Problem 1:- Oscillation (SHM)

F = -kx & F = m$\frac{d^2x}{dt^2}$

Homogenous Linear Differential Equation:-

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad (\ \omega^2 = \frac{k}{m}\ )$$

General Solution: - x = Be^{rt}

$\frac{d^2x}{dt^2} + \omega^2$x = r^2 Be^{rt} + ω^2 Be^{rt}

$r^2 + \omega^2 = 0 \rightarrow$ Auxiliary Equation

Solution will be (x must be real),

x = $B_1\, e^{iwt} + B_2\, e^{-iwt}$

x* = $B_1{}^* e^{-iwt} + B_2{}^*\, e^{iwt}$ (* means Conjugate of the Complex Number)

x = x* → $B_1 = B_2{}^*$

x = $B_1\, e^{iwt} + B_1{}^*\, e^{-iwt}$

Let us Assume, $B_1 = \alpha e^{-t\delta}$ ($\delta, \alpha \rightarrow$ Real Numbers)

x = $\alpha e^{-t\delta}\, e^{iwt} + \alpha e^{-t\delta}\, e^{-iwt}$

$$= 2\alpha\, \frac{e^{i(wt-\delta)} + e^{-i(wt-\delta)}}{2}$$

x = $2\alpha \cos(\omega t \quad \delta)$ → $\boxed{x = A\cos(\omega t \quad \delta)}$

Linear Drag Force:-

m$\frac{d^2x}{dt^2}$ = -kx - b$\frac{dx}{dt}$

$\omega = \sqrt{\frac{k}{m}}$ & $\beta = \frac{b}{2m}$

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega^2 x = 0$$

By taking General Solution: - $x = Be^{rt}$

We get Auxiliary Equation: - $r^2 + 2\omega r + \omega^2 = 0 \rightarrow r = \quad \beta \quad \sqrt{\beta^2 \quad \omega^2}$

MATLAB Code:-

Main Function:-

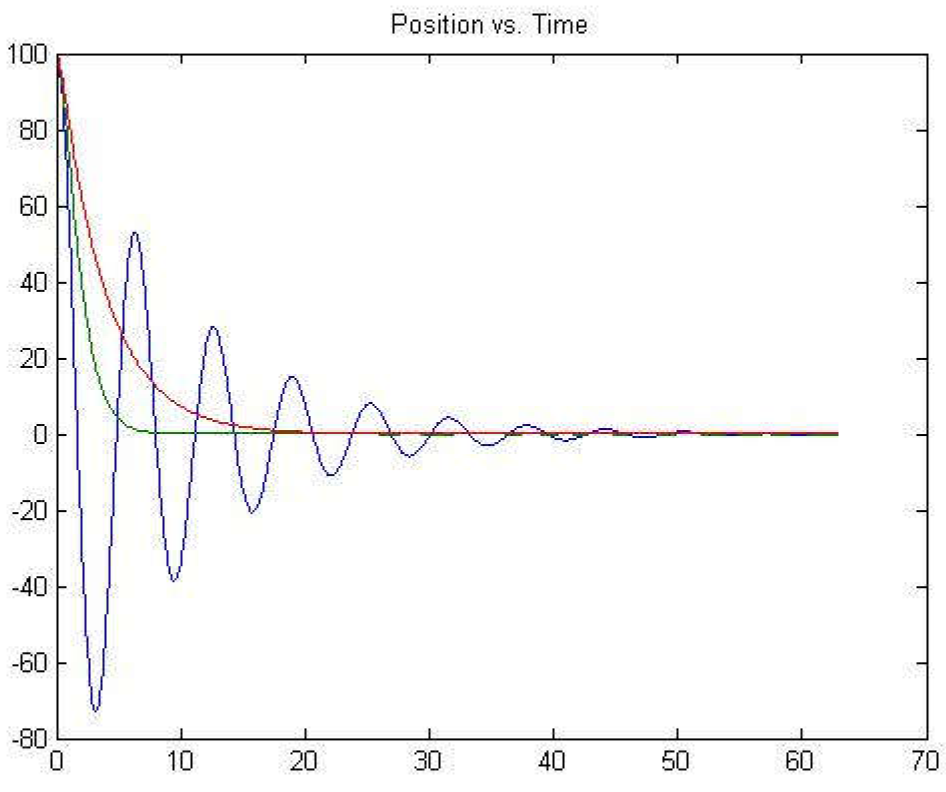
```
clear;
close all;
% declare the pendulam variables to be global and set it
global cnst B;
B=0.5;
B=1;
B=2;
cnst=1;

timescale=2*pi*(1/cnst);
dt=timescale/100;

% set the initial and final times
tstart=0;
tfinal=10*timescale;

% set the initial conditions in the y0 column vector
u0=zeros(2,1);
u0(1)=100;
u0(2)=0; % initial velocity


[t,u]=ode45(@rhs,[tstart:dt:tfinal],u0);

x1=u(:,1);
v1=u(:,2);


[t,u]=ode45(@rhs,[tstart:dt:tfinal],u0);

x2=u(:,1);
v2=u(:,2);


[t,u]=ode45(@rhs,[tstart:dt:tfinal],u0);
x3=u(:,1);
v3=u(:,2);
% plot the position vs. time
plot(t,x1,t,x2,t,x3)
title('Position vs. Time')
```

