Discrete II Notes

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Set Theory Review

Sets are collections of unique elements.

Complement

Everything that's not in the set. Denoted by A', A^C, or $\overline{A}$

Intersection

An operation that takes two sets, and returns the common elements between them. Denoted by $A \cap B$

Union

An operation that takes two sets, and returns a all the elements that are in A, B, and $A \cap B$. Denoted by $A \cup B$.

De Morgan's Laws

$$(A \cup B)^C = A^C \cap B^C$$
$$(A \cap B)^C = A^C \cup B^C$$

Disjoint Sets

Disjoint sets or **Mutually Exclusive** sets, are sets that have no elements in common. More formally: $(A \cup B) = \emptyset$

Intro to Probability

Sample Space of an experiment is the set of all possible outcomes of that experiment. An **event** is any collection (subset) of outcomes contained in the sample space S. An event is said to be **simple** if it consists of exactly one outcome and **compound** if it consists of more than one outcome.

Counting

For an ordered pair defined by (x, y) where x can be selected in n_1 ways, and y can be selected in n_2 ways, the number of pairs is $n_1 n_2$. Can be extended to k dimensions. This is known as the **Multiplication rule**.

Permutations

For k selections made **with replacement** on n distinct elements, there are n^k possible outcomes.
Without replacement however, there are n options for the first selection, $n-1$ choices for the next selection, and $n-k+1$ choice(s) for the k^{th} selection. This yeilds.

$$_nP_k = n(n-1)(n-2)\ldots(n-k+1)$$

Combinations

Given n distinct objects, the number of **unordered** subsets of size k is given by $_nC_k$, or $\binom{n}{k}$ (n choose k).

$$_nC_k = \frac{n!}{(n-k)!(k!)}$$

Overcounting with Groups

For n distinct objects being devided into k groups, $\binom{n}{k}$ over counts by a factor of $k!$. To account for that, we do the following:

$$\frac{\binom{n}{k}}{k!}$$

Bose Einstein

For counting ways to separate n indistinguisable objects into k groups. We can use the following:

$$\binom{n-1+k}{k}$$

Analogy: For splitting n peices of candies between k kids, add $k-1$ placeholder candies. $\binom{n-1+k}{k}$ represents all the ways the dividers can be placed, such that the kids get the candy.

Counting methods summary

	With order	Without Order
With replacement	n^k	$\binom{n-1+k}{k}$
Without Replacement	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

Set inclusion/exclusion principal

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap B| - |B \cap C| + |A \cap B \cap C|$$

Conditional Probability

$P(A|B)$ represents the probability that A happens, given that B *already* happened.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Bayes' Theorem

Sometimes we don't have some of those probabilities. In this situation, we can use Bayes' Theorem:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Law of total probability

For an A, B where $P(B) \neq 0$ and $P(B) \neq 1$

$$P(A) = P(A|B) \cdot P(B) + P(A|B^C) \cdot P(B^C)$$

Multiplication Rule for $P(A \cap B)$

$$P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$$

Independence

Two Events A and B are independent if $P(A|B) = P(A)$, and are dependent otherwise.
A and B are independent if and only if:

$$P(A \cap B) = P(A) \cdot P(B)$$

Multiple levels of independence

Types of independence between n events:

- Pairwise independence all pairs of events within n are independent.
- K-wise independence all k sets of events are independent.
- Mutual independence all k-wise sets are independent.

Random Variables

For a given sample space, a **random variable** is any rule that associates a number with each outcome. Customarily denoted by uppercase letters, such as X and Y, near the end of our alphabet. Lowercase letters are used to represent some particular value of the corresponding random variable.
Any random variable whose only possible values are 0 and 1 is called a **Bernoulli random variable**.
A **discrete** random variable's values are either a finite set of integers. They are countable.
A **continuous** random variable can take on any value within a given interval.

Probability distribution

A **probability distribution** or a **probability mass function** of a discrete random variable is defined for every number x. For every possible value x of the random variable, the pmf specifies the probability of observing that value when the experiment is performed.

Binomial distribution

An experiment where there are 2 outcomes with a given probability, with n trials. Criteria for a binomial distribution:

1. n trials
2. 2 possible outcomes per trial
3. trials are independent
4. a single p that does not change

Cumulative Distribution Function

For a probability distribution at a fixed point x, we wish to compute the probability that the observed value of X will be at most x. We're interested in the sum of all the values until, this point.
The **CDF** is defined by:

$$F(X) = P(X \leq x) = \sum_{y:y \leq x} p(y)$$