IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received March 16, 2020, accepted March 28, 2020, date of publication March 31, 2020, date of current version April 16, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.2984689

Mandelbrot Sets and Julia Sets in Picard-Mann Orbit

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The authors extend their appreciation to the Deanship of Scientific Research at Majmaah University for funding this work under Project Number (RGP-2019-28).

ABSTRACT The purpose of this paper is to introduce the Mandelbrot and Julia sets by using Picard-Mann iteration procedure. Escape criteria is established which plays an important role to generate Mandelbrot and Julia sets. Also, numerous graphical pictures of these sets have been visualized and certain examples have been recognized. Presented results shows that fractal images generated by Picard-Mann iteration procedure are entirely different from those generated in Mann orbit.

INDEX TERMS Julia set, Mandelbrot set, Picard-Mann iteration, escape criterion.

I. INTRODUCTION

The visual beauty, multifaceted nature and self similarity of complex graphics have made a field of extraordinary research these days. In 1918, Julia [1] attained a Julia set by investigating the iteration process for complex function $z_{n+1} = z_n^2 + c$, where c is a complex number. Later on, Mandelbrot extended the idea of Julia sets and introduced the object Mandelbrot set by utilizing c as a complex parameter in complex quadratic function [2]. He coined the name "Fractal" for such best known representations of a highly convoluted chaotic systems produced by a very simple mathematical procedure. Since then many mathematicians have studied different properties of Mandelbrot and Julia sets and proposed various generalizations of those sets. The first and the most obvious generalization was the use of $z^p + c$ function instead of the quadratic one used by Mandelbrot [3], [4]. Then some other types of functions were studied: rational [5], transcendental [6], elliptic [7], anti-polynomials [8] etc. Another step in the studies on Mandelbrot and Julia sets was the extension from complex numbers to other algebras, e.g., quaternions [9], octonions [10], bicomplex numbers [11] etc. In order to generate and generalize Julia and Mandelbrot sets various techniques are used from the fixed point theory. Fixed point theory provides a suitable iterative technique for finding fixed points of a given mapping that depend on the utilization of different feedback iterative procedures.

Fixed point iterative schemes can be found in the generation of the different types of fractals: for example, Iterated Function Systems [12], [13], V-variable fractals [13], Inversion fractals [14] and Biomorphs [15]. Some polynomiographs are types of fractals which can be obtained via different iterative schemes, for more detail see [16]–[19] and references therein. In 2004, Rani and Kumar [20], [21] used Mann iteration, an one-step fixed point iterative procedure, to generate superior Julia and Mandelbrot sets. Relative superior Julia sets and Mandelbrot sets have been displayed in [22] and [23] by means of Ishikawa iteration, a two-step fixed point iterative method. Kang *et al.* [24] introduced relative superior Mandelbrot sets via S-iteration scheme and demonstrated that S-iteration procedure converges faster than Ishikawa iteration procedure in complex plane. These sets have been presented by Rani *et al.* [25] in Noor orbit, Li *et al.* [26] in Jungck-Mann orbit, Kwun *et al.* [27], [28] via Jungck-CR and CR iteration schemes with s-convexity, Gdawiec and Shahid [29] in the s-iteration orbit with s-convexity, Nazeer *et al.* [30] in Jungck-Mann and Jungck-Ishikawa iterations with s-convexity and Cho *et al.* [31] in Noor orbit and s-convexity. In this paper we introduce and visualize a new class of fractals in Picard-Mann orbit. Many

The associate editor coordinating the review of this manuscript and approving it for publication was Abdel-Hamid Soliman.

graphical patterns of Mandelbrot and Julia sets are presented by using proposed two step iterative procedure.

This paper is organized as follows. Section II contains some basic definitions. In section III the escape criteria for Mandelbrot and Julia sets are presented. In section IV we generate Mandelbrot sets by using Picard-Mann and Mann iteration schemes. Section V contains Julia sets generated in Picard-Mann and Mann orbits. At last, section VI has some concluded remarks.

II. PRELIMINARIES

Definition 1 (Julia set [32]): Let $f : C \longrightarrow C$ symbolize a polynomial of degree ≥ 2 where C be a complex plane. Let F_f be the set of points in C whose orbits do not converge to the point at infinity. That is, $F_f = \{x \in C : \{|f^n(x)|, n$ varies from 0 to $\infty\}$ is bounded$\}$. F_f is called as filled Julia set of the polynomial f. The boundary points of F_f are called as the points of Julia set of the polynomial f or simply the Julia set.

Definition 2 (Mandelbrot set [33]): The Mandelbrot set M for the quadratic $Q_c(z) = z^2+c$ is defined as the collection of all $c \in C$ for which the orbit of the point 0 is bounded, that is

$$M = \{c \in C : \{Q_c^n(0)\}; n = 0, 1, 2, \ldots \text{is bounded}\}.$$

We choose the initial point 0, as 0 is the only critical point of Q_c.