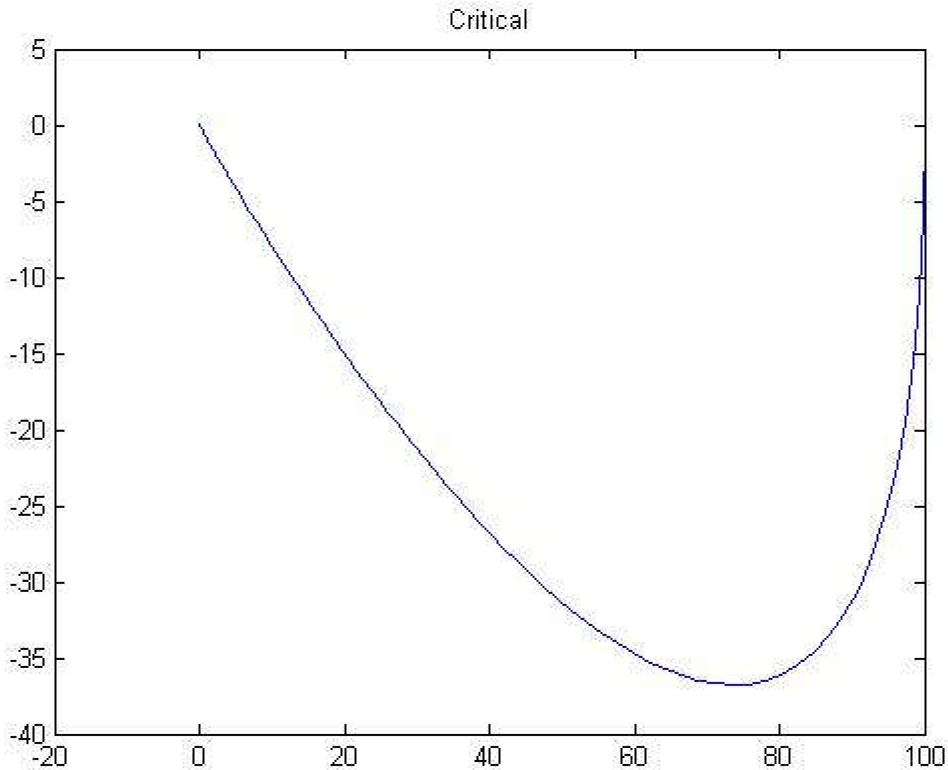
```
% make a "phase-space" plot of v vs. x
figure();
plot(x1,v1)
title('Underdamp');
figure();
plot(x2,v2)
title('Critical');
figure();
plot(x3,v3)
title('Overdamp');
```

