^3 + \frac{1}{4}\lambda v^2 h^2 - \frac{1}{16}\lambda v^4\,. \tag{4.38}$$

It is typically said that the Goldstone boson is *eaten* by the longitudinal polarization of the now massive gauge boson and the physical mass spectrum of the theory is simply given by

$$m_A^2 = q^2 v^2 \qquad m_h^2 = \tfrac{1}{2}\lambda v^2\,, \tag{4.39}$$

both proportional to the VEV v. Note that for the symmetric minimum represented by the potential shape a) the VEV is $v = 0$ and the gauge boson is massless. In fact, while the

[6] Recall that a physical particle spectrum contains ONLY diagonal bilinear, or mass, terms.

observation of massive gauge bosons in nature is a sign of spontaneously broken symmetries, massless vector bosons indicate conserved symmetries. For example, the electromagnetic theory, and more generally quantum electrodynamics (QED), is a gauge U(1) theory containing massless photons.

Let us summarize the most important findings:

- The mechanism of spontaneous symmetry breaking generates a mass term to the gauge field A_μ, massless in the unbroken U(1) phase, as well as a massive real scalar, the Higgs boson;
- In the presence of a local (gauge) symmetry the Goldstone boson is absorbed into the longitudinal modes of the massive gauge boson;
- For the case of a global symmetry, since the scalar degrees of freedom are conserved before and after the breaking, the Goldstone boson is a physical massless sate;
- After the breaking new purely-scalar and scalar-gauge interactions are generated as depicted in Fig. 8.

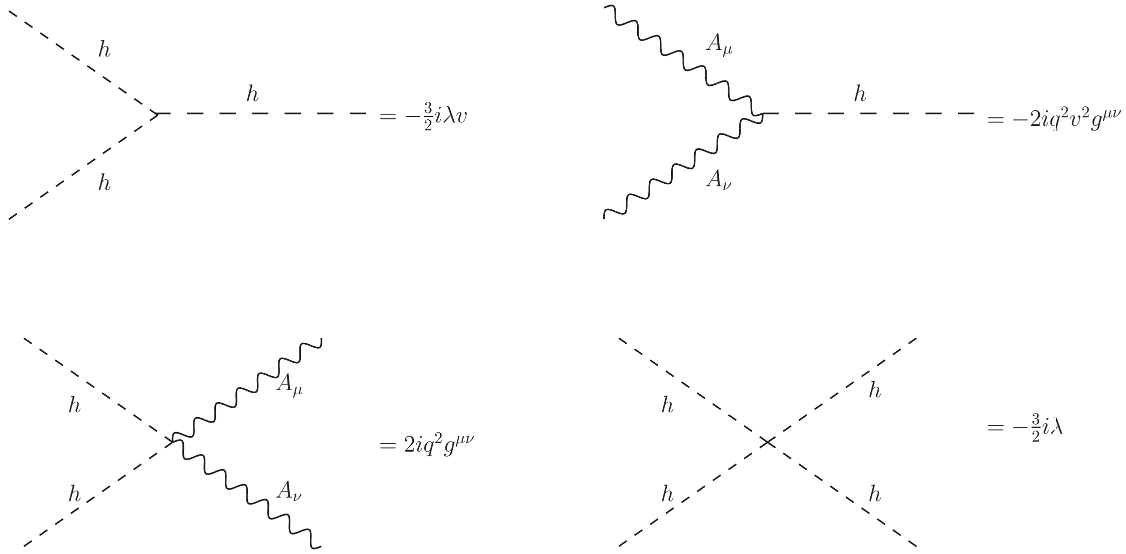


Figure 8: *Feynman diagrams and Feynman rules at tree-level for the* U(1)*-broken phase.*

4.3 The abelian Standard Model: fermion mass generation

In the Standard Model, the masses of fermions are also generated dynamically with the Higgs field. The reason why we cannot write directly fermion mass terms in the Lagrangian for the SM is due to very tight constraints on the possible type of field contractions that are allowed. These constraints, in turn, result from the various gauge symmetries that we are forced to impose to describe the various fundamental forces (electromagnetic, strong nuclear force and weak nuclear force). The fermion fields, being “charged” under such symmetries, can only appear in the Lagrangian in neutral combinations (called singlet operators). The Dirac mass terms will turn out not to be allowed and a different mechanism is called to the rescue using the Higgs boson in Yukawa terms involving one Higgs field and two fermion fields.

We start by illustrating the principle with a simplified model, denoted here as the abelian Standard Model, where one complex scalar and two Dirac fermions transform under a gauge $\mathrm{U}(1)_\mathrm{B} \times \mathrm{U}(1)_\mathrm{C}$ symmetry according to the quantum numbers provided in table 1. The

Field	$\mathrm{U}(1)_\mathrm{B}$	$\mathrm{U}(1)_\mathrm{C}$
Φ	1	1
ψ	1	0
χ	0	-1

Table 1: *Field charges in the abelian Standard Model*

charges present in table 1 indicate how the fields transform under each U(1). In particular we have

$$\begin{aligned}
&\Phi \to \Phi e^{i(g\alpha(x)+g'\beta(x))} \\
&\psi \to \psi e^{ig\alpha(x)} \\
&\chi \to \chi e^{-ig'\beta(x)}
\end{aligned} \tag{4.40}$$

where the constants g and g' were introduced in order to distinguish $\mathrm{U}(1)_\mathrm{B}$ and $\mathrm{U}(1)_\mathrm{C}$ transformations respectively. Such constants are denoted as **gauge couplings** and their size characterizes the interaction strength of each U(1). For example, if we set $g \gg g'$ it means that $\mathrm{U}(1)_\mathrm{B}$ is a strong interaction while $\mathrm{U}(1)_\mathrm{C}$ is a weak one. We also cast the $\mathrm{U}(1)_\mathrm{B}$ and $\mathrm{U}(1)_\mathrm{C}$ gauge fields as B_μ and C_μ respectively and the corresponding field strength tensors as $B_{\mu\nu}$ and $C_{\mu\nu}$.

When building a theory, we first start by defining the transformation properties as in table 1, which is the same as saying that we define the symmetries, and then write ALL allowed interactions with dimension up to four. (Note that scalar fields and gauge boson fields have both dimension 1 and fermions have dimension 2/3). The Lagrangian for this system is then

$$\begin{aligned}
\mathcal{L} = &\left[\mathfrak{D}^\mu \Phi\right]^* \mathfrak{D}_\mu \Phi - V(\Phi) \\
&- \frac{1}{4} B^{\mu\nu} B_{\mu\nu} - \frac{1}{4} C^{\mu\nu} C_{\mu\nu} \\
&+ i\overline{\psi}\gamma^\mu d_\mu \psi + i\overline{\chi}\gamma^\mu \mathcal{D}_\mu \chi \\
&+ y\overline{\psi}\Phi\chi + c.c.
\end{aligned} \tag{4.41}$$

with the covariant derivatives given by

$$\begin{aligned}\mathfrak{D}_\mu &\equiv \partial_\mu + igB_\mu + ig'C_\mu \\ d_\mu &\equiv \partial_\mu + igB_\mu \\ \mathcal{D}_\mu &\equiv \partial_\mu - ig'C_\mu\end{aligned} \tag{4.42}$$

On the first line of the Lagrangian (4.41) we have the scalar sector, whose potential is identical to the two cases studied above, on the second line the pure gauge sector, on the third line two Dirac fermions gauge-kinetic terms and on the fourth line the Yukawa interactions that are allowed by the gauge symmetries.

