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$f(R)$ theories

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Chapter 1

Introduction - Margarida Baptista

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1.1 Historical Overview

Gravity is an important fundamental force which rules the Universe.

Since the 17th century, Isaac Newton was already studying the gravitational mass attration.

With the discovery of General relativity in 1915, developed by Albert Einstein, which brought a paradigm shift in the whole field of physics. It obeys a deeply reasoning which plays a crucial role to understand gravity and describe the Universe on a large scale. Introduces the curvature of spacetime due to the presence of mass and energy.

This theory was proven when for the first time it was detected gravitational waves from a binary black hole in 2015 by LIGO.

Despite GR being a huge success it had a few pendent questions to solve. The occurrence of singularities at the centers of blackholes suggests that GR is inapplicable because of the breakdown of the equivalence principle at the singularities.

Theories such as $f(R)$ theories propose modifications in gravity, implementing a arbitrary function $f(R)$, which describes how matter and energy curve in spacetime. Unlike GR, whose field equations contain only up to second-order derivatives, the modified theories with higher derivative Ricci/Riemann tensor gravity models include higher derivatives.

1.2 Motivation

The motivation of the work reside in the necessary of explore new approaches beyond General Relativity. Modified gravity theories propose other way to give answers about observational phenomena beyond GR. The concept of $f(r)$ theories are a set of modified gravity theories for an arbitrary function, in a scalar curvature, R.

In this essay we will approach modified gravity, exploring f(R) theories, which appears, of course, during the 20st century, then explore its equivalence with other theories such as Brans-Dicke, developed in 1961, then explore conformal transformations of this theories, explain the difference between the Jordan and Einstein frames, and the appearance of a propagating field in the conformal $f(R)$ which will be the main topic in the last chapter while introducing the main approach physicist use to explain the late-time acceleration of the universe, dark energy and hence the cosmological constant.

Therefore, in this work we will start with a brief explanation about the importance of modified gravity and why it's importance to explore.

1.2.1 Why and How

Is important to clarify why modifications to Einstein's General Theory of Relativity (GR) may be necessary.

Einstein's General Theory of Relativity is based on an action that is linearly dependent on the curvature scalar R. However, there are four reasons why this linear dependence might not be valid.

- The first reason is that gravity is not tested at all scales. While General Relativity is well-tested at solar and stellar scales, it has not been extensively tested at cosmic scales.
- The second reason is that, in principle, in addition to the metric field, other gravitational fields, such as scalar fields, may exist. If these fields have a weaker influence compared to gravity, it could lead to a violation of the Weak Equivalence Principle [12].

- The third reason is that, according to GR, gravity should follow an inverse square law, but this law may not be conclusive at all scales, as demonstrated by Yukawa-type gravity interaction experiments [1].
- The fourth reason is that with improved observational and experimental data, there may be a need for corrections in situations of strong gravity and at very large scales, leading to modifications in GR.

These reasons support the idea that modified theories of gravity, like $f(R)$, may be the correct way to solve these problems.

The standard cosmological model, ΛCDM (Lambda Cold Dark Matter model), which assumes that General Relativity accurately describes gravity on cosmological scales. This model postulates the existence of dark matter and dark energy (represented by the cosmological constant Λ) as the dominant components of the universe's energy budget.

Despite the success of the ΛCDM model in explaining various observations, there are several issues it faces:

- Fine Tuning Problem: The current energy density of the cosmological constant Λ is estimated to be extremely small, which contradicts the predicted vacuum energy density from quantum field theory. This requires fine-tuning to reconcile the observed value with theoretical expectations.
- Coincidence Problem: The present value of the dark energy density parameter and the matter-energy density parameter are approximately equal, which is considered coincidental and would require fine-tuning in the early Universe. This is known as the cosmological coincidence problem.
- Tensions between Observations: While the ΛCDM model aligns well with both Cosmic Microwave Background (CMB) observations and late-time universe observations (related to the acceleration of the universe), there are growing tensions between these two sets of data, this tensions are discrepancies between the values of the Hubble parameter.

These problems also motivate the necessaty for other ways to explain the expansion of the universe, for example quintessence, which uses a scalar field (fifth force), to accelerate the universe.

1.3 $f(R)$ gravity

First of all, we need to understand by the Lovelock's Theorem that GR is unique in four dimensions, imposing restrictions such as the linearity of the equations of motion, absence of additional degrees of freedom, covariance, locality, and the limitation to four-dimensional spacetime. These conditions, while limiting modifications to GR, open up space for investigating alternatives when relaxed. When the conditions of Lovelock's Theorem are relaxed, opportunities arise for constructing consistent modified theories. By extending the analysis to higher dimensions, we observe that Lovelock's Theorem loses its rigidity, allowing for the existence of more diverse gravitational theories [6].

There are four conditions to passby the Lovelock's Theorem. In particular, the "Beyond second order", where include the $f(R)$ theories.

1.3.1 Action principle

The action for the $f(R)$ gravity as mentioned in the introduction, assumes an arbitrary proportionality between the Lagrangian and the Ricci curvature scalar, and is given by [9],

$$S = \int_{\mathcal{M}} d^D x \mathcal{L} = \int_{\mathcal{M}} d^D x \sqrt{-g} \left(\frac{f(R)}{2k} + \mathcal{L}_m \right), \tag{1.1}$$

where $\mathcal{L}_m$ is the matter Lagrangian, and $f(R)$ an arbitrary function of the Ricci curvature scalar, R, and D is an arbitrary dimension, usual 4 space and time, but for the sake of generality we will use D.

By the principle of action, we can derive the field equations as follows,

$$\begin{aligned} \delta S &= \int_{\mathcal{M}} d^D x \left[\delta\sqrt{-g} \left(\frac{f(R)}{2k} \right) + \sqrt{-g} \frac{\delta(f(R))}{2k} \right] + \int_{\mathcal{M}} d^D x \delta(\sqrt{-g}\mathcal{L}_m) \\ &= \int_{\mathcal{M}} d^D x \left[-\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \left(\frac{f(R)}{2k} \right) + \sqrt{-g} \frac{1}{2k} \frac{\partial f(R)}{\partial R} \delta R \right] - \int_{\mathcal{M}} d^D x \frac{\sqrt{-g}}{2} T^{(M)}_{\mu\nu} \delta g^{\mu\nu}, \end{aligned} \tag{1.2}$$

where in the last it was used A.1 and A.4.

The second term can be computed as,

$$\begin{aligned} \frac{\partial f(R)}{\partial R}\delta R &= F(R)\delta(R_{\mu\nu}g^{\mu\nu}) \\ &= F(R)\left(R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right) \\ &= F(R)\left(R_{\mu\nu}\delta g^{\mu\nu} + \nabla^\mu\nabla^\nu\delta g_{\mu\nu} - g^{\mu\nu}\Box\delta g_{\mu\nu}\right), \end{aligned} \tag{1.3}$$

where $F(R)$ is the derivative of f, and in the last step A.3.

Then the action is,

$$\begin{aligned} &\int_{\mathcal{M}} d^D x\frac{\sqrt{-g}}{2}\left[-g_{\mu\nu}\delta g^{\mu\nu}\left(\frac{f(R)}{2k}\right) - T^{(M)}_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{k}F(R)\left(R_{\mu\nu}\delta g^{\mu\nu} + \nabla^\mu\nabla^\nu\delta g_{\mu\nu} - g^{\mu\nu}\Box\delta g_{\mu\nu}\right)\right] \\ &= \left(\int_{\mathcal{M}} d^D x\frac{\sqrt{-g}}{2k}\left[-g_{\mu\nu}\left(\frac{f(R)}{2}\right) - kT^{(M)}_{\mu\nu} + F(R)R_{\mu\nu}\right]\delta g^{\mu\nu} + \int_{\mathcal{M}} d^D x\frac{\sqrt{-g}}{2k}F(R)\left(\nabla^\mu\nabla^\nu\delta g_{\mu\nu} - g^{\mu\nu}\Box\delta g_{\mu\nu}\right)\right). \end{aligned} \tag{1.4}$$

By integration by parts the second integral is equal to,

$$\begin{aligned} \int_{\mathcal{M}} d^D x\frac{\sqrt{-g}}{2k}F(R)\left(\nabla^\mu\nabla^\nu\delta g_{\mu\nu} - g^{\mu\nu}\Box\delta g_{\mu\nu}\right) = \int_{\mathcal{M}} d^D x\frac{\sqrt{-g}}{2k}&\Big[\nabla^\mu\nabla^\nu\left(F(R)\delta g_{\mu\nu} - g_{\mu\nu}g^{\alpha\beta}\delta g_{\alpha\beta}\right) \\ &- \left(\nabla^\mu\nabla^\nu F(R)\right)\left(\delta g_{\mu\nu} - g_{\mu\nu}g^{\alpha\beta}\delta g_{\alpha\beta}\right)\Big] \end{aligned} \tag{1.5}$$

$$= \int_{\mathcal{M}} d^D x\frac{\sqrt{-g}}{2k}\left(g^{\mu\nu}\Box - \nabla^\mu\nabla^\nu\right)F(R)\delta g_{\mu\nu} + \int_{\mathcal{M}} d^D x\nabla^\mu\left[\frac{\sqrt{-g}}{2k}\nabla^\nu\left(F(R)\delta g_{\mu\nu} - g_{\mu\nu}g^{\alpha\beta}\delta g_{\alpha\beta}\right)\right] \tag{1.6}$$

$$= \int_{\mathcal{M}} d^D x\frac{\sqrt{-g}}{2k}\left(g^{\mu\nu}\Box - \nabla^\mu\nabla^\nu\right)F(R)\delta g_{\mu\nu} + \int_{\partial\mathcal{M}} d\Sigma_x\frac{\sqrt{-g}}{2k}n^\mu\nabla^\nu\left(F(R)\delta g_{\mu\nu} - g_{\mu\nu}g^{\alpha\beta}\delta g_{\alpha\beta}\right) \tag{1.7}$$

$$= \int_{\mathcal{M}} d^D x\frac{\sqrt{-g}}{2k}\left(g^{\mu\nu}\Box - \nabla^\mu\nabla^\nu\right)F(R)\delta g_{\mu\nu}, \tag{1.8}$$

here we assume the Gibbons–Hawking–York boundary term is zero, and we will assume it is zero throughout this essay.

Going back to the action we get,

$$\int_{\mathcal{M}} d^D x\frac{\sqrt{-g}}{2k}\left[-g_{\mu\nu}\left(\frac{f(R)}{2}\right) - kT^{(M)}_{\mu\nu} + F(R)R_{\mu\nu} + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)F(R)\right]\delta g^{\mu\nu}. \tag{1.9}$$

Therefore, the equation of motion with respect to the metric is [9],

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \Leftrightarrow -g_{\mu\nu}\left(\frac{f(R)}{2}\right) + F(R)R_{\mu\nu} + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)F(R) = kT^{(M)}_{\mu\nu}.$$

Another way to write it,

$$kT_{\mu\nu} = G_{\mu\nu} - G_{\mu\nu} - g_{\mu\nu}\left(\frac{f(R)}{2}\right) + F(R)R_{\mu\nu} + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)F(R) \Leftrightarrow \tag{1.10}$$

$$\Leftrightarrow G_{\mu\nu} = kT^{(M)}_{\mu\nu} + \left[(1 - F(R))R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}(f(R) - R) + (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)F(R)\right], \tag{1.11}$$

where $G_{\mu\nu}$ is the Einstein tensor, which is given by,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{1.12}$$

We can see this yields a new energy-momentum tensor, given by

$$kT^{(\text{curve})}_{\mu\nu} = (1 - F(R))R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}(f(R) - R) + (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)F(R), \tag{1.13}$$

which we will call the curvature energy-momentum tensor. Traditionally this tensor wouldn't exist in Hilbert-Einstein gravity.

The continuity equation for the matter energy-momentum tensor, $\nabla^\mu T^{(M)}_{\mu\nu} = 0$, and also knowing that $\nabla^\mu G_{\mu\nu} = 0$, we can conclude the curvature energy-momentum tensor also follows the continuity equation, $\nabla^\mu T^{(\text{curve})}_{\mu\nu} = 0$.

The trace of the field equation 1.11,

$$g^{\mu\nu}G_{\mu\nu} = kT^{(M)}_{\mu\nu}g^{\mu\nu} + \left[(1-F(R))R_{\mu\nu}g^{\mu\nu} + \frac{1}{2}g_{\mu\nu}g^{\mu\nu}(f(R)-R) + (g^{\mu\nu}\nabla_\mu\nabla_\nu - g^{\mu\nu}g_{\mu\nu}\Box)F(R)\right] \Leftrightarrow \tag{1.14}$$

$$\Leftrightarrow g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R = kT^{(M)} + \left[(1-F(R))R + \frac{D}{2}(f(R)-R) + (g^{\mu\nu}\nabla_\mu\nabla_\nu - D\Box)F(R)\right] \Leftrightarrow \tag{1.15}$$

$$\Leftrightarrow 0 = kT^{(M)} - F(R)R + \frac{D}{2}f(R) + \Box F(R) - D\Box F(R) \Leftrightarrow \tag{1.16}$$

$$\Leftrightarrow F(R)R + (D-1)\Box F(R) - \frac{D}{2}f(R) = kT^{(M)}. \tag{1.17}$$

In Hilbert-Einstein gravity, without a cosmological constant, is the same as having $f(R) = R$, so that the term $F(R) = 1$, vanishes. In the traditional gravity we have that $R = -2kT^{(M)}$ and hence the Ricci curvature scalar R is directly determined by the matter energy-momentum tensor. In modified gravity the term $F(R)$ does not vanish, which means that there is a propagating scalar, $\varphi = F(R)$. The equation obtained by the trace determines the dynamics of the scalar field, called “scalaron”. This scalaron will appear when treating the conformal transformation of the $f(R)$ theory, which we will check later.

1.4 Brans-Dicke theory

So, Brans was a graduate student in theoretical physics. He approached Dicke. Dicke suggested exploring a general relativistic variation of Einstein's theory, ensuring the conservation of matter. Brans introduced a new scalar field, ϕ, coupled to mass and geometry, with the introduction of a dimensionless constant, ω, determined experimentally. The resulting theory, known as the Brans-Dicke (BD) formulation, incorporates a new constant ω and a scalar field ϕ into Einstein's formalism. The field equation for ϕ is related to the Ricci curvature, suggesting that the theory can be interpreted as adding a scalar field to the metric to produce the total gravitational field.

The Brans-Dicke Theory (BD) extends Einstein's general theory of relativity by introducing a dynamic scalar field to accommodate for a variable gravitational constant and incorporate Mach's principle. In Einstein's standard theory, the spacetime metric tensor, or more precisely, geometry, is the sole quantity describing gravity. Dicke suggested the addition of another field, a scalar field. This scalar field, denoted as ϕ, much like gravity, has all matter as its source. Thus, in some semantic sense, it can be described as an extension of the gravitational field from purely geometric to geometric plus scalar, hence the term "scalar-tensor."

In Brans-Dicke, the gravitation constant is assumed not to be constant and be replaced by a scalar field, $\phi = 1/G$, with the $k = 8\pi G/c^D = 8\pi/(\phi c^D)$, we will define a new constant $k' = 8\pi/c^D$ as [2],

$$S = \int d^Dx\sqrt{-g}\left(\frac{1}{2k'}\left[R\phi - \omega\frac{\phi_{,\rho}\phi^{,\rho}}{\phi}\right] + \mathcal{L}_m\right). \tag{1.18}$$

The scalar aspect of ϕ is important since it cannot be "gauged" away as the metric and connection components can be, at least locally [2].

In this theory, the gravitational constant becomes time-dependent, varying inversely with a scalar field coupled to gravity with a coupling parameter ω. The Mach's principle posits that the measure of mass is not absolute but results from interactions with neighbouring bodies. Mach's Principle is incorporated into B-D by considering that the inertial mass of an object is not intrinsic but results from its interaction with surrounding matter. This represents a significant conceptual shift in our understanding of gravity [13].

The scalar field, represented by ϕ, is crucial in BD, influencing gravitational dynamics. The coupling parameter ω determines the strength of this interaction. As w approaches infinity, the theory recovers General Relativity. Observational constraints set minimum limits for ω, with large values providing adequate inflation and small/negative values explaining cosmic acceleration and structure formation.

Now, we will focus in understanding the influence of the parameter ω. This point helps us to visualize when the Brans-Dicke theory and General Relativity are similar. We already know that when ω tends to infinite these two theories became similar, however when w is a finite number, Brans-Dicke theory introduces distinct effects compared to General Relativity. It is also similar when ω tends to zero, to $f(R)$ theories.

In these situations, the scalar field ϕ plays a significant role, influencing gravitational dynamics and introducing specific features not present in General Relativity.

1.4.1 Action principle

By the action principle,

$$\delta S = \int_{\mathcal{M}} d^D x \delta \left\{ \sqrt{-g} \frac{1}{2k'} \left[R\phi - \omega \frac{\phi_{,\rho}\phi^{,\rho}}{\phi} \right] + \sqrt{-g}\mathcal{L}_m \right\} \tag{1.19}$$

$$= \int_{\mathcal{M}} d^D x \left\{ \delta\sqrt{-g} \frac{1}{2k'} \left[R\phi - \omega \frac{\phi_{,\rho}\phi^{,\rho}}{\phi} \right] + \sqrt{-g} \frac{1}{2k'} \delta \left[R\phi - \omega \frac{\phi_{,\rho}\phi^{,\rho}}{\phi} \right] \right\} + \int_{\mathcal{M}} d^D x \delta \left(\sqrt{-g}\mathcal{L}_m \right) \tag{1.20}$$

$$= \int_{\mathcal{M}} d^D x \left\{ -\frac{1}{2} g_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu} \frac{1}{2k'} \left[R\phi - \omega \frac{\phi_{,\rho}\phi^{,\rho}}{\phi} \right] + \sqrt{-g} \frac{1}{2k'} \left[\delta(R_{\alpha\beta} g^{\alpha\beta}\phi) - \omega\delta \left(g^{\mu\nu} \frac{\phi_{,\mu}\phi_{,\nu}}{\phi} \right) \right] \right\} \tag{1.21}$$

$$- \int_{\mathcal{M}} d^D x \frac{\sqrt{-g}}{2} \delta g^{\mu\nu} T^{(M)}_{\mu\nu} \tag{1.22}$$

$$= \int_{\mathcal{M}} d^D x \left(-\frac{1}{2} T^{(M)}_{\mu\nu} - \frac{g_{\mu\nu}}{4k'} \left[R\phi - \omega \frac{\phi_{,\rho}\phi^{,\rho}}{\phi} \right] \right) \sqrt{-g}\delta g^{\mu\nu} \tag{1.23}$$

$$+ \int_{\mathcal{M}} d^D x \sqrt{-g} \frac{1}{2k'} \left[g^{\alpha\beta}\phi\delta R_{\alpha\beta} + R_{\alpha\beta}\phi\delta g^{\alpha\beta} + R_{\alpha\beta} g^{\alpha\beta}\delta\phi - \omega \left[\frac{\phi_{,\mu}\phi^{,\nu}}{\phi} \delta g^{\mu\nu} + g^{\mu\nu} \frac{\delta(\phi_{,\mu}\phi_{,\nu})\phi - \phi_{,\mu}\phi_{,\nu}\delta\phi}{\phi^2} \right] \right] \tag{1.24}$$

$$= \int_{\mathcal{M}} d^D x \left(R_{\mu\nu} \frac{\phi}{2k'} - \frac{\omega}{2k'} \frac{\phi_{,\mu}\phi_{,\nu}}{\phi} - \frac{1}{2} T^{(M)}_{\mu\nu} - \frac{g_{\mu\nu}}{4k'} \left[R\phi - \omega \frac{\phi_{,\rho}\phi^{,\rho}}{\phi} \right] \right) \sqrt{-g}\delta g^{\mu\nu} \tag{1.25}$$

$$+ \int_{\mathcal{M}} d^D x \sqrt{-g} \frac{1}{2k'} \left[\phi \left(\nabla^\alpha\nabla^\beta \delta g_{\alpha\beta} - g^{\alpha\beta}\Box\delta g_{\alpha\beta} \right) + R_{\alpha\beta} g^{\alpha\beta}\delta\phi - \omega g^{\mu\nu} \left[\frac{\delta(\phi_{,\mu}\phi_{,\nu})}{\phi} - \frac{\phi_{,\mu}\phi_{,\nu}}{\phi^2}\delta\phi \right] \right]. \tag{1.26}$$

Taking a similar approach as 1.8, the term,

$$\int_{\mathcal{M}} d^D x \sqrt{-g} \frac{1}{2k'} \phi \left(\nabla^\alpha\nabla^\beta \delta g_{\alpha\beta} - g^{\alpha\beta}\Box\delta g_{\alpha\beta} \right) = \int_{\mathcal{M}} d^D x \frac{\sqrt{-g}}{2k'} \left(g^{\mu\nu}\Box - \nabla^\mu\nabla^\nu \right) \phi\delta g_{\mu\nu}. \tag{1.27}$$

And the term, can also be treated as, remembering the Euler-Lagrange equations

$$-\omega g^{\mu\nu} \frac{\delta(\phi_{,\mu}\phi_{,\nu})}{\phi} = -g^{\mu\nu} \frac{\omega}{\phi} \frac{\partial(\phi_{,\mu}\phi_{,\nu})}{\partial\phi}\delta\phi = -g^{\mu\nu} \frac{\omega}{\phi} \partial_\rho \left(\frac{\partial(\phi_{,\mu}\phi_{,\nu})}{\partial(\phi_{,\rho})} \right) \delta\phi = -2\frac{\omega}{\phi} \partial_\rho \left(\phi^{,\rho} \right) \delta\phi = -2\frac{\omega}{\phi} \Box\phi\delta\phi. \tag{1.28}$$

Finally our action is,

$$\delta S = \int_{\mathcal{M}} d^D x \left(R_{\mu\nu}\phi + \left(g^{\mu\nu}\Box - \nabla^\mu\nabla^\nu \right)\phi - \omega \frac{\phi_{,\mu}\phi_{,\nu}}{\phi} - k' T^{(M)}_{\mu\nu} - \frac{g_{\mu\nu}}{2} \left[R\phi - \omega \frac{\phi_{,\rho}\phi^{,\rho}}{\phi} \right] \right) \frac{\sqrt{-g}}{2k'} \delta g^{\mu\nu} \tag{1.29}$$

$$+ \int_{\mathcal{M}} d^D x \sqrt{-g} \frac{1}{2k'} \left[R - 2\frac{\omega}{\phi}\Box\phi + \omega g^{\mu\nu} \frac{\phi_{,\mu}\phi_{,\nu}}{\phi^2} \right] \delta\phi. \tag{1.30}$$

The field equations of the Brans-Dicke theory are [2],

$$G_{\mu\nu} = \frac{1}{\phi} k' T_{\mu\nu} + \frac{1}{\phi} \left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi \right) + \frac{\omega}{\phi^2} \left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2} g_{\mu\nu} (\nabla\phi)^2 \right] \tag{1.31}$$

$$\Box\phi = \frac{\phi_{,\alpha}\phi^{,\alpha}}{2\phi} - \frac{\phi}{2\omega} R, \tag{1.32}$$

This Klein-Gordon type equation relates ϕ to the geometry expressed in the Ricci curvature, so that perhaps this theory should be interpreted as adding a scalar field to the metric to produce the total gravitational field.

In this spirit the variation with respect to the metric and the wave equation gives

$$(3 + 2\omega)\Box\phi = k' T. \tag{1.33}$$

1.5 Equivalence of Brans-Dicke with $f(R)$ theories

Changing the $f(R)$ action in the following way [9],

$$S = \int_{\mathcal{M}} d^D x \mathcal{L} = \int_{\mathcal{M}} d^4 x \sqrt{-g} \left(\frac{f(R)}{2k} + \mathcal{L}_m \right) = \int_{\mathcal{M}} d^4 x \sqrt{-g} \left(\frac{RF(R) - RF(R)}{2k} + \frac{f(R)}{2k} + \mathcal{L}_m \right) \tag{1.34}$$

$$= \int_{\mathcal{M}} d^D x \sqrt{-g} \left(\frac{RF(R)}{2k} + \underbrace{\frac{f(R) - RF(R)}{2k}}_{Y} + \mathcal{L}_m \right) = \int_{\mathcal{M}} d^D x \sqrt{-g} \left(\frac{RF(R)}{2k} + Y + \mathcal{L}_m \right), \tag{1.35}$$

where $Y = \frac{f(R) - RF(R)}{2k}$.

Remembering the Brans-Dicke action is

$$S = \int_{\mathcal{M}} d^4 x \sqrt{-g} \left(\frac{R\phi}{2k'} - \omega \frac{\phi_{,\mu} \phi^{,\mu}}{\phi} + \mathcal{L}_m \right). \tag{1.36}$$

If we set the parameters as [9],

$$\omega = 0, \quad F(R) = G\phi, \quad f(R) = GR\phi. \tag{1.37}$$

And the field equations of $f(R)$ are,

$$G_{\mu\nu} = kT_{\mu\nu} + \left[(1 - F(R))R_{\mu\nu} + \frac{1}{2} g_{\mu\nu}(f(R) - R) + (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)F(R) \right], \tag{1.38}$$

and by substituting the parameters we get,

$$G_{\mu\nu} = kT_{\mu\nu} + \left[(1 - G\phi)R_{\mu\nu} + \frac{1}{2} g_{\mu\nu}(GR\phi - R) + (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)G\phi \right] \tag{1.39}$$

$$= kT_{\mu\nu} + \left[(1 - G\phi)R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu}(G\phi - 1) + G(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)\phi \right], \tag{1.40}$$

now if we substitute the gravitational constant $G = \frac{1}{\phi}$, we get,

$$G_{\mu\nu} = \frac{1}{\phi} k' T_{\mu\nu} + \frac{1}{\phi}(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)\phi, \tag{1.41}$$

which corresponds to the Brans-Dicke field equation when the parameter $w = 0$.

Chapter 2

Conformal transformations - Bruno Parracho

Bruno Parracho

2.1 Introduction

A conformal transformations are a local change of scale that translates to a local change in the metric of our manifold, and it's given by [9],

$$\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}. \tag{2.1}$$

This transformation is often used in **scalar-tensor theories**, where a scalar field, λ, is coupled directly with the scalar curvature, an example where this happens is the previously studied **Brans-Dicke** gravity. This new transformation intrinsically gives us a new frame of reference often called the **Einstein frame**, since in this new frame the equations take the original **Hilbert-Einstein**, hence the action in this new frame is linear with the conformal curvature scalar.

2.2 Conformal Riemann and Ricci tensors

This section will be based on [18]. In this section, we will start by determining the transformed Riemann and Ricci tensors, which will be necessary to derive the field equations for the different actions being studied.

First off, the christoffel symbol is given by

$$\begin{aligned}
\Gamma^\lambda_{\mu\nu} &= \frac{1}{2} g^{\lambda\alpha}\left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right) = \frac{1}{2}\Omega^2 \tilde{g}^{\lambda\alpha}\left(\partial_\mu\left(\Omega^{-2}\tilde{g}_{\nu\alpha}\right) + \partial_\nu\left(\Omega^{-2}\tilde{g}_{\alpha\mu}\right) - \partial_\alpha\left(\Omega^{-2}\tilde{g}_{\mu\nu}\right)\right) \\
&= \frac{1}{2}\Omega^2\tilde{g}^{\lambda\alpha}\left(\Omega^{-2}\right)\left[\partial_\mu\tilde{g}_{\nu\alpha} + \partial_\nu\tilde{g}_{\alpha\mu} - \partial_\alpha\tilde{g}_{\mu\nu}\right] + \frac{1}{2}\Omega^2\tilde{g}^{\lambda\alpha}\left(\tilde{g}_{\nu\alpha}\partial_\mu\Omega^{-2} + \tilde{g}_{\alpha\mu}\partial_\nu\Omega^{-2} - \tilde{g}_{\mu\nu}\partial_\alpha\Omega^{-2}\right) \\
&= \widetilde{\Gamma}^\lambda_{\mu\nu} + \frac{1}{2}\Omega^2\tilde{g}^{\lambda\alpha}\left(-2\Omega^{-3}\right)\left(\tilde{g}_{\nu\alpha}\partial_\mu\Omega + \tilde{g}_{\alpha\mu}\partial_\mu\Omega - \tilde{g}_{\mu\nu}\partial_\alpha\Omega\right) \\
&= \widetilde{\Gamma}^\lambda_{\mu\nu} - \tilde{g}^{\lambda\alpha}\left(\tilde{g}_{\nu\alpha}\partial_\mu\ln\Omega + \tilde{g}_{\alpha\mu}\partial_\nu\ln\Omega - \tilde{g}_{\mu\nu}\partial_\alpha\ln\Omega\right) \\
&= \widetilde{\Gamma}^\lambda_{\mu\nu} - \left(2\delta^\lambda_{(\mu}\nabla_{\nu)}\ln\Omega - \tilde{g}^{\lambda\sigma}\tilde{g}_{\mu\nu}\nabla_\sigma\ln\Omega\right) \\
&= \widetilde{\Gamma}^\lambda_{\mu\nu} - C^\lambda_{\mu\nu}.
\end{aligned}$$

Given this lets find the relation between ∇_a and $\tilde{\nabla}_a$ for a vector w which will prove useful,

$$\tilde{\nabla}_a w_b = \partial_a w_b - \widetilde{\Gamma}^c_{ab} w_c = \left(\nabla_a w_b + \Gamma^c_{ab} w_c\right) - \widetilde{\Gamma}^c_{ab} w_c = \nabla_a w_b + \left(\Gamma^c_{ab} - \widetilde{\Gamma}^c_{ab}\right) w_c = \nabla_a w_b - C^c_{ab} w_c, \tag{2.2}$$

notice that $\nabla_a A = \partial_a A$, where A is a function.

Now lets find the relation for a tensor field of rank 2,

$$\tilde{\nabla}_a h_{bc} = \partial_a h_{bc} - \widetilde{\Gamma}^e_{ab} h_{ec} - \widetilde{\Gamma}^e_{ac} h_{be} = \left(\nabla_a h_{bc} + \Gamma^e_{ab} h_{ec} + \Gamma^e_{ac} h_{be}\right) - \widetilde{\Gamma}^e_{ab} h_{ec} - \widetilde{\Gamma}^e_{ac} h_{be} \tag{2.3}$$

$$= \nabla_a h_{bc} + \left(\Gamma^e_{ab} - \widetilde{\Gamma}^e_{ab}\right) h_{ec} + \left(\Gamma^e_{ac} - \widetilde{\Gamma}^e_{ac}\right) h_{be} = \nabla_a h_{bc} - C^e_{ab} h_{ec} - C^e_{ac} h_{be}. \tag{2.4}$$

From the definition of the Riemann tensor, for all dual vector fields, w_c,

$$R^d_{abc} w_c = \nabla_a \nabla_b w_c - \nabla_b \nabla_a w_c. \tag{2.5}$$

Hence the conformal Riemann tensor is given by

$$\tilde{R}^d_{abc}w_d = \tilde{\nabla}_a\tilde{\nabla}_b w_c - \tilde{\nabla}_b\tilde{\nabla}_a w_c = \tilde{\nabla}_a\left(\nabla_b w_c - C^d_{bc}w_d\right) - \tilde{\nabla}_b\left(\nabla_a w_c - C^d_{ac}w_d\right) \tag{2.6}$$

$$= \left(\tilde{\nabla}_a\nabla_b w_c - \tilde{\nabla}_b\nabla_a w_c\right) + \left(C^d_{ac}\tilde{\nabla}_b w_d - C^d_{bc}\tilde{\nabla}_a w_d\right) + \left(w_d\tilde{\nabla}_b C^d_{ac} - w_d\tilde{\nabla}_a C^d_{bc}\right) \tag{2.7}$$

$$= \left[\left(\nabla_a\nabla_b w_c - C^e_{ab}\nabla_e w_c - C^e_{ac}\nabla_b w_e\right) - \left(\nabla_b\nabla_a w_c - C^e_{ba}\nabla_e w_c - C^e_{bc}\nabla_a w_e\right)\right] \tag{2.8}$$

$$+ \left(C^d_{ac}\left(\nabla_b w_d - C^e_{bd}w_e\right)\right) - C^d_{bc}\left(\nabla_a w_d - C^e_{ad}w_e\right)\Big) + 2w_d\tilde{\nabla}_{[b}C^d_{a]c} \tag{2.9}$$

$$= \left(\nabla_a\nabla_b w_c - \nabla_b\nabla_a w_c\right) + \left(C^e_{bc}\nabla_a w_e + C^d_{ac}\nabla_b w_d - C^e_{ac}\nabla_b w_e - C^d_{bc}\nabla_a w_d\right) \tag{2.10}$$

$$+ \left(C^d_{cb}C^e_{ad} - C^d_{ca}C^e_{bd}\right)w_e + 2w_d\tilde{\nabla}_{[b}C^d_{a]c} \tag{2.11}$$

$$= R^d_{abc}w_d + 2C^e_{c[b}C^d_{a]e}w_d + 2w_d\tilde{\nabla}_{[b}C^d_{a]c}. \tag{2.12}$$

Substituting the tensor C into the conformal Riemann tensor A.5,

$$\tilde{R}^d_{abc} = R^d_{abc} + \left(2\delta^d_{[a}\widetilde{\nabla}_{b]}\ln\Omega\widetilde{\nabla}_c + 2\tilde{g}^{ef}\delta^d_{[b}\tilde{g}_{a]c}\widetilde{\nabla}_f\ln\Omega\widetilde{\nabla}_e + 2\tilde{g}^{dg}\tilde{g}_{c[b}\widetilde{\nabla}_{a]}\ln\Omega\widetilde{\nabla}_g\right]\ln\Omega \tag{2.13}$$

$$+ \left(2\delta^d_{[a}\widetilde{\nabla}_{b]}\widetilde{\nabla}_c + 2\tilde{g}^{df}\tilde{g}_{c[b}\widetilde{\nabla}_{a]}\widetilde{\nabla}_f\right)\ln\Omega. \tag{2.14}$$

Therefore the conformal Ricci tensor is given by A.5,

$$\tilde{R}_{ac} \equiv \tilde{R}^b_{abc} = R^b_{abc} + \left[(2-D)\,\widetilde{\nabla}_a\ln\Omega\widetilde{\nabla}_c + \tilde{g}_{ca}\,(D-2)\,\tilde{g}^{ef}\widetilde{\nabla}_f\ln\Omega\widetilde{\nabla}_e\right]\ln\Omega + \left((2-D)\,\widetilde{\nabla}_a\widetilde{\nabla}_c - \tilde{g}_{ac}\tilde{g}^{bf}\widetilde{\nabla}_b\widetilde{\nabla}_f\right)\ln\Omega. \tag{2.15}$$

And so the conformal Ricci curvature scalar is,

$$\tilde{R} \equiv \tilde{R}_{ac}\tilde{g}^{ac} = \tilde{g}^{ac}R^b_{abc} + \left[(2-D)\,\tilde{g}^{ac}\widetilde{\nabla}_a\ln\Omega\widetilde{\nabla}_c + \tilde{g}^{ac}\tilde{g}_{ca}\,(D-2)\,\tilde{g}^{ef}\widetilde{\nabla}_f\ln\Omega\widetilde{\nabla}_e\right]\ln\Omega \tag{2.16}$$

$$+ \left((2-D)\,\tilde{g}^{ac}\widetilde{\nabla}_a\widetilde{\nabla}_c - \tilde{g}^{ac}\tilde{g}_{ac}\tilde{g}^{bf}\widetilde{\nabla}_b\widetilde{\nabla}_f\right)\ln\Omega \tag{2.17}$$

$$= \Omega^{-2}R + (D-1)\,(D-2)\,\tilde{g}^{ac}\widetilde{\nabla}_a\ln\Omega\widetilde{\nabla}_c\ln\Omega - 2\,(D-1)\,\tilde{g}^{ac}\widetilde{\nabla}_a\widetilde{\nabla}_c\ln\Omega. \tag{2.18}$$

And so the full Ricci curvature scalar,

$$R = \Omega^2\left[\tilde{R} + 2(D-1)\tilde{g}^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\ln\Omega - (D-2)(D-1)\tilde{g}^{\alpha\beta}\widetilde{\nabla}_\alpha\ln\Omega\widetilde{\nabla}_\beta\ln\Omega\right]. \tag{2.19}$$

2.3 Conformal Matter energy-momentum tensor

The conformal matter energy-momentum tensor is given by,

$$\widetilde{T}^{(M)}_{\mu\nu} = -\frac{2}{\sqrt{-\tilde{g}}}\frac{\delta(\sqrt{-\tilde{g}}\mathcal{L}_m)}{\delta(\tilde{g}^{\mu\nu})}, \tag{2.20}$$

and is related with the matter energy-momentum tensor by,

$$\widetilde{T}^{(M)}_{\mu\nu} = -\frac{2}{\sqrt{-\tilde{g}}}\frac{\delta(\sqrt{-\tilde{g}}\mathcal{L}_m)}{\delta(\tilde{g}^{\mu\nu})} = -\frac{2}{\Omega^{-D}\sqrt{-g}}\frac{\delta(\Omega^{-D}\sqrt{-g}\mathcal{L}_m)}{\delta(\Omega^{-2}g^{\mu\nu})} = -\frac{2}{\Omega^{-D}\sqrt{-g}}\Omega^{-D}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\Omega^{-2}\delta g^{\mu\nu}} = \Omega^2 T^{(M)}_{\mu\nu}. \tag{2.21}$$

The continuity equation in the Jordan frame is $\nabla^\mu T^{(M)}_{\mu\nu} = 0$, but the conformal continuity equation, from 2.4, is given by,

$$
\begin{aligned}
\widetilde{\nabla}^{\mu}\widetilde{T}^{(M)}_{\mu\nu} &= \tilde{g}^{\mu\lambda}\widetilde{\nabla}_{\lambda}\widetilde{T}^{(M)}_{\mu\nu} = \tilde{g}^{\mu\lambda}\left(\nabla_{\lambda}\widetilde{T}^{(M)}_{\mu\nu} - C^{\rho}_{\lambda\mu}\widetilde{T}^{(M)}_{\rho\nu} - C^{\rho}_{\lambda\nu}\widetilde{T}^{(M)}_{\mu\rho}\right) = \\
&= \Omega^{-2}\left(g^{\mu\lambda}\nabla_{\lambda}\Omega^{2}T^{(M)}_{\mu\nu} - g^{\mu\lambda}C^{\rho}_{\lambda\mu}\Omega^{2}T^{(M)}_{\rho\nu} - g^{\mu\lambda}C^{\rho}_{\lambda\nu}\Omega^{2}T^{(M)}_{\mu\rho}\right) = \\
&= T^{(M)}_{\mu\nu}\Omega^{-2}\nabla^{\mu}\Omega^{2} + \Omega^{-2}\left(\Omega^{2}\nabla^{\mu}T^{(M)}_{\mu\nu} - g^{\mu\lambda}C^{\rho}_{\lambda\mu}\Omega^{2}T^{(M)}_{\rho\nu} - g^{\mu\lambda}C^{\rho}_{\lambda\nu}\Omega^{2}T^{(M)}_{\mu\rho}\right) = \\
&= 2T^{(M)}_{\mu\nu}\nabla^{\mu}\ln\Omega - \left(g^{\mu\lambda}\left(2\delta^{\rho}_{(\lambda}\nabla_{\mu)}\ln\Omega - \tilde{g}^{\rho\sigma}\tilde{g}_{\lambda\mu}\nabla_{\sigma}\ln\Omega\right)T^{(M)}_{\rho\nu} + g^{\mu\lambda}\left(2\delta^{\rho}_{(\lambda}\nabla_{\nu)}\ln\Omega - \tilde{g}^{\rho\sigma}\tilde{g}_{\lambda\nu}\nabla_{\sigma}\ln\Omega\right)T^{(M)}_{\mu\rho}\right) \\
&= 2T^{(M)}_{\mu\nu}\nabla^{\mu}\ln\Omega - g^{\mu\lambda}\left(\delta^{\rho}_{\lambda}\nabla_{\mu}\ln\Omega + \delta^{\rho}_{\mu}\nabla_{\lambda}\ln\Omega - \tilde{g}^{\rho\sigma}\tilde{g}_{\lambda\mu}\nabla_{\sigma}\ln\Omega\right)T^{(M)}_{\rho\nu} \\
&- g^{\mu\lambda}\left(\delta^{\rho}_{\lambda}\nabla_{\nu}\ln\Omega + \delta^{\rho}_{\nu}\nabla_{\lambda}\ln\Omega - \tilde{g}^{\rho\sigma}\tilde{g}_{\lambda\nu}\nabla_{\sigma}\ln\Omega\right)T^{(M)}_{\mu\rho} \\
&= 2T^{(M)}_{\mu\nu}\nabla^{\mu}\ln\Omega - \left(g^{\rho\mu}\nabla_{\mu}\ln\Omega + g^{\rho\lambda}\nabla_{\lambda}\ln\Omega - D\Omega^{2}\tilde{g}^{\rho\sigma}\nabla_{\sigma}\ln\Omega\right)T^{(M)}_{\rho\nu} \\
&- \left(g^{\mu\rho}\nabla_{\nu}\ln\Omega + g^{\mu\lambda}\delta^{\rho}_{\nu}\nabla_{\lambda}\ln\Omega - \delta^{\mu}_{\nu}\Omega^{2}\tilde{g}^{\rho\sigma}\nabla_{\sigma}\ln\Omega\right)T^{(M)}_{\mu\rho} \\
&= 2T^{(M)}_{\mu\nu}\nabla^{\mu}\ln\Omega - 2g^{\rho\mu}T^{(M)}_{\rho\nu}\nabla_{\mu}\ln\Omega + Dg^{\rho\sigma}T^{(M)}_{\rho\nu}\nabla_{\sigma}\ln\Omega \\
&- T^{(M)}\nabla_{\nu}\ln\Omega - g^{\mu\lambda}T^{(M)}_{\mu\nu}\nabla_{\lambda}\ln\Omega + g^{\rho\sigma}T^{(M)}_{\nu\rho}\nabla_{\sigma}\ln\Omega \\
&= Dg^{\rho\sigma}T^{(M)}_{\rho\nu}\nabla_{\sigma}\ln\Omega - T^{(M)}\nabla_{\nu}\ln\Omega,
\end{aligned}
$$

with this we see that the equation is not conformally invariant. This also describes an exchange of energy and momentum between matter and the scalar field Ω, reflecting the fact that matter and the geometric factor Ω are directly coupled in the Einstein frame. Since the geodesic equation ruling the motion of free particles in General Relativity can be derived from the continuity equation, it follows that timelike geodesics of the original metric $g_{\alpha\beta}$ are not geodesics of the rescaled metric $\tilde{g}_{\alpha\beta}$ and vice-versa [3],

2.4 Jordan vs Einstein frame

This section is to introduce Jordan frame and Einstein frame in a generic way.

As we already know, scalar-tensor theories of gravity, like Brans-Dicke theory, are competitors to Einstein theory (GR). This scalar-tensor theories are motivated in unifying gravity with other interactions.

There are two fundamental scalar-tensor theories: Jordan frame and Einstein frame [8].

In Jordan frame, gravity is entirely described by the metric tensor and in Einstein frame, the scalar field acts as a source for the metric tensor and formally plays the role of an external matter field.

The Jordan frame formulation of a scalar-tensor theory is impractical due to the energy density of the gravitational scalar field, which is not bounded from below (violation of the weak energy condition). Consequently, the system is unstable and continuously decays towards lower and lower energy states) [10].

The Einstein frame formulation of scalar-tensor theories addresses the previously mentioned problem. However, in the Einstein frame, there is a violation of the equivalence principle due to the anomalous coupling of the scalar field to ordinary matter (this violation is small and compatible with available tests of the equivalence principle). Indeed, it is considered an important low-energy manifestation of compacted theories [17].

In a general opinion, gravitational physics community are more in favour of Einstein frame, because of the abstract way of formulation, which comparative to the violation of weak energy conditions by scalar-tensor in Jordan frame, the violation of the equivalence principal is less relevant. So Jordan frame became unreliable to describe classical gravitation.

2.5 Weyl tensor

The Weyl tensor tells us how the shape changes, however contrary to the Riemann curvature tensor, in which it gives information about how the volume of a body changes, under a tidal force (force that tends to distort objects), the Weyl tensor doesn't convey that information, and since the trace of the Riemann tensor is the volume of the body, the Weyl is **traceless**. This is useful in **vacuum solutions**, $T^{M}_{\mu\nu} = 0$, which allows us to see that considering the Hilbert-Einstein field equation without a cosmological constant,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} = 0 \Leftrightarrow \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0\right)g^{\mu\nu} \tag{2.22}$$

$$\Leftrightarrow R - \frac{D}{2}R = 0 \Leftrightarrow R = 0, \tag{2.23}$$

hence the Ricci tensor is traceless, which resembles the Weyl tensor, in fact it's the Weyl tensor that governs the propagating of gravitational waves in regions devoid of matter.

There are two ways we can get the Weyl tensor, one is by separating the Riemann tensor into its trace, and trace-free part, $C^{\alpha}_{\beta\gamma\delta}$, another route, is by determining a tensor which will be conformally invariant [7].

Going back in history Weyl wanted for this tensor to have the same symmetries as the Riemann tensor but such that, the conformal transformation of this new tensor would be the same as the normal tensor as,

$$\widetilde{C}^{\alpha}_{\beta\gamma\delta} = C^{\alpha}_{\beta\gamma\delta} \Leftrightarrow \delta C^{\alpha}_{\beta\gamma\delta} = 0. \tag{2.24}$$

Determining the Weyl tensor is easier with lowered indexes,

$$C_{\alpha\beta\gamma\delta} = C^{\epsilon}_{\beta\gamma\delta} g_{\epsilon\alpha}, \tag{2.25}$$

however we can notice with the lowered indices,

$$\delta C_{\alpha\beta\gamma\delta} = C^{\epsilon}_{\beta\gamma\delta}\delta g_{\epsilon\alpha} = -C^{\epsilon}_{\beta\gamma\delta}\left(\tilde{g}_{\epsilon\alpha} - g_{\epsilon\alpha}\right) = -C^{\epsilon}_{\beta\gamma\delta}\left(\Omega^2 g_{\epsilon\alpha} - g_{\epsilon\alpha}\right) = C_{\alpha\beta\gamma\delta}\left(1-\Omega^2\right). \tag{2.26}$$

The Weyl tensor is given by [7],

$$C^{\alpha}_{\beta\gamma\delta} = R^{\alpha}_{\beta\gamma\delta} - \frac{1}{D-2}\left(\delta^{\alpha}_{\gamma}R_{\beta\delta} - \delta^{\alpha}_{\delta}R_{\beta\gamma} - g_{\beta\gamma}R^{\alpha}_{\delta} + g_{\beta\delta}R^{\alpha}_{\gamma}\right) + \frac{1}{(D-1)(D-2)}\delta^{\alpha}_{[\gamma}g_{\delta]\beta}R, \tag{2.27}$$

notice that when $D \leq 2$ the tensor has to be zero, and when $D = 3$ it's surprisingly zero, one way to see this is by doing the divergence of the Weyl tensor,

$$\nabla_{\alpha}C^{\alpha}_{\beta\gamma\delta} = 2(D-3)\nabla_{[\gamma}S_{\delta]\beta} \tag{2.28}$$

where $S_{\alpha\beta}$ is the Schouten tensor [15], where the contracted Bianchi identity of the Ricci tensor is useful,

$$\nabla^{\mu}G_{\mu\nu} = 0 \Leftrightarrow \nabla^{\mu}R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}\nabla^{\mu}R \Leftrightarrow \nabla_{\mu}R^{\mu}_{\nu} = \frac{1}{2}\nabla_{\nu}R. \tag{2.29}$$

If the Weyl tensor vanishes in the Jordan frame then it also vanishes in the Einstein frame.

2.6 $f(R)$ theory

This section will be based on [16]. Applying a conformal transformation to the $f(R)$ gravity action. Inserting the tensor computed in the previous section into the action, knowing the determinant $g = \Omega^{-2D}\tilde{g}$,

$$S = \int_{\mathcal{M}} d^D x\sqrt{-\tilde{g}}\Omega^{-D}\left(\frac{F(R)}{2k}\right)\Omega^2\left(\tilde{R} + 2(D-1)\tilde{g}^{\alpha\beta}\widetilde{\nabla}_{\alpha}\widetilde{\nabla}_{\beta}\ln\Omega - (D-2)(D-1)\tilde{g}^{\alpha\beta}\widetilde{\nabla}_{\alpha}\ln\Omega\widetilde{\nabla}_{\beta}\ln\Omega\right) \tag{2.30}$$

$$+ \int_{\mathcal{M}} d^D x\sqrt{-\tilde{g}}\Omega^{-D}\left(Y + \mathcal{L}_m\left(\Omega^{-2}\tilde{g}_{\mu\nu}, \psi\right)\right) \tag{2.31}$$

$$= \int_{\mathcal{M}} d^D x\sqrt{-\tilde{g}}\Omega^{-D+2}\left(\frac{F(R)}{2k}\right)\left(\tilde{R} + 2(D-1)\tilde{g}^{\alpha\beta}\widetilde{\nabla}_{\alpha}\widetilde{\nabla}_{\beta}\ln\Omega - (D-2)(D-1)\tilde{g}^{\alpha\beta}\widetilde{\nabla}_{\alpha}\ln\Omega\widetilde{\nabla}_{\beta}\ln\Omega\right) \tag{2.32}$$

$$+ \int_{\mathcal{M}} d^D x\sqrt{-\tilde{g}}\Omega^{-D}\left(Y + \mathcal{L}_m\left(\Omega^{-2}\tilde{g}_{\mu\nu}, \psi\right)\right). \tag{2.33}$$

Notice that the Matter Lagrangian will also be rescaled, the particle masses will also change accordingly, to maintain the Matter Lagrangian invariant.

Making $F(R) = \Omega^{D-2}$, we will finally be in the Einstein frame because the action will be proportional to $\tilde{R}$, then $\Omega = F(R)^{1/D-2}$,

$$S = \int_{\mathcal{M}} d^D x\frac{\sqrt{-\tilde{g}}}{2k}\left(\tilde{R} + 2(D-1)\tilde{g}^{\alpha\beta}\widetilde{\nabla}_{\alpha}\widetilde{\nabla}_{\beta}\ln\Omega - (D-2)(D-1)\tilde{g}^{\alpha\beta}\widetilde{\nabla}_{\alpha}\ln\Omega\widetilde{\nabla}_{\beta}\ln\Omega\right) \tag{2.34}$$

$$+ \int_{\mathcal{M}} d^D x\sqrt{-\tilde{g}}F(R)^{-D/D-2}\left(Y + \mathcal{L}_m\left(F(R)^{-2/D-2}\tilde{g}_{\mu\nu}, \psi\right)\right). \tag{2.35}$$

The second RHS term can be computed into this,

$$\int_{\mathcal{M}} d^D x\sqrt{-\tilde{g}}\tilde{g}^{\alpha\beta}\widetilde{\nabla}_{\alpha}\widetilde{\nabla}_{\beta}\ln\Omega = \int_{\mathcal{M}} d^D x\widetilde{\nabla}_{\alpha}\left(\sqrt{-\tilde{g}}\tilde{g}^{\alpha\beta}\widetilde{\nabla}_{\beta}\ln\Omega\right) - \int_{\mathcal{M}} d^D x\widetilde{\nabla}_{\alpha}\left(\sqrt{-\tilde{g}}\tilde{g}^{\alpha\beta}\right)\widetilde{\nabla}_{\beta}\ln\Omega. \tag{2.36}$$

The first RHS term is a total derivative assuming $\delta g^{\mu\nu}$ is zero at the boundary, $\partial\mathcal{M}$,

$$\int_{\mathcal{M}} d^D x \widetilde{\nabla}_\alpha \left(\sqrt{-\tilde{g}} \tilde{g}^{\alpha\beta} \widetilde{\nabla}_\beta \ln \Omega \right) = \int_{\partial\mathcal{M}} d\Sigma_\alpha \sqrt{-\tilde{g}} \tilde{g}^{\alpha\beta} \widetilde{\nabla}_\beta \ln \Omega = 0, \tag{2.37}$$

and the second RHS term is zero due to the metric compatibility, $\widetilde{\nabla}_\alpha(\sqrt{-\tilde{g}}\tilde{g}^{\alpha\beta}) = 0$, and so

$$\int_{\mathcal{M}} d^D x \sqrt{-\tilde{g}} \tilde{g}^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \ln \Omega = 0. \tag{2.38}$$

Therefore the action,

$$S = \int_{\mathcal{M}} d^D x \sqrt{-\tilde{g}} \left[\frac{1}{2k} \left(\tilde{R} - (D-2)(D-1) \tilde{g}^{\alpha\beta} \widetilde{\nabla}_\alpha \ln \Omega \widetilde{\nabla}_\beta \ln \Omega \right) + F(R)^{-D/D-2} \left(Y + \mathcal{L}_m \left(F(R)^{-2/D-2} \tilde{g}_{\mu\nu}, \psi \right) \right) \right]. \tag{2.39}$$

2.6.1 Scalaron

This subsection will be based on [9]. As previously mentioned, the $f(R)$ equations of motion, in the Jordan frame, include a term, $\Box F(R)$, which resembles a propagating field, in fact in the conformally transformed we can see there arises some kind of propagating field. Therefore lets go back to the conformal action,

$$S = \int_{\mathcal{M}} d^D x \sqrt{-\tilde{g}} \left[\frac{1}{2k} \left(\tilde{R} - \underbrace{(D-2)(D-1) \tilde{g}^{\alpha\beta} \widetilde{\nabla}_\alpha \ln \Omega \widetilde{\nabla}_\beta \ln \Omega}_{I} \right) + F(R)^{-D/D-2} \left(Y + \mathcal{L}_m \left(F(R)^{-2/D-2} \tilde{g}_{\mu\nu}, \psi \right) \right) \right]. \tag{2.40}$$

Notice that transforming the I term into some propagation term of some scalar field, φ, which we will call **Scalaron**, yields,

$$\begin{aligned} \frac{1}{2} \tilde{g}^{\alpha\beta} \widetilde{\nabla}_\alpha \varphi \widetilde{\nabla}_\beta \varphi &= \frac{1}{2k} (D-2)(D-1) \tilde{g}^{\alpha\beta} \widetilde{\nabla}_\alpha \ln \Omega \widetilde{\nabla}_\beta \ln \Omega = \tilde{g}^{\alpha\beta} \widetilde{\nabla}_\alpha \sqrt{\frac{1}{2k}(D-2)(D-1)} \ln \Omega \widetilde{\nabla}_\beta \sqrt{\frac{1}{2k}(D-2)(D-1)} \ln \Omega \\ &= \frac{1}{2} \tilde{g}^{\alpha\beta} \widetilde{\nabla}_\alpha \sqrt{\frac{1}{k}(D-2)(D-1)} \ln \Omega \widetilde{\nabla}_\beta \sqrt{\frac{1}{k}(D-2)(D-1)} \ln \Omega, \end{aligned}$$

and so the scalaron is,

$$\varphi = \sqrt{\frac{1}{k}(D-2)(D-1)} \ln \Omega = \sqrt{\frac{(D-1)}{k(D-2)}} \ln F(R), \tag{2.41}$$

from which it follows that,

$$F(R) = \exp \left\{ \sqrt{\frac{k(D-2)}{(D-1)}} \varphi \right\}, \tag{2.42}$$

And we notice the term,

$$V(\varphi) = F(R)^{-D/D-2}(-Y) = \frac{f(R) - RF(R)}{2kF(R)^{D/D-2}}, \tag{2.43}$$

is some kind of potential of the scalaron.

With this we see the action is equivalent to

$$\begin{aligned} S &= \int_{\mathcal{M}} d^D x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{2k} - \frac{1}{2} \tilde{g}^{\alpha\beta} \widetilde{\nabla}_\alpha \varphi \widetilde{\nabla}_\beta \varphi - V(\varphi) + \mathcal{L}_m \left(F(\varphi)^{-2/D-2} \tilde{g}_{\mu\nu}, \psi \right) \right] &(2.44) \\ &= \int_{\mathcal{M}} d^D x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{2k} + \mathcal{L}_\varphi + \mathcal{L}_m \left(F(\varphi)^{-2/D-2} \tilde{g}_{\mu\nu}, \psi \right) \right], &(2.45) \end{aligned}$$

where the scalaron Lagrangian is given by,

$$\mathcal{L}_\varphi = -\frac{1}{2} \tilde{g}^{\alpha\beta} \widetilde{\nabla}_\alpha \varphi \widetilde{\nabla}_\beta \varphi - V(\varphi). \tag{2.46}$$

In 2.44, we see that the scalaron is coupled with the matter Lagrangian, this becomes more apparent when using the Euler-Lagrange equations,

$$\partial_\mu \left(\frac{\partial \left(\sqrt{-\tilde{g}}(\mathcal{L}_\varphi + \mathcal{L}_m)\right)}{\partial \left(\partial_\mu \varphi\right)} \right) = \frac{\partial \left(\sqrt{-\tilde{g}}(\mathcal{L}_\varphi + \mathcal{L}_m)\right)}{\partial \varphi}. \tag{2.47}$$

The LHD is,

$$\frac{\partial \left(\sqrt{-\tilde{g}}(\mathcal{L}_\varphi + \mathcal{L}_m)\right)}{\partial \left(\partial_\mu \varphi\right)} = \sqrt{-\tilde{g}} \frac{\partial \left(\mathcal{L}_\varphi\right)}{\partial \left(\partial_\mu \varphi\right)} = \sqrt{-\tilde{g}} \frac{\partial \left(-\frac{1}{2}\tilde{g}^{\alpha\beta}\widetilde{\nabla}_\alpha \varphi \widetilde{\nabla}_\beta \varphi - V(\varphi)\right)}{\partial \left(\partial_\mu \varphi\right)} = \tag{2.48}$$

$$= -\frac{1}{2}\tilde{g}^{\alpha\beta}\sqrt{-\tilde{g}} \frac{\partial \left(\widetilde{\nabla}_\alpha \varphi \widetilde{\nabla}_\beta \varphi\right)}{\partial \left(\partial_\mu \varphi\right)} - \sqrt{-\tilde{g}} \frac{\partial V(\varphi)}{\partial \left(\partial_\mu \varphi\right)} = -\sqrt{-\tilde{g}}\tilde{g}^{\mu\beta}\widetilde{\nabla}_\beta \varphi = -\sqrt{-\tilde{g}}\tilde{g}^{\mu\beta}\widetilde{\nabla}_\beta \varphi. \tag{2.49}$$

The first term in the RHS,

$$\frac{\partial \left(\sqrt{-\tilde{g}}\mathcal{L}_\varphi\right)}{\partial \varphi} = -\frac{\partial \left(\sqrt{-\tilde{g}}V(\varphi)\right)}{\partial \varphi} = -\sqrt{-\tilde{g}}\frac{\partial V(\varphi)}{\partial \varphi}.$$

The second RHS terms is,

$$\begin{aligned}
\frac{\partial \left(\sqrt{-\tilde{g}}\mathcal{L}_m\right)}{\partial \varphi} &= \frac{\partial \left(\sqrt{-\tilde{g}}\mathcal{L}_m\right)}{\partial g^{\tilde{\mu}\nu}} \frac{\partial \tilde{g}^{\mu\nu}}{\partial \varphi} = \left(-\frac{1}{2}\sqrt{-\tilde{g}}\widetilde{T}^{(M)}_{\mu\nu}\right) g^{\mu\nu} \frac{\partial F(R)^{-2/D-2}}{\partial \varphi} \\
&= \left(-\frac{1}{2}\sqrt{-\tilde{g}}\widetilde{T}^{(M)}_{\mu\nu}\right) \left(F(R)^{2/D-2}\tilde{g}^{\mu\nu}\right) \frac{-2}{D-2} F(R)^{-1-2/D-2} \frac{\partial F(R)}{\partial \varphi} = \\
&= \sqrt{-\tilde{g}}\, \underbrace{\frac{1}{D-2}\frac{1}{F(R)}\frac{\partial F(R)}{\partial \varphi}}_{kQ} \tilde{g}^{\mu\nu}\widetilde{T}^{(M)}_{\mu\nu} = \sqrt{-\tilde{g}}kQ\tilde{g}^{\mu\nu}\widetilde{T}^{(M)}_{\mu\nu} = \sqrt{-\tilde{g}}kQ\widetilde{T}^{(M)},
\end{aligned}$$

where Q is the strength coupling, which is given by,

$$kQ = \frac{1}{D-2}\frac{1}{F(R)}\frac{\partial F(R)}{\partial \varphi} \Leftrightarrow (D-2)kQd\varphi = \frac{dF(R)}{F(R)} \Leftrightarrow F(R) = \exp\{(D-2)kQ\varphi\}, \tag{2.50}$$

we can see this strength coupling term, determines how the scalaron couples with $F(R)$.

Putting all of this together we get,

$$\partial_\mu(-\sqrt{-\tilde{g}}\tilde{g}^{\mu\beta}\widetilde{\nabla}_\beta\varphi) = -\sqrt{-\tilde{g}}\frac{\partial V(\varphi)}{\partial \varphi} + \sqrt{-\tilde{g}}kQ\widetilde{T}^{(M)} \Leftrightarrow \frac{1}{\sqrt{-\tilde{g}}}\partial_\mu(\sqrt{-\tilde{g}}\tilde{g}^{\mu\beta}\widetilde{\nabla}_\beta\varphi) + \frac{\partial V(\varphi)}{\partial \varphi} - kQ\widetilde{T}^{(M)} = 0. \tag{2.51}$$

We see this is a Klein-Gordon type of field equation for a scalaron, φ [9],

$$\widetilde{\Box}\varphi + \frac{\partial V(\varphi)}{\partial \varphi} - kQ\widetilde{T}^{(M)} = 0, \tag{2.52}$$

where $\widetilde{\Box} = \partial_\mu(\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\widetilde{\nabla}_\nu)/\sqrt{-\tilde{g}}$.

This equation shows the direct relationship between this scalaron and the matter, which reinforces the idea that the factor Ω in this conformally transformed "world" acts as an effective matter. This shows that the vacuum in the Jordan frame is not the same as in the Einstein frame, so the physical meaning of matter is frame-dependent.

2.7 Brans-Dicke gravity

This section will be based on [2]. Using the Brans-Dicke theory and $f(R)$ theory equivalence, such that going back to 1.37, and from the stated made in the last section, $F(R) = \Omega^{D-2}$,

$$F(R) = \Omega^{D-2} = G\phi \Leftrightarrow \Omega = (G\phi)^{1/D-2}. \tag{2.53}$$

From the Brans-Dicke gravity action, omitting the Matter action part

$$\begin{aligned}
S_\phi &= \frac{1}{2k'}\int_{\mathcal{M}} d^Dx\sqrt{-\tilde{g}}\Omega^{-D}\left[\Omega^2\left(\tilde{R} + 2(D-1)\tilde{g}^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\ln\Omega - (D-2)(D-1)\tilde{g}^{\alpha\beta}\widetilde{\nabla}_\alpha\ln\Omega\widetilde{\nabla}_\beta\ln\Omega\right)\phi - \omega\frac{\nabla_\mu\phi\nabla^\mu\phi}{\phi}\right] \\
&= \frac{1}{2k'}\int_{\mathcal{M}} d^Dx\sqrt{-\tilde{g}}\left[\Omega^{-D+2}\left(\tilde{R} + 2(D-1)\tilde{g}^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\ln\left((G\phi)^{1/D-2}\right)\right.\right. \\
&\left.\left. -(D-2)(D-1)\tilde{g}^{\alpha\beta}\widetilde{\nabla}_\alpha\ln\left((G\phi)^{1/D-2}\right)\widetilde{\nabla}_\beta\ln\left((G\phi)^{1/D-2}\right)\right)\phi - \omega\Omega^{-D}\frac{\nabla_\mu\phi\nabla^\mu\phi}{\phi}\right]
\end{aligned}$$

$$= \frac{1}{2k'} \int_{\mathcal{M}} d^D x \sqrt{-\tilde{g}} \left[(G\phi)^{-1} \left(\tilde{R} + 2\frac{(D-1)}{D-2} \tilde{g}^{\alpha\beta} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \ln\phi - \frac{(D-1)}{D-2} \tilde{g}^{\alpha\beta} \widetilde{\nabla}_\alpha \ln\phi \widetilde{\nabla}_\beta \ln\phi \right) \phi - \omega \,(G\phi)^{-D/D-2} \frac{\nabla_\mu \phi \nabla^\mu \phi}{\phi} \right]$$
$$= \frac{1}{2k'} \int_{\mathcal{M}} d^D x \sqrt{-\tilde{g}} \left[\frac{1}{G} \left(\tilde{R} + \tilde{g}^{\alpha\beta} \frac{(D-1)}{D-2} \left(2\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \ln\phi - \widetilde{\nabla}_\alpha \ln\phi \widetilde{\nabla}_\beta \ln\phi \right) \right) - \omega \,(G\phi)^{-D/D-2} \frac{\nabla_\mu \phi \nabla^\mu \phi}{\phi} \right].$$

We can see that the term,

$$\frac{1}{2k'} \int_{\mathcal{M}} d^D x \sqrt{-\tilde{g}} \frac{2}{G} \tilde{g}^{\alpha\beta} \frac{(D-1)}{D-2} \widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \ln\phi = 0 \tag{2.54}$$
$$\widetilde{\nabla}_\alpha \ln\phi \widetilde{\nabla}_\beta \ln\phi = \frac{\widetilde{\nabla}_\alpha \phi}{\phi} \frac{\widetilde{\nabla}_\beta \phi}{\phi} = \frac{\widetilde{\nabla}_\alpha \phi \widetilde{\nabla}_\beta \phi}{\phi^2}. \tag{2.55}$$

Going back,

$$= \frac{1}{2k'} \int_{\mathcal{M}} d^D x \sqrt{-\tilde{g}} \left[\frac{1}{G} \left(\tilde{R} + \frac{(D-1)}{D-2} \left(-\frac{\nabla_\mu \phi \nabla^\mu \phi}{\phi^2} \right) \right) - \omega \,(G\phi)^{-D/D-2} \frac{\nabla_\mu \phi \nabla^\mu \phi}{\phi} \right]$$
$$= \frac{1}{2k'} \int_{\mathcal{M}} d^D x \sqrt{-\tilde{g}} \left[\frac{R}{G} - \left(G^{-1} \frac{D-1}{D-2} \phi^{-2} + \omega \,(G\phi)^{-D/D-2} \right) \frac{\nabla_\mu \phi \nabla^\mu \phi}{\phi} \right]$$
$$= \int_{\mathcal{M}} d^D x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{2k} - \frac{1}{2k} \left(\frac{D-1}{D-2} \phi^{-1} + \omega \,(G\phi)^{1-D/D-2} \right) \frac{\nabla_\mu \phi \nabla^\mu \phi}{\phi^2} \right].$$

Setting the dimension of our manifold to be 4-dimensional, $D = 4$,

$$S = \int_{\mathcal{M}} d^4 x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{2k} - \frac{1}{2k} \left(\frac{3}{2\phi} + \frac{\omega}{G\phi} \right) \frac{\nabla_\mu \phi \nabla^\mu \phi}{\phi^2} + \widetilde{\mathcal{L}}_m \left(\tilde{g}_{\mu\nu}, \psi \right) \right] \tag{2.56}$$
$$\int_{\mathcal{M}} d^4 x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{2k} - \frac{1}{2k} \left(\frac{3}{2} + \frac{\omega}{G} \right) \frac{\nabla_\mu \phi \nabla^\mu \phi}{\phi^3} + \widetilde{\mathcal{L}}_m \left(\tilde{g}_{\mu\nu}, \psi \right) \right] \tag{2.57}$$
$$\int_{\mathcal{M}} d^4 x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{2k} - \frac{1}{2k} \left(\frac{3G + 2\omega}{2G} \right) \frac{\nabla_\mu \phi \nabla^\mu \phi}{\phi^3} + \widetilde{\mathcal{L}}_m \left(\tilde{g}_{\mu\nu}, \psi \right) \right]. \tag{2.58}$$

Notice that the matter Lagrangian locally changes, due to a conformal transformation, for example,

$$\mathcal{L}_\Phi = g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + m^2 \Phi^2, \tag{2.59}$$

we can see that the conformal transformation gives,

$$\widetilde{\mathcal{L}}_\Phi = \tilde{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \tilde{m}^2 \Phi^2 = \Omega^{-2} \left(g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \Omega^2 \tilde{m}^2 \Phi^2 \right) = \Omega^{-2} \mathcal{L}_\Phi \Leftrightarrow \Omega^2 \tilde{m}^2 = m^2 \Leftrightarrow \tilde{m} = \Omega^{-1} m, \tag{2.60}$$

beware the mass of a particle changes under a conformal transformation.

As mentioned in the conformal transformation of the energy-momentum tensor, it should also be noted that the non-null geodesics are not conformally invariant, and so the path the particles take will be different as seen the conformal Lagrangian derived in 2.58 and the unmodified Brans-Dicke Lagrangian.

Chapter 3

Cosmological constant vs $f(R)$ gravity - Tiago Lopes

TIAGO LOPES

3.1 Cosmological constant in Hilbert-Einstein action

One of the most fundamental principles in theoretical physics is encapsulated by the remarkable Principle of Action Minimisation. Through the formalism of the variational principle, it is possible to elegantly derive the equations characteristic of a specific phenomenon. In this context, we will explore this technique to deduce the famous Einstein equations that describe gravity. Later, we will extend our analysis to include more extensive formulations, such as those found in the F(R) theory of gravity.

3.1.1 Action principle

We define Hilbert-Einsein as

$$S = \frac{1}{2k}\int_{\mathcal{M}} \sqrt{-g}(R-2\Lambda)d^Dx + \int_{\mathcal{M}} \sqrt{-g}\mathcal{L}_M\, d^Dx \tag{3.1}$$

We are going to vary the action inside an infinitesimal region $\mathcal{M}$, letting the variation of the metric and its derivative vanish on the boundary of the region. Then we calculate the variation of the action integrals, and deduce Einstein's field equations from the requirement that $\delta S = 0$ for arbitrary variations of the metric. And so,

By the principal action,

$$\begin{aligned}\delta S &= \delta\left[\frac{1}{2k}\int_{\mathcal{M}} d^Dx\sqrt{-g}(R-2\Lambda) + \int_{\mathcal{M}} d^Dx\sqrt{-g}\mathcal{L}_m\right] = \frac{1}{2k}\int_{\mathcal{M}} d^Dx\delta\left(\sqrt{-g}(R-2\Lambda\right) + \int_{\mathcal{M}} d^Dx\delta(\sqrt{-g}\mathcal{L}_m) \\ &= \frac{1}{2k}\int_{\mathcal{M}} d^Dx\left[\delta\sqrt{-g}(R-2\Lambda) + \sqrt{-g}\left(\delta R - 2\delta\Lambda\right)\right] - \int_{\mathcal{M}} d^Dx\frac{1}{2}\sqrt{-g}T^{(M)}_{\mu\nu}\delta g^{\mu\nu}\end{aligned}$$

where in the last step it was used the result derived in A.4. The term $\delta\sqrt{-g}$, and δR,

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{3.2}$$

$$\delta R = \delta(R_{\mu\nu}g^{\mu\nu}) = g^{\mu\nu}\delta R_{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu} = g^{\mu\nu}\left(\nabla_\alpha\nabla_\beta - g_{\alpha\beta}\Box\right)\delta g^{\alpha\beta} + R_{\mu\nu}\delta g^{\mu\nu} \tag{3.3}$$

this results were derived in A.1 and A.3. And knowing the cosmological constant is a constant, $\delta\Lambda = 0$.

Putting this in the action

$$= \frac{1}{2k}\int_{\mathcal{M}} d^Dx\left[\left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\right)(R-2\Lambda) + \sqrt{-g}\left(g^{\mu\nu}\left(\nabla_\alpha\nabla_\beta - g_{\alpha\beta}\Box\right)\delta g^{\alpha\beta} + R_{\mu\nu}\delta g^{\mu\nu}\right) - \sqrt{-g}kT^{(M)}_{\mu\nu}\delta g^{\mu\nu}\right]$$

We can see the term,

$$\frac{1}{2k}\int_{\mathcal{M}} d^Dx\sqrt{-g}g^{\mu\nu}\left(\nabla_\alpha\nabla_\beta - g_{\alpha\beta}\Box\right)\delta g^{\alpha\beta} \overset{\nabla_\alpha g^{\mu\nu}=0}{\overset{\downarrow}{=}} \frac{1}{2k}\int_{\mathcal{M}} d^Dx\left(\nabla_\alpha\nabla_\beta - g_{\alpha\beta}\Box\right)\sqrt{-g}g^{\mu\nu}\delta g^{\alpha\beta} \tag{3.4}$$

$$= \frac{1}{2k}\int_{\mathcal{M}} d^Dx\nabla_\gamma\left(\delta^\gamma_\alpha\nabla_\beta - g_{\alpha\beta}g^{\gamma\delta}\nabla_\delta\right)\sqrt{-g}g^{\mu\nu}\delta g^{\alpha\beta} = \frac{1}{2k}\int_{\partial\mathcal{M}} d\Sigma_x n_\gamma\left(\delta^\gamma_\alpha\nabla_\beta - g_{\alpha\beta}g^{\gamma\delta}\nabla_\delta\right)\sqrt{-g}g^{\mu\nu}\delta g^{\alpha\beta} \tag{3.5}$$

where in the last step we invoked the Stokes theorem where we get a Σ_x which is the $D-1$ manifold, and as done previously we assume the Gibbons Hawking York term, the boundary term, is zero, which means at the boundary of the manifold, $\delta g_{\mu\nu} = 0$. Going back to the action we get,

$$\delta S = \frac{1}{2k}\int_{\mathcal{M}} d^D x \left[-\frac{1}{2}\sqrt{-g}g_{\mu\nu}(R-2\Lambda) + \sqrt{-g}R_{\mu\nu} - \sqrt{-g}kT^{(M)}_{\mu\nu}\right]\delta g^{\mu\nu}$$

$$= \frac{1}{2k}\int_{\mathcal{M}} d^D x \left[\underbrace{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}}_{G_{\mu\nu}} + g_{\mu\nu}\Lambda - kT^{(M)}_{\mu\nu}\right]\sqrt{-g}\delta g^{\mu\nu}$$

Thus the field equation with respect to the metric is,

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = kT^{(M)}_{\mu\nu} \tag{3.6}$$

These are the famous *Einstein field equations.*

The next step is evaluate R
We can calculate the trace of equation 3.6 by multiplying it by $g^{\mu\nu}$

$$g^{\mu\nu}\left(G_{\mu\nu} + g_{\mu\nu}\Lambda = kT^{(M)}_{\mu\nu}\right)$$

$$= g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \lambda g_{\mu\nu}) = g^{\mu\nu}(kT^{(M)}_{\mu\nu}) \Leftrightarrow \underbrace{g^{\mu\nu}R_{\mu\nu}}_{R} - \frac{1}{2}R\underbrace{g^{\mu\nu}g_{\mu\nu}}_{D} + \Lambda\underbrace{g^{\mu\nu}g_{\mu\nu}}_{D} = k\underbrace{g^{\mu\nu}T^{(M)}_{\mu\nu}}_{T} \Leftrightarrow \tag{3.7}$$

$$\Leftrightarrow R - \frac{1}{2}RD + \Lambda D = kT \Leftrightarrow R = \frac{kT - D\Lambda}{1-\frac{D}{2}} \tag{3.8}$$

If now we substituated the last expression R in the equaction 3.6

$$R_{\mu\nu} - \frac{1}{2}\left(\frac{kT - D\Lambda}{1-\frac{D}{2}}\right)g_{\mu\nu} + \Lambda g_{\mu\nu} = kT_{\mu\nu} \Leftrightarrow R_{\mu\nu} - \left(\frac{kT - D\Lambda}{2-D}\right)g_{\mu\nu} + \Lambda g_{\mu\nu} = kT_{\mu\nu} \Leftrightarrow \tag{3.9}$$

$$\Leftrightarrow R_{\mu\nu} = kT_{\mu\nu} + \left(\frac{kT - D\Lambda}{2-D}\right)g_{\mu\nu} - \Lambda g_{\mu\nu} \tag{3.10}$$

Gives to us the new view of the Einstein equation

$$R_{\mu\nu} = k\left(T_{\mu\nu} + \frac{Tg_{\mu\nu}}{2-D}\right) - \frac{2\Lambda g_{\mu\nu}}{2-D} \tag{3.11}$$

So far we have dealt with our action for the D-dimensional case. Let's now restrict our problem to the D=4 case. The Einstein field equation becomes

$$R_{\mu\nu} = k\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) + \Lambda g_{\mu\nu} \tag{3.12}$$

This equation reflects a symmetry in the equations, the Ricci tensor and the energy-momentum tensor is invariant under a permutation between the two tensors. The vacuum equations with a cosmological constant are

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{3.13}$$

With $\Lambda = 0$ this equation says that Ricci tensor must vanish for a vacuum space-time without a cosmologic constant.

$$R_{\mu\nu} = 0 \tag{3.14}$$

We would emphasize however, that this does not mean that such a space-time is flat.The reason for this is that the Riemann tensor consists of basically two parts, one gives the contribution to the Ricci tensor under contraction. The other part, the trace-free part of the Riemann tensor, will not give any contribution to the Ricci tensor. Hence, it is not determined by the Einstein equations.

In the presence of matter and energy, the Ricci tensor is determined from the momentum-energy tensor

$$R_{\mu\nu} = k\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) \tag{3.15}$$

3.1.2 Friedmann equations

Using again the Friedman-Lemaitre-Robertson-Walker (FLRW) metric, which characterises an homogeneous and isotropic universe,

$$ds^2 = -dt^2 + a(t)^2 d\mathcal{M}^2, \tag{3.16}$$

where $\mathcal{M}$ represents the space, which can be several types, in our case curved space time,

$$d\mathcal{M}^2 = \frac{dr^2}{1-kr^2} + r^2\left(\sin^2(\theta)d\phi^2 + d\theta^2\right), \tag{3.17}$$

and the $a(t)$ represents the spatial component's dependence on time, in our case the scale factor, that relates to the expansion of the space itself.

The Ricci curvature scalar for the FLRW metric A.6,

$$R = 6\left(\frac{\ddot{a}(t)}{a(t)} + \left(\frac{\dot{a}(t)}{a(t)}\right)^2 + \frac{k}{a^2(t)}\right). \tag{3.18}$$

The universe is not empty, so we are not interested in vacuum solutions to Einstein's equations. We will choose to model the matter and energy in the universe by a perfect fluid.

The energy-momentum tensor for a perfect fluid can be written

$$T_{\mu\nu} = (\rho + p)\,U_\mu U_\nu + pg_{\mu\nu}, \tag{3.19}$$

where ρ and p are the energy density and pressure and U^μ is the four-velocity of the fluid.

As the matter energy-momentum tensor is a tensor we can write it in the comoving frame where the four-velocity is,

$$U^\mu = (1, 0, 0, 0), \tag{3.20}$$

then can write the velocity as $U_\mu = \delta^0_\mu$. So we can write the energy-tensor 3.19 as,

$$T_{\mu\nu} = (\rho + p)\delta^0_\mu\delta^0_\nu + pg_{\mu\nu}. \tag{3.21}$$

The more convenient form for the energy-momentum , [4]

$$T^\mu_\nu = \mathrm{diag}(-\rho, p, p, p). \tag{3.22}$$

Note that the trace is given by

$$T = T^\mu_\mu = 3p - \rho. \tag{3.23}$$

Hence we can write the terms on the right-hand side of the field equations 3.12 that depend on the energy-momentum as,

$$T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu} = (\rho + p)\delta^0_\mu\delta^0_\nu + pg_{\mu\nu} - \frac{1}{2}(3p - \rho)g_{\mu\nu} = (\rho + p)\delta^0_\mu\delta^0_\nu + pg_{\mu\nu} - \frac{3}{2}pg_{\mu\nu} + \frac{1}{2}\rho g_{\mu\nu} \tag{3.24}$$

$$= (\rho + p)\delta^0_\mu\delta^0_\nu + pg_{\mu\nu}(1 - \frac{3}{2}) + \frac{1}{2}\rho g_{\mu\nu} = (\rho + p)\delta^0_\mu\delta^0_\nu - pg_{\mu\nu}\frac{1}{2} + \frac{1}{2}\rho g_{\mu\nu}. \tag{3.25}$$

Thus we have

$$T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu} = (\rho + p)\delta^0_\mu\delta^0_\nu - \frac{1}{2}(p - \rho)g_{\mu\nu}. \tag{3.26}$$

By including the cosmological constant term, we observe that the right-hand side of the Einstein field equations 3.6 becomes zero for $\mu \neq \nu$. This result holds when considering the FLRW metric tensors, as detailed in Appendix A.6. From the equations 3.26,3.12,A.18 and with the equations mentioned in the appendix for the Ricci tensor in the FLRW metric we will obtain the Friedmann equations.

We will get the following equations for the following indices. For the component $\mu\nu = 00$,

$$R_{00} = R_{tt} = k\left(T_{00} - \frac{1}{2}Tg_{00}\right) + \Lambda g_{00} \Leftrightarrow -3\frac{\ddot{a}(t)}{a(t)} = k\left(\rho + p + \frac{1}{2}p - \frac{1}{2}\rho\right) - \Lambda \Leftrightarrow \tag{3.27}$$

$$\Leftrightarrow -3\frac{\ddot{a}(t)}{a(t)} = \frac{1}{2}k\left(\rho + 3p\right) - \Lambda \Leftrightarrow \frac{\ddot{a}(t)}{a(t)} = -\frac{1}{6}k\left(\rho + 3p\right) + \frac{\Lambda}{3}. \tag{3.28}$$

For the component $\mu\nu = 11$,

$$R_{11} = R_{rr} = k(T_{11} - \frac{1}{2}Tg_{11}) + \Lambda g_{11} \Leftrightarrow \frac{\ddot{a}(t)a(t) + 2\dot{a}^2(t) + 2\kappa}{1 - \kappa r^2} = k\left(-\frac{1}{2}(p - \rho)\frac{a^2(t)}{1 - \kappa r^2}\right) + \Lambda\frac{a^2(t)}{1 - \kappa r^2} \Leftrightarrow \tag{3.29}$$

$$\Leftrightarrow \ddot{a}(t)a(t) + 2\dot{a}^2(t) + 2\kappa = \left(-\frac{k}{2}(p - \rho) + \Lambda\right)a^2(t). \tag{3.30}$$

For the component $\mu\nu = 22$,

$$R_{22} = R_{\phi\phi} = k(T_{22} - \frac{1}{2}Tg_{22}) + \Lambda g_{22} \Leftrightarrow \left(\ddot{a}(t)a(t) + 2\dot{a}^2(t) + 2\kappa\right)r^2\sin^2\theta = \left(-\frac{k}{2}(p - \rho) + \Lambda\right)a^2(t)r^2\sin^2\theta \Leftrightarrow \tag{3.31}$$

$$\Leftrightarrow \ddot{a}(t)a(t) + 2\dot{a}^2(t) + 2\kappa = \left(-\frac{k}{2}(p - \rho) + \Lambda\right)a^2(t) \tag{3.32}$$

For the component $\mu\nu = 33$,

$$R_{33} = R_{\theta\theta} = k(T_{33} - \frac{1}{2}Tg_{33}) + \Lambda g_{33} \Leftrightarrow \left(\ddot{a}(t)a(t) + 2\dot{a}^2(t) + 2\kappa\right)r^2 = \left(-\frac{k}{2}(p - \rho) + \Lambda\right)a^2(t)r^2 \Leftrightarrow \tag{3.33}$$

$$\Leftrightarrow \ddot{a}(t)a(t) + 2\dot{a}^2(t) + 2\kappa = \left(-\frac{k}{2}(p - \rho) + \Lambda\right)a^2(t). \tag{3.34}$$

As we can see, all the field equations applied to the FLRW metric are the same except when $uv = 00$. The Friedmann equations are thus obtained

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{1}{6}k\left(\rho + 3p\right) + \frac{\Lambda}{3} \tag{3.35}$$

$$\ddot{a}(t)a(t) + 2\dot{a}^2(t) + 2\kappa = \left(-\frac{k}{2}(p - \rho) + \Lambda\right)a^2(t) \tag{3.36}$$

We can arrive at a more simplified solution to the Friedman equations by eliminating the second derivative . We can solve both equations, but let's take a more trivial approach, where we use the Ricci scalar using the equation 3.6

$$R_{tt} - \frac{1}{2}Rg_{tt} = kT_{tt} - \Lambda g_{tt} \Leftrightarrow \left(-3\frac{\ddot{a}}{a}\right) + \frac{6}{2}\left(\frac{\ddot{a}(t)}{a(t)} + \left(\frac{\dot{a}(t)}{a(t)}\right)^2 + \frac{\kappa}{a^2(t)}\right) = k\rho + \Lambda \Leftrightarrow \tag{3.37}$$

$$\Leftrightarrow 3\left(\frac{\dot{a}(t)}{a(t)}\right)^2 + \frac{\kappa}{a^2(t)} = k\rho + \Lambda \Leftrightarrow \left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{k}{3}\rho + \frac{\Lambda}{3} - \frac{\kappa}{a^2(t)} \tag{3.38}$$

where the traditional way this equation is shown is by defining $H = \frac{\dot{a}}{a}$ which will be called the Hubble parameter.

3.1.3 Debate about Cosmologic constant

While working on the equations for the gravitational field, Einstein introduced the cosmological constant, initially to create a static universe, a kind of antidote to his theory, which at the time suggested an expanding or contracting universe.However, in 1929, astronomer Edwin Hubble made a discovery that shook the foundations of the static view of the universe. His observations revealed that distant galaxies were moving away from each other, indicating a cosmic expansion. Einstein, initially reluctant to abandon his cosmological constant, recognised that he had missed an opportunity by disregarding the cosmological term from his equations.

Meanwhile, the Russian mathematician and physicist Alexander Friedmann, independently of Einstein's work, was exploring possible solutions to the equations of general relativity. His solutions indicated a dynamic universe, contradicting the static vision previously proposed by Einstein.

It was only reluctantly that Einstein abandoned his cosmological constant, calling it his "biggest mistake". However, his original equations, now adjusted for the expansion of the universe, formed the basis for Friedmann's equations, which would describe the dynamics of cosmic expansion.

Thus, the cosmological constant, initially an attempt by Einstein to maintain a static universe, ended up playing a crucial role in driving the development of the Friedmann equations. This journey from Einstein's initial interpretation to the dynamic Friedmann equations reflects the intrepid nature of the human quest to understand the cosmos and how we adjust our theories in the light of observational evidence.

Despite these challenges, the cosmological constant has remained a topic of interest. Recent empirical data strongly suggests the existence of a positive cosmological constant in our Universe. However, explaining its small but positive value presents a theoretical puzzle known as the cosmological constant problem.

We know from the previous chapter that

$$H^2 \equiv \left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{k}{3}\rho + \frac{\Lambda}{3} - \frac{\kappa}{a^2(t)} \tag{3.39}$$

Here ρ is the mass density; $\kappa = -1, 0, 1$ for a Universe that is respetively open, "flat", closed. (colocar referencia)

Equation 3.39 says that three competing terms drive the universal expansion: a matter term, a cosmological constant term, and a curvature term. It is convenient to assign symbols to thier respective fractional contribuitions at the present epoch. We define

$$\Omega_M \equiv \frac{k}{3H^2} \tag{3.40}$$

$$\Omega_\Lambda \equiv \frac{\Lambda}{3H^2} \tag{3.41}$$

$$\Omega_\kappa \equiv -\frac{\kappa}{a^2(t)H^2} \tag{3.42}$$

The equation 3.39 take the form

$$1 = \Omega_M + \Omega_\Lambda + \Omega_\kappa \tag{3.43}$$

It sometimes convenient to define $\Omega_{tot} = \Omega_M + \Omega_\Lambda = 1 - \Omega_\kappa$. It is an observational question whether a nonzero Ω_Λ is required to achieve consistency in 3.43. This is the astronomer's cosmologic constant problem.

In conclusion, the equation 3.43 offers a fundamental insight into the composition and evolution of the universe in cosmological models. The different solutions derived from this equation reflect the distinct characteristics of the conditions of the universe in terms of its global geometry and the relative contributions of matter, cosmological constant and curvature.

For a more detailed analysis of this topic, visit the following references [5] [11]

3.2 Cosmology with $f(R)$ gravity

This section will be based on [14], and we will be using natural units. Cosmology is one of the primary applications of f(R) gravity, hence in the Einstein frame we will consider again a FLWR flat metric 3.16, where $k = 0$, which is given by,

$$\tilde{g}_{\mu\nu} = diag\left(-1, \tilde{a}^2, \tilde{a}^2, \tilde{a}^2\right), \tag{3.44}$$

where $\tilde{a}$ will be the scale factor in the Einstein frame. Which will represent a quintessence field (fifth force) in Einstein gravity, linear proportionality to the Ricci scalar. From the Lagrangian of the scalaron 2.46, we can obtain a scalaron energy-momentum tensor,

$$\widetilde{T}^{(\varphi)}_{\mu\nu} = -\frac{2}{\sqrt{-\tilde{g}}}\frac{\partial(\sqrt{-\tilde{g}}\mathcal{L}_\varphi)}{\partial\tilde{g}^{\mu\nu}} = -\frac{2}{\sqrt{-\tilde{g}}}\frac{\partial\sqrt{-\tilde{g}}}{\partial\tilde{g}^{\mu\nu}}\mathcal{L}_\varphi - 2\frac{\partial\mathcal{L}_\varphi}{\partial\tilde{g}^{\mu\nu}} = \tilde{g}_{\mu\nu}\mathcal{L}_\varphi - 2\frac{\partial\left(-\frac{1}{2}\tilde{g}^{\alpha\beta}\widetilde{\nabla}_\alpha\varphi\widetilde{\nabla}_\beta\varphi - V(\varphi)\right)}{\partial\tilde{g}^{\mu\nu}} \tag{3.45}$$

$$= \tilde{g}_{\mu\nu}\mathcal{L}_\varphi + \widetilde{\nabla}_\mu\varphi\widetilde{\nabla}_\nu\varphi, \tag{3.46}$$

We can see that the components of this scalaron stress tensor are

$$\widetilde{T}_{tt} = (-1)\left(-\frac{1}{2}\tilde{g}^{\alpha\beta}\widetilde{\nabla}_\alpha\varphi\widetilde{\nabla}_\beta\varphi - V(\varphi)\right) + \dot{\varphi}^2 = \left(-\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\left(\frac{\widetilde{\nabla}\varphi}{a}\right)^2 + V(\varphi)\right) + \dot{\varphi}^2 \tag{3.47}$$

$$= \frac{1}{2}\left(\dot{\varphi}^2 + \left(\frac{\widetilde{\nabla}\varphi}{a}\right)^2\right) + V(\varphi) \tag{3.48}$$

$$\widetilde{T}_{ij} = \tilde{g}_{ij}\mathcal{L}_\varphi + \widetilde{\nabla}_i\varphi\widetilde{\nabla}_j\varphi = \tilde{g}_{ij}\left(-\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\left(\frac{\widetilde{\nabla}\varphi}{a}\right)^2 + V(\varphi)\right) + \widetilde{\nabla}_i\varphi\widetilde{\nabla}_j\varphi. \tag{3.49}$$

From the definition of a perfect fluid, the energy-momentum tensor we know the component (t,t), $\widetilde{T}_{tt} = \rho_\varphi$, and the components, (i,j), $P_\varphi = -\frac{1}{3}g^{ij}\widetilde{T}_{ij}$, which allow us to conclude,

$$\rho_\varphi = \frac{1}{2}\left(\dot{\varphi}^2 + \left(\frac{\widetilde{\nabla}\varphi}{a}\right)^2\right) + V(\varphi) \tag{3.50}$$

$$P_\varphi = -\frac{1}{3}g^{ij}\widetilde{T}_{ij} = \frac{1}{3}g^{ij}\left[\tilde{g}_{ij}\left(-\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\left(\frac{\widetilde{\nabla}\varphi}{a}\right)^2 + V(\varphi)\right) + \widetilde{\nabla}_i\varphi\widetilde{\nabla}_j\varphi\right] = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{6}\left(\frac{\widetilde{\nabla}\varphi}{a}\right)^2 - V(\varphi). \tag{3.51}$$

From this we can see different equation of state depending on which component of energy is dominant (energy, matter or dark energy),

- If the kinetic term is dominant, $\dot{\varphi}^2$, then we will have $P_\varphi = \rho_\varphi$, $\tilde{w} = 1$
- If the gradient term is dominant, $\widetilde{\nabla}\varphi$, then $P_\varphi = -\frac{1}{3}\rho_\varphi$, $\tilde{w} = -\frac{1}{3}$, which resembles the radiation domination;
- If the potential dominates, $V(\varphi)$, then $P_\varphi = \rho_\varphi$, $\tilde{w} = -1$, this case may mimic the behaviour of a cosmological constant and yield an accelerated expanding universe,

where $\tilde{w}$ is a parameter of the equation of state given by,

$$\tilde{w}_\varphi(\tilde{a}) = \frac{P_\varphi(\tilde{a})}{\rho_\varphi(\tilde{a})}. \tag{3.52}$$

Assuming the gradient term is negligible, due to radiation having a small contribution to the energy density when compared to matter and dark energy, according to the ΛCDM cosmological model. The equation of state is then given by,

$$\tilde{w}_\varphi(a) = \frac{P_\varphi(a)}{\rho_\varphi(a)} = \frac{\frac{1}{2}\dot{\varphi}^2 + V(\varphi)}{\frac{1}{2}\dot{\varphi}^2 - V(\varphi)}. \tag{3.53}$$

We can also express, $P_\varphi = \tilde{w}_\varphi\rho_\varphi$,

$$\begin{cases} \rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \\ P_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \end{cases} \Leftrightarrow \begin{cases} \rho_\varphi + P_\varphi = \rho_\varphi(1+\tilde{w}_\varphi) = \dot{\varphi}^2 \\ \rho_\varphi - P_\varphi = \rho_\varphi(1-\tilde{w}_\varphi) = 2V(\varphi) \end{cases} \Leftrightarrow \begin{cases} \dot{\varphi}^2 = \rho_\varphi(1+\tilde{w}_\varphi) \\ V(\varphi) = \frac{1}{2}\rho_\varphi(1-\tilde{w}_\varphi) \end{cases}. \tag{3.54}$$

From the continuity equation, and the component $\nu = t$, and assuming our quintessence represents a perfect fluid, $T^\alpha_\alpha = 3P_\varphi - \rho_\varphi$,

$$\widetilde{\nabla}^\mu\widetilde{T}^\varphi_{\mu t} = \tilde{g}^{\mu\alpha}\widetilde{\nabla}_\alpha\widetilde{T}^\varphi_{\mu t} = \widetilde{\nabla}_\alpha\widetilde{T}^\alpha_t = \partial_\alpha\widetilde{T}^\alpha_t + \widetilde{\Gamma}^\alpha_{\alpha\rho}\widetilde{T}^\rho_t - \widetilde{\Gamma}^\rho_{\alpha t}\widetilde{T}^\alpha_\rho = \partial_t\widetilde{T}^t_t + \widetilde{\Gamma}^\alpha_{\alpha t}\widetilde{T}^t_t - \widetilde{\Gamma}^\alpha_{\alpha t}\widetilde{T}^\alpha_\alpha \tag{3.55}$$

$$= -\dot{\rho}_\varphi + \left(3\frac{\dot{\tilde{a}}}{\tilde{a}}\right)(-\rho) - \widetilde{\Gamma}^t_{tt}\widetilde{T}^t_t - \widetilde{\Gamma}^i_{it}\widetilde{T}^i_i = -\dot{\rho}_\varphi - 3\left(\frac{\dot{\tilde{a}}}{\tilde{a}}\right)\rho_\varphi - 3\left(\frac{\dot{\tilde{a}}}{\tilde{a}}\right)P_\varphi = 0 \Leftrightarrow \tag{3.56}$$

$$\Leftrightarrow \dot{\rho}_\varphi + 3\left(\frac{\dot{\tilde{a}}}{\tilde{a}}\right)(\rho_\varphi + P_\varphi) = \dot{\rho}_\varphi + 3\left(\frac{\dot{\tilde{a}}}{\tilde{a}}\right)\rho_\varphi\left(1 + w_\varphi\right) = 0. \tag{3.57}$$

Assuming the parameter w_φ doesn't depend on time,

$$\frac{d\rho_\varphi}{dt}+3\frac{d\tilde{a}}{dt}\left(\frac{1}{\tilde{a}}\right)\rho_\varphi\left(1+w_\varphi\right)=0\Leftrightarrow\frac{d\rho_\varphi}{\rho_\varphi}=-3\frac{d\tilde{a}}{\tilde{a}}\left(1+w_\varphi\right)\Leftrightarrow\ln\rho_\varphi=-3(1+w_\varphi)\ln\tilde{a}+C\Leftrightarrow \tag{3.58}$$

$$\Leftrightarrow\rho_\varphi=\rho_\varphi^0 a^{-3(1+w_\varphi)}, \tag{3.59}$$

where ρ^0 is the value of the density when $a=1$.

From the Friedmann equation 3.38, that was determined previously,

$$\left(\frac{\dot{\tilde{a}}}{\tilde{a}}\right)^2=\frac{k}{3}\rho_\varphi\Leftrightarrow\rho_\varphi=\frac{3}{k}\left(\frac{\dot{\tilde{a}}}{\tilde{a}}\right)^2, \tag{3.60}$$

we arrive to the following dependence,

$$\dot{\varphi}^2=\rho_\varphi(1+\tilde{w}_\varphi)=(1+\tilde{w}_\varphi)\frac{3}{k}\left(\frac{\dot{\tilde{a}}}{\tilde{a}}\right)^2\Leftrightarrow\dot{\varphi}=\sqrt{(1+\tilde{w}_\varphi)\frac{3}{k}}\left(\frac{\dot{\tilde{a}}}{\tilde{a}}\right)\Leftrightarrow \tag{3.61}$$

$$d\varphi=\sqrt{(1+\tilde{w}_\varphi)\frac{3}{k}}\left(\frac{d\tilde{a}}{\tilde{a}}\right)\Leftrightarrow\varphi+\varphi_0=\sqrt{(1+\tilde{w}_\varphi)\frac{3}{k}}\left(\ln\tilde{a}\right) \tag{3.62}$$

$$\Leftrightarrow\tilde{a}=\exp\left\{\sqrt{\frac{k}{3(1+w_\varphi)}}(\varphi+\varphi_0)\right\}. \tag{3.63}$$

Thus finally our quintessence potential is,

$$V(\varphi)=\frac{1}{2}\rho_\varphi(1-\tilde{w}_\varphi)=\frac{1}{2}(1-\tilde{w}_\varphi)\rho_\varphi^0\tilde{a}^{-3(1+w_\varphi)}=\frac{1}{2}(1-\tilde{w}_\varphi)\rho_\varphi^0\exp\left\{-\sqrt{3k(1+w_\varphi)}(\varphi+\varphi_0)\right\}. \tag{3.64}$$

Picking up from the **Scalaron** section, we got a scalar field given by, for this case $D=4$,

$$\varphi=\sqrt{\frac{3}{2k}}\ln F(R), \tag{3.65}$$

and with this we modify our potential into,

$$V(\varphi)=\frac{1}{2}(1-\tilde{w}_\varphi)\rho_\varphi^0\exp\left\{-\sqrt{3k(1+w_\varphi)}\sqrt{\frac{3}{2k}}\ln F(R)\right\}=\frac{1}{2}(1-\tilde{w}_\varphi)\rho_\varphi^0F^{-3\sqrt{(1+w_\varphi)/2}}. \tag{3.66}$$

Remembering the original scalaron potential in 2.43,

$$V(\varphi)=\frac{f(R)-RF(R)}{2kF(R)^2}. \tag{3.67}$$

In order for the quintessence potential be generated by the $f(R)$ function, we need to equal the scalaron potential,

$$\frac{f(R)-RF(R)}{2kF(R)^2}=\frac{1}{2}(1-\tilde{w}_\varphi)\rho_\varphi^0F^{-3\sqrt{(1+w_\varphi)/2}}\Leftrightarrow \tag{3.68}$$

$$\Leftrightarrow f(R)-RF(R)=k(1-\tilde{w}_\varphi)\rho_\varphi^0F^{-3\sqrt{(1+w_\varphi)/2}+2}, \tag{3.69}$$

by making,

$$A=k(1-\tilde{w}_\varphi)\rho_\varphi^0 \tag{3.70}$$

$$b=-3\sqrt{\frac{(1+w_\varphi)}{2}}+2, \tag{3.71}$$

the differentiable equation to solve is,

$$f-RF=AF^b, \tag{3.72}$$

and it can be solved analytically and it has a trivial linear solution [14],

$$f(R)=C_1R-AC_1^b,$$

where C_1 is an integration constant. The conformal parameter in this case becomes a constant, C_1, thus we recover the cosmological constant model. The non-trivial solution of (3.7) has a power law form,

$$f(R)=\frac{b-1}{b\,(Ab)^{\frac{1}{b-1}}}R^{\frac{b}{b-1}},$$

where $A,b\neq0$. Interestingly, this $f(R)$ model never reduces to the cosmological constant model for any finite value of the parameter b.

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Appendix A

Appendix

A.1 Determinant

The variation of $\sqrt{-g}$ where $g = \det\big(g_{\mu\nu}\big)$, and using the Jacobi formula, $\delta \det(A) = \det(A) Tr(A^{-1}\delta A)$.

$$\delta\sqrt{-g} = -\frac{\delta \det\big(g_{\mu\nu}\big)}{2\sqrt{-g}} = \frac{\sqrt{-g}}{2} g^{\mu\nu}\delta g_{\nu\mu} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu} \tag{A.1}$$

where in last step it was $\delta(g_{\mu\nu}g^{\mu\nu}) = 0 \Leftrightarrow g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu}$

A.2 Christoffel Symbol

Knowing that the Christoffel symbol with no torsion is given by,

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(g_{\beta\mu,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}\right) \tag{A.2}$$

Its variation is given by

$$\begin{aligned} \delta\Gamma^{\alpha}_{\mu\nu} &= \frac{1}{2}\delta g^{\alpha\beta}\left(g_{\beta\mu,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}\right) + \frac{1}{2} g^{\alpha\beta}\left(\delta g_{\beta\mu,\nu} + \delta g_{\nu\beta,\mu} - \delta g_{\mu\nu,\beta}\right) \\ &= -g^{\lambda\alpha}\delta g_{\lambda\delta}\Gamma^{\delta}_{\mu\nu} + \frac{1}{2} g^{\alpha\beta}\left(\delta g_{\beta\mu,\nu} + \delta g_{\nu\beta,\mu} - \delta g_{\mu\nu,\beta}\right) \end{aligned} \tag{A.3}$$

Using the definition of the covariant derivative, we can arrange the normal derivative

$$\nabla_\alpha A_{\mu\nu} = A_{\mu\nu,\alpha} - \Gamma^{\rho}_{\alpha\mu} A_{\rho\nu} - \Gamma^{\rho}_{\alpha\nu} A_{\mu\rho} \Leftrightarrow \tag{A.4}$$

$$\Leftrightarrow A_{\mu\nu,\alpha} = \nabla_\alpha A_{\mu\nu} + \Gamma^{\rho}_{\alpha\mu} A_{\rho\nu} + \Gamma^{\rho}_{\alpha\nu} A_{\mu\rho} \tag{A.5}$$

Returning to the point,

$$\begin{aligned} \delta\Gamma^{\alpha}_{\mu\nu} &= -g^{\lambda\alpha}\Gamma^{\delta}_{\mu\nu}\delta g_{\lambda\delta} \\ &\quad + \frac{1}{2} g^{\alpha\beta}\Big[\Big(\nabla_\nu \delta g_{\beta\mu} + \Gamma^{\rho}_{\nu\beta}\delta g_{\rho\mu} + \Gamma^{\rho}_{\nu\mu}\delta g_{\beta\rho}\Big) + \Big(\nabla_\mu \delta g_{\nu\beta} + \Gamma^{\rho}_{\mu\nu}\delta g_{\rho\beta} + \Gamma^{\rho}_{\mu\beta}\delta g_{\nu\rho}\Big) - \Big(\nabla_\beta \delta g_{\mu\nu} + \Gamma^{\rho}_{\beta\mu}\delta g_{\rho\nu} + \Gamma^{\rho}_{\beta\nu}\delta g_{\mu\rho}\Big)\Big] \\ &= \frac{1}{2} g^{\alpha\beta}\left(\nabla_\nu \delta g_{\beta\mu} + \nabla_\mu \delta g_{\nu\beta} - \nabla_\beta \delta g_{\mu\nu}\right) \\ &\quad + \frac{1}{2} g^{\alpha\beta}\Big(-2 g_{\beta\eta} g^{\lambda\eta}\Gamma^{\delta}_{\mu\nu}\delta g_{\lambda\delta} + \Gamma^{\rho}_{\nu\beta}\delta g_{\rho\mu} + \Gamma^{\rho}_{\nu\mu}\delta g_{\beta\rho} + \Gamma^{\rho}_{\mu\nu}\delta g_{\rho\beta} + \Gamma^{\rho}_{\mu\beta}\delta g_{\nu\rho} - \Gamma^{\rho}_{\beta\mu}\delta g_{\rho\nu} - \Gamma^{\rho}_{\beta\nu}\delta g_{\mu\rho}\Big) \\ &= \frac{1}{2} g^{\alpha\beta}\left(\nabla_\nu \delta g_{\beta\mu} + \nabla_\mu \delta g_{\nu\beta} - \nabla_\beta \delta g_{\mu\nu}\right) + \frac{1}{2} g^{\alpha\beta}\Big(-2\delta^{\lambda}_{\beta}\Gamma^{\delta}_{\mu\nu}\delta g_{\lambda\delta} + \Gamma^{\rho}_{\nu\mu}\delta g_{\beta\rho} + \Gamma^{\rho}_{\mu\nu}\delta g_{\rho\beta}\Big) \\ &= \frac{1}{2} g^{\alpha\beta}\left(\nabla_\nu \delta g_{\beta\mu} + \nabla_\mu \delta g_{\nu\beta} - \nabla_\beta \delta g_{\mu\nu}\right) \end{aligned} \tag{A.6}$$

where we eliminate most of the terms and use the following property, $\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu}$.

This could be immediatly seen, by knowing $\delta\Gamma$ was the difference between 2 connections and so it would transform like a tensor, $\partial \to \nabla$.

A.3 Ricci Tensor

By the definition of the Ricci tensor $R_{\mu\nu} \equiv R^{\rho}_{\mu\rho\nu}$, using the Palatini identity, and the compatibility with the metric, $\nabla_\sigma g^{\mu\nu}$,

$$
\begin{aligned}
g^{\mu\nu}\delta R_{\mu\nu} &\equiv g^{\mu\nu}\delta R^{\rho}_{\mu\rho\nu} \\
&= g^{\mu\nu}\left(\nabla_\rho \delta\Gamma^{\rho}_{\nu\mu} - \nabla_\nu \delta\Gamma^{\rho}_{\rho\mu}\right) \\
&= \frac{g^{\mu\nu}}{2}\nabla_\rho\left[g^{\rho\beta}\left(\nabla_\mu \delta g_{\beta\nu} + \nabla_\nu \delta g_{\mu\beta} - \nabla_\beta \delta g_{\nu\mu}\right)\right] - \frac{g^{\mu\nu}}{2}\nabla_\nu\left[g^{\rho\beta}\left(\nabla_\mu \delta g_{\beta\rho} + \nabla_\rho \delta g_{\mu\beta} - \nabla_\beta \delta g_{\rho\mu}\right)\right] \\
&= \frac{g^{\mu\nu}}{2}\left[g^{\rho\beta}\nabla_\rho\left(\nabla_\mu \delta g_{\beta\nu} + \nabla_\nu \delta g_{\mu\beta} - \nabla_\beta \delta g_{\nu\mu}\right)\right] - \frac{g^{\mu\nu}}{2}\left[g^{\rho\beta}\nabla_\nu\left(\nabla_\mu \delta g_{\beta\rho} + \nabla_\rho \delta g_{\mu\beta} - \nabla_\beta \delta g_{\rho\mu}\right)\right] \\
&= \frac{g^{\mu\nu}}{2}\left[g^{\rho\beta}\left(\nabla_\rho\nabla_\mu \delta g_{\beta\nu} + \nabla_\rho\nabla_\nu \delta g_{\mu\beta} - \nabla_\rho\nabla_\beta \delta g_{\nu\mu} - \nabla_\nu\nabla_\mu \delta g_{\beta\rho} - \nabla_\nu\nabla_\rho \delta g_{\mu\beta} + \nabla_\nu\nabla_\beta \delta g_{\rho\mu}\right)\right] \\
&= \frac{g^{\mu\nu}}{2}\left[g^{\rho\beta}\left(\nabla_\rho\nabla_\mu \delta g_{\beta\nu} - \nabla_\rho\nabla_\beta \delta g_{\nu\mu} - \nabla_\nu\nabla_\mu \delta g_{\beta\rho} + \nabla_\nu\nabla_\beta \delta g_{\rho\mu}\right)\right] \\
&= \frac{g^{\mu\nu}}{2}\left[\nabla^\beta\nabla_\mu \delta g_{\beta\nu} - \nabla^\rho\nabla_\rho \delta g_{\nu\mu} - g^{\rho\beta}\nabla_\nu\nabla_\mu \delta g_{\beta\rho} + \nabla_\nu\nabla^\rho \delta g_{\rho\mu}\right] \\
&= \frac{g^{\mu\nu}}{2}\left[-\nabla^\rho\nabla_\rho \delta g_{\nu\mu} - g^{\rho\beta}\nabla_\nu\nabla_\mu \delta g_{\beta\rho}\right] + g^{\mu\nu}\nabla_\nu\nabla^\rho \delta g_{\rho\mu} \\
&= -g^{\mu\nu}\nabla^\rho\nabla_\rho \delta g_{\mu\nu} + \nabla^\mu\nabla^\nu \delta g_{\mu\nu} \\
&= \nabla^\mu\nabla^\nu \delta g_{\mu\nu} - g^{\mu\nu}\Box \delta g_{\mu\nu}
\end{aligned}
\tag{A.7}
$$

A.4 Matter

Using the matter energy-momentum tensor,

$$T^{(M)}_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \Leftrightarrow \delta(\sqrt{-g}\mathcal{L}_m) = -\frac{\sqrt{-g}}{2}T^{(M)}_{\mu\nu}\delta g^{\mu\nu} \tag{A.8}$$

A.5 Conformal transformation

$$\tilde{R}^d_{abc} = R^d_{abc} + \left(C^e_{cb}C^d_{ae} - C^e_{ca}C^d_{be}\right) + \left(\tilde{\nabla}_b C^d_{ac} - \tilde{\nabla}_a C^d_{bc}\right) \tag{A.9}$$

$$= R^d_{abc} + \left[\left(2\delta^e_{(b}\tilde{\nabla}_{c)} - \tilde{g}^{ef}\tilde{g}_{bc}\tilde{\nabla}_f\right)\ln\Omega\left(2\delta^d_{(a}\tilde{\nabla}_{e)} - \tilde{g}^{dg}\tilde{g}_{ae}\tilde{\nabla}_g\right)\right. \tag{A.10}$$

$$\left. - \left(2\delta^e_{(c}\tilde{\nabla}_{a)} - \tilde{g}^{ef}\tilde{g}_{ca}\tilde{\nabla}_f\right)\ln\Omega\left(2\delta^d_{(b}\tilde{\nabla}_{e)} - \tilde{g}^{dg}\tilde{g}_{be}\tilde{\nabla}_g\right)\right]\ln\Omega \tag{A.11}$$

$$+ \left(\tilde{\nabla}_b\left(2\delta^d_{(a}\tilde{\nabla}_{c)} - \tilde{g}^{df}\tilde{g}_{ac}\tilde{\nabla}_f\right) - \tilde{\nabla}_a\left(2\delta^d_{(b}\tilde{\nabla}_{c)} - \tilde{g}^{df}\tilde{g}_{bc}\tilde{\nabla}_f\right)\right)\ln\Omega \tag{A.12}$$

$$= R^d_{abc} + \left(\delta^e_b\tilde{\nabla}_c\ln\Omega\left(\delta^d_a\tilde{\nabla}_e + \delta^d_e\tilde{\nabla}_a - \tilde{g}^{dg}\tilde{g}_{ae}\tilde{\nabla}_g\right) + \delta^e_c\tilde{\nabla}_b\ln\Omega\left(\delta^d_a\tilde{\nabla}_e + \delta^d_e\tilde{\nabla}_a - \tilde{g}^{dg}\tilde{g}_{ae}\tilde{\nabla}_g\right)\right. \tag{A.13}$$

$$-\tilde{g}^{ef}\tilde{g}_{bc}\tilde{\nabla}_f\ln\Omega\left(\delta^d_a\tilde{\nabla}_e + \delta^d_e\tilde{\nabla}_a - \tilde{g}^{dg}\tilde{g}_{ae}\tilde{\nabla}_g\right) - \delta^e_c\tilde{\nabla}_a\ln\Omega\left(\delta^d_b\tilde{\nabla}_e + \delta^d_e\tilde{\nabla}_b - \tilde{g}^{dg}\tilde{g}_{be}\tilde{\nabla}_g\right) \tag{A.14}$$

$$\left.-\delta^e_a\tilde{\nabla}_c\ln\Omega\left(\delta^d_b\tilde{\nabla}_e + \delta^d_e\tilde{\nabla}_b - \tilde{g}^{dg}\tilde{g}_{be}\tilde{\nabla}_g\right) + \tilde{g}^{ef}\tilde{g}_{ca}\tilde{\nabla}_f\ln\Omega\left(\delta^d_b\tilde{\nabla}_e + \delta^d_e\tilde{\nabla}_b - \tilde{g}^{dg}\tilde{g}_{be}\tilde{\nabla}_g\right)\right]\ln\Omega \tag{A.15}$$

$$+ \left(\delta^d_a\tilde{\nabla}_b\tilde{\nabla}_c + \delta^d_c\tilde{\nabla}_b\tilde{\nabla}_a - \tilde{g}^{df}\tilde{g}_{ac}\tilde{\nabla}_b\tilde{\nabla}_f - \delta^d_b\tilde{\nabla}_a\tilde{\nabla}_c - \delta^d_c\tilde{\nabla}_a\tilde{\nabla}_b + \tilde{g}^{df}\tilde{g}_{bc}\tilde{\nabla}_a\tilde{\nabla}_f\right)\ln\Omega \tag{A.16}$$