The escape criterion for Julia and Mandelbrot sets [33] is given by:

Theorem 1: For $Q_c(z) = z^2+c$, $c \in C$, if there exists $n \geq 0$ such that

$$\left|Q_c^n(z)\right| > \max\{|c|, 2\},$$

then $\left|Q_c^n(z)\right| \to \infty$ as $n \to \infty$

The term $\max\{|c|, 2\}$ is also known as escape radius threshold. The escape radius varies in each iteration. Now, we define some well known iterative procedures. Let C be a subset of complex numbers and $T : C \to C$ and $x_0 \in C$. The Picard orbit [33] is a sequence $\{x_n\}$ is given by

$$x_{n+1} = T(x_n) \tag{1}$$

where $n \geq 0$. The Mann iteration process [34] is illustrated by the following sequence $\{x_n\}$:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\zeta)x_n + \zeta T x_n, \quad n \geq 0, \end{cases} \tag{2}$$

where $\zeta \in (0, 1]$.

Definition 3 (see [35], Picard-Mann Orbit): Let C be a subset of complex numbers and $T : C \to C$. Consider a sequence $\{x_n\}$ of iterates for initial point $x_0 \in C$ such that

$$\begin{cases} x_{n+1} = T(y_n), \\ y_n = (1-\zeta)x_n + \zeta T(x_n); \quad n \geq 0, \end{cases} \tag{3}$$

where $\zeta \in (0, 1]$. The above sequence of iterates is called Picard-Mann orbit, which is a function of three arguments (T, x_0, ζ) which can be written as $PMO(T, x_0, \zeta)$.

III. MAIN RESULTS

In this section we established the escape criterion which performed an important job to generate the Julia and Mandelbrot sets. We take $z_\circ = z \in C$ then we construct the Picard-Mann iteration scheme (3) in the following manner:

$$\begin{cases} z_{n+1} = Q_c(u_n), \\ u_n = (1-\zeta)z_n + \zeta Q_c(z_n), \quad n \geq 0, \end{cases} \tag{4}$$

where $Q_c(z_n)$ is a quadratic, cubic or kth degree complex polynomial and $\zeta \in (0, 1]$.

A. ESCAPE CRITERION FOR HIGHER DEGREE FUNCTIONS

An escape criterion for polynomials $Q_c(z) = z^k + c$, where $k \geq 2$, has been presented in the following result.

Theorem 2: Assume that $|z| \geq |c| > (\frac{2}{\zeta})^{\frac{1}{k-1}}$ where $k \geq 2$, $0 < \zeta \leq 1$ and c be a complex number. Define

$$\begin{cases} z_{n+1} = Q_c(u_n) \\ u_n = (1-\zeta)z_n + \zeta Q_c(z_n), \quad n \geq 0, \end{cases}$$

then $|z_n| \to \infty$ as $n \to \infty$.

Proof: Let $z = z_\circ$ and $u = u_\circ$. For $Q_c(z) = z^k + c$

$$\begin{aligned} |u| &= |(1-\zeta)z + \zeta Q_c(z)| \\ &= \left|(1-\zeta)z + \zeta(z^k + c)\right| \\ &\geq \left|\zeta z^k + (1-\zeta)z\right| - |\zeta c| \\ &\geq \left|\zeta z^k\right| - |(1-\zeta)z| - |\zeta z| \because |z| \geq |c| \\ &\geq \zeta \left|z^k\right| - |z| + |\zeta z| - |\zeta z| \\ &= |z| (\zeta |z|^{k-1} - 1). \end{aligned}$$

Also for

$$\begin{aligned} |z_1| &= |Q_c(u)| \\ &= \left|u^k + c\right| \\ &\geq \left|(|z| (\zeta \ |z|^{k-1} - 1))^k + c\right| \\ &\geq \left|(|z| (\zeta \ |z|^{k-1} - 1))^k\right| - |c| \\ &\geq (|z| (\zeta \ |z|^{k-1} - 1))^k - |z| \because |z| \geq |c| \end{aligned} \tag{5}$$

Since $|z| \geq |c| > (\frac{2}{\zeta})^{\frac{1}{k-1}}$ implies

$$\begin{aligned} \zeta \ |z|^{k-1} - 1 &\geq 1 \\ (\zeta \ |z|^{k-1} - 1)^k &\geq 1 \\ |z|^k (\zeta \ |z|^{k-1} - 1)^k &\geq |z|^k \end{aligned} \tag{6}$$

Substituting in (5), we get

$$\begin{aligned} |z_1| &\geq |z|^k - |z| \\ &= |z| (|z|^{k-1} - 1) \end{aligned}$$

Since $|z| > (\frac{2}{\zeta})^{\frac{1}{k-1}} > 2^{\frac{1}{k-1}}$ it follows $|z|^{k-1} - 1 > 1$. Hence there exists $\gamma > 0$m such that $|z|^{k-1} - 1 > 1 + \gamma > 1$. Consequently

$$|z_1| \geq (1+\gamma) |z|.$$

We can apply the similar argument repeatedly to obtain:

$$|z_2| \geq (1+\gamma)^2 |z|,$$
$$\vdots$$
$$|z_n| \geq (1+\gamma)^n |z|.$$

Hence $|z_n| \to \infty$ as $n \to \infty$. □

The following corollary gives the escape criteria to generate the Julia and Mandelbrot sets in Picard-Mann orbit.

