Formulary

Lecture
Data Analysis for Risk and Security Management
(M+I815)

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Content

1. Review of Statistical Techniques

1.1 Mean

$$\bar{x} = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

1.2 Mode

The mode is the value that appears most frequently in a data set. A set of data may have one mode, more than one mode, or no mode at all.

1.3 Median

$$x_{\text{med}} = \begin{cases} x_{\left(\frac{n+1}{2}\right)} & \text{; if n is not even numbered} \\ \frac{1}{2} \cdot \left[x_{\left(\frac{n}{2}\right)} + x_{\left(\frac{n}{2}+1\right)} \right] & \text{; if n is even numbered} \end{cases}$$

1.4 Sample Variance

$$s^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \bar{x})^2$$

1.5 Standard Deviation

$$s = \sqrt{s^2} = \sqrt{\frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \bar{x})^2}$$

1.6 Kurtosis Coefficient (KC) and Kurtosis

$$\text{KC} = \frac{\frac{1}{n} \ \sum_{i=1}^{n} (x_i - \bar{x})^4}{s^4}$$

$$Kurtosis = \frac{(n+1) * n}{(n-1) * (n-2) * (n-3)} * \frac{\sum_{i=1}^{n} (x_i - \bar{x})^4}{s^4} - 3 * \frac{(n-1)^2}{(n-2) * (n-3)}$$

1.7 Skewness Coefficient (SC) and Skewness

$$\mathrm{SC} = \frac{\frac{1}{n}\ \sum_{i=1}^{n}(x_i - \bar{x})^3}{s^3}$$

$$Skewness = \frac{n}{(n-1)*(n-2)} * \ \frac{\sum_{i=1}^{n}(x_i - \bar{x})^3}{s^3}$$

1.8 Confidence Intervals

Confidence Intervals (σ^2 is known)

$$P\left(\bar{x} + z_{\alpha/2} * \left(\frac{\sigma}{\sqrt{n}}\right) \geq \mu \geq \bar{x} - z_{\alpha/2} * \left(\frac{\sigma}{\sqrt{n}}\right)\right) = 1 - \alpha$$

Confidence Intervals (σ^2 is unknown)

$$P\left(\bar{x} + t_{(\alpha/2)}^{[n-1]} * \left(\frac{s}{\sqrt{n}}\right) \geq \mu \geq \bar{x} - t_{(\alpha/2)}^{[n-1]} * \left(\frac{s}{\sqrt{n}}\right)\right) = 1 - \alpha$$

1.9 Transformation

Transformation of a random variable X (theoretically)

$$X \Rightarrow \ \frac{X - \mu}{\sigma} = Z$$

Transformation of a concrete value x_i (empirically)

$$x_i \Rightarrow \ \frac{x_i - \bar{x}}{s} = z_i$$

1.10 Histogram

The following guidelines and terminology should be used to group data into classes of equal length:

(1) Perform a z-transformation of the available data.

(2) Determine the smallest (minimum) and largest (maximum) observations.

(3) Select the classes according to the z-transformed values, which are not overlapping intervals, usually of equal length. These classes should cover the entire interval from the minimum to the maximum.

(4) The intervals are called class intervals and the boundaries are called class boundaries.

(5) Finally, we assign the total number of values to the relevant classes.

1.11 Covariance

$$s_{xy} = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \bar{x}) \cdot (y_i - \bar{y})$$

1.12 Correlation

$$r = \frac{s_{xy}}{s_x \cdot s_y} = \frac{\sum_{i=1}^{n} (x_i - \bar{x}) \cdot (y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2} \cdot \sqrt{\sum_{i=1}^{n} (y_i - \bar{y})^2}}$$

