

# Discrete II Notes

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## Set Theory Review

Sets are collections of unique elements.

### Complement

Everything that's not in the set. Denoted by  $A'$ ,  $A^C$ , or  $\overline{A}$

### Intersection

An operation that takes two sets, and returns the common elements between them. Denoted by  $A \cap B$

### Union

An operation that takes two sets, and returns all the elements that are in  $A$ ,  $B$ , and  $A \cap B$ . Denoted by  $A \cup B$ .

### De Morgan's Laws

$$(A \cup B)^C = A^C \cap B^C$$

$$(A \cap B)^C = A^C \cup B^C$$

### Disjoint Sets

**Disjoint** sets or **Mutually Exclusive** sets, are sets that have no elements in common. More formally:  $(A \cap B) = \emptyset$

## Intro to Probability

**Sample Space** of an experiment is the set of all possible outcomes of that experiment. An **event** is any collection (subset) of outcomes contained in the sample space  $S$ . An event is said to be **simple** if it consists of exactly one outcome and **compound** if it consists of more than one outcome.

## Counting

For an ordered pair defined by  $(x, y)$  where  $x$  can be selected in  $n_1$  ways, and  $y$  can be selected in  $n_2$  ways, the number of pairs is  $n_1 n_2$ . Can be extended to  $k$  dimensions. This is known as the **Multiplication rule**.

## Permutations

For  $k$  selections made **with replacement** on  $n$  distinct elements, there are  $n^k$  possible outcomes.

**Without replacement** however, there are  $n$  options for the first selection,  $n - 1$  choices for the next selection, and  $n - k + 1$  choice(s) for the  $k^{th}$  selection. This yields.

$${}_n P_k = n(n-1)(n-2) \dots (n-k+1)$$

## Combinations

Given  $n$  distinct objects, the number of **unordered** subsets of size  $k$  is given by  ${}_n C_k$ , or  $\binom{n}{k}$  ( $n$  choose  $k$ ).

$${}_n C_k = \frac{n!}{(n-k)!(k!)}$$

## Overcounting with Groups

For  $n$  distinct objects being divided into  $k$  groups,  $\binom{n}{k}$  over counts by a factor of  $k!$ . To account for that, we do the following:

$$\frac{\binom{n}{k}}{k!}$$

## Bose Einstein

For counting ways to separate  $n$  indistinguishable objects into  $k$  groups. We can use the following:

$$\binom{n-1+k}{k}$$

Analogy: For splitting  $n$  pieces of candies between  $k$  kids, add  $k - 1$  placeholder candies.  $\binom{n-1+k}{k}$  represents all the ways the dividers can be placed, such that the kids get the candy.

## Counting methods summary

	With order	Without Order
With replacement	$n^k$	$\binom{n-1+k}{k}$
Without Replacement	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

## Set inclusion/exclusion principal

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

## Conditional Probability

$P(A|B)$  represents the probability that  $A$  happens, given that  $B$  *already* happened.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

## Bayes' Theorem

Sometimes we don't have some of those probabilities. In this situation, we can use Bayes' Theorem:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

## Law of total probability

For an  $A, B$  where  $P(B) \neq 0$  and  $P(B) \neq 1$

$$P(A) = P(A|B) \cdot P(B) + P(A|B^C) \cdot P(B^C)$$

## Multiplication Rule for $P(A \cap B)$

$$P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$$

## Independence

Two Events  $A$  and  $B$  are independent if  $P(A|B) = P(A)$ , and are dependent otherwise.

$A$  and  $B$  are independent if and only if:

$$P(A \cap B) = P(A) \cdot P(B)$$

## Multiple levels of independence

Types of independence between  $n$  events:

- Pairwise independence all pairs of events within  $n$  are independent.
- $K$ -wise independence all  $k$  sets of events are independent.
- Mutual independence all  $k$ -wise sets are independent.

## Random Variables

For a given sample space, a **random variable** is any rule that associates a number with each outcome. Customarily denoted by uppercase letters, such as  $X$  and  $Y$ , near the end of our alphabet. Lowercase letters are used to represent some particular value of the corresponding random variable.

Any random variable whose only possible values are 0 and 1 is called a **Bernoulli random variable**. (1 usually indicates success)

A **discrete** random variable's values are either a finite set of integers. They are countable.

A **continuous** random variable can take on any value within a given interval.

## Probability distribution

A **probability distribution** or a **probability mass function** of a discrete random variable is defined for every number  $x$ . For every possible value  $x$  of the random variable, the pmf specifies the probability of observing that value when the experiment is performed.

## Binomial distribution

An experiment where there are 2 outcomes with a given probability, with  $n$  trials. Criteria for a binomial distribution:

1.  $n$  trials
2. 2 possible outcomes per trial
3. trials are independent
4. a single  $p$  that does not change

$$Binom(x, n, p) = \binom{n}{x} p^x (1 - p)^{n-x}$$

## Cumulative Distribution Function

For a probability distribution at a fixed point  $x$ , we wish to compute the probability that the observed value of  $X$  will be at most  $x$ . We're interested in the sum of all the values until, this point.

The **CDF** is defined by:

$$F(X) = P(X \leq x) = \sum_{y: y \leq x} p(y)$$