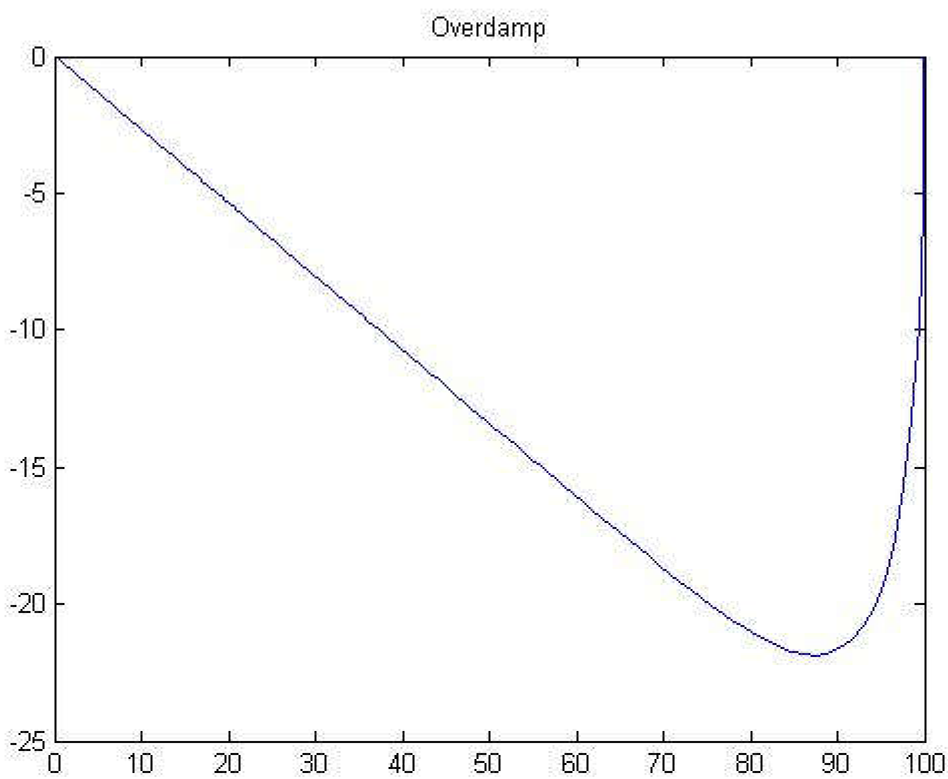
RHS Function:-

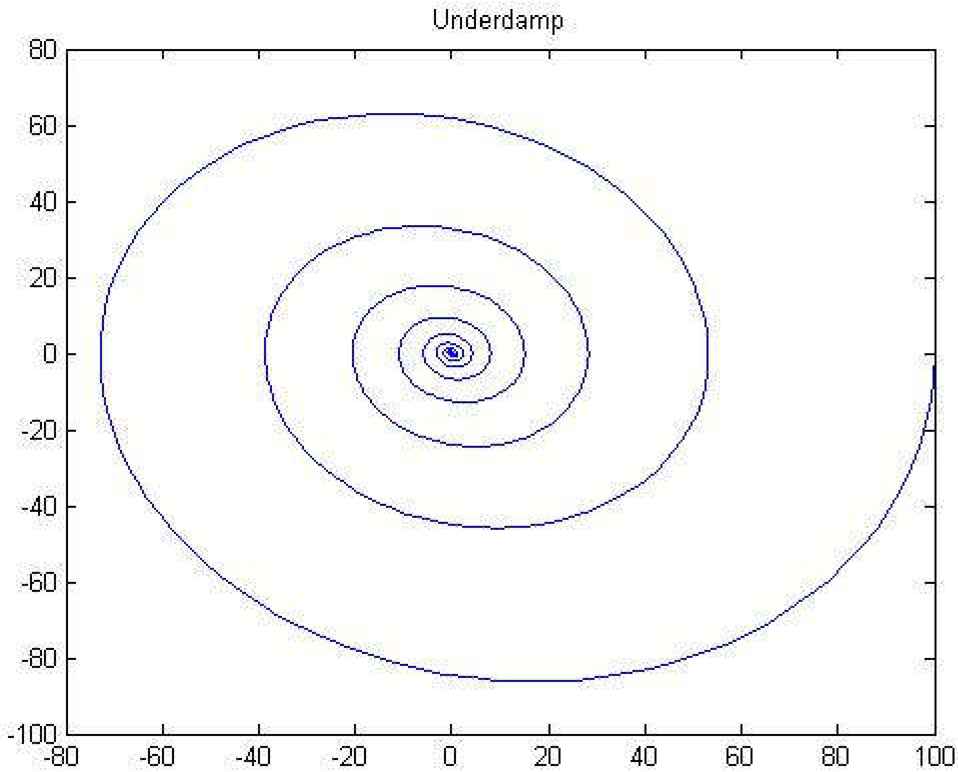
```
function F=rhs(t,u)
global cnst B;
F=zeros(length(u),1);
F(1)=u(2);
F(2)=-cnst*u(1)-2*B*u(2);
```

Output:-




Critical
5
0
-5
-10
-15
-20
-25
-30
-35
-40
-20
0
20
40
60
80
100


Overdamp
0
-5
-10
-15
-20
-25
0
10
20
30
40
50
60
70
80
90
100



Observation:-

According to the values of β & ω we get 3 Cases:-

1) Under Damping $\omega^2 > \beta^2$

   $\omega 1 = \sqrt{\omega^2 \quad \beta^2}$

   Equation:-

   $$X = Ae^{-\beta t}\cos(\omega 1 t \quad \delta)$$

   In this case, Amplitude of sinusoidal wave decreases exponentially.