Corollary 1: Suppose $|z| > \max\left\{|c|, \left(\frac{2}{\zeta}\right)^{\frac{1}{k-1}}\right\}$ *then* $|z_n| \to \infty$ *as* $n \to \infty$.

IV. VISUALIZATION OF MANDELBROT SETS

In this section some Mandelbrot sets are presented for quadratic, cubic and higher degree polynomials of the form $Q_c(z) = z^k + c$ for $k \geq 2$ via Picard-Mann iteration and Mann iteration procedures. The escape criterion with escape time algorithm executed in the software Mathematica 9.0 to produce the graphics. Pseudocode of the Mandelbrot set generation algorithms in Picard-Mann orbit and Mann orbit are exhibited in Algorithm 1 and Algorithm 2 respectively.

Algorithm 1 Generation of Mandelbrot set in Picard-Mann Orbit

Input: $Q_c(z) = z^k + c$, where $k \geq 2$, $A \subset \mathbb{C}$ – area, K – iterations, $\zeta \in (0, 1]$ – parameter for Picard-Mann iteration process, *colourmap*$[0..C-1]$ – with C colours.
Output: Mandelbrot set for area A.

for $c \in A$ **do**
  $R = \max\{|c|, (\frac{2}{\zeta})^{1/k-1}\}$
  $n = 0$
  $z_0 = 0$
  while $n \leq K$ **do**
    $u_n = (1-\zeta)z_n + \zeta Q_c(z_n)$,
    $z_{n+1} = Q_c(u_n)$
    if $|z_{n+1}| > R$ **then**
      break
    $n = n + 1$
  $i = \lfloor (C-1)\frac{n}{K} \rfloor$
  colour c with *colourmap*$[i]$

A. MANDELBROT SETS FOR $Q_c(z) = z^2 + c$

In Fig. 1 quadratic Mandelbrot set is presented in Picrd orbit. While in Figs. 2–9, quadratic Mandelbrot sets are presented in Picard-Mann orbit and Mann orbit by choosing greatest number of iterations 30. We have experienced following observations:

- In Fig. 2 with $A = [-7.4, 2.1] \times [-3.2, 3.2]$ Mandelbrot set generated in Picard-Mann orbit and in Fig. 3 with $A = [-7.7, 2.1] \times [-7.2, 7.2]$ Mandelbrot set generated in Mann orbit for same value of $\zeta = 0.2$.
- In Fig. 4 with $A = [-5.4, 1.1] \times [-2.2, 2.2]$ Mandelbrot set visualized in Picard-Mann orbit and in Fig. 5 with $A = [-5.7, 2.1] \times [-5.2, 5.2]$ Mandelbrot set generated in Mann orbit for same value of $\zeta = 0.3$.

Algorithm 2 Generation of Mandelbrot set in Mann Orbit

Input: $Q_c(z) = z^k + c$, where $k \geq 2$, $A \subset \mathbb{C}$ – area, K – iterations, $\zeta \in (0, 1]$ – parameter for Mann iteration process, *colourmap*$[0..C-1]$ – with C colours.
Output: Mandelbrot set for area A.

for $c \in A$ **do**
  $R = \max\{|c|, (\frac{2}{\zeta})^{1/k-1}\}$
  $n = 0$
  $z_0 = 0$
  while $n \leq K$ **do**
    $z_{n+1} = (1-\zeta)z_n + \zeta Q_c(z_n)$
    if $|z_{n+1}| > R$ **then**
      break
    $n = n + 1$
  $i = \lfloor (C-1)\frac{n}{K} \rfloor$
  colour c with *colourmap*$[i]$

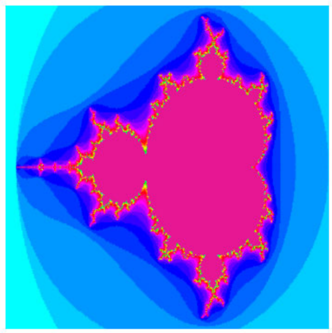

FIGURE 1. Mandelbrot set generated in picard orbit.

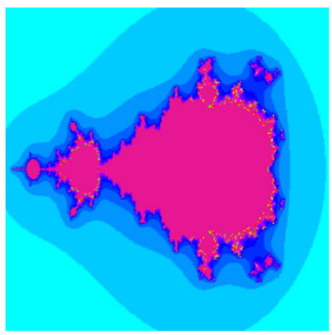

FIGURE 2. Quadratic mandelbrot set generated in picard-mann orbit for $\zeta = 0.2$.

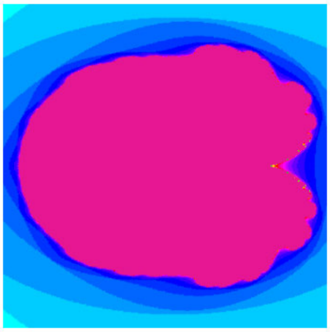

FIGURE 3. Quadratic mandelbrot set generated in mann orbit for $\zeta = 0.2$.