Once again we make use of the shift in the complex scalar Φ in order to expand the first line in (4.41):

$$\begin{aligned}&[\mathfrak{D}^\mu\Phi]^*\,\mathfrak{D}_\mu\Phi = \\ =&\frac{1}{2}\left[\left(\partial^\mu - igB^\mu - ig'C^\mu\right)(v+h(x)-iG(x))\right]\left[\left(\partial_\mu + igB_\mu + ig'C_\mu\right)(v+h(x)+iG(x))\right] \\ =&\tfrac{1}{2}v^2\left(g^2B^\mu B_\mu + g'^2C^\mu C_\mu + gg'B^\mu C_\mu + g'gC^\mu B_\mu\right) \\ &+ gv\left(\partial^\mu G\right)B_\mu + g'v\left(\partial^\mu G\right)C_\mu + \text{interaction terms}\end{aligned} \tag{4.43}$$

Contrary to the previous example the gauge fields do not form diagonal mass terms meaning that we are not in the mass basis. To determine the masses of the physical gauge bosons we need to calculate the eigenvalues of the mass form

$$\mathbf{M^2} = v^2\begin{pmatrix} g^2 & gg' \\ gg' & g'^2\end{pmatrix} \longrightarrow \mathbf{m^2} = v^2\begin{pmatrix} 0 & 0 \\ 0 & g^2+g'^2\end{pmatrix}\,. \tag{4.44}$$

We have then generated one massive gauge boson, which we might denote as a Z-like boson in analogy to the SM, as well as a massless one which we call photon-like

$$m_A^2 = 0 \qquad m_Z^2 = v^2\left(g^2+g'^2\right)\,. \tag{4.45}$$

Since we are mixing two states the eigenvectors in the physical basis can be expressed as

$$\begin{aligned}A_\mu &= \sin\theta\,B_\mu + \cos\theta\,C_\mu \\ Z_\mu &= \cos\theta\,B_\mu - \sin\theta\,C_\mu\end{aligned} \tag{4.46}$$

with θ the mixing angle that transforms the gauge basis into the mass basis. We may also recast the gauge couplings as

$$\begin{aligned}g &= \frac{e}{\sin\theta} \\ g' &= \frac{e}{\cos\theta}\end{aligned} \tag{4.47}$$

for later use. Note that the presence of a massless gauge boson indicates that the minimum of the potential does not break the full symmetry and a residual $\mathrm{U(1)_A}$ is still present in the theory. We can then represent such a breaking scheme as

$$\mathrm{U(1)_B}\times\mathrm{U(1)_C}\to\mathrm{U(1)_A}\,. \tag{4.48}$$

But this is not yet the full story. Once more, the last line in eq. (4.43) contains bilinear terms involving the original gauge fields and the Goldstone boson. To fix this we can choose a gauge where we set the transformation on the scalar field as in eq (4.34). This implies that

$$\tfrac{G(x)}{v} = g\alpha(x) + g'\beta(x)\,. \tag{4.49}$$

For simplicity we may also choose $\alpha(x) = \beta(x)$ such that

$$\alpha(x) = \frac{G(x)}{(g+g')\,v}\,. \tag{4.50}$$

With this choice we can remove $G(x)$ from the theory as we did before. Note that we have **only one** Goldstone boson to be absorbed into longitudinal polarizations of the physical gauge fields. Therefore, only one of the gauge bosons may become massive in agreement with the mass spectrum (4.45).

Finally we can look to the fermion sector. First, note that the usual Dirac mass terms of the form $m\left(\overline{\psi}\chi + \text{c.c.}\right)$ are forbidden by the gauge symmetry as can be seen from the charges in table 1. On the other hand, an interaction with the Higgs boson via the Yukawa term in the last line of Lagrangian (4.41) is invariant under $\mathrm{U}(1)_\mathrm{B} \times \mathrm{U}(1)_\mathrm{C}$.

Using the Higgs boson expansion for physical states $\Phi = (v+h)$, the gauge boson eigenvectors in the mass basis as given by eq. (4.46) and the gauge coupling redefinition as in (4.47), we can expand the last two lines of (4.41) to obtain

$$\begin{aligned}\mathcal{L}_f = {} & i\overline{\chi}\gamma^\mu\partial_\mu\chi + i\overline{\psi}\gamma^\mu\partial_\mu\psi + e\gamma^\mu A_\mu\overline{\chi}\chi + e\gamma^\mu A_\mu\overline{\psi}\psi \\ & - e\cot\theta\gamma^\mu Z_\mu\overline{\psi}\psi + e\tan\theta\gamma^\mu Z_\mu\overline{\chi}\chi + \frac{y}{\sqrt{2}}h\overline{\psi}\chi \\ & + \frac{y}{\sqrt{2}}v\overline{\psi}\chi + \text{c.c.}\end{aligned} \tag{4.51}$$

Similarly to what we have observed for the gauge bosons the last line in (4.51) is a mass term for fermions generated in the broken phase, that is, in the symmetric vacuum (origin of the potential) fermions are massless, once the complex Higgs develops a VEV fermion mass terms proportional to the VEV v are generated

$$m_f = \frac{y}{\sqrt{2}}v\,. \tag{4.52}$$

Typically, Ψ and χ represent distinct fermionic chiralities, that is, one is right-handed and the other left-handed under Lorentz transformations. Note that the bar inverts the chirality making $\overline{\chi}\psi$ a Lorentz invariant. While in the massless limit fermions are purely chiral, the physical massive states contain components of both chiralities.

We can also read from the Lagrangian (4.51) the conserved and broken charges of the fermions. The conserved ones, or electric-like, are $+e$ for both ψ and χ while the broken charges read $-e\cot\theta$ for ψ and $e\tan\theta$ for χ. We can also read from (4.51) five fermion-boson interactions vertices whose Feynman diagrams are represented in Fig. **??**.

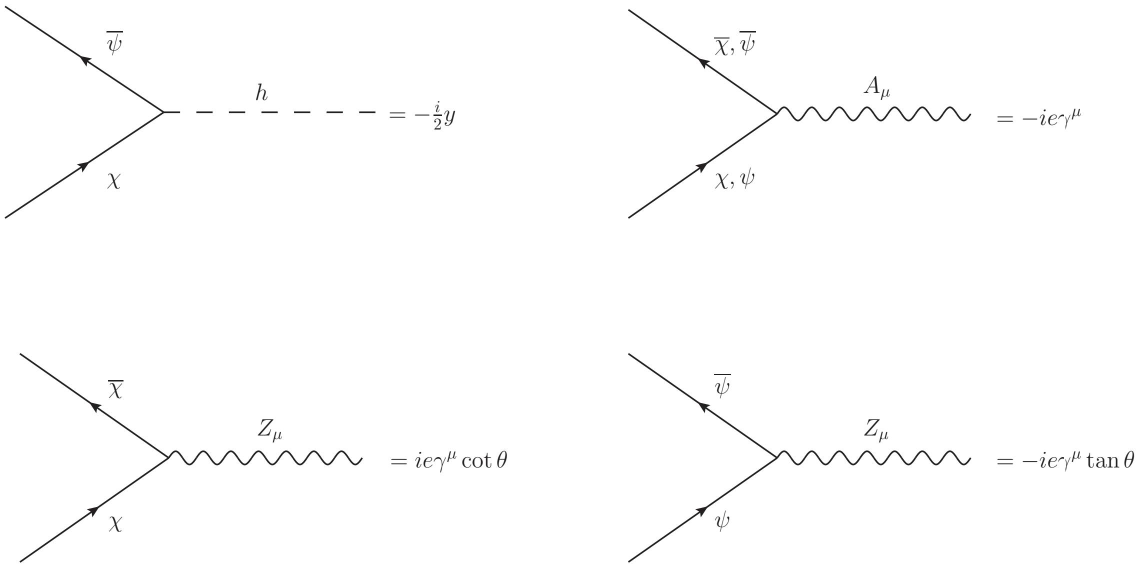


Figure 9: *Feynman diagrams and Feynman rules at tree-level the* $\mathrm{U}(1)_\mathrm{A}$ *phase.*

4.4 The Higgs mechanism in the Standard Model

4.4.1 The non-abelian SU(2) group and Yang-Mills theories

One defines a multiplicative group as being a set G such that the binary operation $ab = c$: $G \times G \to G$ obeys the following conditions:

- (associativity) for all a, b, $c \in G\,(ab)\,c = a\,(bc)$
- (identity) there is an element $e \in G$ such that $ea = a = ae$
- (inverse element) for each $a \in G$ there is an element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$.