2) Critical Damping $\omega^2 = \beta^2$

   $\omega 1 = 0$

   Equation:-

   $$X = Ae^{-\beta t}$$

   In this case, No sinusoidal wave. Directly Amplitude decreases exponentially.

3) Over Damping $\omega^2 < \beta^2$

   $\omega 1 = \sqrt{\beta^2 \quad \omega^2}$

   Equation:-

$$X = B_1 e^{-(\beta+\omega 1)t} + B_2 e^{-(\beta-\omega 1)t}$$

In this case, Amplitude also decreases exponentially but in slower rate than critical damp.

As we can see in the graphs that Critical damping reaches faster at Equilibrium than Over-damping and Under-Damping.

Time to reach at EQLBM:-

Critical Damping $<$ Over Damping $<$ Under Damping

Problem 2:- Driven Oscillation

If the periodic Force at another frequency is applied, the oscillation will be forced to occur at the applied frequency, i.e. Forced/Driven Oscillation.

Forced Oscillation is small because driven frequency is close to Natural Frequency. Resonance will occur at the point where Natural frequency is equals to driven Frequency.

We have seen the equation of Damping Oscillation:-

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Here, Another Forcing function that depends on time will be added So equation will look like,

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t) \quad \rightarrow \quad \frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega 0^2 x = f(t)$$

$$\left(\text{Where } \beta = \frac{b}{2m} \quad \& \quad f(\text{t}) = \frac{F(t)}{m}\right)$$

LHS is Linear in Derivative of x. So we can define a Differential Operator,

$$\text{D} = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega 0^2 \quad \rightarrow \quad \text{D}x = f$$

General Solution,

For homogeneous Equation :- $X_h = C_1\, e^{r1\,t} + C_2\, e^{r2\,t}$

For Non- Homogeneous Equation :- X_p (Assume)

Assume Here the Solution will be, $(X_h + X_p)$

$f(\text{t}) = f_0 \cos(\omega t)$

Where $f_0 \rightarrow$ Amplitude of the Driven Force

$\omega \rightarrow$ Driven Frequency

$\omega_0 \rightarrow$Natural/Resonant Frequency

Now the equation for Driven Oscillation will be,

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega0^2x = f0\ \cos(\omega t)$$

To solve the above equation we can consider x as a real part of complex solution of z = x + iy.

Equation will be:- $\frac{d^2z}{dt^2} + 2\beta\frac{dz}{dt} + \omega0^2z = f0\, e^{i\omega t}$

General Solution:- z = C $e^{i\omega\, t}$

($\omega^2 + 2i\beta\omega + \omega0^2$) C $e^{i\omega\, t}$ = $f_0\, e^{i\omega\, t}$

C = $\frac{f0}{\omega - \omega0 + 2i\beta\omega}$ = A$e^{-t\delta}$ (Where A→ $Amplitude$ & $\delta \rightarrow P\ \ ase$)

As we know C.C* = A^2

A^2 = $\frac{f0^2}{(\omega0^2 - \omega^2)^2 + 4\beta^2\omega^2}$ & $\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega0^2\ \ \omega^2}\right)$

z(t) = A$e^{i(\omega t - \delta)}$ Real Part → x_p = Acos($\omega t\ \ \ \delta$)

Solution for the Driven Oscillation is,

X(t) = $X_h + X_p$ = $C_1\, e^{r1\, t} + C_2\, e^{r2\, t}$ + Acos($\omega t\ \ \ \delta$)