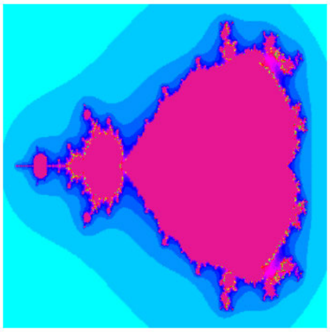

FIGURE 4. Quadratic mandelbrot set generated in picard-mann orbit for $\zeta = 0.3$.

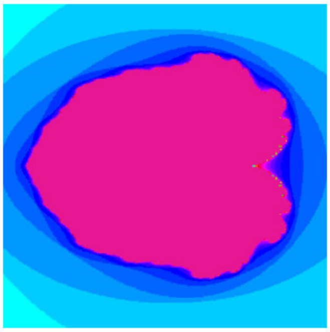

FIGURE 5. Quadratic mandelbrot set generated in mann orbit for $\zeta = 0.3$.

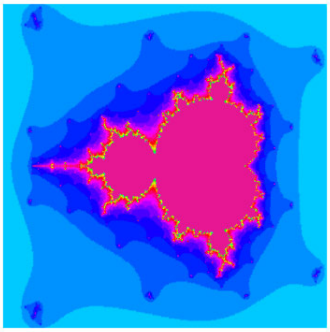

FIGURE 6. Quadratic mandelbrot set generated in picard-mann orbit for $\zeta = 0.05$.

- In Fig. 6 with $A = [-2.8, 1.3] \times [-1.8, 1.8]$ Mandelbrot set obtained in Picard-Mann orbit and in Fig. 7 with $A = [-30.8, 8.3] \times [-30.8, 30.8]$ Mandelbrot set presented in Mann orbit for same value of $\zeta = 0.05$.

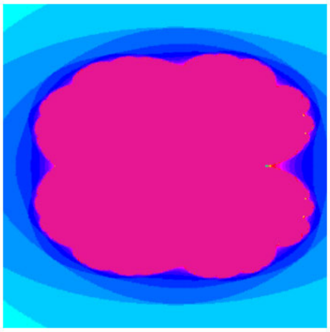

FIGURE 7. Quadratic mandelbrot set generated in mann orbit for $\zeta = 0.05$.

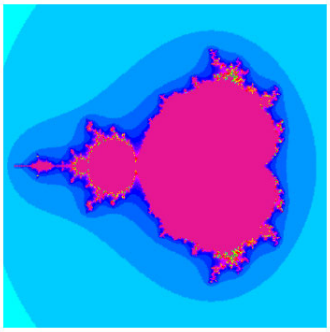

FIGURE 8. Quadratic mandelbrot set generated in picard-mann orbit for $\zeta = 0.6$.

FIGURE 9. Quadratic mandelbrot set generated in mann orbit for $\zeta = 0.6$.

- In Fig. 8 with $A = [-3.1, 1.1] \times [-2.0, 2.0]$ Mandelbrot set obtained in Picard-Mann orbit and in Fig. 9 with $A = [-3.1, 1.1] \times [-2.0, 2.0]$ Mandelbrot set visualized in Mann orbit for similar value of $\zeta = 0.6$.

It is observed that quadratic Mandelbrot sets presented in Figs. 2–9, maintain symmetry along x-axis. Quadratic Mandelbrot sets generated in Picard-Mann orbit for different values of ζ contain one main lobe and few small lobes attached to it whereas Mandelbrot sets generated in Mann orbit for similar value of ζ have only main lobe and there are no small lobes observed attach to it. A clear variation of shapes of Mandelbrot sets is observed generated in Picard-Mann orbit and Mann orbit for same value of ζ (see Figs. 2–3, Figs. 4–5, Figs. 6–7, Figs. 8–9).

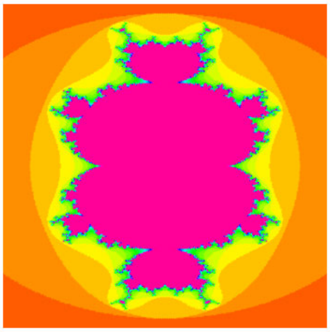

FIGURE 10. Cubic mandelbrot set generated in picard orbit.

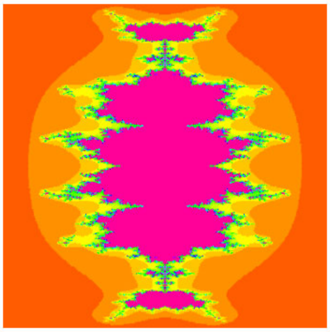

FIGURE 11. Cubic mandelbrot set generated in picard-mann orbit for $\zeta = 0.1$.

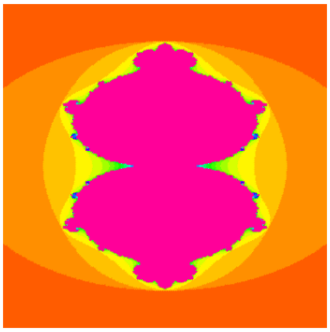

FIGURE 12. Cubic mandelbrot set generated in mann orbit for $\zeta = 0.1$.