2. Regression Analysis

2.1 The Single Linear Regression Model

Estimation

$$b_1 = \frac{n * \sum x_i * y_i - \sum x_i * \sum y_i}{n * \sum(x_i^2) - (\sum x_i)^2} = \frac{\sum x_i * y_i - \frac{\sum x_i * \sum y_i}{n}}{\sum(x_i^2) - \frac{(\sum x_i)^2}{n}} = \frac{SS_{xy}}{SS_{xx}}$$

$$b_0 = \bar{y} - b_1 * \bar{x}$$

Coefficient of Determination

$$R^2 = \frac{\sum(\hat{y}_i - \bar{y})^2}{\sum(y_i - \bar{y})^2} = 1 - \frac{\sum(y_i - \hat{y}_i)^2}{\sum(y_i - \bar{y})^2}$$

Error Term

$$y_i = b_0 + b_1 * x + e_i = \hat{y}_i + e_i$$

$$\Leftrightarrow e_i = y_i - \hat{y}_i$$

SSE (Sum of Squared Residuals) and Standard Error

$$SSE = \sum e_i^2 = \sum (y_i - \hat{y}_i)^2 = \sum (y_i - b_0 - b_1 * x_i)^2$$

$$s^2 = \frac{SSE}{n-2} \Rightarrow s = \sqrt{\frac{SSE}{n-2}} = \sqrt{\frac{\sum(y_i - \hat{y}_i)^2}{n-2}}$$

Hypothesis Test for the Slope Parameter

$$H_0: \beta_1 = 0 \text{ against } H_1: \beta_1 \neq 0$$

$$t_{b_1} = \frac{b_1}{s_{b_1}} = \frac{b_1}{\frac{s}{\sqrt{\sum(x_i - \bar{x})^2}}}$$

with

$$s_{b_1} = \frac{s}{\sqrt{SS_{xx}}} = \frac{s}{\sqrt{\sum(x_i - \bar{x})^2}}$$

Reject H_0if

$$|t| > t_{[\alpha/2]}^{(n-2)}$$

Confidence Interval for the slope parameter

$$\left[b_1 \pm t_{[\alpha/2]}^{(n-2)} * s_{b_1}\right]$$

Hypotheses Test for the y-intercept

$$H_0\text{: } \beta_0 = 0 \text{ against } H_1\text{: } \beta_0 \neq 0$$

$$t_{b_0} = \frac{b_0}{s_{b_0}}$$

with

$$s_{b_0} = s * \sqrt{\frac{1}{n} + \frac{\bar{x}^2}{SS_{xx}}}$$

Reject H_0if

$$|t| > t_{[\alpha/2]}^{(n-2)}$$

Confidence Interval for the y-intercept

$$\left[b_0 \pm t_{[\alpha/2]}^{(n-2)} * s_{b_0}\right]$$

Confidence and Prediction Intervals for Predicted Values of y

$$DV = \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{SS_{xx}}$$

$$\left[\hat{y} \pm t_{[\alpha/2]}^{(n-2)} * s * \sqrt{DV}\right]$$

$$\left[\hat{y} \pm t_{[\alpha/2]}^{(n-2)} * s * \sqrt{1 + DV}\right]$$

2.2 The Multiple Linear Regression Model

Estimation

$$\boldsymbol{y}_{(nx1)} = \boldsymbol{X}_{(nxk)} * \boldsymbol{\beta}_{(kx1)} + \boldsymbol{\epsilon}_{(nx1)}$$

$$\text{SSE} = \sum_{i=1}^{n} \varepsilon_i^2 = \boldsymbol{\varepsilon}'\boldsymbol{\varepsilon} = [\varepsilon_1 \quad (\dots) \quad \epsilon_n] * \begin{bmatrix} \varepsilon_1 \\ (\dots) \\ \varepsilon_n \end{bmatrix} = \epsilon_1^2 + \varepsilon_2^2 + (\dots) + \varepsilon_n^2$$

$$\Leftrightarrow \boldsymbol{\varepsilon}'\boldsymbol{\varepsilon} = \text{SSE}(\boldsymbol{\beta})$$

$$\boldsymbol{b} = (\boldsymbol{X}'\boldsymbol{X})^{-\mathbf{1}}\boldsymbol{X}'\boldsymbol{y}$$

