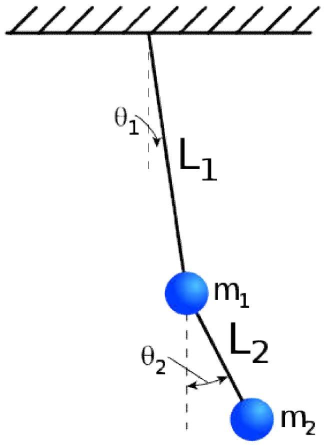
Problem: - Double Pendulum

Derive the governing differential equation of a pendulum. Solve the derived differential equation and plot the values of $\theta1$ and $\theta2$ with respect to time.



Translational kinetic energies of the centres of mass of the two limbs are given by:

$$\begin{aligned} T_{1,trans} &= \tfrac{1}{2}m_1\left(\dot{x_1}^2+\dot{y_1}^2\right) \\ &= \tfrac{1}{2}m_1 l_1^2\dot{\theta_1}^2 \\ T_{2,trans} &= \tfrac{1}{2}m_1\left(\dot{x_2}^2+\dot{y_2}^2\right) \\ &= \tfrac{1}{2}m_2L^2\dot{\theta_1}^2+\tfrac{1}{2}m_2l_2^2\dot{\theta_2}^2+m_2Ll_2\cos(\theta_1-\theta_2)\dot{\theta_1}\dot{\theta_2} \end{aligned}$$

Rotational kinetic energies of the limbs around their respective centres of mass are given by

$$\begin{aligned} T_{1,rot} &= \tfrac{1}{2}I_1\dot{\theta_1}^2 \\ T_{2,rot} &= \tfrac{1}{2}I_2\dot{\theta_2}^2 \end{aligned}$$

Hence the total kinetic energy of the system is

$$T=\tfrac{1}{2}m_1l_1^2\dot{\theta_1}^2+\tfrac{1}{2}m_2L^2\dot{\theta_1}^2+\tfrac{1}{2}m_2l_2^2\dot{\theta_2}^2+m_2Ll_2\cos(\theta_1-\theta_2)\dot{\theta_1}\dot{\theta_2}+\tfrac{1}{2}I_1\dot{\theta_1}^2+\tfrac{1}{2}I_2\dot{\theta_2}^2$$

The gravitational potential energies of the two limbs are

$$\begin{aligned} V_1 &= -gm_1l_1\cos(\theta_1) \\ V_2 &= -gm_2L\cos(\theta_1)-gm_2l_2\cos(\theta_2) \end{aligned}$$

and the Lagrangian is

$$\begin{aligned} L &= T - V \\ &= c_1\dot{\theta}_1^{\,2} + c_2\dot{\theta}_2^{\,2} + c_3\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + c_4\cos(\theta_1) + c_5\cos(\theta_2) \end{aligned}$$

where

$$\begin{aligned} c_1 &= \frac{m_1 l_1^2}{2} + \frac{I_1}{2} + \frac{m_2 L^2}{2} \\ c_2 &= \frac{m_2 l_2^2}{2} + \frac{I_2}{2} \\ c_3 &= m_2 L l_2 \\ c_4 &= g\left(m_1 l_1 + m_2 L\right) \\ c_5 &= g m_2 l_2 \end{aligned}$$

The evolution of the system is determined by the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = 0$$

For the current system, this gives us two coupled second order ordinary differential equations

$$\begin{aligned} c_4\sin(\theta_1) + 2c_1\ddot{\theta}_1 + c_3\ddot{\theta}_2\cos(\theta_1-\theta_2) + c_3\dot{\theta}_2^{\,2}\sin(\theta_1-\theta_2) &= 0 \\ c_5\sin(\theta_2) + 2c_2\ddot{\theta}_2 + c_3\ddot{\theta}_1\cos(\theta_1-\theta_2) - c_3\dot{\theta}_1^{\,2}\sin(\theta_1-\theta_2) &= 0 \end{aligned}$$

$$\begin{bmatrix} (m_2+m_1)l_1 & m_2 l_2\cos(\theta_2-\theta_1) \\ m_2 l_1\cos(\theta_2-\theta_1) & m_2 l_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} -(w_1+w_2)\sin\theta_1 + m_2 l_2\dot{\theta}_2^{\,2}\sin(\theta_2-\theta_1) \\ -w_2\sin\theta_2 - m_1 l_1\dot{\theta}_1^{\,2}\sin(\theta_2-\theta_1) \end{Bmatrix}$$