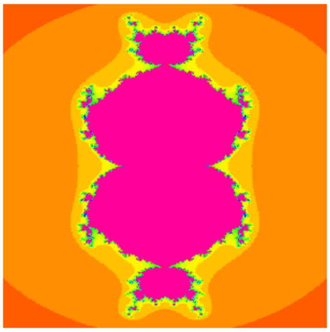

FIGURE 13. Cubic mandelbrot set generated in picard-mann orbit for $\zeta = 0.5$.

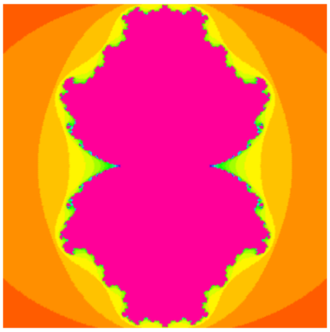

FIGURE 14. Cubic mandelbrot set generated in mann orbit for $\zeta = 0.5$.

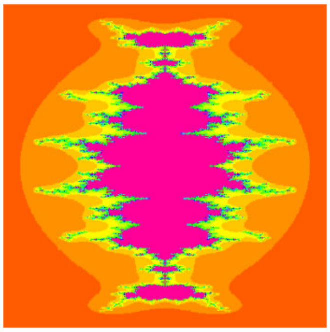

FIGURE 15. Cubic mandelbrot set generated in picard-mann orbit for $\zeta = 0.08$.

FIGURE 16. Cubic mandelbrot set generated in mann orbit $\zeta = 0.08$.

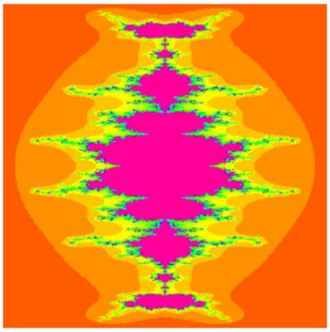

FIGURE 17. Cubic mandelbrot set generated in picard-mann orbit for $\zeta = 0.07$.

B. MANDELBROT SETS FOR $Q_c(z) = z^3 + c$

In Fig. 10 cubic Mandelbrot set is given in Picard orbit and in Figs. 11–18, Mandelbrot sets are visualized for cubic function in Picard-Mann orbit and in Mann orbit by choosing greatest number of iterations 30 and shifting parameters are following:

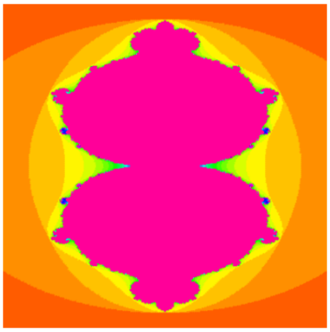

FIGURE 18. Cubic mandelbrot set generated in mann orbit $\zeta = 0.07$.

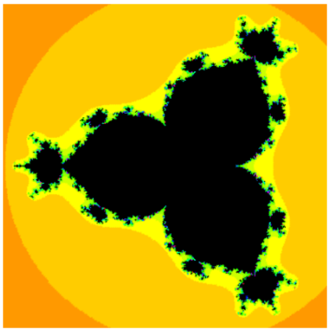

FIGURE 19. Mandelbrot set generated in picard-mann orbit for $\zeta = 0.3$ and $k = 4$.

FIGURE 20. Mandelbrot set generated in mann orbit for $\zeta = 0.3$ and $k = 4$.

- For Fig. 11 Mandelbrot set in Picard-Mann orbit with $A = [-1.5, 1.5] \times [-3.8, 3.8]$ while for Fig. 12 Mandelbrot set in Mann orbit with $A = [-3.5, 3.5] \times [-5.8, 5.8]$ and $\zeta = 0.1$.
- For Fig. 13 Mandelbrot set in Picard-Mann orbit and Fig. 14 Mandelbrot set in Mann orbit with $\zeta = 0.5$, and $A = [-1.5, 1.5] \times [-1.9, 1.9]$.
- For Fig. 15 Mandelbrot set in Picard-Mann orbit with $A = [-1.5, 1.5] \times [-4.5, 4.5]$ also for Fig. 16 Mandelbrot set in Mann orbit with $A = [-3.5, 3.5] \times [-5.5, 5.5]$ and $\zeta = 0.08$.
- For Fig. 17 Mandelbrot set in Picard-Mann orbit with $A = [-1.5, 1.5] \times [-4.5, 4.5]$ and for Fig. 18 Mandelbrot set in Mann orbit with $A = [-3.5, 3.5] \times [-5.5, 5.5]$ for same value of $\zeta = 0.07$.

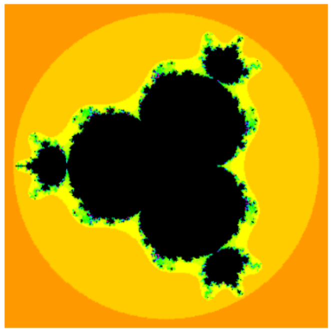

FIGURE 21. Mandelbrot set generated in picard-mann orbit for $\zeta = 0.7$ and $k = 4$.