The inverse $(\boldsymbol{X}'\boldsymbol{X})^{-1}$ must be calculated. The Gauß algorithm is by far too complicated; therefore, we calculate the determinant and the adjoint for a (2x2) $\boldsymbol{X}$ matrix:

$$\boldsymbol{X}^{-1} = \frac{1}{\det(\boldsymbol{X})} * Adj(\boldsymbol{X}) = \frac{1}{x_{11} * x_{22} - x_{12} * x_{21}} * \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix}$$

with

$$\boldsymbol{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

$$\rightarrow \det(\boldsymbol{X}) = x_{11} * x_{22} - x_{12} * x_{21}$$

and

$$Adj(\boldsymbol{X}) = \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix}$$

The inverse $(\boldsymbol{X}'\boldsymbol{X})^{-1}$ must be calculated. The Gauß algorithm is by far too complicated; therefore, we calculate the determinant and the adjoint for a (3x3) $\boldsymbol{X}$ matrix:

$$\boldsymbol{X}^{-\mathbf{1}} = \frac{1}{\det(\boldsymbol{X})} * Adj(\boldsymbol{X})$$

The general procedure for finding the inverse of a square matrix **X** thus involves the following steps:

(1) Find det (**X**) if det(**X**)≠0, but for det(**X**)=0 the inverse will be undefined.

(2) Find the cofactors of all the elements of **X**, and arrange them as a cofactor matrix $\boldsymbol{C} = [|\boldsymbol{C}_{ij}|]$.

(3) Take the transpose of **C** to get Adj(**X**).

(4) Divide Adj(**X**) by the determinant det(**X**).

A determinant of order 3 is associated with a (3x3) matrix; given

$$\boldsymbol{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

the determinant has the value

$$\det(\boldsymbol{X}) = x_{11} * \begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix} - x_{12} * \begin{vmatrix} x_{21} & x_{23} \\ x_{31} & x_{33} \end{vmatrix} + x_{13} * \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix}$$

$$\Leftrightarrow x_{11}x_{22}x_{33} - x_{11}x_{23}x_{32} - x_{12}x_{21}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{13}x_{22}x_{31} \text{ (=a scalar)}$$

A concept closely related to the minor is that of the cofactor. A cofactor, denoted by $|\boldsymbol{C}_{ij}|$, is a minor with prescribed algebraic sign attached to it:

$$|\boldsymbol{C}_{ij}| \equiv (-1)^{i+j} * |\boldsymbol{M}_{ij}| \Rightarrow \boldsymbol{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \Rightarrow \boldsymbol{C} = \begin{bmatrix} |\boldsymbol{C}_{11}| & -|\boldsymbol{C}_{12}| & |\boldsymbol{C}_{13}| \\ -|\boldsymbol{C}_{21}| & |\boldsymbol{C}_{22}| & -|\boldsymbol{C}_{23}| \\ |\boldsymbol{C}_{31}| & -|\boldsymbol{C}_{32}| & |\boldsymbol{C}_{33}| \end{bmatrix}$$

$$\Rightarrow \boldsymbol{C} = \begin{bmatrix} |\boldsymbol{C}_{11}| & -|\boldsymbol{C}_{12}| & |\boldsymbol{C}_{13}| \\ -|\boldsymbol{C}_{21}| & |\boldsymbol{C}_{22}| & -|\boldsymbol{C}_{23}| \\ |\boldsymbol{C}_{31}| & -|\boldsymbol{C}_{32}| & |\boldsymbol{C}_{33}| \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix} & -\begin{vmatrix} x_{21} & x_{23} \\ x_{31} & x_{33} \end{vmatrix} & \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix} \\ -\begin{vmatrix} x_{12} & x_{13} \\ x_{32} & x_{33} \end{vmatrix} & \begin{vmatrix} x_{11} & x_{13} \\ x_{31} & x_{33} \end{vmatrix} & -\begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{vmatrix} \\ \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} & -\begin{vmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{vmatrix} & \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \end{bmatrix}$$