These three conditions define a mutiplicative symmetry group. A special case is an abelian group, i.e. if all elements

$$a_1, \cdots, a_n$$

commute, i.e.,

$$[a_i, a_j] = 0\,,$$

otherwise the group is non-abelian. We will be interested in continuous Lie groups (for intuition think for example of the group of rotations in 2D that depends on a continuous parameter the angle α but in general we may have higher dimensional groups/parameters). Another set of quantities that relate to the structure of the Lie group are the **structure constants** which, for **matrix groups**, are defined from the Lie Algebra as follows:

The elements of the Lie algebra can be defined as the set of matrices that when exponentiated give an element of the Lie group, i.e.

$$a = e^{-i\tau}$$

(where $-i$ is conventional). Then the structure constants f_{ijk} are defined such that for elements $\tau_1, \tau_2, \ldots$ of the Lie algebra we have

$$[\tau_i, \tau_j] = i \sum_k f_{ijk} \tau_k \,. \tag{4.53}$$

For clarity, note that the set τ_i denotes elements of the Lie algebra whereas a_i are elements of the Lie Group. An **abelian** group is defined by a commuting Lie algebra, that is, the commutator (4.53) is zero. On the other, a **non-abelian** group is defined by a non-commuting Lie algebra which means the commutator (4.53) is non-zero. A Lie algebra can be geometrically regarded as the tangent space of the Lie group close (i.e. continuously connected) to the identity, so it can be thought of as the matrices used to expand the exponential close to the identity matrix (i.e. $e^x \simeq 1 + x + \ldots$). We will see this with a specific example below. We should also add here that we can sub-divide groups in two categories: continuous or discrete. While the former contain an infinite amount of elements, e.g. $a = e^{-i\omega\tau}$ with ω a continuous real parameter, whereas the latter contain a finite number of elements. As example of abelian discrete groups we can think of $\mathbb{Z}_N$ whose elements are given by

$$a_n = e^{i\frac{2n\pi}{N}} \text{ with } n = 1, \cdots, N \,. \tag{4.54}$$

For $N = 2$ we have the set

$$\{a_1, a_2\} = \{e^{i\pi}, e^{2i\pi}\} = \{-1, 1\} \,. \tag{4.55}$$

For $N = 3$ there are three elements that read

$$\{a_1, a_2, a_3\} = \{e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}, e^{2i\pi}\} \,. \tag{4.56}$$

Note that $a_3 = e$ is the identity and the above properties for multiplicative groups apply. Let us now look into continuous Lie groups.

SU(N) groups

The SU(N) group is defined as the set of N by N unitary complex matrices U with unit determinant, i.e.

$$U^\dagger U = \mathbf{1} \qquad \text{and} \qquad \det U = 1 \,. \tag{4.57}$$

The first condition defines a matrix group to be unitary, U(N), and the added condition on the determinant makes it a special unitary group, SU(N). Now, let us think of a generic complex $N \times N$ matrix. Since we have $2N^2$ independent components we can span it in terms of $2N^2$ real parameters ω_n, i.e.

$$M = \omega_1 \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} + \omega_2 \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} + \cdots + i\omega_{N^2+1} \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} + \cdots + i\omega_{2N^2} \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} . \tag{4.58}$$

However, the constraints given by eq. (4.57) imply the following:

$$\sum_k \left(U^\dagger\right)_{ik} U_{kj} = 0 \quad \text{for} \quad i \neq j$$
$$\sum_k \left(U^\dagger\right)_{ik} U_{kj} = 1 \quad \text{for} \quad i = k \tag{4.59}$$

which results in one condition per entry, that is, N^2 constraints. Furthermore, the last condition, $detU = 1$, offers one more constraint which means that in total, the number of independent real parameters needed to parametrize a general SU(N) matrix is

$$2N^2 - N^2 - 1 = N^2 - 1\,, \tag{4.60}$$

which corresponds to the number of independent generators τ_k of the Lie algebra.

SU(2) group

Let us now study the $N = 2$ case, which will be crucial for understanding electroweak interactions. The first thing to note is that we will have $N^2 - 1 = 3$ independent generators defining an algebra of the form of (4.53). It is from basic knowledge on Quantum Mechanics that such an algebra, which describes spin, is

$$[\tau_i, \tau_j] = i \sum_k \varepsilon_{ijk} \tau_k \quad \text{with} \quad \tau_i = \tfrac{\sigma_i}{2} \tag{4.61}$$

and σ_i the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{4.62}$$

So, we see that the structure constants of SU(2) are simply the elements of the Levi-Civita tensor $f_{ijk} = \varepsilon_{ijk}$. Equation (4.61) also defines a non-abelian algebra which means that SU(2) is a non-abelian group. This will have important physical consequences as we will see ahead. Elements of the group SU(2) can be generically written as

$$U(\boldsymbol{\omega}) = e^{-i\boldsymbol{\omega}\cdot\frac{\boldsymbol{\sigma}}{2}} \tag{4.63}$$

with $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ a vector of continuous parameters. Just as a side note, if we consider the SU(3) group, which describes strong interactions, we would have $N^2 - 1 = 8$ generators which are the well known Gell-Mann matrices.

Connection to Physics: Yang-Mills theories

The massive Yang-Mills Lagrangian reads

$$\begin{aligned} \mathcal{L}_{YM} &= i\overline{\psi}\gamma^\mu D_\mu \psi - m\overline{\psi}\psi - \tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \\ &= i\overline{\psi}_A \gamma^\mu \left[D_\mu\right]^A{}_B\, \psi^B - m\overline{\psi}_A \psi^A - \tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a\,, \end{aligned} \tag{4.64}$$

where ψ^A is a multiplet of Dirac spinors (in the first line we have suppressed the indices[7]). By multiplet we denote a representation of SU(N) group which can be written as

$$\psi^A = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \vdots \\ \psi^N \end{pmatrix} . \tag{4.65}$$

For the case of SU(2) we can simply write

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} , \tag{4.66}$$

where ψ_+ and ψ_- are **isospin** eigenstates, in analogy to spin eigenstates.

Once we take into account SU(N) algebra and promote it to a gauge symmetry, that is

$$\psi \to \psi e^{i\omega_a(x)\tau^a} \quad \text{with} \quad a = 1, 2, \cdots , N^2 - 1 \tag{4.67}$$

the gauge covariant derivative takes the form:

$$\begin{aligned} D_\mu &= \partial_\mu - ig\tau_a A_\mu^a \\ \Leftrightarrow [D_\mu]^A{}_B &= \delta^A_B \partial_\mu - ig\,[\tau_a]^A{}_B\, A_\mu^a\,. \end{aligned} \tag{4.68}$$

To determine the non-abelian field strength let us recall the abelian U(1) covariant derivative, $D_\mu = \partial_\mu - igA_\mu$ and compute the commutator $[D_\mu, D_\nu]$:

$$\begin{aligned} [D_\mu, D_\nu] &= [\partial_\mu - igA_\mu, \partial_\nu - igA_\nu] \\ &= [\partial_\mu, \partial_\nu - igA_\nu] - [igA_\mu, \partial_\nu - igA_\nu] \\ &= [\partial_\mu, \partial_\nu] - ig\,[\partial_\mu, A_\nu] - ig\,[A_\mu, \partial_\nu] + i^2 g^2\,[A_\mu, A_\nu] \\ &= -ig\,([\partial_\mu, A_\nu] + [A_\mu, \partial_\nu]) \\ &= -igF_{\mu\nu} \end{aligned} \tag{4.69}$$

[7]Also note that we are using the Einstein summation convention for these indices that are assumed to be Euclidean, so up and down is the same, the position of the indices up or down is only used to make it clear when there are dummy indices being summed over.