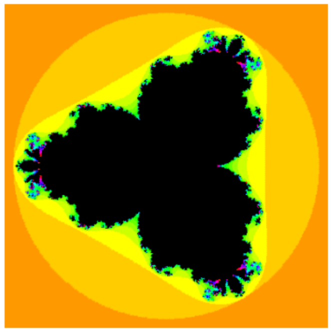

FIGURE 22. Mandelbrot set generated in mann orbit for $\zeta = 0.7$ and $k = 4$.

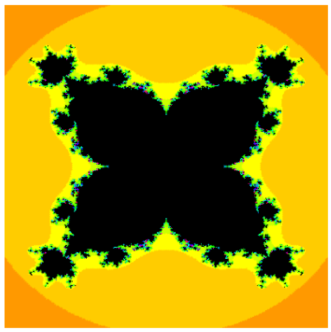

FIGURE 23. Mandelbrot set generated in picard-mann orbit for $\zeta = 0.2$ and $k = 5$.

We observed that cubic Mandelbrot sets generated in Picard-Mann orbit and Mann orbit (Figs. 11–18) there is a reflection symmetry along x-axis as well as y-axis. Further, we observed that Mandelbrot sets for cubic functions took the shape of fascinating Coupled Urn. Mandelbrot sets generated in Picard-Mann orbit visualized in Figs. 11, 15, 17, become more decorated Coupled Urns. It is noticed interesting changes in the figures for different values of parameter ζ. It is also seen that for similar value of ζ there is clear variation of figures visualized in Picard-Mann orbit and Mann orbit.

C. MANDELBROT SETS FOR $Q_c(z) = z^k + c, k \geq 4$

In Figs. 19–28, Mandelbrot sets are presented for $Q_c(z) = z^k + c$, where $k \geq 4$ in Picard-Mann orbit and Mann orbit

FIGURE 24. Mandelbrot set generated in mann orbit for $\zeta = 0.2$ and $k = 5$.

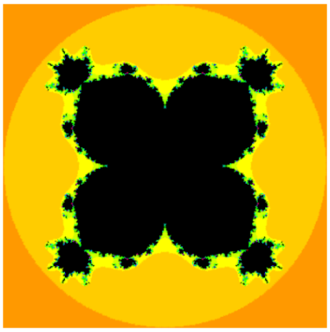

FIGURE 25. Mandelbrot set generated in picard-mann orbit for $\zeta = 0.4$ and $k = 5$.

by choosing greatest number of iterations 30 with shifting parameters are following:

- For Fig. 19 and Fig. 20: $\zeta = 0.3, A = [-1.9, 1.4] \times [-1.7, 1.7]$ and $k = 4$,
- For Fig. 21 and Fig. 22: $\zeta = 0.7, A = [-1.5, 1.5] \times [-1.5, 1.5]$ and $k = 4$,
- For Fig. 23 and Fig. 24: $\zeta = 0.2, A = [-1.5, 1.5] \times [-1.7, 1.7]$ and $k = 5$,
- For Fig. 25 and Fig. 26: $\zeta = 0.4, A = [-1.5, 1.5] \times [-1.5, 1.5]$ and $k = 5$,
- For Fig. 27 and Fig. 28: $\zeta = 0.5, A = [-1.5, 1.5] \times [-1.5, 1.5]$ and $k = 6$,

Mandelbrot sets generated for higher order polynomials in Picard-Mann orbit and Mann orbit presented in Figs. 19–28, we observed that the polynomial $z^k + c, k \geq 4$ generates the Mandelbrot sets that have $(k - 1)$ lobes. Mandelbrot set have reflection symmetry along real (x-axis) and imaginary axis (y-axis) when k is odd and symmetry preserved only x-axis when k is even also rotational symmetry along center of the complex plane. In Figs. 19, 21, 23, 25 and 27 few small lobes are attached to main lobes of Mandelbrot sets. It is seen that for same value of ζ different Mandelbrot sets have been visualized in Picard-Mann orbit and Mann orbit.

V. VISUALIZATION OF JULIA SETS

In this section some Julia sets are visualized for quadratic and cubic polynomials with respect to Picard-Mann and Mann iteration schemes. Algorithm 3 presents the pseudocode

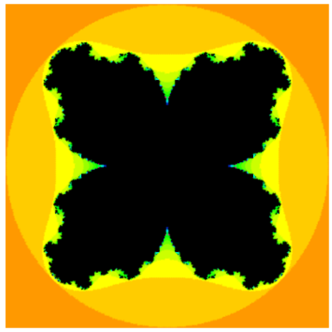

FIGURE 26. Mandelbrot set generated in mann orbit for $\zeta = 0.4$ and $k = 5$.

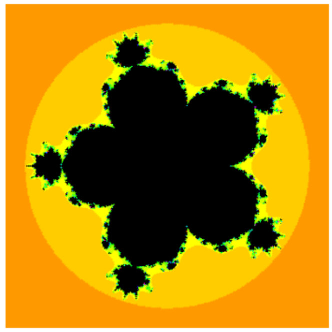

FIGURE 27. Mandelbrot set generated in picard-mann orbit for $\zeta = 0.5$ and $k = 6$.