SSE (Sum of Squared Residuals) and Standard Error

$$\text{SSE} = \textstyle\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

$$s^2 = \frac{SSE}{n-(k+1)} \rightarrow s = \sqrt{\frac{SSE}{n-(k+1)}} = \sqrt{\frac{\sum(y_i - \hat{y}_i)^2}{n-(k+1)}}$$

Hypotheses Test of the Regression Parameters

$$H_0\text{: } \beta_j = 0 \text{ against } H_1\text{: } \beta_j \neq 0$$

$$t = \frac{b_j - \beta_j}{s_{b_j}} = \frac{b_j}{s * \sqrt{c_{ij}}}$$

with

$$s_{b_j} = s * \sqrt{c_{ij}}$$

Reject H_0if

$$|t| > t_{[\alpha/2]}^{(n-(\mathrm{k}+1))}$$

Independent Variable	b_j	$s_{b_j} = s * \sqrt{c_{ij}}$	$t = \dfrac{b_j}{s * \sqrt{c_{ij}}}$
Intercept	b_0	$s_{b_0} = s * \sqrt{c_{00}}$	$t = \dfrac{b_0}{s * \sqrt{c_{00}}}$
x_1	b_1	$s_{b_1} = s * \sqrt{c_{11}}$	$t = \dfrac{b_1}{s * \sqrt{c_{11}}}$
x_2	b_2	$s_{b_2} = s * \sqrt{c_{22}}$	$t = \dfrac{b_2}{s * \sqrt{c_{22}}}$

We compute s_{b_j} as the standard error of the estimate b_j, while c_{ij} is the j-th diagonal element of $(\boldsymbol{X}'\boldsymbol{X})^{-1}$.

Confidence Intervals for b_j

$$\left[b_j \pm t_{[\alpha/2]}^{(n-(k+1))} * s_{b_j}\right] = \left[b_j \pm t_{[\alpha/2]}^{(n-(k+1))} * s * \sqrt{c_{ij}}\right]$$

Confidence and Prediction Intervals for Predicted Values of y

$$DV = \boldsymbol{x}_0'(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{x}_0$$

$$\hat{y} \pm t_{[\frac{\alpha}{2}]}^{[n-(k+1)]} * s * \sqrt{Distance\ value}$$

$$\hat{y} \pm t_{[\frac{\alpha}{2}]}^{[n-(k+1)]} * s * \sqrt{1 + Distance\ value}$$

2.3 Time Series Regression

Trend Models

The trend model is

$$y_t = TR_t + \epsilon_t$$

Some useful trends are:

No Trend	$TR_t = \beta_0$
Linear Trend	$TR_t = \beta_0 + \beta_1 * t$
Quadratic Trend	$TR_t = \beta_0 + \beta_1 * t + \beta_2 * t^2$
Polynomial Trend of p-th order	$TR_t = \beta_0 + \beta_1 * t + \beta_2 * t^2 + \cdots + \beta_p * t^p$

Hypothesis Test regarding Positive Autocorrelation

H_0: The error terms are not positively autocorrelated

versus

H_1: The error terms are positively autocorrelated

$$d = \frac{\sum_{t=2}^{n}(e_t - e_{t-1})^2}{\sum_{t=1}^{n} e_t^2}$$

- If $d < d_{L,\alpha}$, we reject H_0.
- If $d > d_{U,\alpha}$, we do not reject H_0.
- If $d_{L,\alpha} \le d \le d_{U,\alpha}$, the test is inconclusive (no decision possible).