MATLAB Code:-

Main Function:-

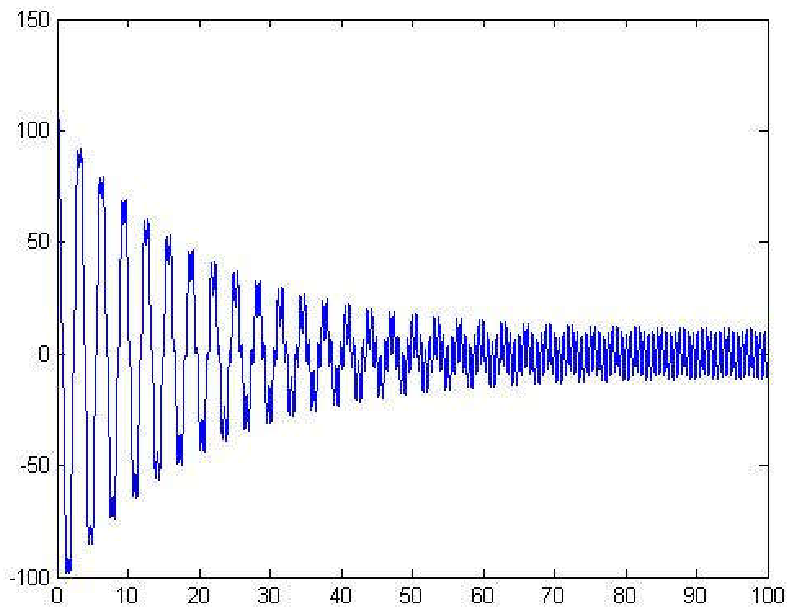
```
clear all;
close all;

global f0;
global Wdriven;
global beta;
global W0;
global A;
A = 0;
f0=1000;
beta = 0.05;
W0 = 2;
w = 0:0.1:5*W0;
a = zeros(1,length(w));
present = 1;
Wdriven = 10;
v0 = 0;
x0 = 100;

dt = 1e-2;
u0 = zeros(2,1);
u0(1) = x0;
u0(2) = v0;
[t,u] = ode45(@rhs,0:dt:100,u0);

x = u(:,1);
v = u(:,2);

plot(t,x);
```

RHS Function:-

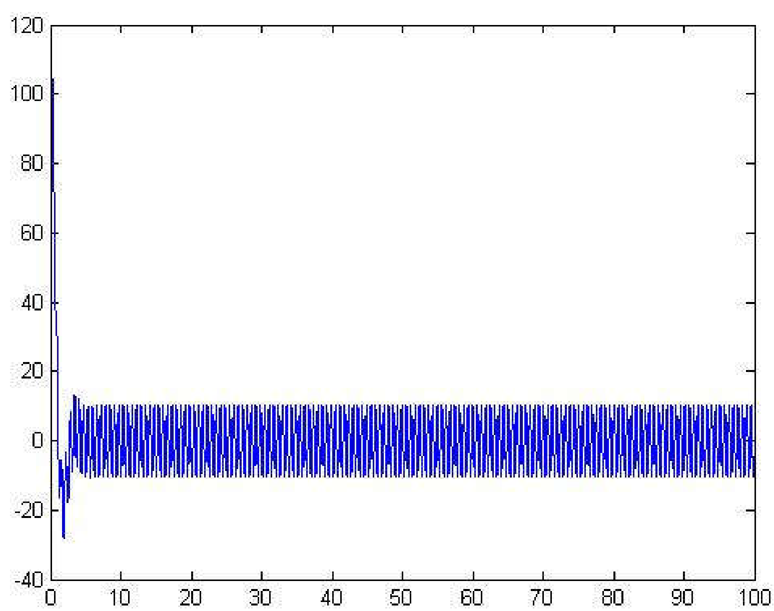
```
function F = rhs(t,u)
global f0;
global Wdriven;
global beta;
global W0;
global A;
F = zeros(length(u),1);
F(1) = u(2);
F(2) = f0*cos(Wdriven*t) - 2*beta*u(2) - W0*W0*u(1);
```