$$\tag{A.17}$$

$$
\begin{aligned}
&= R^d_{abc} + \Big(\delta^d_a\delta^e_b\widetilde{\nabla}_c \ln\Omega\widetilde{\nabla}_e + \delta^d_e\delta^e_b\widetilde{\nabla}_c \ln\Omega\widetilde{\nabla}_a - \tilde{g}^{dg}\tilde{g}_{ae}\delta^e_b\widetilde{\nabla}_c \ln\Omega\widetilde{\nabla}_g + \delta^d_a\delta^e_c\widetilde{\nabla}_b \ln\Omega\widetilde{\nabla}_e + \delta^d_e\delta^e_c\widetilde{\nabla}_b \ln\Omega\widetilde{\nabla}_a \\
&-\tilde{g}^{dg}\tilde{g}_{ae}\delta^e_c\widetilde{\nabla}_b \ln\Omega\widetilde{\nabla}_g - \delta^d_a\tilde{g}^{ef}\tilde{g}_{bc}\widetilde{\nabla}_f \ln\Omega\widetilde{\nabla}_e - \delta^d_e\tilde{g}^{ef}\tilde{g}_{bc}\widetilde{\nabla}_f \ln\Omega\widetilde{\nabla}_a + \tilde{g}^{dg}\tilde{g}_{ae}\tilde{g}^{ef}\tilde{g}_{bc}\widetilde{\nabla}_f \ln\Omega\widetilde{\nabla}_g \\
&-\delta^d_b\delta^e_c\widetilde{\nabla}_a \ln\Omega\widetilde{\nabla}_e - \delta^d_e\delta^e_c\widetilde{\nabla}_a \ln\Omega\widetilde{\nabla}_b + \tilde{g}^{dg}\tilde{g}_{be}\delta^e_c\widetilde{\nabla}_a \ln\Omega\widetilde{\nabla}_g - \delta^d_b\delta^e_a\widetilde{\nabla}_c \ln\Omega\widetilde{\nabla}_e - \delta^d_e\delta^e_a\widetilde{\nabla}_c \ln\Omega\widetilde{\nabla}_b + \tilde{g}^{dg}\tilde{g}_{be}\delta^e_a\widetilde{\nabla}_c \ln\Omega\widetilde{\nabla}_g \\
&+\delta^d_b\tilde{g}^{ef}\tilde{g}_{ca}\widetilde{\nabla}_f \ln\Omega\widetilde{\nabla}_e + \delta^d_e\tilde{g}^{ef}\tilde{g}_{ca}\widetilde{\nabla}_f \ln\Omega\widetilde{\nabla}_b - \tilde{g}^{dg}\tilde{g}_{be}\tilde{g}^{ef}\tilde{g}_{ca}\widetilde{\nabla}_f \ln\Omega\widetilde{\nabla}_g\Big] \ln\Omega \\
&+ \Big(\delta^d_a\widetilde{\nabla}_b\widetilde{\nabla}_c + \delta^d_c\widetilde{\nabla}_b\widetilde{\nabla}_a - \tilde{g}^{df}\tilde{g}_{ac}\widetilde{\nabla}_b\widetilde{\nabla}_f - \delta^d_b\widetilde{\nabla}_a\widetilde{\nabla}_c - \delta^d_c\widetilde{\nabla}_a\widetilde{\nabla}_b + \tilde{g}^{df}\tilde{g}_{bc}\widetilde{\nabla}_a\widetilde{\nabla}_f\Big) \ln\Omega \\
&= R^d_{abc} + \Big(\delta^d_a\widetilde{\nabla}_c \ln\Omega\widetilde{\nabla}_b + \delta^d_b\widetilde{\nabla}_c \ln\Omega\widetilde{\nabla}_a - \tilde{g}^{dg}\tilde{g}_{ab}\widetilde{\nabla}_c \ln\Omega\widetilde{\nabla}_g + \delta^d_a\widetilde{\nabla}_b \ln\Omega\widetilde{\nabla}_c + \delta^d_c\widetilde{\nabla}_b \ln\Omega\widetilde{\nabla}_a \\
&-\tilde{g}^{dg}\tilde{g}_{ac}\widetilde{\nabla}_b \ln\Omega\widetilde{\nabla}_g - \delta^d_a\tilde{g}^{ef}\tilde{g}_{bc}\widetilde{\nabla}_f \ln\Omega\widetilde{\nabla}_e - \tilde{g}^{df}\tilde{g}_{bc}\widetilde{\nabla}_f \ln\Omega\widetilde{\nabla}_a + \tilde{g}^{dg}\tilde{g}_{bc}\widetilde{\nabla}_a \ln\Omega\widetilde{\nabla}_g \\
&-\delta^d_b\widetilde{\nabla}_a \ln\Omega\widetilde{\nabla}_c - \delta^d_c\widetilde{\nabla}_a \ln\Omega\widetilde{\nabla}_b + \tilde{g}^{dg}\tilde{g}_{bc}\widetilde{\nabla}_a \ln\Omega\widetilde{\nabla}_g - \delta^d_b\widetilde{\nabla}_c \ln\Omega\widetilde{\nabla}_a - \delta^d_a\widetilde{\nabla}_c \ln\Omega\widetilde{\nabla}_b + \tilde{g}^{dg}\tilde{g}_{ba}\widetilde{\nabla}_c \ln\Omega\widetilde{\nabla}_g \\
&+\delta^d_b\tilde{g}^{ef}\tilde{g}_{ca}\widetilde{\nabla}_f \ln\Omega\widetilde{\nabla}_e + \tilde{g}^{df}\tilde{g}_{ca}\widetilde{\nabla}_f \ln\Omega\widetilde{\nabla}_b - \tilde{g}^{dg}\tilde{g}_{ca}\widetilde{\nabla}_b \ln\Omega\widetilde{\nabla}_g\Big] \ln\Omega \\
&+ \Big(\delta^d_a\widetilde{\nabla}_b\widetilde{\nabla}_c + \delta^d_c\widetilde{\nabla}_b\widetilde{\nabla}_a - \tilde{g}^{df}\tilde{g}_{ac}\widetilde{\nabla}_b\widetilde{\nabla}_f - \delta^d_b\widetilde{\nabla}_a\widetilde{\nabla}_c - \delta^d_c\widetilde{\nabla}_a\widetilde{\nabla}_b + \tilde{g}^{df}\tilde{g}_{bc}\widetilde{\nabla}_a\widetilde{\nabla}_f\Big) \ln\Omega \\
&= R^d_{abc} + \Big(\delta^d_a\widetilde{\nabla}_b \ln\Omega\widetilde{\nabla}_c + \delta^d_c\widetilde{\nabla}_b \ln\Omega\widetilde{\nabla}_a - \delta^d_a\tilde{g}^{ef}\tilde{g}_{bc}\widetilde{\nabla}_f \ln\Omega\widetilde{\nabla}_e + \tilde{g}^{dg}\tilde{g}_{bc}\widetilde{\nabla}_a \ln\Omega\widetilde{\nabla}_g \\
&-\delta^d_c\widetilde{\nabla}_a \ln\Omega\widetilde{\nabla}_b - \delta^d_b\widetilde{\nabla}_c \ln\Omega\widetilde{\nabla}_a + \delta^d_b\tilde{g}^{ef}\tilde{g}_{ca}\widetilde{\nabla}_f \ln\Omega\widetilde{\nabla}_e - \tilde{g}^{dg}\tilde{g}_{ca}\widetilde{\nabla}_b \ln\Omega\widetilde{\nabla}_g\Big] \ln\Omega \\
&+ \Big(\delta^d_a\widetilde{\nabla}_b\widetilde{\nabla}_c + \delta^d_c\widetilde{\nabla}_b\widetilde{\nabla}_a - \tilde{g}^{df}\tilde{g}_{ac}\widetilde{\nabla}_b\widetilde{\nabla}_f - \delta^d_b\widetilde{\nabla}_a\widetilde{\nabla}_c - \delta^d_c\widetilde{\nabla}_a\widetilde{\nabla}_b + \tilde{g}^{df}\tilde{g}_{bc}\widetilde{\nabla}_a\widetilde{\nabla}_f\Big) \ln\Omega \\
&= R^d_{abc} + \Big(\Big(\delta^d_a\widetilde{\nabla}_b \ln\Omega - \delta^d_b\widetilde{\nabla}_a \ln\Omega\Big)\widetilde{\nabla}_c + \tilde{g}^{ef}\Big(\delta^d_b\tilde{g}_{ca} - \delta^d_a\tilde{g}_{bc}\Big)\widetilde{\nabla}_f \ln\Omega\widetilde{\nabla}_e + \tilde{g}^{dg}\Big(\tilde{g}_{bc}\widetilde{\nabla}_a \ln\Omega - \tilde{g}_{ca}\widetilde{\nabla}_b \ln\Omega\Big)\widetilde{\nabla}_g\Big] \ln\Omega \\
&+ \Big(\Big(\delta^d_a\widetilde{\nabla}_b - \delta^d_b\widetilde{\nabla}_a\Big)\widetilde{\nabla}_c - \delta^d_c\Big(\widetilde{\nabla}_a\widetilde{\nabla}_b - \widetilde{\nabla}_b\widetilde{\nabla}_a\Big) + \tilde{g}^{df}\Big(\tilde{g}_{bc}\widetilde{\nabla}_a\widetilde{\nabla}_f - \tilde{g}_{ac}\widetilde{\nabla}_b\widetilde{\nabla}_f\Big)\Big) \ln\Omega \\
&= R^d_{abc} + \Big(2\delta^d_{[a}\widetilde{\nabla}_{b]} \ln\Omega\widetilde{\nabla}_c + 2\tilde{g}^{ef}\delta^d_{[b}\tilde{g}_{a]c}\widetilde{\nabla}_f \ln\Omega\widetilde{\nabla}_e + 2\tilde{g}^{dg}\tilde{g}_{c[b}\widetilde{\nabla}_{a]} \ln\Omega\widetilde{\nabla}_g\Big] \ln\Omega + \Big(2\delta^d_{[a}\widetilde{\nabla}_{b]}\widetilde{\nabla}_c + 2\tilde{g}^{df}\tilde{g}_{c[b}\widetilde{\nabla}_{a]}\widetilde{\nabla}_f\Big) \ln\Omega
\end{aligned}
$$

Therefore the Ricci tensor,

$$
\begin{aligned}
\tilde{R}_{ac} \equiv \tilde{R}^b_{abc} &= R^b_{abc} + \Big(\Big(\delta^b_a\widetilde{\nabla}_b \ln\Omega - \delta^b_b\widetilde{\nabla}_a \ln\Omega\Big)\widetilde{\nabla}_c + \tilde{g}^{ef}\Big(\delta^b_b\tilde{g}_{ca} - \delta^b_a\tilde{g}_{bc}\Big)\widetilde{\nabla}_f \ln\Omega\widetilde{\nabla}_e + \tilde{g}^{bg}\Big(\tilde{g}_{bc}\widetilde{\nabla}_a \ln\Omega - \tilde{g}_{ca}\widetilde{\nabla}_b \ln\Omega\Big)\widetilde{\nabla}_g\Big] \ln\Omega \\
&+ \Big(\Big(\delta^b_a\widetilde{\nabla}_b - \delta^b_b\widetilde{\nabla}_a\Big)\widetilde{\nabla}_c - \delta^b_c\Big(\widetilde{\nabla}_a\widetilde{\nabla}_b - \widetilde{\nabla}_b\widetilde{\nabla}_a\Big) + \tilde{g}^{bf}\Big(\tilde{g}_{bc}\widetilde{\nabla}_a\widetilde{\nabla}_f - \tilde{g}_{ac}\widetilde{\nabla}_b\widetilde{\nabla}_f\Big)\Big) \ln\Omega \\
&= R^b_{abc} + \Big(\Big(\widetilde{\nabla}_a \ln\Omega - D\widetilde{\nabla}_a \ln\Omega\Big)\widetilde{\nabla}_c + \tilde{g}^{ef}\left(D\tilde{g}_{ca} - \tilde{g}_{ca}\right)\widetilde{\nabla}_f \ln\Omega\widetilde{\nabla}_e + \Big(\widetilde{\nabla}_a \ln\Omega\widetilde{\nabla}_c - \tilde{g}^{bg}\tilde{g}_{ca}\widetilde{\nabla}_b \ln\Omega\widetilde{\nabla}_g\Big)\Big] \ln\Omega \\
&+ \Big(\Big(\widetilde{\nabla}_a - D\widetilde{\nabla}_a\Big)\widetilde{\nabla}_c + \Big(\widetilde{\nabla}_a\widetilde{\nabla}_c - \tilde{g}^{bf}\tilde{g}_{ac}\widetilde{\nabla}_b\widetilde{\nabla}_f\Big)\Big) \ln\Omega \\
&= R^b_{abc} + \Big(\left(1-D\right)\widetilde{\nabla}_a \ln\Omega\widetilde{\nabla}_c + \tilde{g}_{ca}\left(D-1\right)\tilde{g}^{ef}\widetilde{\nabla}_f \ln\Omega\widetilde{\nabla}_e + \Big(\widetilde{\nabla}_a \ln\Omega\widetilde{\nabla}_c - \tilde{g}_{ca}\tilde{g}^{bg}\widetilde{\nabla}_b \ln\Omega\widetilde{\nabla}_g\Big)\Big] \ln\Omega \\
&+ \Big(\left(1-D\right)\widetilde{\nabla}_a\widetilde{\nabla}_c + \Big(\widetilde{\nabla}_a\widetilde{\nabla}_c - \tilde{g}_{ac}\tilde{g}^{bf}\widetilde{\nabla}_b\widetilde{\nabla}_f\Big)\Big) \ln\Omega \\
&= R^b_{abc} + \Big[\left(2-D\right)\widetilde{\nabla}_a \ln\Omega\widetilde{\nabla}_c + \tilde{g}_{ca}\left(D-2\right)\tilde{g}^{ef}\widetilde{\nabla}_f \ln\Omega\widetilde{\nabla}_e\Big] \ln\Omega + \Big(\left(2-D\right)\widetilde{\nabla}_a\widetilde{\nabla}_c - \tilde{g}_{ac}\tilde{g}^{bf}\widetilde{\nabla}_b\widetilde{\nabla}_f\Big) \ln\Omega
\end{aligned}
$$

A.6 FLRW metric tensors

The respective metric is,

$$g_{\mu\nu} = diag\left(-1, \frac{a^2(t)}{1-kr^2}, a^2(t)r^2\sin^2(\theta), a^2(t)r^2\right) \tag{A.18}$$

and the inverse is,

$$g^{\mu\nu} = diag\left(-1, \frac{1-kr^2}{a^2(t)}, \frac{1}{a^2(t)r^2\sin^2(\theta)}, \frac{1}{a^2(t)r^2}\right) \tag{A.19}$$

which allows us to see some properties, $g_{\alpha\beta} = 0$, $\alpha \neq \beta$

$$\begin{aligned}
&\partial_\phi g_{\alpha\alpha} = 0 \\
&\partial_\gamma g_{tt} = 0 \\
&\partial_t g_{rr} = \frac{2a(t)\dot{a}(t)}{1-kr^2} = 2\frac{\dot{a}(t)}{a(t)} g_{rr} \\
&\partial_r g_{rr} = \frac{2kra^2(t)}{(1-kr^2)^2} = \frac{2kr}{1-kr^2} g_{rr} \\
&\partial_\theta g_{rr} = 0
\end{aligned}$$

$$\begin{aligned}
&\partial_t g_{\phi\phi} = 2a(t)\dot{a}(t)r^2\sin^2(\theta) = \frac{2\dot{a}(t)}{a(t)} g_{\phi\phi} \\
&\partial_r g_{\phi\phi} = 2a^2(t)r\sin^2(\theta) = \frac{2}{r} g_{\phi\phi} \\
&\partial_\theta g_{\phi\phi} = 2a^2(t)r^2\cos(\theta)\sin(\theta) = 2\frac{\cos(\theta)}{\sin(\theta)} g_{\phi\phi} \\
&\partial_t g_{\theta\theta} = 2a(t)\dot{a}(t)r^2 = \frac{2\dot{a}(t)}{a(t)} g_{\theta\theta} \\
&\partial_r g_{\theta\theta} = 2a^2(t)r = \frac{2}{r} g_{\theta\theta}
\end{aligned}$$

Then from the Christoffel symbol is,

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) \tag{A.20}$$

We see that some components of the Christoffel symbol, for $\lambda \neq \mu \neq \nu$ is,

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu}\right) = \tag{A.21}$$

$$= \frac{1}{2} g^{\lambda t}\left(\partial_\mu g_{\nu t} + \partial_\nu g_{t\mu}\right) + \frac{1}{2} g^{\lambda r}\left(\partial_\mu g_{\nu r} + \partial_\nu g_{r\mu}\right) + \frac{1}{2} g^{\lambda\phi}\left(\partial_\mu g_{\nu\phi} + \partial_\nu g_{\phi\mu}\right) + \frac{1}{2} g^{\lambda\theta}\left(\partial_\mu g_{\nu\theta} + \partial_\nu g_{\theta\mu}\right) \tag{A.22}$$

notice μ nor ν can be ϕ and notice that if $\lambda \neq \mu$ not $\lambda \neq \nu$, then $\Gamma^\lambda_{\mu\nu} = 0$

We can also see that some components of the Christoffel symbol, is,

$$\Gamma^\lambda_{\mu\lambda} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\lambda\rho} + \partial_\lambda g_{\rho\mu} - \partial_\rho g_{\mu\lambda}\right) = \frac{1}{2}\left(g^{\lambda\rho}\partial_\mu g_{\lambda\rho} + g^{\lambda\rho}\partial_\lambda g_{\rho\mu} - g^{\lambda\rho}\partial_\rho g_{\mu\lambda}\right) = \frac{1}{2} g^{\lambda\rho}\partial_\mu g_{\lambda\rho} = \tag{A.23}$$

$$= \frac{1}{2} g^{\lambda t}\partial_\mu g_{\lambda t} + \frac{1}{2} g^{\lambda r}\partial_\mu g_{\lambda r} + \frac{1}{2} g^{\lambda\phi}\partial_\mu g_{\lambda\phi} + \frac{1}{2} g^{\lambda\theta}\partial_\mu g_{\lambda\theta} = \tag{A.24}$$

$$= \frac{1}{2} g^{tt}\partial_\mu g_{tt} + \frac{1}{2} g^{rr}\partial_\mu g_{rr} + \frac{1}{2} g^{\phi\phi}\partial_\mu g_{\phi\phi} + \frac{1}{2} g^{\theta\theta}\partial_\mu g_{\theta\theta} = \tag{A.25}$$

$$= \frac{1}{2} g^{rr}\partial_\mu g_{rr} + \frac{1}{2} g^{\phi\phi}\partial_\mu g_{\phi\phi} + \frac{1}{2} g^{\theta\theta}\partial_\mu g_{\theta\theta} \tag{A.26}$$

$$\Gamma^\lambda_{t\lambda} = \frac{1}{2} g^{rr}\partial_t g_{rr} + \frac{1}{2} g^{\phi\phi}\partial_t g_{\phi\phi} + \frac{1}{2} g^{\theta\theta}\partial_t g_{\theta\theta} = g^{rr}\frac{\dot{a}(t)}{a(t)} g_{rr} + g^{\phi\phi}\frac{\dot{a}(t)}{a(t)} g_{\phi\phi} + g^{\theta\theta}\frac{\dot{a}(t)}{a(t)} g_{\theta\theta} = 3\frac{\dot{a}(t)}{a(t)} \tag{A.27}$$

$$\Gamma^\lambda_{r\lambda} = \frac{1}{2} g^{rr}\partial_r g_{rr} + \frac{1}{2} g^{\phi\phi}\partial_r g_{\phi\phi} + \frac{1}{2} g^{\theta\theta}\partial_r g_{\theta\theta} = g^{rr}\frac{kr}{1-kr^2} g_{rr} + g^{\phi\phi}\frac{1}{r} g_{\phi\phi} + g^{\theta\theta}\frac{1}{r} g_{\theta\theta} = \frac{kr}{1-kr^2} + \frac{2}{r} \tag{A.28}$$

$$\Gamma^\lambda_{\phi\lambda} = \frac{1}{2} g^{rr}\partial_\phi g_{rr} + \frac{1}{2} g^{\phi\phi}\partial_\phi g_{\phi\phi} + \frac{1}{2} g^{\theta\theta}\partial_\phi g_{\theta\theta} = 0 \tag{A.29}$$

$$\Gamma^\lambda_{\theta\lambda} = \frac{1}{2} g^{rr}\partial_\theta g_{rr} + \frac{1}{2} g^{\phi\phi}\partial_\theta g_{\phi\phi} + \frac{1}{2} g^{\theta\theta}\partial_\theta g_{\theta\theta} = g^{\phi\phi}\frac{\cos(\theta)}{\sin(\theta)} g_{\phi\phi} = \cot(\theta) \tag{A.30}$$

$$\begin{aligned}
&\Gamma^t_{\mu t} = 0 \\
&\Gamma^r_{tr} = \Gamma^\phi_{t\phi} = \Gamma^\theta_{t\theta} = \frac{\dot{a}(t)}{a(t)} \\
&\Gamma^\phi_{r\phi} = \Gamma^\theta_{r\theta} = \frac{1}{r}
\end{aligned}$$

$$\begin{aligned}
&\Gamma^\lambda_{\lambda\phi} = 0 \\
&\Gamma^r_{r\theta} = \Gamma^\theta_{\theta\theta} = 0 \\
&\Gamma^\phi_{\phi\theta} = \cot(\theta)
\end{aligned}$$

And the components of the Christoffel symbol, for $\lambda \neq \mu$ is,

$$\Gamma^\lambda_{\mu\mu} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\mu\rho} + \partial_\mu g_{\rho\mu} - \partial_\rho g_{\mu\mu}\right) = \frac{1}{2}\left(g^{\lambda\rho}\partial_\mu g_{\mu\rho} + g^{\lambda\rho}\partial_\mu g_{\rho\mu} - g^{\lambda\rho}\partial_\rho g_{\mu\mu}\right) = \frac{1}{2}\left(2g^{\lambda\rho}\partial_\mu g_{\mu\rho} - g^{\lambda\rho}\partial_\rho g_{\mu\mu}\right) \tag{A.31}$$

$$= \frac{1}{2}\left(g^{\lambda\rho}\partial_\mu g_{\mu\rho} + g^{\lambda\rho}\partial_\mu g_{\rho\mu} - g^{\lambda\rho}\partial_\rho g_{\mu\mu}\right) = -\frac{1}{2} g^{\lambda\rho}\partial_\rho g_{\mu\mu} = -\frac{1}{2} g^{\lambda t}\partial_t g_{\mu\mu} - \frac{1}{2} g^{\lambda r}\partial_r g_{\mu\mu} - \frac{1}{2} g^{\lambda\theta}\partial_\theta g_{\mu\mu} \tag{A.32}$$

$$\tag{A.33}$$

$$\Gamma^{\lambda}_{tt} = 0 \tag{A.34}$$
$$\Gamma^{t}_{rr} = -\frac{1}{2}g^{tt}\partial_t g_{rr} = -g^{tt}\frac{\dot{a}(t)}{a(t)}g_{rr} = \frac{\dot{a}(t)a(t)}{1-kr^2} \tag{A.35}$$
$$\Gamma^{\phi}_{\mu\mu} = 0 \tag{A.36}$$
$$\Gamma^{\theta}_{rr} = 0 \tag{A.37}$$
$$\Gamma^{t}_{\phi\phi} = -\frac{1}{2}g^{tt}\partial_t g_{\phi\phi} = -g^{tt}\frac{\dot{a}(t)}{a(t)}g_{\phi\phi} = \dot{a}(t)a(t)r^2\sin^2(\theta) \tag{A.38}$$
$$\Gamma^{r}_{\phi\phi} = -\frac{1}{2}g^{rr}\partial_r g_{\phi\phi} = -g^{rr}\frac{1}{r}g_{\phi\phi} = -(1-kr^2)r\sin^2(\theta) \tag{A.39}$$
$$\Gamma^{\theta}_{\phi\phi} = -\frac{1}{2}g^{\theta\theta}\partial_\theta g_{\phi\phi} = -g^{\theta\theta}\frac{\cos(\theta)}{\sin(\theta)}g_{\phi\phi} = -\cos(\theta)\sin(\theta) \tag{A.40}$$
$$\Gamma^{t}_{\theta\theta} = -\frac{1}{2}g^{tt}\partial_t g_{\theta\theta} = -g^{tt}\frac{\dot{a}(t)}{a(t)}g_{\theta\theta} = \dot{a}(t)a(t)r^2 \tag{A.41}$$
$$\Gamma^{r}_{\theta\theta} = -\frac{1}{2}g^{rr}\partial_r g_{\theta\theta} = -g^{rr}\frac{1}{r}g_{\theta\theta} = -(1-kr^2)r \tag{A.42}$$

The component of the Christoffel symbol,

$$\Gamma^{\mu}_{\mu\mu} = \frac{1}{2}g^{\mu\rho}\left(\partial_\mu g_{\mu\rho} + \partial_\mu g_{\rho\mu} - \partial_\rho g_{\mu\mu}\right) = \frac{1}{2}\left(g^{\mu\rho}\partial_\mu g_{\mu\rho} + g^{\mu\rho}\partial_\mu g_{\rho\mu} - g^{\mu\rho}\partial_\rho g_{\mu\mu}\right) = \frac{1}{2}\left(2g^{\mu\rho}\partial_\mu g_{\mu\rho} - g^{\mu\rho}\partial_\rho g_{\mu\mu}\right) \tag{A.43}$$
$$= \frac{1}{2}g^{\mu\mu}\partial_\mu g_{\mu\mu} \tag{A.44}$$

$$\Gamma^{t}_{tt} = 0 \tag{A.45}$$
$$\Gamma^{r}_{rr} = \frac{1}{2}g^{rr}\partial_r g_{rr} = g^{rr}\frac{kr}{1-kr^2}g_{rr} = \frac{kr}{1-kr^2} \tag{A.46}$$
$$\Gamma^{\phi}_{\phi\phi} = 0 \tag{A.47}$$
$$\Gamma^{\theta}_{\theta\theta} = 0 \tag{A.48}$$

The Ricci curvature scalar, $R \equiv R_{\mu\nu}g^{\mu\nu}$ where the Ricci tensor is given by,

$$R_{\mu\nu} = \partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\mu \Gamma^{\lambda}_{\nu\lambda} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{\mu\nu} - \Gamma^{\lambda}_{\mu\rho}\Gamma^{\rho}_{\lambda\nu} \tag{A.49}$$

we can clearly see by the definition of the Ricci scalar, being the trace of the Ricci tensor, the only terms needed to be determined will be the diagonal components.

The component (t,t),

$$R_{tt} = \partial_\lambda \Gamma^{\lambda}_{tt} - \partial_t \Gamma^{\lambda}_{t\lambda} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{tt} - \Gamma^{\lambda}_{t\rho}\Gamma^{\rho}_{\lambda t} = -\partial_t\left(3\frac{\dot{a}(t)}{a(t)}\right) - \left(\Gamma^{\lambda}_{tt}\Gamma^{t}_{\lambda t} + \Gamma^{\lambda}_{tr}\Gamma^{r}_{\lambda t} + \Gamma^{\lambda}_{t\phi}\Gamma^{\phi}_{\lambda t} + \Gamma^{\lambda}_{t\theta}\Gamma^{\theta}_{\lambda t}\right) \tag{A.50}$$
$$= -3\frac{\ddot{a}(t)a(t) - \dot{a}^2(t)}{a^2(t)} - \left(\Gamma^{r}_{tr}\Gamma^{r}_{rt} + \Gamma^{\phi}_{t\phi}\Gamma^{\phi}_{\phi t} + \Gamma^{\theta}_{t\theta}\Gamma^{\theta}_{\theta t}\right) = -3\frac{\ddot{a}(t)a(t) - \dot{a}^2(t)}{a^2(t)} - \left(\Gamma^{r}_{tr}\right)^2 - \left(\Gamma^{\phi}_{\phi t}\right)^2 - \left(\Gamma^{\theta}_{\theta t}\right)^2 \tag{A.51}$$
$$= -3\frac{\ddot{a}(t)a(t) - \dot{a}^2(t)}{a^2(t)} - \left(\frac{\dot{a}(t)}{a(t)}\right)^2 - \left(\frac{\dot{a}(t)}{a(t)}\right)^2 - \left(\frac{\dot{a}(t)}{a(t)}\right)^2 = -3\frac{\ddot{a}(t)a(t) - \dot{a}^2(t)}{a^2(t)} - 3\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = -3\frac{\ddot{a}(t)}{a(t)} \tag{A.52}$$

The component (r, r)

$$R_{rr} = \partial_\lambda \Gamma^\lambda_{rr} - \partial_r \Gamma^\lambda_{r\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{rr} - \Gamma^\lambda_{r\rho}\Gamma^\rho_{\lambda r} \tag{A.53}$$

$$= \left(\partial_t \Gamma^t_{rr} + \partial_r \Gamma^r_{rr}\right) - \partial_r \left(\frac{kr}{1-kr^2} + \frac{2}{r}\right) + \left(\Gamma^\lambda_{\lambda t}\Gamma^t_{rr} + \Gamma^\lambda_{\lambda r}\Gamma^r_{rr}\right) - \left(\Gamma^\lambda_{rt}\Gamma^t_{\lambda r} + \Gamma^\lambda_{rr}\Gamma^r_{\lambda r} + \Gamma^\lambda_{r\theta}\Gamma^\theta_{\lambda r}\right) \tag{A.54}$$

$$= \partial_t \left(\frac{\dot{a}(t)a(t)}{1-kr^2}\right) + \partial_r \left(\frac{kr}{1-kr^2}\right) - \partial_r \left(\frac{kr}{1-kr^2} + \frac{2}{r}\right) + \left(\left(3\frac{\dot{a}(t)}{a(t)}\right)\left(\frac{\dot{a}(t)a(t)}{1-kr^2}\right) + \left(\frac{kr}{1-kr^2} + \frac{2}{r}\right)\left(\frac{kr}{1-kr^2}\right)\right) \tag{A.55}$$

$$- \left(\left(\frac{\dot{a}(t)}{a(t)}\right)\left(\frac{\dot{a}(t)a(t)}{1-kr^2}\right) + \Gamma^t_{rr}\Gamma^r_{tr} + \Gamma^r_{rr}\Gamma^r_{rr} + \Gamma^r_{r\theta}\Gamma^\theta_{rr} + \Gamma^\theta_{r\theta}\Gamma^\theta_{\theta r}\right) \tag{A.56}$$

$$= \partial_t \left(\frac{\dot{a}(t)a(t)}{1-kr^2}\right) - \partial_r \left(\frac{2}{r}\right) + \left(\left(2\frac{\dot{a}(t)}{a(t)}\right)\left(\frac{\dot{a}(t)a(t)}{1-kr^2}\right) + \left[\frac{kr}{1-kr^2}\left(\frac{kr}{1-kr^2}\right) + \frac{2}{r}\left(\frac{kr}{1-kr^2}\right)\right]\right) \tag{A.57}$$

$$- \left(\left(\frac{\dot{a}(t)a(t)}{1-kr^2}\right)\left(\frac{\dot{a}(t)}{a(t)}\right) + \left(\frac{kr}{1-kr^2}\right)^2 + 2\left(\frac{1}{r}\right)^2\right) \tag{A.58}$$

$$= \left(\frac{\ddot{a}(t)a(t) + \dot{a}^2(t)}{1-kr^2}\right) + \left(\frac{2}{r^2}\right) + \left(\left(\frac{\dot{a}(t)}{a(t)}\right)\left(\frac{\dot{a}(t)a(t)}{1-kr^2}\right) + \frac{2}{r}\left(\frac{kr}{1-kr^2}\right)\right) - 2\left(\frac{1}{r}\right)^2 \tag{A.59}$$

$$= \left(\frac{\ddot{a}(t)a(t) + \dot{a}^2(t)}{1-kr^2}\right) + \left(\frac{\dot{a}^2(t)}{1-kr^2}\right) + \left(\frac{2k}{1-kr^2}\right) = \frac{\ddot{a}(t)a(t) + 2\dot{a}^2(t) + 2k}{1-kr^2} \tag{A.60}$$

The component (ϕ, ϕ)

$$R_{\phi\phi} = \partial_\lambda \Gamma^\lambda_{\phi\phi} - \partial_\phi \Gamma^\lambda_{\phi\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\phi\phi} - \Gamma^\lambda_{\phi\rho}\Gamma^\rho_{\lambda\phi} = \left(\partial_t \Gamma^t_{\phi\phi} + \partial_r \Gamma^r_{\phi\phi} + \partial_\theta \Gamma^\theta_{\phi\phi}\right) + \left(\Gamma^\lambda_{\lambda t}\Gamma^t_{\phi\phi} + \Gamma^\lambda_{\lambda r}\Gamma^r_{\phi\phi} + \Gamma^\lambda_{\lambda\theta}\Gamma^\theta_{\phi\phi}\right) \tag{A.61}$$

$$- \left(\Gamma^\lambda_{\phi t}\Gamma^t_{\lambda\phi} + \Gamma^\lambda_{\phi r}\Gamma^r_{\lambda r} + \Gamma^\lambda_{\phi\phi}\Gamma^\phi_{\lambda\phi} + \Gamma^\lambda_{\phi\theta}\Gamma^\theta_{\lambda\phi}\right) \tag{A.62}$$

$$= \partial_t \left(\dot{a}(t)a(t)r^2\sin^2(\theta)\right) + \partial_r \left(-(1-kr^2)r\sin^2(\theta)\right) + \partial_\theta \left(-\cos(\theta)\sin(\theta)\right) \tag{A.63}$$

$$+ \left[\left(3\frac{\dot{a}(t)}{a(t)}\right)\left(\dot{a}(t)a(t)r^2\sin^2(\theta)\right) + \left(\frac{kr}{1-kr^2} + \frac{2}{r}\right)\left(-(1-kr^2)r\sin^2(\theta)\right) + \left(\cot(\theta)\right)\left(-\cos(\theta)\sin(\theta)\right)\right] \tag{A.64}$$

$$- \left[\left(\Gamma^t_{\phi t}\Gamma^t_{t\phi} + \Gamma^\phi_{\phi t}\Gamma^t_{\phi\phi}\right) + \left(\Gamma^r_{\phi r}\Gamma^r_{rr} + \Gamma^\phi_{\phi r}\Gamma^r_{\phi r}\right) + \left(\Gamma^t_{\phi\phi}\Gamma^\phi_{t\phi} + \Gamma^r_{\phi\phi}\Gamma^\phi_{r\phi} + \Gamma^\theta_{\phi\phi}\Gamma^\phi_{\theta\phi}\right) + \left(\Gamma^\phi_{\phi\theta}\Gamma^\theta_{\phi\phi} + \Gamma^\theta_{\phi\theta}\Gamma^\theta_{\theta\phi}\right)\right] \tag{A.65}$$

$$= \left(\ddot{a}(t)a(t) + \dot{a}^2(t)\right)r^2\sin^2(\theta) - \left((1-kr^2) - 2kr^2\right)\sin^2(\theta) - \left(\cos^2(\theta) - \sin^2(\theta)\right) \tag{A.66}$$

$$+ \left[3\left(\dot{a}^2(t)r^2\sin^2(\theta)\right) - \left(kr^2 + 2(1-kr^2)\right)\left(\sin^2(\theta)\right) - \cos^2(\theta)\right] \tag{A.67}$$

$$- \left[\dot{a}(t)a(t)r^2\sin^2(\theta)\frac{\dot{a}(t)}{a(t)} + \left(\left(-(1-kr^2)r\sin^2(\theta)\right)\left(\frac{1}{r}\right) + \left(-\cos(\theta)\sin(\theta)\right)\cot(\theta)\right) + \cot(\theta)(-\cos(\theta)\sin(\theta))\right] \tag{A.68}$$

$$= \ddot{a}(t)a(t)r^2\sin^2(\theta) - \left(2(1-kr^2) - kr^2\right)\sin^2(\theta) + \sin^2(\theta) + 4\left(\dot{a}^2(t)r^2\sin^2(\theta)\right) - \left(\dot{a}^2(t)r^2\sin^2(\theta)\right) \tag{A.69}$$

$$= \left[\ddot{a}(t)a(t)r^2 + 2kr^2 + 2\left(\dot{a}^2(t)r^2\right)\right]\sin^2(\theta) = \left[\ddot{a}(t)a(t) + 2k + 2\dot{a}^2(t)\right]r^2\sin^2(\theta) \tag{A.70}$$

The component (θ, θ),

$$R_{\theta\theta} = \partial_\lambda \Gamma^\lambda_{\theta\theta} - \partial_\theta \Gamma^\lambda_{\theta\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\theta\theta} - \Gamma^\lambda_{\theta\rho}\Gamma^\rho_{\lambda\theta} = \left(\partial_t \Gamma^t_{\theta\theta} + \partial_r \Gamma^r_{\theta\theta}\right) - \partial_\theta \cot(\theta) + \left(\Gamma^\lambda_{\lambda t}\Gamma^t_{\theta\theta} + \Gamma^\lambda_{\lambda r}\Gamma^r_{\theta\theta}\right) \tag{A.71}$$

$$- \left(\Gamma^\lambda_{\theta t}\Gamma^t_{\lambda\theta} + \Gamma^\lambda_{\theta r}\Gamma^r_{\lambda\theta} + \Gamma^\lambda_{\theta\phi}\Gamma^\phi_{\lambda\theta} + \Gamma^\lambda_{\theta\theta}\Gamma^\theta_{\lambda\theta}\right) \tag{A.72}$$

$$= \left(\partial_t(\dot{a}(t)a(t)r^2) + \partial_r(-(1-kr^2)r)\right) + \frac{1}{\sin^2(\theta)} + \left(\left(3\frac{\dot{a}(t)}{a(t)}\right)(\dot{a}(t)a(t)r^2) + \left(\frac{kr}{1-kr^2} + \frac{2}{r}\right)(-(1-kr^2)r)\right) \tag{A.73}$$

$$- \left[\left(\Gamma^t_{\theta t}\Gamma^t_{t\theta} + \Gamma^\theta_{\theta t}\Gamma^t_{\theta\theta}\right) + \left(\Gamma^r_{\theta r}\Gamma^r_{r\theta} + \Gamma^\theta_{\theta r}\Gamma^r_{\theta\theta}\right) + \Gamma^\phi_{\theta\phi}\Gamma^\phi_{\phi\theta} + \left(\Gamma^t_{\theta\theta}\Gamma^\theta_{t\theta} + \Gamma^r_{\theta\theta}\Gamma^\theta_{r\theta}\right)\right] \tag{A.74}$$

$$= \left((\ddot{a}(t)a(t) + \dot{a}^2(t))r^2 - ((1-kr^2) - 2kr^2)\right) + \frac{1}{\sin^2(\theta)} + \left(3(\dot{a}^2(t)r^2) - \left(kr^2 + 2(1-kr^2)\right)\right) \tag{A.75}$$

$$- \left[\left(\frac{\dot{a}(t)}{a(t)}\right)\left(\dot{a}(t)a(t)r^2\right) + \left(\frac{1}{r}(-(1-kr^2)r)\right) + \cot^2(\theta) + \left((\dot{a}(t)a(t)r^2)\frac{\dot{a}(t)}{a(t)} + (-(1-kr^2)r)\frac{1}{r}\right)\right] \tag{A.76}$$

$$= \left((\ddot{a}(t)a(t) + 4\dot{a}^2(t))r^2 - ((1-kr^2) - 2kr^2)\right) + \frac{1}{\sin^2(\theta)} - \left(2 - kr^2\right) \tag{A.77}$$

$$- \left[\left(\dot{a}^2(t)r^2 - (1-kr^2)\right) + \cot^2(\theta) + \left(\dot{a}^2(t)r^2 - (1-kr^2)\right)\right] \tag{A.78}$$

$$= \left(\ddot{a}(t)a(t)r^2 + 2\dot{a}(t)a(t)r^2 - 1 + 3kr^2 - 2 + kr^2 + 1 - kr^2 + 1 - kr^2 + 1\right) \tag{A.79}$$

$$= \left(\ddot{a}(t)a(t) + 2\dot{a}^2(t) + 2k\right) r^2 \tag{A.80}$$

The Ricci curvature scalar,

$$R = R_{tt}g^{tt} + R_{rr}g^{rr} + R_{\phi\phi}g^{\phi\phi} + R_{\theta\theta}g^{\theta\theta} = \left(-3\frac{\ddot{a}(t)}{a(t)}\right)(-1) + \left(\frac{\ddot{a}(t)a(t) + 2\dot{a}^2(t) + 2k}{1-kr^2}\right)\left(\frac{1-kr^2}{a^2(t)}\right) \tag{A.81}$$

$$+ \left(\left[\ddot{a}(t)a(t) + 2k + 2\dot{a}^2(t)\right] r^2 \sin^2(\theta)\right)\left(\frac{1}{a^2(t)r^2\sin^2(\theta)}\right) + \left(\left(\ddot{a}(t)a(t) + 2\dot{a}^2(t) + 2k\right) r^2\right)\left(\frac{1}{a^2(t)r^2}\right) \tag{A.82}$$

$$= 3\frac{\ddot{a}(t)}{a(t)} + \left(\frac{\ddot{a}(t)a(t) + 2\dot{a}^2(t) + 2k}{a^2}\right) + \left(\frac{\ddot{a}(t)a(t) + 2k + 2\dot{a}^2(t)}{a^2(t)}\right) + \left(\frac{\ddot{a}(t)a(t) + 2\dot{a}^2(t) + 2k}{a^2(t)}\right) \tag{A.83}$$

$$= 3\frac{\ddot{a}(t)}{a(t)} + 3\left(\frac{\ddot{a}(t)a(t)}{a^2} + \frac{2\dot{a}^2(t)}{a^2(t)} + \frac{2k}{a^2(t)}\right) = 3\frac{\dot{a}(t)}{a(t)} + 3\left(\frac{\ddot{a}(t)}{a(t)} + 2\left(\frac{\dot{a}(t)}{a(t)}\right)^2 + 2\frac{k}{a^2}\right) \tag{A.84}$$

$$= 6\left(\frac{\ddot{a}(t)}{a(t)} + \left(\frac{\dot{a}(t)}{a(t)}\right)^2 + \frac{k}{a^2(t)}\right) \tag{A.85}$$