Hypothesis Test regarding Negative Autocorrelation

H_0: The error terms are not negatively autocorrelated

versus

H_1: The error terms are negatively autocorrelated

$$d = \frac{\sum_{t=2}^{n}(e_t - e_{t-1})^2}{\sum_{t=1}^{n} e_t^2}$$

- If $(4-\mathrm{d}) < \mathrm{d}_{\mathrm{L},\alpha}$, we reject H_0.
- If $(4-\mathrm{d}) > \mathrm{d}_{\mathrm{U},\alpha}$, we do not reject H_0.
- If $\mathrm{d}_{\mathrm{L},\alpha} \leq (4-\mathrm{d}) \leq \mathrm{d}_{\mathrm{U},\alpha}$, the test is inconclusive (no decision possible).

2.4 Dummy Regression

Model

Model with constant seasonal variation

$$y_t = TR_t + SN_t + \varepsilon_t$$

with

$$SN_t = \beta_{s1} * x_{s1,t} + \beta_{s2} * x_{s2,t} + \cdots + \beta_{s(L-1)} * x_{s(L-1),t}$$

where $x_{s1,t}$, $x_{s2,t}$,..., $x_{s(L-1),t}$ are dummy variables that are defined as follows:

$$x_{s1,t} = \begin{cases} 1 & if\ time\ period\ t\ is\ season\ 1 \\ 0 & otherwise \end{cases}$$

$$x_{s2,t} = \begin{cases} 1 & if\ time\ period\ t\ is\ season\ 2 \\ 0 & otherwise \end{cases}$$

$$(\ldots)$$

$$x_{s(L-1),t} = \begin{cases} 1 & if\ time\ period\ t\ is\ season\ (L-1) \\ 0 & otherwise \end{cases}$$

2.5 Trigonometric Models

First model

$$y_t^* = \beta_0 + \beta_1 * t + \beta_2 * \sin\left(\frac{2\pi t}{L}\right) + \beta_3 * \cos\left(\frac{2\pi t}{L}\right) + \varepsilon_t$$

$$\text{with} \quad y_t^* = ln(y_t)$$

Second model

$$y_t^* = \beta_0 + \beta_1 * t + \beta_2 * \sin\left(\frac{2\pi t}{L}\right) + \beta_3 * \cos\left(\frac{2\pi t}{L}\right) + \beta_4 * \sin\left(\frac{4\pi t}{L}\right) + \beta_5 * \cos\left(\frac{4\pi t}{L}\right) + \varepsilon_t$$

$$\text{with} \quad y_t^* = ln(y_t)$$

3. Time Series

3.1 Simple Exponential Smoothing

Model

$$\vartheta_t = \alpha * y_t + (1-\alpha) * \vartheta_{t-1}$$

Standard Error

$$s = \sqrt{\frac{SSE}{T-1}} = \sqrt{\frac{\sum_{t=1}^{T}[y_t - \hat{y}_t(t-1)]^2}{T-1}} = \sqrt{\frac{\sum_{t=1}^{T}[y_t - \vartheta_{t-1}]^2}{T-1}}$$

Point Forecast

$$\hat{y}_{t+\tau}(t) = \vartheta_t \text{ mit } \tau = 1,\ 2,\ 3, \ldots \text{ (Tau)}$$

Prediction Intervals

If $\tau=1$, then a 95% prediction interval computed in time period t for y_{t+1} is

$$\left[\vartheta_t \pm z_{[0,025]} * s\right]$$

If $\tau=2$, then a 95% prediction interval computed in time period t for y_{t+2} is

$$\left[\vartheta_t \pm z_{[0,025]} * s * \sqrt{1+\alpha^2}\right]$$

In general for any τ, a 95% prediction interval computed in time period t for $y_{t+\tau}$ is