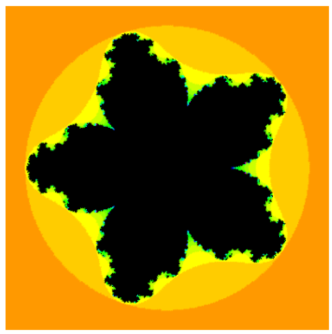

FIGURE 28. Mandelbrot set generated in mann orbit for $\zeta = 0.5$ and $k = 6$.

for the Julia set generation in Picard-Mann orbit while Algorithm 4 demonstrates the pseudocode for the Julia set generation in Mann orbit.

A. JULIA SETS FOR $Q_c(z) = z^2 + c$

Julia sets for the function $Q_c(z) = z^2 + c$ are presented in Picard-Mann orbit and Mann orbit in Figs. 29–36. The maximum number of iterations to generate the images are 30. Moreover, the shifting parameters are the following:

- For Fig. 29 and Fig. 30: $\zeta = 0.5, A = [-3.0, 2.0] \times [-1.7, 1.7]$ and $c = -1.58 - 0.2\mathbf{i}$,
- For Fig. 31 and Fig. 32: $\zeta = 0.6, A = [2.8, 2.0] \times [-1.7, 1.7]$ and $c = -1.58 - 0.2\mathbf{i}$,

Algorithm 3 Generation of Julia set in Picard-Mann Orbit

Input: $Q_c(z) = z^k + c$, where $k \geq 2$, $A \subset \mathbb{C}$ – area, K – iterations, $\zeta \in (0, 1]$ – parameter for Picard-Mann iteration process, $colourmap[0..C-1]$ – with C colours.

Output: Julia set for area A.

$R = \max\{|c|, (\frac{2}{\zeta})^{1/k-1}\}$
for $z_0 \in A$ **do**
    $n = 0$
    while $n \leq K$ **do**
        $u_n = (1-\zeta)z_n + \zeta Q_c(z_n)$,
        $z_{n+1} = Q_c(u_n)$
        if $|z_{n+1}| > R$ **then**
            break
        $n = n + 1$
    $i = \lfloor (C-1)\frac{n}{K} \rfloor$
    colour z_0 with $colourmap[i]$

Algorithm 4 Generation of Julia set in Mann Orbit

Input: $Q_c(z) = z^k + c$, where $k \geq 2$, $A \subset \mathbb{C}$ – area, K – iterations, $\zeta \in (0, 1]$ – parameter for Mann iteration process, $colourmap[0..C-1]$ – with C colours.

Output: Julia set for area A.

$R = \max\{|c|, (\frac{2}{\zeta})^{1/k-1}\}$
for $z_0 \in A$ **do**
    $n = 0$
    while $n \leq K$ **do**
        $z_{n+1} = (1-\zeta)z_n + \zeta Q_c(z_n)$,
        if $|z_{n+1}| > R$ **then**
            break
        $n = n + 1$
    $i = \lfloor (C-1)\frac{n}{K} \rfloor$
    colour z_0 with $colourmap[i]$

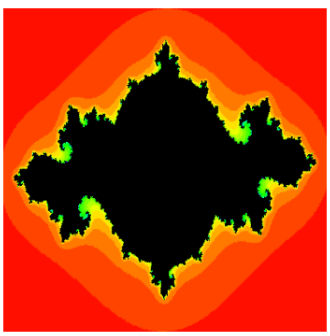

FIGURE 29. Quadratic julia set generated in picard-mann orbit for $\zeta = 0.5$.

- For Fig. 33 and Fig. 34: $\zeta = 0.4, A = [-3.7, 2.1] \times [-1.7, 1.7]$ and $c = -1.78 + 0.2\mathbf{i}$,
- For Fig. 35 and Fig. 36: $\zeta = 0.6, A = [-2.7, 2.0] \times [-1.6, 1.6]$ and $c = -1.78 + 0.2\mathbf{i}$.

FIGURE 30. Quadratic julia set generated in mann orbit for $\zeta = 0.5$.

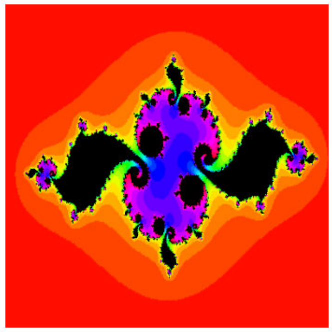

FIGURE 31. Quadratic julia set generated in picard-mann orbit for $\zeta = 0.6$.

FIGURE 32. Quadratic julia set generated in mann orbit for $\zeta = 0.6$.

FIGURE 33. Quadratic julia set generated in picard-mann orbit for $\zeta = 0.4$.

It is observed from the graphical representation of quadratic Julia sets generated in Picard-Mann orbit and Mann orbit that the shape difference between Fig. 29 and 30, 31 and 32, 33 and 34, 35 and 36 for same value of ζ and c is

FIGURE 34. Quadratic julia set generated in mann orbit for $\zeta = 0.4$.