Output:-

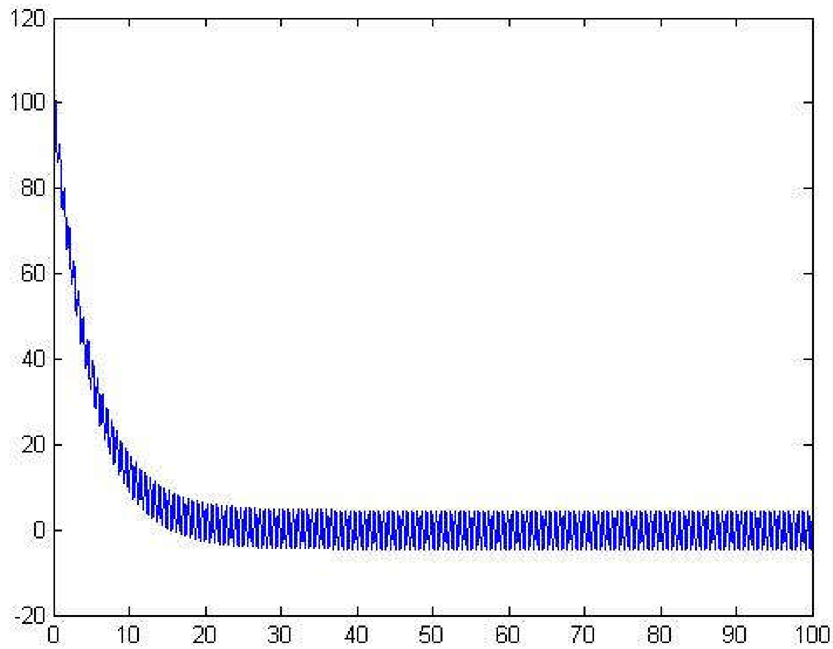
1) Beta = 0.05



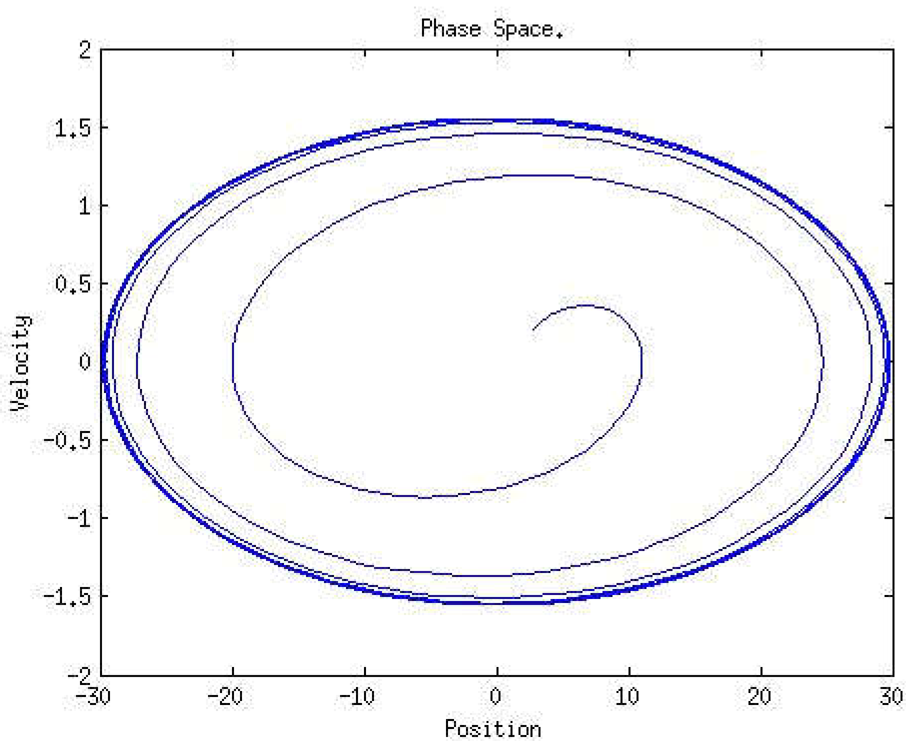
2) Beta = 1



3) Beta = 10



Phase Space Diagram:-



Observation:-

Equation of Driven Oscillation:- (From the Above)

$$X = C_1\, e^{r1\,t} + C_2\, e^{r2\,t} + A\cos(\omega t \quad \delta)$$

Here in the above equation C1 & C2 dies out exponentially with the time. So, Behavior at long time is just set by driving force. We can see this in the graph too.