MATLAB Code:-

"Main" Function:-

```
tspan=[0 10];
y0=[1;0;1;0];
options = odeset('mass','M(t,y)');
[t,y]=ode113('indmot_ode',tspan,y0,options);
subplot(2,1,1)
plot(t,y(:,1),'r')
grid
xlabel('Time')
ylabel('Theta1')
subplot(2,1,2)
plot(t,y(:,3),'b')
grid
xlabel('Time')
ylabel('Theta2')
figure;
```

```
plot(y(:,1),y(:,2),'r');
ylabel('Angular Velocity')
xlabel('Theta1')
title('Phase Space (Pendulum-1)')
figure;
plot(y(:,3),y(:,4),'b')
ylabel('Angular Velocity')
xlabel('Theta2')
title('Phase Space (Pendulum-2)')
```

“mass” Function:-

```
function m = mass(t,y)
M1=5;
M2=5;
g=9.81;
l1=1;
l2=1;
m1=[1 0 0 0];
m2=[0 (M1+M2)*l1 0 M2*l2*cos(y(3)-y(1))];
m3=[0 0 1 0];
m4=[0 M2*l1*cos(y(3)-y(1)) 0 M2*l2];
m=[m1;m2;m3;m4];
```

“pend” Function:-

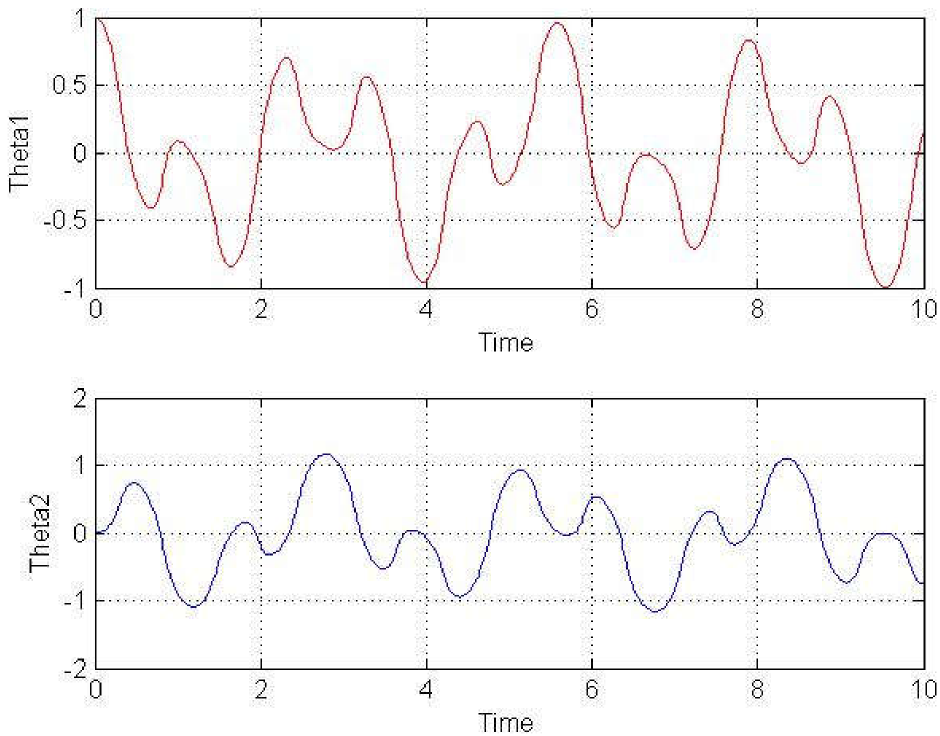
```
function yp= pend(t,y)
M1=5;
M2=5;
g=9.81;
l1=1;
l2=1;
w2=M2*9.81;
w1=M1*9.81;
yp=zeros(4,1);
yp(1)=y(2);
yp(2)=-(w1+w2)*sin(y(1))+M2*l2*(y(4)^2)*sin(y(3)-y(1));
yp(3)=y(4);
yp(4)=-w2*sin(y(3))-M2*l1*(y(2)^2)*sin(y(3)-y(1));
```

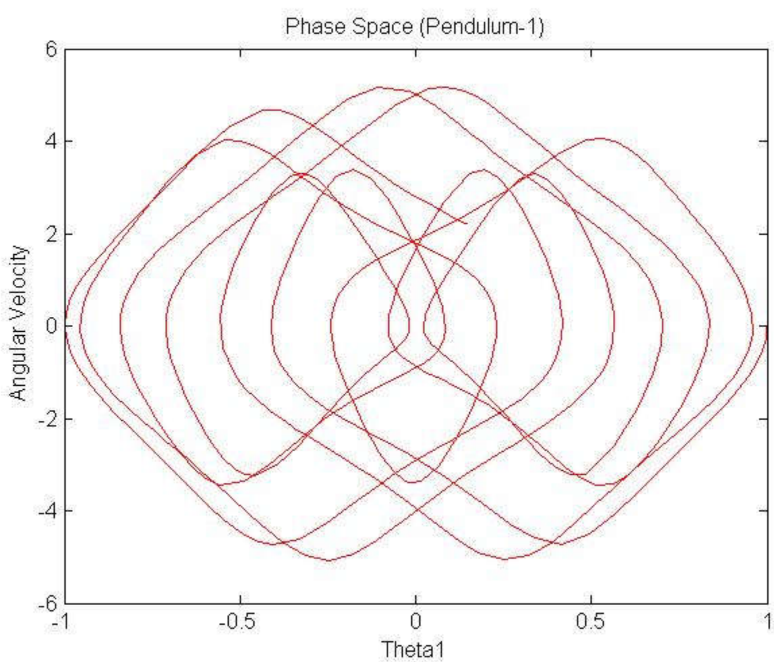
“indmot_ode” Function:-

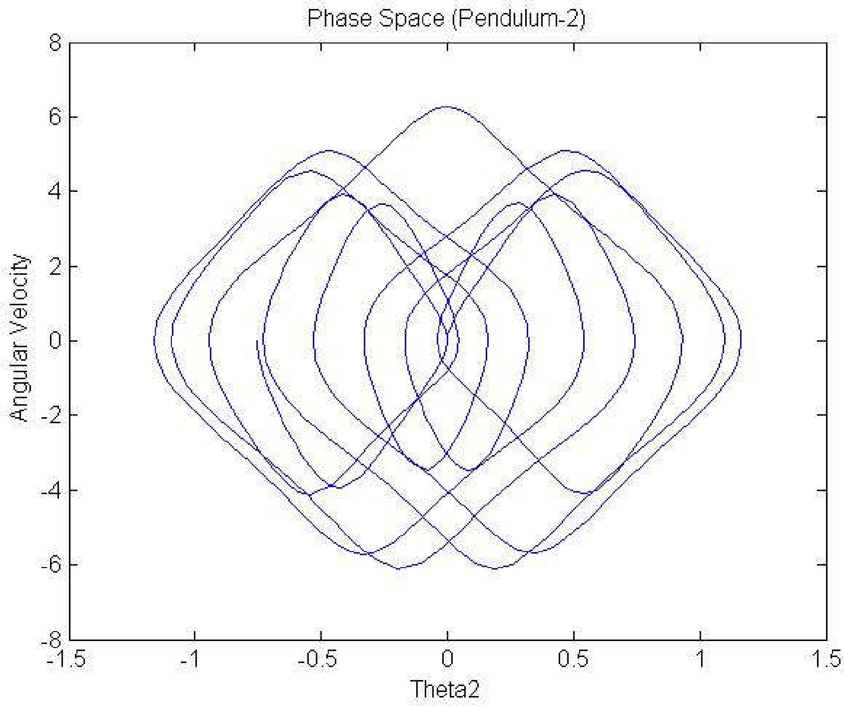
```
function varargout=indmot_ode(t,y,flag)
switch flag
case ''
varargout{1}=pend(t,y);
case 'mass'
varargout{1}=mass(t,y);
otherwise
error(['unknown flag ''' flag '''.']);
end
```

Output:-

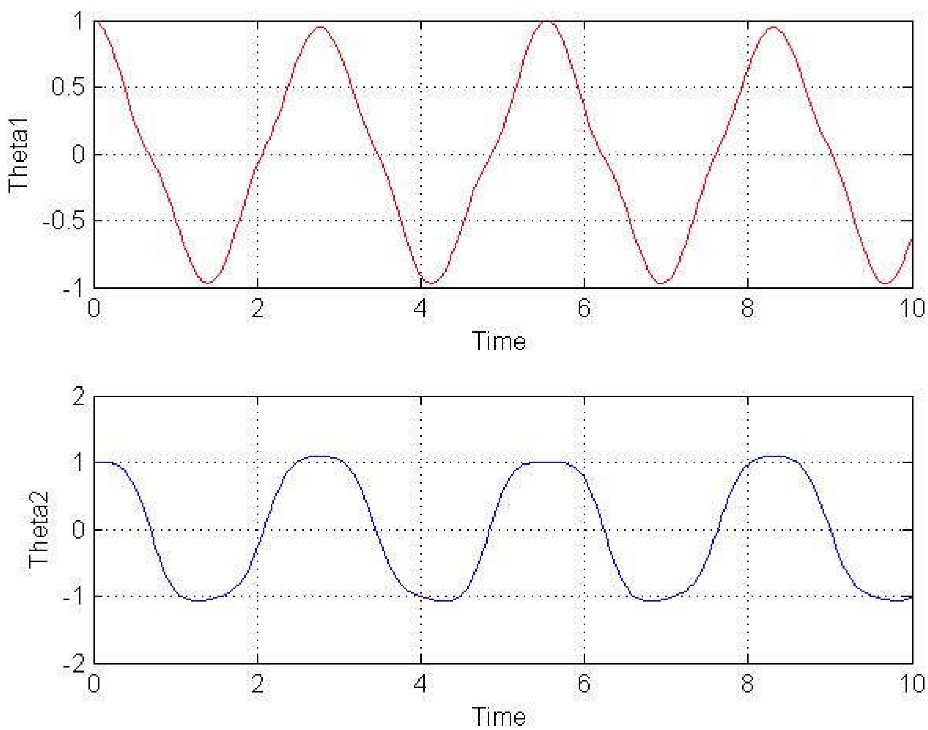
1) **M1=5 & M2=15 and $\boldsymbol{\theta 1 = 1}$ & $\theta 2 = 0$**