$$\left[\vartheta_t \pm z_{[0,025]} * s * \sqrt{1+(\tau-1)*\alpha^2}\right]$$

Error Correction Form

$$\vartheta_t = \vartheta_{t-1} + \alpha(y_t - \vartheta_{t-1})$$

3.2 Holt's Trend Corrected Exponential Smoothing

Model

$$\vartheta_t = \alpha * y_t + (1-\alpha) * [\vartheta_{t-1} + b_{t-1}]$$

$$b_t = \gamma * [\vartheta_t - \vartheta_{t-1}] + (1-\gamma) * b_{t-1}$$

Standard Error

$$s = \sqrt{\frac{SSE}{T-2}} = \sqrt{\frac{\sum_{t=1}^{T}[y_t - \hat{y}_t(t-1)]^2}{T-2}}$$

$$\Leftrightarrow \sqrt{\frac{\sum_{t=1}^{T}[y_t - (\vartheta_{t-1} + b_{t-1})]^2}{T-2}}$$

Point Forecast

$$\hat{y}_{t+\tau}(t) = \vartheta_t + \tau * b_t \quad (\tau = 1, 2, \ldots)$$

Prediction Intervals

If $\tau = 1$, then a 95% prediction interval computed in time period t for y_{t+1} is

$$\left[(\vartheta_t + b_t) \pm z_{[0,025]} * s\right]$$

If $\tau = 2$, then a 95% prediction interval computed in time period t for y_{t+2} is

$$\left[(\vartheta_t + 2b_t) \pm z_{[0,025]} * s * \sqrt{1 + \alpha^2 * (1+\gamma)^2}\right]$$

If $\tau = 3$, then a 95% prediction interval computed in time period t for y_{t+3} is

$$\left[(\vartheta_t + 3b_t) \pm z_{[0,025]} * s * \sqrt{1 + \alpha^2 * (1+\gamma)^2 + \alpha^2(1+2\gamma)^2}\right]$$

3.3 Additive Holt-Winters Method

Model

$$\vartheta_t = \alpha * (y_t - sn_{t-L}) + (1-\alpha) * (\vartheta_{t-1} + b_{t-1})$$

$$b_t = \gamma * (\vartheta_t - \vartheta_{t-1}) + (1-\gamma) * b_{t-1}$$

$$sn_t = \delta * (y_t - \vartheta_t) + (1-\delta) * sn_{t-L}$$

Standard Error

$$s = \sqrt{\frac{SSE}{T-3}} = \sqrt{\frac{\sum_{t=1}^{T}[y_t - \hat{y}_t * (t-1)]^2}{T-3}}$$

$$\Leftrightarrow \sqrt{\frac{\sum_{t=1}^{T}[y_t - (\vartheta_{t-1} + b_{t-1} + sn_{t-L})]^2}{T-3}}$$

Point Forecast

$$\hat{y}_{t+\tau}(t) = \vartheta_t + \tau * b_t + \; sn_{t+\tau-L} \quad (\tau = 1,\ 2, \dots)$$

Prediction Intervals

If $\tau = 1$, then $c_1 = 1$

$$\left[\hat{y}_{t+\tau}(t) \pm z_{[0,025]} * s * \sqrt{c_\tau}\ \right]$$

If $2 \leq \tau \leq L$

$$c_\tau = \left[1 + \sum_{j=1}^{\tau-1} \alpha^2 * (1 + j * \gamma)^2\right]$$

Error Correction Forms

$$\vartheta_t = \vartheta_{t-1} + b_{t-1} + \alpha * [y_t - (\vartheta_{t-1} + b_{t-1} + sn_{t-L})]$$

$$b_t = b_{t-1} + \alpha * \gamma * [y_t - (\vartheta_{t-1} + b_{t-1} + sn_{t-L})]$$

$$sn_t = sn_{t-L} + (1-\alpha) * \delta * [y_t - (\vartheta_{t-1} + b_{t-1} + sn_{t-L})]$$

3.4 Multiplicative Holt-Winters Method

Model

$$\vartheta_t = \alpha * (y_t/sn_{t-L}) + (1-\alpha) * (\vartheta_{t-1} + b_{t-1})$$

$$\mathrm{b_t} = \gamma * (\vartheta_t - \vartheta_{t-1}) + (1-\gamma) * \mathrm{b}_{t-1}$$

$$\mathrm{sn_t} = \delta * (\mathrm{y_t}/\vartheta_t) + (1-\delta) * \mathrm{sn}_{t-L}$$