Now, let us repeat the exercise for the case of SU(N) groups:

$$
\begin{aligned}
[D_\mu, D_\nu] &= \left[\partial_\mu - ig\tau_a A^a_\mu, \partial_\nu - ig\tau_b A^b_\nu\right] \\
&= \left[\partial_\mu, \partial_\nu - ig\tau_b A^b_\nu\right] + \left[-ig\tau_a A^a_\mu, \partial_\nu - ig\tau_b A^b_\nu\right] \\
&= [\partial_\mu, \partial_\nu] + \left[\partial_\mu, -ig\tau_b A^b_\nu\right] + \left[-ig\tau_a A^a_\mu, \partial_\nu\right] + \left[-ig\tau_a A^a_\mu, -ig\tau_b A^b_\nu\right] \\
&= -ig\tau_a \left([\partial_\mu, A^a_\nu] + \left[A^a_\mu, \partial_\nu\right]\right) + i^2 g^2 \left[\tau_a, \tau_b\right] A^a_\mu A^b_\nu \\
&= -ig\tau_a \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) - ig^2 f_{ab}{}^c \tau_c A^a_\mu A^b_\nu \\
&= -ig\tau_a \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) - ig^2 f_{cb}{}^a \tau_a A^c_\mu A^b_\nu \\
&= -ig\tau_a \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{cb}{}^a A^c_\mu A^b_\nu\right) \\
&= -ig\tau_a \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{bc}{}^a A^b_\mu A^c_\nu\right) \\
&= -ig\tau_a \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^a{}_{bc} A^b_\mu A^c_\nu\right) \\
&= -ig\tau_a F^a_{\mu\nu}
\end{aligned}
\tag{4.70}
$$

where A^a_μ is the Yang Mills non-abelian field and its field strength is

$$
F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^a{}_{bc} A^b_\mu A^c_\nu \,. \tag{4.71}
$$

This result reveals an outstanding consequence for the physics of gauge bosons. While abelian gauge bosons do not self-interact, the Yang-Mills Lagrangian contains the following contributions

$$
\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a = -g f^{abc} \left(-ik^\nu\right) g^{\mu\rho} A^a_\mu A^b_\nu A^c_\rho - \frac{1}{4} g^2 f^{eab} f^{ecd} g^{\mu\rho} g^{\nu\lambda} A^a_\mu A^b_\mu A^c_\rho A^d_\lambda + \cdots \tag{4.72}
$$

which contains triple and quartic self interactions as depicted in the diagrams of Fig. 10. Note that such self-interactions are only possible due to the non-commutative character of the SU(N) algebra which induces an extra term in the field strength tensor. In the SM, while photons are the mediators of an abelian U(1) symmetry they do not contain self-interacting vertices such as in Fig. 10. However, W and Z bosons, which transform according to an SU(2) algebra, as well as 8 gluons, which are the mediators of the strong force described by a non-abelian SU(3) symmetry, do allow triple and quartic interaction vertices as in Fig. 10. The Feynman rules for the two vertices in Fig. 10 can be read from the Lagrangian (4.72) taking into account all possible contraction of indices. Note also that the momentum dependency in the triple vertex is coming from a Fourier transform from space-time coordinates to momentum space and reads $\partial_\mu \to -ik_\mu$. It is left as exercise to the reader to determine the full form of the Feynman rules for such vertices (see bibliography text books).

4.4.2 The electroweak sector in the SM

The construction of the SM Lagrangian follows largely the principles outlined in the previous sections. The main difference compared to the Abelian model of section 1.1 that was used to illustrate the basic principles (gauge symmetries, Higgs mechanism and mass generation

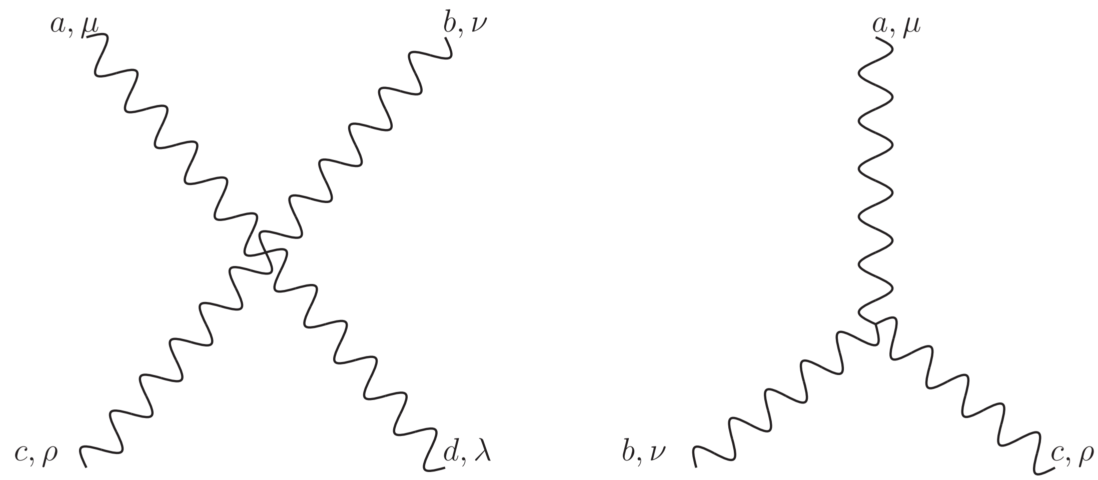


Figure 10: Vector boson self-interactions in non-abelian gauge theories.

for fermions by choosing appropriate Yukawa terms) are the same. The main difference is that the weak forces (and also QCD) have to be described by non-abelian gauge theories. This requires choosing an appropriate set of fields, one for each particle, in appropriate representations of the gauge symmetries. To make the construction clearer it is simpler, again, to discuss first the bosonic sector.

The bosonic sector

The Standard Model of particle physics is a theory invariant under the symmetry $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$ denoted as **electroweak symmetry** (EW). The Lagrangian of this theory reads

$$\mathcal{L}_{EW} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - V\left(\Phi^\dagger \Phi\right) - \tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - \tfrac{1}{4} B_{\mu\nu} B^{\mu\nu} \tag{4.73}$$

where the scalar potential is

$$V\left(\Phi^\dagger \Phi\right) = \lambda \left(\Phi^\dagger \Phi\right)^2 + \mu^2 \Phi^\dagger \Phi \tag{4.74}$$

and the Higgs field, Φ, is an $\mathrm{SU}(2)_\mathrm{L}$ complex scalar $\mathrm{SU}(2)_\mathrm{L}$ doublet given by

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \equiv \tfrac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} . \tag{4.75}$$

The notation ϕ^+ and ϕ^0 will be explained below. The gauge covariant derivative reads

$$D_\mu = \partial_\mu \mathbb{1} - ig\tau_a A^a_\mu - ig' Y \tfrac{\mathbb{1}}{2} B_\mu \tag{4.76}$$

where Y is the hypercharge which is equal to 1 for the Higgs doublet, that is $Y_\Phi = 1$. Recall that the scalar Φ transforms both under $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{U}(1)_\mathrm{Y}$ since it is a doublet with hypercharge non-zero. Note that a field is said to be a singlet (neutral) of $\mathrm{SU}(2)_\mathrm{L}$ if it transforms trivially under such symmetry. In practice it would be just a number (not a matrix) like the complex scalar studied previously in the abelian Standard Model. The transformations properties that leave the potential $V\left(\Phi^\dagger\Phi\right)$ invariant read

$$\Phi \to \Phi e^{i\left(\omega_a(x)\tau^a + \omega_4(x)\frac{Y}{2}\mathbb{1}\right)} \tag{4.77}$$

Finally, the field strength tensors are

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\varepsilon^a{}_{bc}A^b_\mu A^c_\nu \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu\,. \end{aligned} \tag{4.78}$$

As we have done before for the abelian model, we can calculate the minimization condition for $V\left(\Phi^\dagger\Phi\right)$ which follows from the stationarity equations

$$\begin{aligned} \frac{\partial V}{\partial\phi_1} &= \mu^2\phi_1 + \lambda\left(\phi_1^3 + \phi_1\phi_2^2 + \phi_1\phi_3^2 + \phi_1\phi_4^2\right) = 0 \\ \frac{\partial V}{\partial\phi_2} &= \mu^2\phi_2 + \lambda\left(\phi_2^3 + \phi_2\phi_1^2 + \phi_2\phi_3^2 + \phi_2\phi_4^2\right) = 0 \\ \frac{\partial V}{\partial\phi_3} &= \mu^2\phi_3 + \lambda\left(\phi_3^3 + \phi_3\phi_1^2 + \phi_3\phi_2^2 + \phi_3\phi_4^2\right) = 0 \\ \frac{\partial V}{\partial\phi_4} &= \mu^2\phi_4 + \lambda\left(\phi_4^3 + \phi_4\phi_1^2 + \phi_4\phi_2^2 + \phi_4\phi_3^2\right) = 0 \end{aligned} \tag{4.79}$$

which results in

$$\mu^2 = -\lambda\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right) \equiv -\lambda v^2 \tag{4.80}$$

We can now determine the Hessian matrix in the minimum of the potential

$$\left[\mathbf{M^2}\right]_{ij} = 2\lambda\begin{pmatrix} \phi_1^2 & \phi_1\phi_2 & \phi_1\phi_3 & \phi_1\phi_4 \\ \phi_2\phi_1 & \phi_2^2 & \phi_2\phi_3 & \phi_2\phi_4 \\ \phi_3\phi_1 & \phi_3\phi_2 & \phi_3^2 & \phi_3\phi_4 \\ \phi_4\phi_1 & \phi_4\phi_2 & \phi_4\phi_3 & \phi_4^2 \end{pmatrix}. \tag{4.81}$$

whose eigenvalues are

$$m_{G^1} = m_{G^2} = m_{G^3} = 0 \;\text{ and }\; m_h^2 = 2\lambda v^2\,, \tag{4.82}$$

that is, we have now obtained three Goldstone bosons and one massive Higgs boson. Recalling the Goldstone's theorem states that there is one massless particle per broken generator, the result above indicates that out of four generators, three for $\mathrm{SU(2)_L}$ and one of $\mathrm{U(1)_Y}$, we must be left with only one unbroken generator, thus a residual U(1) symmetry. In fact, we can align, without loss of generality, the SM vacuum as

$$\langle\Phi\rangle = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}\,, \tag{4.83}$$

and verify which of the $\mathrm{SU(2)_L \times U(1)_Y}$ generators are broken. **Note: A generator $\hat{\Omega}$ leaves the vacuum invariant whenever $\hat{\Omega}\langle\Phi\rangle = 0$.**

Let us then verify what happens with the generator combinations τ_1, τ_2, $Q = \tau^3 + \frac{Y}{2}\mathbb{1}$ and $Q^\perp = \tau^3 - \frac{Y}{2}\mathbb{1}$

$$\tau_1\langle\Phi\rangle = \tfrac{1}{2\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix} \quad \tau_2\langle\Phi\rangle = \tfrac{i}{2\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix} \quad Q^\perp\langle\Phi\rangle = -\tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \quad Q\langle\Phi\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.84}$$

which shows that out of four generator combinations, three are broken and one is left unbroken. This means that the SM vacuum breaks the electroweak symmetry but leaves a residual $\mathrm{U}(1)_\mathrm{Q}$ symmetry unbroken, which is nothing but the electromagnetic theory. So, the Higgs mechanims breaks the electroweak symmetry reducing it as

$$\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y} \to \mathrm{U}(1)_\mathrm{Q} \tag{4.85}$$

with $Q = \tau^3 + \frac{Y}{2}\mathbb{1}$ the **electric charge generator**. We can now understand the notation used in (4.75). An $\mathrm{SU}(2)_\mathrm{L}$ doublet is a weak isospin representation and, similarly to the theory of spin in Quantum Mechanics, the eigenstates of τ^3 are $\left|\pm\frac{1}{2}\right\rangle$ corresponding to the isospin eigenvalues $I_3 = \pm\frac{1}{2}$. Regarding the hypercharge generator, since the whole doublet has $Y = 1$, then both components are eigenstates of $\frac{\mathbb{1}}{2}$ with eigenvalue $\frac{1}{2}$. Therefore, the Higgs doublet can be written in terms of the vectors of the basis $\{|I_3\rangle \otimes \left|\frac{Y}{2}\right\rangle\} = \{\left|I_3, \frac{Y}{2}\right\rangle\}$ as

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \left|\frac{1}{2}, \frac{1}{2}\right\rangle \\ \left|-\frac{1}{2}, \frac{1}{2}\right\rangle \end{pmatrix}. \tag{4.86}$$

Now, provided that the unbroken generator is $Q = \tau^3 + \frac{Y}{2}\mathbb{1}$, then we can write the components of the Higgs doublet in terms of the electric charge $q = I_3 + \frac{Y}{2}$, that is

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \left|\frac{1}{2} + \frac{1}{2}\right\rangle \\ \left|-\frac{1}{2} + \frac{1}{2}\right\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} |1\rangle \\ |0\rangle \end{pmatrix} \tag{4.87}$$

which shows that in the chosen basis the upper component has electric charge $+1$, thus denoted as ϕ^+, whereas the lower component is neutral, thus denoted as ϕ^0.

Similarly to what we have done before, and noting that the Goldstone bosons are simply phases in the scalar potential, we fix the gauge transformations

$$\begin{aligned} \Phi(x) &\to e^{i\frac{\tau_a \xi^a(x)}{\sqrt{2}v}}\Phi(x) \\ A^a_\mu(x) &\to A^a_\mu(x) + \frac{1}{g\sqrt{2}v}\partial_\mu \xi^a(x) \end{aligned} \tag{4.88}$$

bringing the scalar field into the simpler form

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \tag{4.89}$$

This is called the unitary gauge which we assume in the remainder. Note that this choice is equivalent to setting the 3 Goldstone field fluctuations to zero and we can eliminate them directly from the scalar potential which yields

$$V(h) = \frac{1}{4}\lambda h^4 + \lambda v h^3 + \lambda v^2 h^2 - \frac{1}{4}\lambda v^4\,, \tag{4.90}$$

in agreement with the Higgs mass calculated above $m_h^2 = 2\lambda v^2$.

Let us now analyse the first term in the EW Lagrangian (4.73) and expand it around the vacuum state of eq. (4.89) by using the matrix form of the covariant derivative (4.76), i.e.

$$D_\mu = \left[\begin{pmatrix}\partial_\mu & 0\\ 0 & \partial_\mu\end{pmatrix} - \frac{1}{2}ig\begin{pmatrix}0 & A_\mu^1\\ A_\mu^1 & 0\end{pmatrix} - \frac{1}{2}ig\begin{pmatrix}0 & -iA_\mu^2\\ iA_\mu^2 & 0\end{pmatrix} - \frac{1}{2}ig\begin{pmatrix}A_\mu^3 & 0\\ 0 & -A_\mu^3\end{pmatrix} - \frac{Y}{2}ig'\begin{pmatrix}B_\mu & 0\\ 0 & B_\mu\end{pmatrix}\right] \tag{4.91}$$

which, provided that $Y = 1$ for the Higgs doublet, results in

$$\begin{aligned}
\mathcal{L}_{EW} &= (D_\mu\Phi)^\dagger (D^\mu\Phi)\\
&= \frac{1}{2}\left(\frac{gv}{2}\right)^2 \left(A_\mu^1 A^{1\mu} + A_\mu^2 A^{2\mu}\right) + \frac{1}{2}\frac{v^2}{4}\left(g^2 A_\mu^3 A^{3\mu} + g'^2 B_\mu B^\mu - gg' A_\mu^3 B^\mu - gg' B_\mu A^{3\mu}\right)\\
&+ \frac{1}{8}g^2\left(h^2 A_\mu^1 A^{1\mu} + h^2 A_\mu^2 A^{2\mu}\right) + \frac{1}{8}\left(g^2 h^2 A_\mu^3 A^{3\mu} + g'^2 h^2 B_\mu B^\mu - gg' h^2 A_\mu^3 B^\mu - gg' h^2 B_\mu A^{3\mu}\right)\\
&+ \frac{1}{4}g^2 v\left(h A_\mu^1 A^{1\mu} + h A_\mu^2 A^{2\mu}\right) + \frac{v}{4}\left(g^2 h A_\mu^3 A^{3\mu} + g'^2 h B_\mu B^\mu - gg' h A_\mu^3 B^\mu - gg' B_\mu h A^{3\mu}\right)\\
&+ \cdots
\end{aligned} \tag{4.92}$$

where the second line corresponds to quadratic terms from where we will determine the gauge boson masses in the electroweak vacuum, the third line represents quartic interactions between the Higgs and the gauge bosons and the fourth line cubic Higgs-gauge interactions. The current Lagrangian is not yet in the mass basis as the second term on the second line of contains crossed terms among the A_μ^3 and B_μ fields. The first step is then to diagonalize the theory by rotating to the mass basis. Similarly to what was done for the abelian Standard Model we determine the eigenvalues and eigenvectors of the mass matrix by performing the following diagonalization

$$\mathbf{M^2} = \frac{v^2}{4}\begin{pmatrix}g^2 & gg'\\ gg' & g'^2\end{pmatrix} \longrightarrow \mathbf{m^2} = \frac{v^2}{4}\begin{pmatrix}0 & 0\\ 0 & g^2 + g'^2\end{pmatrix}\,, \tag{4.93}$$

which yields

$$\begin{aligned}
\mathcal{A}_\mu &= \frac{1}{\sqrt{g^2+g'^2}}\left(g' A_\mu^3 + g B_\mu\right) \qquad m_{\mathcal{A}} = 0\,,\\
Z_\mu &= \frac{1}{\sqrt{g^2+g'^2}}\left(g A_\mu^3 - g' B_\mu\right) \qquad m_Z = \sqrt{g^2+g'^2}\,\frac{v}{2}\,,
\end{aligned} \tag{4.94}$$

from where we can promptly identify $\mathcal{A}_\mu$ with the photon.

The first term in the second line of the electroweak Lagrangian is already diagonalized. However, it is common practice to redefine A^1_μ and A^2_μ introducing two complex vector fields W^+_μ and W^-_μ as

$$\begin{aligned} A^1_\mu &= \tfrac{1}{\sqrt{2}}\left(W^+_\mu + W^-_\mu\right) \\ A^2_\mu &= \tfrac{i}{\sqrt{2}}\left(W^+_\mu - W^-_\mu\right) \end{aligned} \tag{4.95}$$

from where the mass terms become

$$\mathcal{L}_{m_W} = \frac{1}{2}\left(\frac{gv}{2}\right)^2 \left(W^+_\mu W^{-\mu} + W^-_\mu W^{+\mu}\right) . \tag{4.96}$$

This redefinition will become clear in the discussion that follows about electric charge. In addition to Z_μ we find two additional massive states $W^\pm_\mu$ with mass

$$m_W = g\frac{v}{2} . \tag{4.97}$$

Note that we have obtained in eq. (4.82) three massless Goldstone bosons, G^1, G^2 and G^3, that were phased away from the Lagrangian. As we have seen in the abelian Standard Model, with the emergence of three massive gauge bosons, $W^\pm_\mu$ and Z_μ, that acquire longitudinal polarizations, the number of scalar degrees of freedom in the theory is preserved in both the gauge and mass basis.

With the diagonalization procedure we have discovered one massless state, the photon, and three massive ones. However, to properly identify each of them we need to determine their electric charges. To see this we first redefine the gauge couplings g' and g in terms of the so called **weak mixing angle** or **Weinberg angle** and the electric charge e:

$$\begin{aligned} g &= \frac{e}{\sin\theta_W} \\ g' &= \frac{e}{\cos\theta_W} \end{aligned} \tag{4.98}$$

From here it is straightforward to see that

$$\begin{aligned} \mathcal{A}_\mu &= \sin\,\theta_W\, A^3_\mu + \cos\,\theta_W\, B_\mu \\ Z_\mu &= \cos\,\theta_W\, A^3_\mu - \sin\,\theta_W\, B_\mu \end{aligned} \tag{4.99}$$

or inverting the system

$$\begin{aligned} A^3_\mu &= \sin\,\theta_W\, \mathcal{A}_\mu + \cos\,\theta_W\, Z_\mu \\ B_\mu &= \cos\,\theta_W\, \mathcal{A}_\mu - \sin\,\theta_W\, Z_\mu\, . \end{aligned} \tag{4.100}$$

A possible way to determine the electric charges of the Z_μ and $W^\pm_\mu$ states is by expanding the cubic interactions in $-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}$ and find how such states couple to the photon $\mathcal{A}_\mu$. Since the abelian part of the pure gauge sector, $B_{\mu\nu}B^{\mu\nu}$, does not contain self-interaction terms all we have to consider is the non-abelian part as follows:

$$\begin{aligned} \tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a &= -g\varepsilon^{abc}\partial^\nu g^{\mu\rho}A^a_\mu A^b_\nu A^c_\rho + \cdots \\ &= ie\partial^\nu g^{\mu\rho}\left(\mathcal{A}_\mu W^+_\nu W^-_\rho - \mathcal{A}_\mu W^+_\rho W^-_\nu + \text{permutations}\right) \\ &= ie\cot\theta_W \partial^\nu g^{\mu\rho}\left(Z_\mu W^+_\nu W^-_\rho - Z_\mu W^+_\rho W^-_\nu + \text{permutations}\right) \end{aligned} \tag{4.101}$$

where permutations is referred to the Lorentz indices μ, ν, ρ. As we can see, the $W_\mu^\pm$ interact with the photon, the electromagnetic interaction mediator, while Z_μ does not. This means that the former are the well known W-bosons, which carry electric charge, and the latter is the neutral Z-boson.

It is left as an exercise to the reader to determine all interactions in the bosonic sector and using the knowledge of previous sections determine the Feynman diagrams and respective Feynman rules.

The fermionic sector

Now we are going to look at the mass generation mechanism for fermions in the SM. In table 2 we have the matter particles in the SM and respective quantum numbers under the gauge symmetries. In this table, the $\mathrm{SU(2)_L}$ doublets are indicated in **2** while the bold face

SM	$\mathrm{SU(3)_C}$	$\mathrm{SU(2)_L}$	$\mathrm{U(1)_Y}$
$Q_{\mathrm{L}i}$	**3**	**2**	1/3
$u_{\mathrm{R}i}$	**3**	**1**	4/3
$d_{\mathrm{R}i}$	**3**	**1**	−2/3
$E_{\mathrm{L}i}$	**1**	**2**	−1
$e_{\mathrm{R}i}$	**1**	**1**	−2
Φ	**1**	**2**	1

Table 2: *Matter particles in the Standard Model and respective quantum numbers.*

1 denotes an $\mathrm{SU(2)_L}$-singlet, i.e. a field that does not couple to the $\mathrm{SU(2)_L}$ gauge fields A_μ^a. Note that left-handed fermions are doublets of $\mathrm{SU(2)_L}$ and are typically identified with the label L. Such a label refers to the fact that the left handed parts of the physical fermions transform under $\mathrm{SU(2)_L}$ and the right handed parts are singlets of $\mathrm{SU(2)_L}$. Their components read

$$E_{\mathrm{L}}^{1,2,3} = \begin{pmatrix} \nu_e \\ e_{\mathrm{L}} \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu_{\mathrm{L}} \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau_{\mathrm{L}} \end{pmatrix} \qquad Q_{\mathrm{L}}^i = \begin{pmatrix} u_{\mathrm{L}} \\ d_{\mathrm{L}} \end{pmatrix}, \begin{pmatrix} c_{\mathrm{L}} \\ s_{\mathrm{L}} \end{pmatrix}, \begin{pmatrix} t_{\mathrm{L}} \\ b_{\mathrm{L}} \end{pmatrix} \tag{4.102}$$

where in E_{L}^i we have introduced the left-handed electron, muon and tau, e_{L}, μ_{L} and τ_{L} and associated neutrinos ν_e, ν_μ and ν_τ respectively. In the same way we have introduced the left-handed up and down quarks u_{L} and d_{L}, the charm and strange quarks c_{L} and s_{L} and the top and bottom quarks t_{L} and b_{L}. Note that fields with a **3** are $\mathrm{SU(3)_C}$ colour triplets while **1** in the same column correspond to states that do not feel the strong force. The $\mathrm{U(1)_Y}$ charges are shown in the last column.