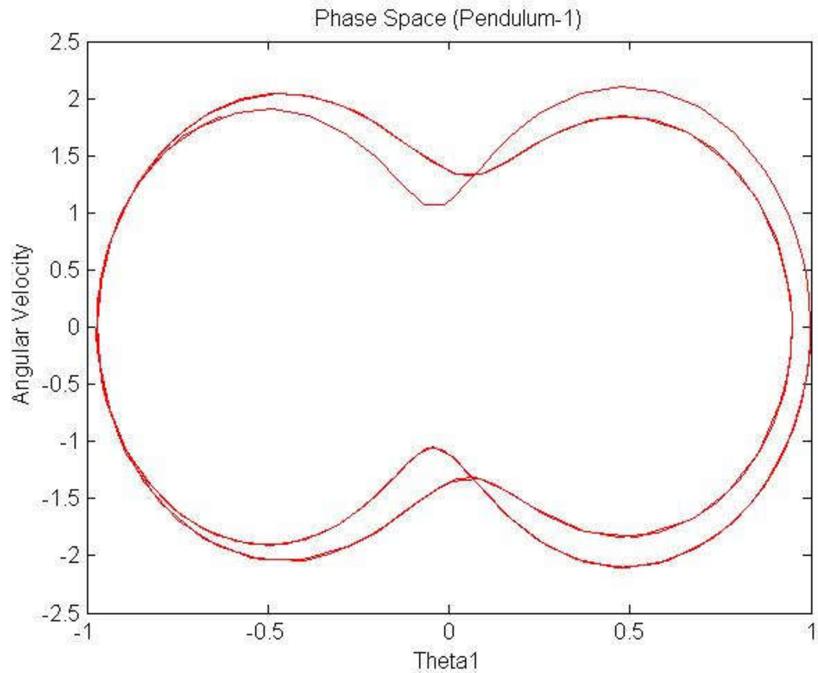


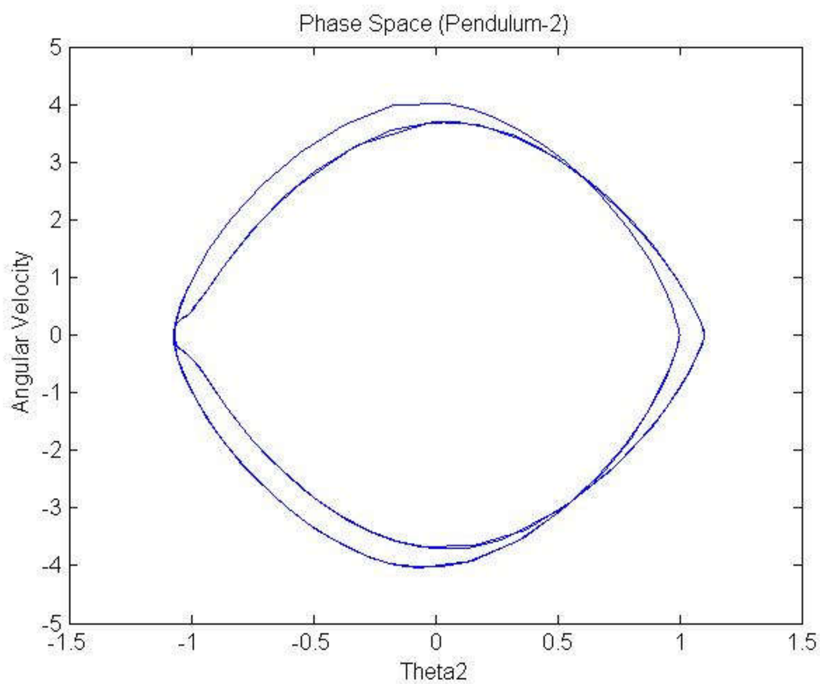



Phase Space (Pendulum-2)
Angular Velocity
Theta2
8
6
4
2
0
-2
-4
-6
-8
-1.5
-1
-0.5
0
0.5
1
1.5