Standard Error

$$s_r = \sqrt{\frac{\sum_{t=1}^{T}\left[\frac{y_t - \hat{y}_t(t-1)}{\hat{y}_t(t-1)}\right]^2}{T-3}} \Leftrightarrow \sqrt{\frac{\sum_{t=1}^{T}\left[\frac{y_t - (\vartheta_{t-1} + b_{t-1}) * sn_{t-L}}{(\vartheta_{t-1} + b_{t-1}) * sn_{t-L}}\right]^2}{T-3}}$$

Point Forecast

$$\hat{y}_{t+\tau}(t) = (\vartheta_t + \tau * b_t) * sn_{t+\tau-L} \; (\tau = 1,\ 2,\ \ldots)$$

Prediction Intervals

An approximate 95% prediction interval computed in time period t for $y_{t+\tau}$ is

$$\left[\hat{y}_{t+\tau} \pm z_{[0{,}025]} * s_r * \sqrt{c_\tau} * sn_{t+\tau-L}\right]$$

If $\tau = 1$ then

$$c_1 = (\vartheta_t + b_t)^2$$

It $\tau = 2$ then

$$c_2 = \alpha^2 * (1+\gamma)^2 * (\vartheta_t + b_t)^2 + (\vartheta_t + 2 * b_t)^2$$

Error Correction Forms

$$\vartheta_t = \vartheta_{t-1} + b_{t-1} + \alpha * \frac{[y_t - (\vartheta_{t-1} + b_{t-1}) * sn_{t-L}]}{sn_{t-L}}$$

$$b_t = b_{t-1} + \alpha * \gamma * \frac{[y_t - (\vartheta_{t-1} + b_{t-1}) * sn_{t-L}]}{sn_{t-L}}$$

$$sn_t = sn_{t-L} + (1-\alpha) * \delta * \frac{[y_t - (\vartheta_{t-1} + b_{t-1}) * sn_{t-L}]}{\vartheta_t}$$

4. Waiting Line Models

4.1 Poisson Distribution

$$P(x) = \frac{\lambda^x}{x!} \quad * \quad e^{-\lambda} \, for \; x = 0,1,2, \ldots$$

- x = the number of arrivals in the time period
- λ = the mean number of arrivals per time period (arrival rate)
- e = 2,71282

4.2 Exponential Distribution

$$f(x) = \begin{cases} \lambda * e^{-\lambda * x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$F(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-\lambda * t} & t \geq 0 \end{cases}$$