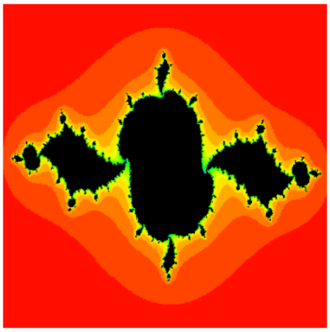

FIGURE 35. Quadratic julia set generated in picard-mann orbit for $\zeta = 0.6$.

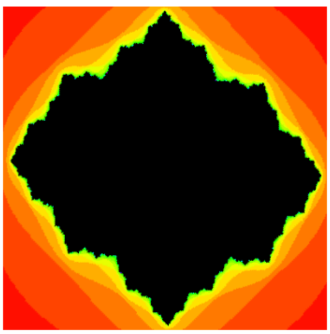

FIGURE 36. Quadratic julia set generated in mann orbit for $\zeta = 0.6$.

FIGURE 37. Cubic julia set generated in picard-mann orbit for $\zeta = 0.5$.

visible in all pairs. It is noticed that for same value of ζ and c quite different Julia sets have been generated in Picard-Mann orbit and Mann orbit. Not only the shape changes, but also the dynamics.

FIGURE 38. Cubic julia set generated in mann orbit for $\zeta = 0.5$.

FIGURE 39. Cubic julia set generated in picard-mann orbit for $\zeta = 0.7$.

FIGURE 40. Cubic julia set generated in mann orbit for $\zeta = 0.7$.

FIGURE 41. Cubic julia set generated in picard-mann orbit for $\zeta = 0.7$.

B. JULIA SETS FOR $Q_c(z) = z^3 + c$

Connected Julia sets for $Q_c(z) = z^3 + c$ are presented in Picard-Mann orbit and Mann orbit in Figs. 37–44.

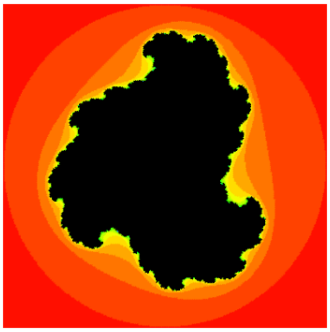

FIGURE 42. Cubic julia set generated in mann orbit for $\zeta = 0.7$.

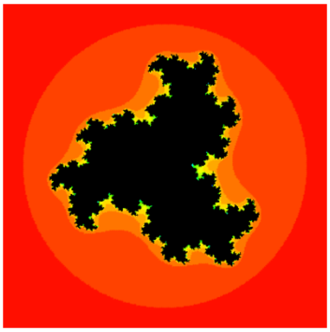

FIGURE 43. Cubic julia set generated in picard-mann orbit for $\zeta = 0.9$.

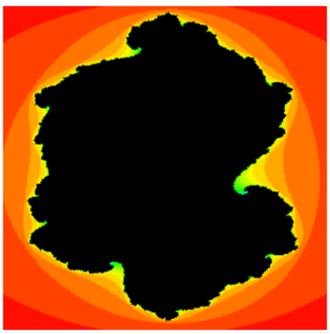

FIGURE 44. Cubic julia set generated in mann orbit for $\zeta = 0.9$.

The maximum number of iterations to generate the images are 30. Moreover, the shifting parameters are the following:

- For Fig. 37 and Fig. 38: $\zeta = 0.5, A = [-1.7, 1.7] \times [-2.0, 2.0]$ and $c = -0.15 + 1.1\mathbf{i}$,
- For Fig. 39 and Fig. 40: $\zeta = 0.7, A = [-1.7, 1.7] \times [-2.0, 2.0]$ and $c = 0.25 + 0.9\mathbf{i}$,
- For Fig. 41 and Fig. 42: $\zeta = 0.7, A = [-1.7, 1.7] \times [-1.7, 1.7]$ and $c = 0.55 + 0.38\mathbf{i}$,
- For Fig. 43 and Fig. 44: $\zeta = 0.9, A = [-1.7, 1.7] \times [-1.7, 1.7]$ and $c = 0.55 + 0.38\mathbf{i}$.

It is observed that Cubic Julia sets generated in Picard-Mann orbit and Mann orbit (Figs. 37–40) maintain symmetry along imaginary axis (y-axis). It is also observed that the shapes of Cubic Julia sets presented in Figs. 37 and 38, 39 and 40, 41 and 42, 43 and 44 have changed in each pair for similar value of ζ and c.

VI. CONCLUSION

In this paper escape criterion for fractals (Mandelbrot and Julia sets) has been presented with respect to Picard-Mann orbit and visualized the new patterns of fractals. Attractive changes can be seen in fractals generated via Picard-Mann iterative procedure for different values of ζ. We have obtained quite different Mandelbrot and Julia sets via Picard-Mann orbit as compared to fractals generated in Mann orbit for similar values of ζ. Variations in the graphics are seen due to change in orbit of iteration scheme. We believe that consequences of this paper will inspire those who are motivating in creating automatically aesthetic designs.

ACKNOWLEDGEMENT

The authors extend their appreciation to the Deanship of Scientific Research at Majmaah University for funding this work under Project Number (RGP-2019-28).

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