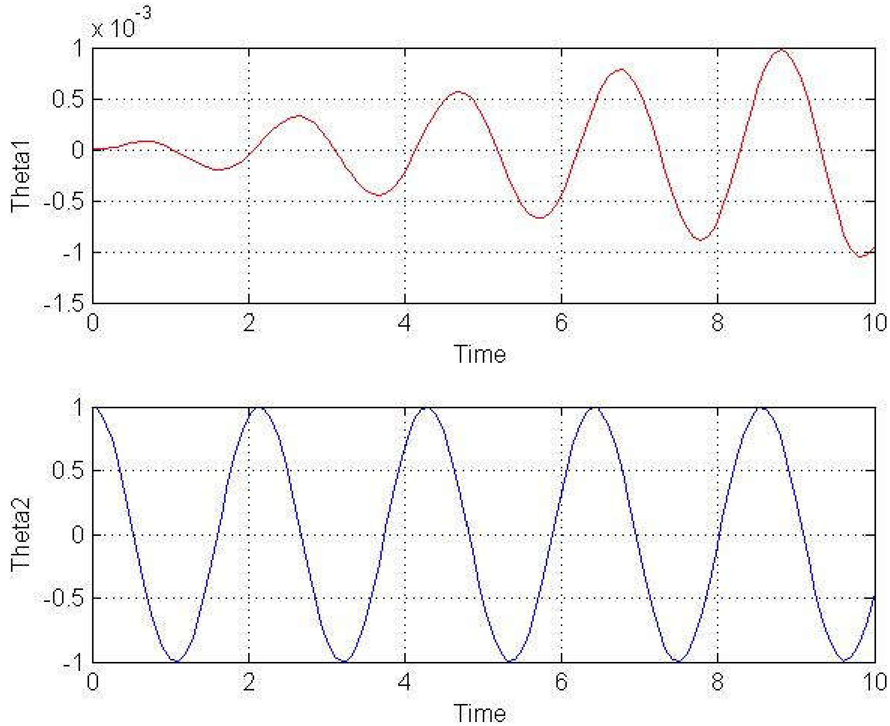
2) M1 = 5 & M2 = 5

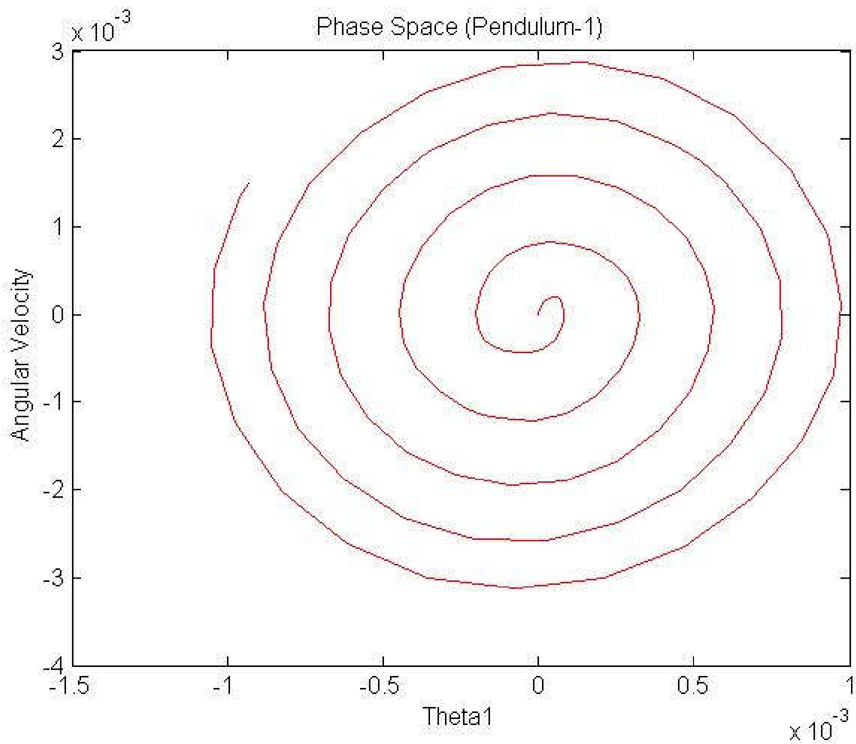

Theta1
Time
Theta2
Time

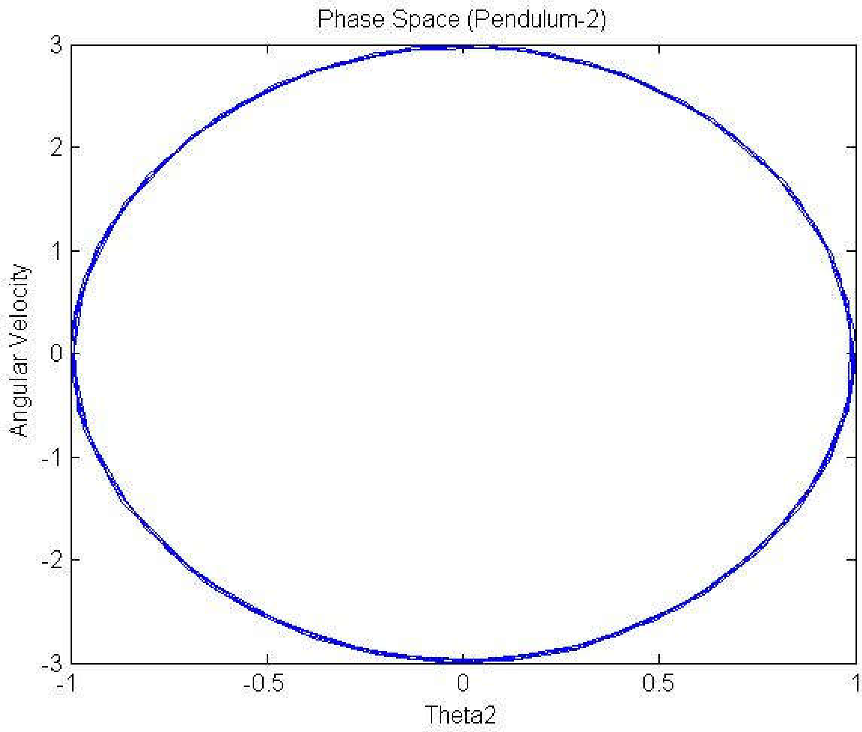

Phase Space (Pendulum-1)
Angular Velocity
Theta1
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-1
-0.5
0
0.5
1


Phase Space (Pendulum-2)
Angular Velocity
Theta2
5
4
3
2
1
0
-1
-2
-3
-4
-5
-1.5
-1
-0.5
0
0.5
1
1.5