4.3 Single server waiting line model with poisson arrivals and exponential service times (M/M/1)

The probability that no units are in the system

$$P_0 = 1 - \frac{\lambda}{\mu}$$

The average number of units in the waiting line

$$L_q = \frac{\lambda^2}{\mu * (\mu - \lambda)}$$

The average number of units in the system

$$L = L_q + \frac{\lambda}{\mu}$$

The average time a unit spends in the waiting line

$$W_q = \frac{L_q}{\lambda}$$

The average time a unit spends in the system

$$W = W_q + \frac{1}{\mu}$$

The probability that an arriving unit has to wait for service

$$P_w = \frac{\lambda}{\mu}$$

The probability of n units in the system

$$P_n = \left(\frac{\lambda}{\mu}\right)^n * P_0$$

Utilization Factor

$$\frac{\lambda}{\mu}\ with\ \mu > \lambda$$

4.4 Multiple server waiting line model with poisson arrivals and exponential service times (M/M/k)

1. The probability that no units are in the system	**5. The average time a unit spends in the system**
$P_0 = \dfrac{1}{\left(\sum_{n=0}^{k-1}\frac{(\lambda/\mu)^n}{n!}\right)+\frac{(\lambda/\mu)^k}{k!}*\left(\frac{k*\mu}{k*\mu-\lambda}\right)}$	$W = W_q + \frac{1}{\mu}$
2. The average number of units in the waiting line	**6. The probability that an arriving unit has to wait for service**
$L_q = \frac{(\lambda/\mu)^k * \lambda * \mu}{(k-1)! * (k*\mu-\lambda)^2} * P_0$	$P_w = \frac{1}{k!} * \left(\frac{\lambda}{\mu}\right)^k * \left(\frac{k*\mu}{k*\mu-\lambda}\right) * P_0$
3. The average number of units in the system	**7. The probability of n units in the system**
$L = L_q + \frac{\lambda}{\mu}$	$P_n = \frac{(\lambda/\mu)^n}{n!} * P_0\ for\ n \le k$ $P_n = \frac{(\lambda/\mu)^n}{k! * k^{(n-k)}} * P_0\ for\ n > k$
4. The average time a units spends in the waiting line	
$W_q = \frac{L_q}{\lambda}$	

4.5 Little's flow equations

1. The average number of units in the system
$L = \lambda * W$
2. The average number of units in the waiting line
$L_q = \lambda * W_q$
3. The average time a unit spends in the waiting line
$W_q = \frac{L_q}{\lambda}$
4. The average time a unit spends in the system
$W = W_q + \frac{1}{\mu}$

4.6 Single-server waiting line model with poisson arrivals and arbitrary service times (M/G/1)

1. The probability that no units are in the system	**5. The average time a unit spends in the system**
$P_0 = 1 - \frac{\lambda}{\mu}$	$W = W_q + \frac{1}{\mu}$
2. The average number of units in the waiting line	**6. The probability that an arriving unit has to wait for service**
$L_q = \frac{\lambda^2 * \sigma^2 + (\lambda/\mu)^2}{2 * (1 - \lambda/\mu)}$	$P_w = \frac{\lambda}{\mu}$
3. The average number of units in the system	**Notation**
$L = L_q + \frac{\lambda}{\mu}$	$\lambda =$ Arrival rate; $\mu = Service\ rate$; $\sigma = Standard\ deviation\ of\ service\ time$ $\frac{\lambda}{\mu} = utilization\ factor\ of\ service\ unit$
4. The average time a units spends in the waiting line	
$W_q = \frac{L_q}{\lambda}$	

4.7 Multiple-server waiting line model with poisson arrivals and arbitrary service times (M/G/k), and no waiting line

Steady-state probabilities that j of the k servers will be busy

$$P_j = \frac{\left(\lambda/\mu\right)^j / j!}{\sum_{i=0}^{k} \left(\lambda/\mu\right)^i / i!}$$

with

- λ = the arrival rate
- μ = the service rate for each server
- k = the number of servers
- P_j = the probability that j of k servers are busy for j=0,1,2,…,k

The average number of units in the system

$$L = \frac{\lambda}{\mu} * (1 - P_k)$$

4.8 Waiting line models with finite calling populations

1. The probability that no units are in the system	**5. The average time a unit spends in the system**
$P_0 = \frac{1}{\sum_{n=0}^{N} \frac{N!}{(N-n)!} * \left(\frac{\lambda}{\mu}\right)^n}$	$W = W_q + \frac{1}{\mu}$
2. The average number of units in the waiting line	**6. The probability that an arriving unit has to wait for service**
$L_q = N - \frac{\lambda + \mu}{\lambda} * (1 - P_0)$	$P_w = 1 - P_0$
3. The average number of units in the system	**7. The probability of n units in the system**
$L = L_q + (1 - P_0)$	$P_n = \frac{N!}{(N-n)!} * \left(\frac{\lambda}{\mu}\right)^n * P_0$ for n = 0,1,…,N
4. The average time a units spends in the waiting line	
$W_q = \frac{L_q}{(N - L) * \lambda}$	