(1)

$$-u''(x) + \pi^2\cos^2(\pi x)u(x) = f(x) \qquad x \in [0, 1] \qquad u(0) = u(1) = 0$$

$$f(x) = \pi^2 \sin\pi x \cosh(\sin\pi x)$$
$$u(x) = \sinh(\sin\pi x)$$

$$u'(x) = \pi\cos\pi x \cosh(\sin\pi x), \quad u''(x) = -\pi^2\sin\pi x\cosh(\sin\pi x) + \pi^2\cos\pi x\cos\pi x\sinh(\sin\pi x)$$

$$-\pi^2\cos^2\pi x\sinh(\sin\pi x) + \pi^2\sin\pi x\cosh(\sin\pi x) + \pi^2\cos^2\pi x\sinh(\sin\pi x)$$

$$= \pi^2\sin\pi x\cosh(\sin\pi x) = f(x) \quad \square$$

(2)

$$g_i' = \frac{g_i - g_{i-1}}{h}, \quad g_{i+1}' = \frac{g_{i+1} - g_i}{h}$$

$$g_i'' = \frac{g_{i+1}' - g_i'}{h} = \frac{\frac{g_{i+1} - g_i}{h} - \frac{g_i - g_{i-1}}{h}}{h} = \frac{g_{i+1} - 2g_i + g_{i-1}}{h^2}$$

$$\Rightarrow g_i'' = \frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} + O(h^2) \quad \square$$

(3)

$$A = \begin{bmatrix} \frac{2}{h^2} & -\frac{1}{h^2} & 0 & 0 & \cdots & 0 \\ -\frac{1}{h^2} & \ddots & \ddots & 0 & 0 & 0 \\ 0 & \ddots & \frac{2}{h^2}+\pi^2\cos^2(\pi x_i) & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & -\frac{1}{h^2} \\ 0 & 0 & 0 & 0 & -\frac{1}{h^2} & \end{bmatrix} \qquad g = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$$

$$A \cdot g = \begin{cases} 0 + g_1\left(\frac{2}{h^2} + \pi^2\cos^2(\pi x_1)\right) - g_2 \cdot \frac{1}{h^2} = \frac{-g_2 + 2g_1 - g_0}{h^2} + \pi^2\cos^2(\pi x_1)g_1 \\ -\frac{1}{h^2} g_1 + g_2\left(\frac{2}{h^2} + \pi^2\cos^2(\pi x_1)\right) - g_3\frac{1}{h^2} = \frac{-g_3 + 2g_2 - g_1}{h^2} + \pi^2\cos^2(\pi x_2)g_2 \\ \vdots \\ -\frac{1}{h^2} g_{n-1} + g_n\left(\frac{2}{h^2} + \pi^2\cos^2(\pi x_n)\right) - g_{n+1}\frac{1}{h^2} = \frac{-g_{n+1} + 2g_n - g_{n-1}}{h^2} + \pi^2\cos^2(\pi x)g_n \end{cases}$$

$$\Rightarrow A \cdot g = f \quad \square$$

(4) As the 4 equation that we proved earlier shows

$$\frac{g_{i+1} - 2g_i + g_{i-1}}{h^2} = g_i'' + O(h^2)$$ so it is second-order accurate.