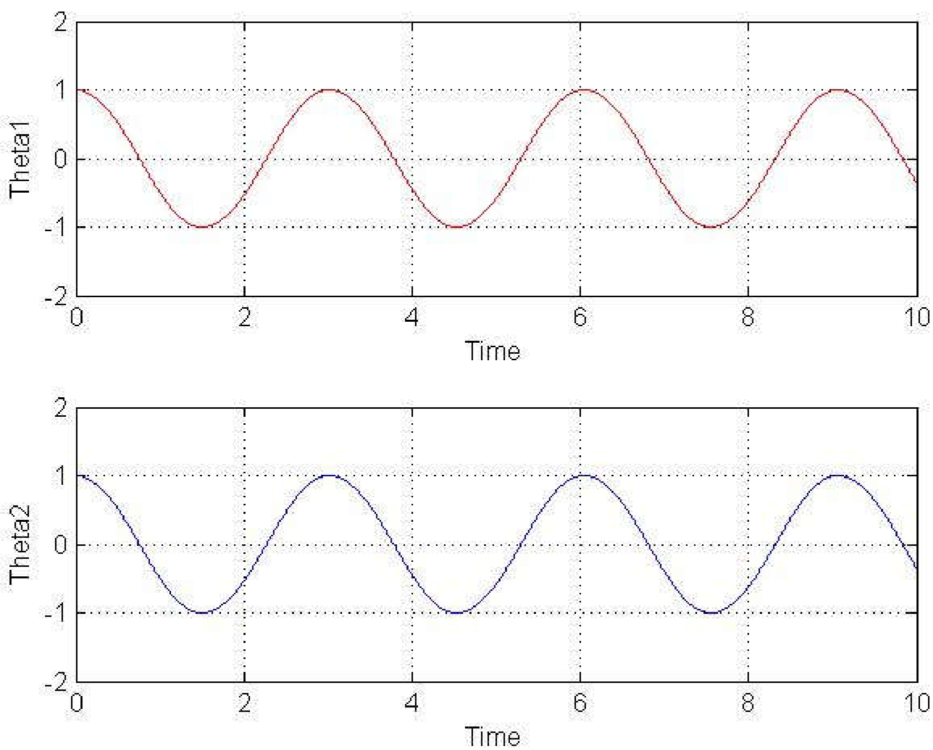
3) M1 = 50000 & M2 = 5

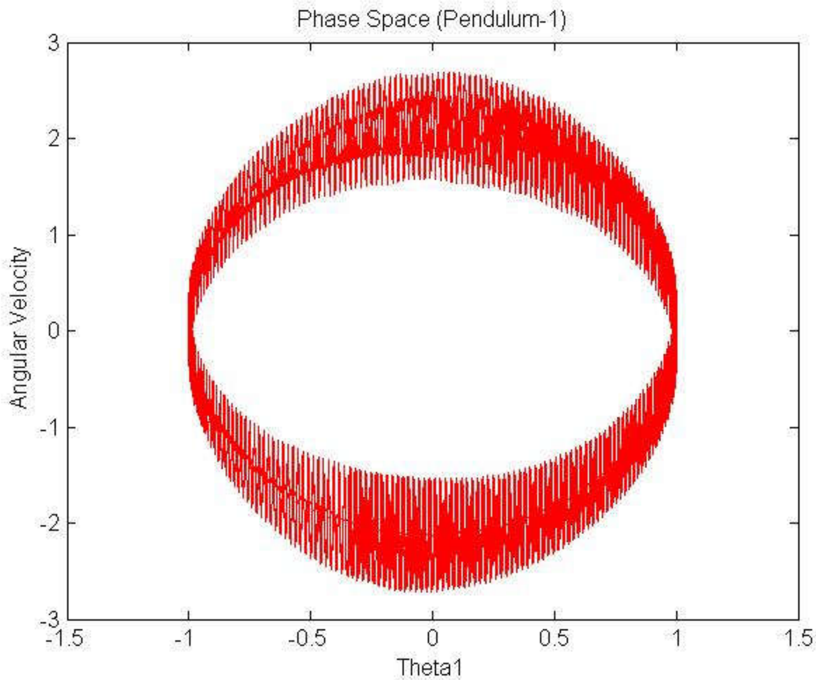


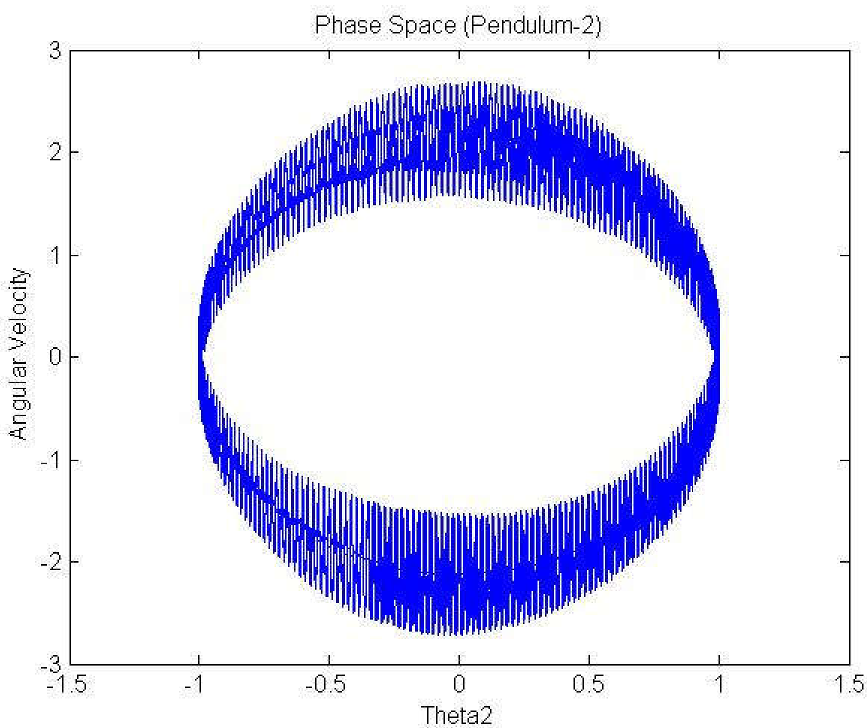




4) M1=5 & M2=50000




Phase Space (Pendulum-1)
Angular Velocity
Theta1
3
2
1
0
-1
-2
-3
-1.5
-1
-0.5
0
0.5
1
1.5


Phase Space (Pendulum-2)
Angular Velocity
Theta2
3
2
1
0
-1
-2
-3
-1.5
-1
-0.5
0
0.5
1
1.5

Observation:-

As we can see from the output that,

In case 3:- M1 >> M2 & $\theta 1 \rightarrow 0$

Second pendulum will do simply a SHM and there is negligible change in position of Pendulum 1.

In case 4:- M1 << M2 & $\theta 1 \rightarrow \theta 2$

Here both pendulums will have almost same trajectory. (Because we can ignore M1- as a part of string)