MATLAB Code:-

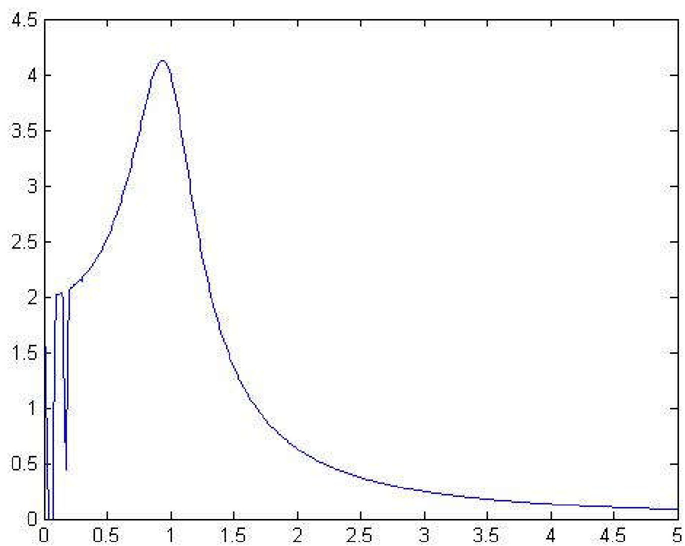
```
global cnst;
global b;
global fdr;
beta = 0.5;
cnst=1;
w0 = cnst;
w0 = round(sqrt(g/l));
timescale=2*pi*sqrt(l/g);
dt=timescale/100;
step=1;
amplitude = zeros(500,1);
tstart=0;
tfinal=20*timescale;
for fdr=0:0.01:5*w0
    u0=zeros(3,1);
    u0(1)=0;
    u0(2)=1;
    u0(3) = 0;
    [t,u]=ode45(@rhs,tstart:dt:tfinal,u0);

    x=u(:,1);
    v=u(:,2);
    max1=0;
    for s=1400:2000
        if(x(s)>max1)
            max1 = x(s);
        end
    end
    amplitude(step) = max1;
    step = step+1;
end
figure;
plot(0:0.01:5*w0,amplitude);
figure;
    plot(t,x);
    title('Position vs. Time(Over damped driven b=2)');
    xlabel('Position');
    ylabel('Velocity');

figure;
plot(x,v);
xlabel('Position');
```

```
ylabel('Velocity');
title('Phase Space.');
```

Rhs Function:

```
function F=rhs(t,u)

global cnst;
global b;
global fdr;
F=zeros(length(u),1);

F(1)=u(2);
u(3) = 2*cos(fdr*t);
F(2)=-cnst*u(1)-b*F(1)+u(3);
```

Output:-

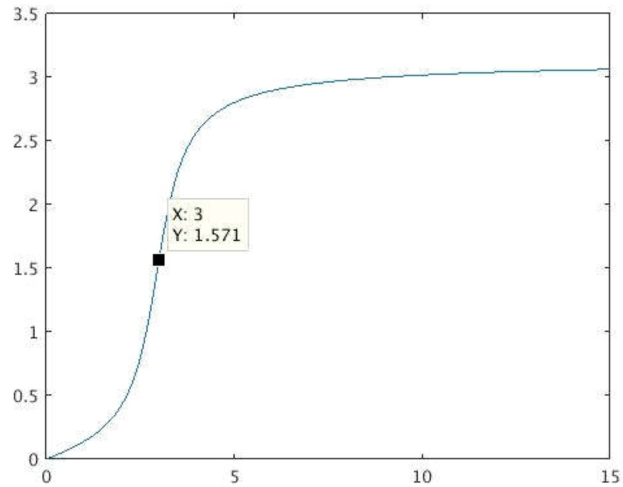


Here we get maximum Amplitude when resonant frequency will become same as Natural Frequency.

MATLAB Code:-

```
clear all;
global cnst;
global beta;
global fdr;
beta = 0.5;
cnst=1;
w0 = cnst;
timescale=2*pi* (1/cnst);
dt=timescale/100;
step=1;

phasediff = zeros(500,1);

% set the initial and final times
tstart=0;
tfinal=20*timescale;
for fdr=0:0.01:5*w0
    % set the initial conditions
    u0=zeros(3,1);
    u0(1)=0;
    u0(2)=1; % initial velocity
    u0(3) = 0;
    [t,u]=ode45(@rhs2,tstart:dt:tfinal,u0);

    drs = 2*cos(fdr*t);

    x=u(:,1);
    v=u(:,2);

    phasediff(step) = atan(2*beta*fdr/(w0^2-fdr^2));
    if( phasediff(step)<0)
        phasediff(step) =  phasediff(step) + pi;
    end

    step = step+1;
end
figure;
plot(0:0.01:5*w0,phasediff);
% plot the position vs. time
```

Output:-
For beta=0.67 Graph look like below:



We know by theory that when driving frequency is equals to natural frequency of the system then the phase difference between them is pi/2.

For beta=1.56 Graph look like below: