Project

Due on April 30th, 2024

Consider the following ordinary differential equation for u:

$$-u''(x) + \pi^2 \cos^2(\pi x) u(x) = f(x) \quad x \in [0,1] \tag{1}$$

with boundary conditions:

$$\begin{aligned} u(0) &= 0, \\ u(1) &= 0. \end{aligned} \tag{2}$$

1. Consider $f(x) = \pi^2 \sin(\pi x) \cosh(\sin(\pi x))$, and check that the function $u(x) = \sinh(\sin(\pi x))$ is the solution to the boundary value problem (BVP) (1)+(2).

2. We want to solve this BVP numerically. We begin by discretizing the interval $[0,1]$. For this, consider the gridpoints:

$$x_i = ih, \quad i = 0, 1, \ldots, n+1, \quad h = \frac{1}{n+1}. \tag{3}$$

Note that $h_i = x_{i+1} - x_i = h$ for all i. Now we approximate the second derivative. Show that if g has four continuous derivatives, then

$$\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} = g_i'' + O(h^2), \tag{4}$$

where $g_i = g(x_i)$.

3. Consider now the linear system of equations

$$-\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} + \pi^2 \cos^2(\pi x_i)\, g_i = f(x_i) \quad i = 1, 2, \ldots, n. \tag{5}$$

Show that this can be rewritten in matrix form as

$$\boldsymbol{A} \cdot \boldsymbol{g} = \boldsymbol{f},$$

where $\boldsymbol{g} = (g_1, \ldots, g_n)^T$, $\boldsymbol{f} = (f_1, \ldots, f_n)^T$, and the matrix $\boldsymbol{A}$ is tridiagonal, with entries:

$$a_{i,j} = \begin{cases} -\frac{1}{h^2} & |i-j| = 1, \\ \frac{2}{h^2} + \pi^2 \cos^2(\pi x_i) & i = j, \\ 0 & \textit{Otherwise}. \end{cases} \tag{6}$$

4. Show that Scheme (5) is second-order accurate.

5. Solve the system of equations (5). Use the following values: $n = 10$, 20, 40, 80, 160, 320. For each $h = 1/(n+1)$, compute the error

$$e(h) = \sup_{1 \le i \le n} |g_i - u(x_i)| \tag{7}$$

and do a log-log plot of $e(h)$, that is, plot $\log(e(h))$ as a function of $\log(h)$. Show, using this plot, that $e(h) = O(h^2)$, consistent with **4**.

6. Consider a neural network solution in the following form

$$u_{\mathrm{NN}}(x) = x(1-x)\left(\sum_{i=1}^{n-1} u_i \sin(w_i x + b_i)\right), \tag{8}$$

which satisfies the boundary condition. The loss function in the least-squares sense is

$$\sum_{i=1}^{n-1} \left(-u''_{\mathrm{NN}}(x_i) + \pi^2 \cos^2(\pi x_i) u_{\mathrm{NN}}(x_i) - f(x_i)\right)^2. \tag{9}$$

Use Adam or stochastic gradient descent method to find the optimal set of parameters $\{u_i, w_i, b_i\}_{i=1}^{n-1}$. Use the following values: $n = 10$, 20, 40, 80, 160, 320. For each n, compute the error

$$e(n) = \sup_{1 \le i \le n} |u_{\mathrm{NN}}(x_i) - u(x_i)|. \tag{10}$$

What is the conclusion we can draw for the neural network solution? Compare this result with that of the second-order difference scheme.

Send your project with your name, your student ID to durui@suda.edu.cn. When sending me your work, you need to send me the pdf file of your work and also all the source files, including the latex files and the code you use to generate the results.

Some issues in mind:

- You must type your work by (La)Tex, CTex, or similar softwares in English;
- I recommend C++ for the programming part and Matlab for visualization;
- You have to finish the project INDEPENDENTLY, and send me your OWN work, your OWN code.