The first thing to note here is that gauge invariance forbids any fermion mass terms in the Lagrangian. For example, if we write a bilinear mass term $m_e \overline{E_{\mathrm{L}i}}\, e_{\mathrm{R}i}$ it violates both $\mathrm{SU(2)_L}$ and $\mathrm{U(1)_Y}$ interactions and therefore another way to generate fermion masses is needed. Let us then consider the Lagrangian

$$\mathcal{L}_\ell = \overline{E}_L^i\,(i\gamma^\mu D_\mu)\,E_{L\,i} + \overline{e}_R^i\,(i\gamma^\mu D_\mu)\,e_{R\,i} - \left[y_e^{ij}\,\overline{E}_{L\,i}\cdot\Phi\,e_{R\,j} + c.c.\right] . \tag{4.103}$$

The first two terms describe interactions among leptons and gauge bosons while the last one is a Yukawa interaction between leptons the Higgs boson. The indices i, j range from 1 to 3 and denote generation number.

Let us first determine the lepton masses. As we have already introduced in the abelian SM all we have to do is to expand the Yukawa terms about the vacuum state. We will assume a flavour diagonal basis that is, $y_e^{11} \equiv y_e$, $y_e^{22} \equiv y_\mu$, $y_e^{33} \equiv y_\tau$ and the remaining $y_e^{ij} = 0$.

$$\begin{aligned}
\mathcal{L}_m &= -\left[y_e^{ij}\,\overline{E}_{L\,i}\cdot\Phi\,e_{R\,j} + c.c.\right] \\
&= y_e\,\overline{E}_{L\,1}\cdot\Phi\,e_R + y_\mu\,\overline{E}_{L\,2}\cdot\Phi\,\mu_R + y_\tau\,\overline{E}_{L\,3}\cdot\Phi\,\tau_R \\
&= \frac{y_e}{\sqrt{2}}\begin{pmatrix}\overline{\nu_e} & \overline{e_{\mathrm{L}}}\end{pmatrix}\cdot\begin{pmatrix}0\\ v+h(x)\end{pmatrix}e_R + \frac{y_\mu}{\sqrt{2}}\begin{pmatrix}\overline{\nu_\mu} & \overline{\mu_{\mathrm{L}}}\end{pmatrix}\cdot\begin{pmatrix}0\\ v+h(x)\end{pmatrix}\mu_R \\
&+ \frac{y_\tau}{\sqrt{2}}\begin{pmatrix}\overline{\nu_\tau} & \overline{\tau_{\mathrm{L}}}\end{pmatrix}\cdot\begin{pmatrix}0\\ v+h(x)\end{pmatrix}\tau_R + \text{c.c.} \\
&= \frac{y_e v}{\sqrt{2}}\overline{e_{\mathrm{L}}}e_{\mathrm{R}} + \frac{y_\mu v}{\sqrt{2}}\overline{\mu_{\mathrm{L}}}\mu_{\mathrm{R}} + \frac{y_\tau v}{\sqrt{2}}\overline{\tau_{\mathrm{L}}}\tau_{\mathrm{R}} + \frac{y_e}{\sqrt{2}}h\overline{e_{\mathrm{L}}}e_{\mathrm{R}} + \frac{y_\mu}{\sqrt{2}}h\overline{\mu_{\mathrm{L}}}\mu_{\mathrm{R}} + \frac{y_\tau}{\sqrt{2}}h\overline{\tau_{\mathrm{L}}}\tau_{\mathrm{R}} \\
&+ \text{c.c.}
\end{aligned} \tag{4.104}$$

from where we obtain the lepton masses in the SM as

$$m_e = \frac{y_e v}{\sqrt{2}} \qquad m_\mu = \frac{y_\mu v}{\sqrt{2}} \qquad m_\tau = \frac{y_\tau v}{\sqrt{2}}\,. \tag{4.105}$$

Note that a full Dirac spinor can be constructed out of the left- and right-handed helicities as

$$\ell = \begin{pmatrix}\ell_{\mathrm{L}}\\ \hline \ell_{\mathrm{R}}\end{pmatrix}\,. \tag{4.106}$$

The lepton masses are proportional to their interaction strength with the Higgs field. In particular, since the electron mass is 0.511 MeV, the muon mass 105 MeV and the tau mass 1.777 GeV, then the Higgs boson interacts more intensely with the third generation leptons that with the second and first generations. On the other hand, note that there are no interactions at all between the Higgs and the neutrinos. This results in massless neutrinos according to the SM theory. However we know that it is not true. Neutrinos were measured to oscillate between different flavours which is only possible if they have mass. In fact, a flavour state is described as a quantum superposition of mass states

$$|\nu_\alpha\rangle = \sum_{k=1}^{3} U_{\alpha k}\,|\nu_k\rangle \tag{4.107}$$

with $\alpha = e, \mu, \tau$ a flavour index, $k = 1, 2, 3$ a mass index denoting the tiny neutrino masses m_1, m_2, m_3 and $U_{\alpha k}$ a unitary mixing matrix. This shown that although correct and outstandingly accurate in most if its predictions, the SM is not complete and fails to explain neutrino masses. Another important comment here is that while we have understood the origin of the lepton masses in the SM, we do not understand the hierarchies between generations. For example, the electron is 100 times lighter than the muon and 1000 times lighter than the tau. The origin of such hierarchies is a modern subject of intense studies in a sub-area of Particle Physics called Flavour Physics.

Another physical quantity that is generated in the theory vacuum is the electric charge. For simplicity we will consider only one generation since the procedure is identical for all three. Let us then expand the first two terms in (4.103) taking into account the hypercharge values for the $e_{\rm L}$ and $e_{\rm R}$ helicities, that is

$$\begin{aligned} &D_\mu E_{\rm L} \supset \\ &\supset \left[-\frac{1}{2}ig\begin{pmatrix}0 & A^1_\mu\\ A^1_\mu & 0\end{pmatrix} - \frac{1}{2}ig\begin{pmatrix}0 & -iA^2_\mu\\ iA^2_\mu & 0\end{pmatrix} - \frac{1}{2}ig\begin{pmatrix}A^3_\mu & 0\\ 0 & -A^3_\mu\end{pmatrix} + \frac{1}{2}ig'\begin{pmatrix}B_\mu & 0\\ 0 & B_\mu\end{pmatrix}\right]\begin{pmatrix}\nu_e\\ e_{\rm L}\end{pmatrix} \end{aligned} \tag{4.108}$$

where we have used $Y=-1$ for $E_{\rm L}$, and

$$D_\mu e_{\rm R} \supset -g'\gamma^\mu B_\mu e_{\rm R} \tag{4.109}$$

where $Y=-2$ for $e_{\rm R}$. Expanding the fermion-gauge interactions and writting the result in the mass basis we get

$$\begin{aligned} &\overline{E}_L\left(i\gamma^\mu D_\mu\right)E_L + \overline{e}_R\left(i\gamma^\mu D_\mu\right)e_R \\ =& -e\gamma^\mu \mathcal{A}_\mu \overline{e_{\rm L}}e_{\rm L} + \frac{1}{2}e\gamma^\mu\left(\tan\theta_W - \cot\theta_W\right)Z_\mu\overline{e_{\rm L}}e_{\rm L} \\ &- e\gamma^\mu\mathcal{A}_\mu\overline{e_{\rm R}}e_{\rm R} + e\gamma^\mu\tan\theta_W Z_\mu\overline{e_{\rm R}}e_{\rm R} \\ &+ \frac{1}{2}e\gamma^\mu\left(\cot\theta_W + \tan\theta_W\right)Z_\mu\overline{\nu_e}\nu_e + \frac{1}{\sqrt{2}}e\gamma^\mu\csc\theta_W W^-_\mu\overline{e_{\rm L}}\nu_e + \frac{1}{\sqrt{2}}e\gamma^\mu\csc\theta_W W^+_\mu\overline{\nu_e}e_{\rm L}\cdots. \end{aligned} \tag{4.110}$$

With this result we observe the following:

1. Only electrons interacts with photons with an interaction strength equal to $-e$. In the same way that the interaction strength with the Higgs boson sets their mass, the interaction strength with photons sets their electric charge.

2. Since neutrinos do not interact with photons they are neutral.

3. Neutrinos only couple to the weak gauge bosons $W^\pm_\mu$ and Z_μ. This means that they only interact via the weak force.

4. While the electromagnetic interaction does not distinguish between left and right helicities, the weak force does. This is manifest in the distinct coefficients in front of $Z_\mu\overline{e_{\rm L}}e_{\rm L}$ and $Z_\mu\overline{e_{\rm R}}e_{\rm R}$.

5. Vertices involving W bosons transform electrons into neutrinos or vice-versa.

The same strategy can be applied to the quark sector in order to obtain their masses. In particular, and considering only one generation, one can write the Yukawa Lagrangian as

$$\mathcal{L}_q = -y_d\,\overline{Q}_L \cdot \Phi\, d_R - y_u\,\overline{Q}_L \cdot \widetilde{\Phi}\, u_R + c.c. \tag{4.111}$$

where the Higgs doublet in the up-sector had to be redefined as

$$\widetilde{\Phi} = i\sigma^2\Phi^\dagger = \begin{pmatrix} \phi^0 \\ -\phi^- \end{pmatrix} \tag{4.112}$$

in order to build a gauge invariant term. Expanding around the vacuum state we get

$$\begin{aligned}\mathcal{L}_q &= \frac{y_d}{\sqrt{2}}\left(\overline{u_{\rm L}}\;\overline{d_{\rm L}}\right)\cdot\begin{pmatrix} 0 \\ v+h(x)\end{pmatrix} d_R + \frac{y_u}{\sqrt{2}}\left(\overline{u_{\rm L}}\;\overline{d_{\rm L}}\right)\cdot\begin{pmatrix} v+h(x) \\ 0\end{pmatrix} u_R \\ &= \frac{y_d v}{\sqrt{2}}\overline{d_{\rm L}}d_{\rm R} + \frac{y_u v}{\sqrt{2}}\overline{u_{\rm L}}u_{\rm R} + \frac{y_d}{\sqrt{2}}h\overline{d_{\rm L}}d_{\rm R} + \frac{y_u}{\sqrt{2}}h\overline{u_{\rm L}}u_{\rm R} + \text{c.c.}\end{aligned} \tag{4.113}$$

from where we take the masses

$$m_u = \frac{y_u v}{\sqrt{2}} \qquad m_d = \frac{y_d v}{\sqrt{2}}\,. \tag{4.114}$$

It is easy to generalize this procedure to three quark generations, however, a non-trivial mixing will emerge in charged currents which we leave for a more advanced course.

As a concluding remark, we have studied the mass and electric charge generation mechanism in the Standard Model. We have seen that the electroweak symmetry is broken in the minimum of the potential from where a remnant U(1) symmetry emerges describing the electromagnetic theory. We have also verified that all masses in the SM are proportional to the VEV v, which is the value that the Higgs field acquires in the electroweak broken phase. However, the SM fails to explain both neutrino masses as well as the huge hierarchies among the Yukawa couplings in order to fit measured masses. These questions are hot-topics in Particle Physics studied both in the domains of flavour and neutrino Physics. In the next section we will have a quick glance at the neutrino mass generation mechanism, the Seesaw mechanism.

5 Extensions of the Standard Model

5.1 The Seesaw mechanism

In the previous section we have identified a problem in the Standard Model, the lack of a mechanism to explain neutrino masses. In fact, the SM only contain left-handed neutrinos which belong to the lepton $\mathrm{SU}(2)_{\rm L}$ doublets

$$E_{\rm L} = \begin{pmatrix} \nu_e \\ e_{\rm L}\end{pmatrix}\,. \tag{5.1}$$

Without a right-handed counterpart there is no way of writing Yukawa terms for neutrinos in order to allow interaction with the Higgs boson, thus mass generation. The existence of neutrino masses is strongly suggesting that there must be right-handed neutrinos in nature, that we shall denote as $N_{\rm R}$. Right handed neutrinos are pure SM singlets which means that they do not interact with any SM gauge bosons. Their quantum numbers can be found in table 3. Since right-handed neutrinos are SM singlets they allow bilinear mass terms

νSM	$\mathrm{SU}(3)_{\rm C}$	$\mathrm{SU}(2)_{\rm L}$	$\mathrm{U}(1)_{\rm Y}$
$Q_{{\rm L}i}$	$\mathbf{3}$	$\mathbf{2}$	$1/3$
$u_{{\rm R}i}$	$\mathbf{3}$	$\mathbf{1}$	$4/3$
$d_{{\rm R}i}$	$\mathbf{3}$	$\mathbf{1}$	$-2/3$
$E_{{\rm L}i}$	$\mathbf{1}$	$\mathbf{2}$	-1
$e_{{\rm R}i}$	$\mathbf{1}$	$\mathbf{1}$	-2
$N_{{\rm R}i}$	$\mathbf{1}$	$\mathbf{1}$	0
Φ	$\mathbf{1}$	$\mathbf{2}$	1

Table 3: *Matter particles in an extension of the Standard Model with right-handed neutrinos and respective quantum numbers.*

of the form $M_N N_{\rm R} N_{\rm R}$. Then, considering only one generation of leptons, the part of the Lagrangian involving right-handed neutrinos reads

$$\begin{aligned}\mathcal{L}_N &= -y_\nu\, \overline{E}_L \cdot \widetilde{\Phi}\, N_{\rm R} + M_N N_{\rm R} N_{\rm R} + c.c. \\ &= \frac{y_\nu}{\sqrt{2}} \begin{pmatrix} \overline{\nu_e} & \overline{e_{\rm L}} \end{pmatrix} \cdot \begin{pmatrix} v + h(x) \\ 0 \end{pmatrix} N_R + M_N N_{\rm R} N_{\rm R} + c.c. \\ &= \frac{y_\nu v}{\sqrt{2}} \overline{\nu_e} N_{\rm R} + M_N N_{\rm R} N_{\rm R} + c.c. + \cdots \end{aligned} \tag{5.2}$$

However, due to the extra term M_N, the Lagrangian is not in the mass basis and should be diagonalized. The mass matrix written in the flavour basis $\{\nu_e, N_{\rm R}\}$ is

$$M_\nu = \begin{pmatrix} 0 & \frac{y_\nu v}{\sqrt{2}} \\ \frac{y_\nu^* v}{\sqrt{2}} & M_N \end{pmatrix} \tag{5.3}$$

whose eigenvalues are

$$m_\pm = \frac{1}{2}\left(M_N \pm \sqrt{M_N^2 + 2 v^2 y_\nu^2} \right) \tag{5.4}$$

However, something interesting should be noted here. Active neutrino masses are tiny, i.e. below eV, however, right handed neutrinos have not yet been observed and can be either very weakly interacting to evade detection or alternatively very heavy. Here we will consider the later case and set $M_N \gg v$. We can then expand the eigenvalues of the mass matrix for large M_N obtaining the following form for the eigenvalues:

$$m_\nu \approx \frac{1}{2} \frac{v^2 y_\nu^2}{M_N} \qquad m_N \approx M_N\,. \tag{5.5}$$

Note that for large M_N the light neutrino mass m_ν will be suppressed by some unknown high scale physics that is in the origin of the right-handed neutrino mass. Let us try some numerics. Let us assume that the scale at which the right-handed neutrino masses are generated is $M_N = 10^{16}$ GeV. Let us also assume that the Yukawa coupling $y_\nu = 1$ and finally let us take the measured value for the Higgs VEV $v = 246$ GeV. Putting all together we get

$$m_\nu = 0.003 \text{ eV} \qquad M_\nu = 10^{16} \text{ GeV} \tag{5.6}$$

in consistency with experimental data for light active neutrinos. So, we have seen that the presence of a large scale induces a remarkable suppression in the neutrino masses without the need for any fine-tuning (note that the Yukawa coupling was even set to 1.) This mechanism of generating tiny masses out of a large mass scale is called the **Seesaw Mechanism** and even though not yet experimentally conformed (no right handed neutrinos have been discovered so far) it is broadly accepted as the mechanism responsible for neutrino mass generation.

References

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