Linear Algebra Homework 3

Parth Mehrotra

- **Section 2.6: 1, 9**
- **Section 3.1: 14, 21, 23**
- **Section 3.2: 5, 21, 33, 63**

2.6.1

Find the LU-Decomposition of:

$$\begin{pmatrix} 2 & 3 & 4 \\ 6 & 8 & 10 \\ -2 & -4 & -3 \end{pmatrix}$$

First, let's find U, the upper triangular matrix.

$$-3R_1 + R_2 \to R_2$$

$$R_1 + R_3 \to R_3$$

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -1 & -12 \\ 0 & -1 & 1 \end{pmatrix}$$

$$-R_2 + R_3 \to R_3$$

$$U = \begin{pmatrix} 2 & 3 & 4 \\ 0 & -1 & -12 \\ 0 & 0 & 13 \end{pmatrix}$$

Now let's use the inverse of these row operations to find L, the lower triangular matrix.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_2 + R_3 \to R_3$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$-R_1 + R_3 \to R_3$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

$$3R_1 + R_2 \to R_2$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

2.6.9

Solve the following system of linear equations:

$$2x_1 + 3x_2 + 4x_3 = 1$$

$$6x_1 + 8x_2 + 10x_3 = 4$$

$$-2x_1 + -4x_2 + -3x_3 = 0$$

From the last problem, we know that:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 2 & 3 & 4 \\ 0 & -1 & -12 \\ 0 & 0 & 13 \end{pmatrix}$$

$$Ly = b$$

$$y_1 = 1$$

$$3y_1 + y_2 = 4$$

$$3(1) + y_2 = 4$$

$$y_2 = 1$$

$$-x_1 + x_2 + x_3 = 0$$

$$-(1) + (1) + x_3 = 0$$

$$x_3 = 0$$

$$Ux = y$$

$$2x_1 + 3x_2 + 4x_3 = 1$$

$$-x_2 - 12x_3 = 1$$

$$13x_3 = 0$$

$$x_1 = 2$$

$$x_2 = -1$$

$$x_3 = 0$$

3.1.14

$$\begin{vmatrix} 1 & -2 & 2 \\ 2 & -1 & 3 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & 3 \\ 1 & -1 \end{vmatrix} - (-2) \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}$$

$$1(1-3) + 2(-2-0) + 2(2-0)$$

$$= 3$$

3.1.21

$$\begin{vmatrix} 4 & -1 & 2 \\ 0 & 3 & 7 \\ 0 & 0 & 5 \end{vmatrix}$$

The determinant of a triangular matrix is the product of it's determinant.

$$4 * 3 * 5 = 60$$

3.1.23

$$\begin{vmatrix} -6 & 0 & 0 \\ 7 & -3 & 2 \\ 2 & 9 & 4 \end{vmatrix}$$

$$\frac{-1}{2} R_3 + R_2 \to R_2$$

$$\begin{vmatrix} -6 & 0 & 0 \\ 6 & \frac{-15}{2} & 0 \\ 2 & 9 & 4 \end{vmatrix}$$

This row operation does not have any affect on the resulting determinant. Since we have formed a triangular matrix, the resulting determinant is the product of the diagonal.

$$-6 * \frac{-15}{2} * 4 = 180$$

3.2.5

Evaluate the following expression by performing cofactor expansion on the third row.

$$\begin{vmatrix} 1 & 3 & 2 \\ 2 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix}$$

$$3 \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix}$$

$$3(9-4) - 1(3-2) + 1(2-6)$$

$$15 - 1 - 4 = 10$$

3.2.21

Find the determinant by only using elementary row operations.

$$\begin{vmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & -1 & 4 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{vmatrix}$$

$$-2R_1 + R_2 \to R_2$$

$$4R_1 + R_3 \to R_3$$

$$-3R_1 + R_4 \to R_4$$

$$\begin{vmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 1 & 7 & -2 \\ 0 & 1 & 4 & -4 \end{vmatrix}$$

$$-R_2 + R_3 \to R_3$$

$$-R_2 + R_4 \to R_4$$

$$\begin{vmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 12 & -4 \\ 0 & 0 & 9 & -8 \end{vmatrix}$$

$$\frac{1}{4} R_3 \to R_3$$

$$\begin{vmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 9 & -8 \end{vmatrix}$$

$$-3R_3 + R_4 \to R_4$$

$$\begin{vmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & -5 \end{vmatrix}$$

The determinant of a triangular matrix is the product of it's diagonal. In this case, to arrive at the triangular form, we multiplied one of the rows by a scalar, so we have to multiply the determinant by that same scalar.

$$det(A) = 1 * 1 * 3 * -5 * (\frac{1}{4}) = \frac{-15}{4}$$

3.2.33

What value of c will make the following matrix not-invertible?

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & c \\ 0 & c & -15 \end{bmatrix}$$

Can be restated as, "For what value of c is the following matrix's determinant 0?"

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & c \\ 0 & c & -15 \end{vmatrix} = 0$$

$$-2R_1 + R_2 \to R_2$$

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & -1 & 2+c \\ 0 & c & -15 \end{vmatrix} = 0$$

A linearly dependent set of vectors, will have a determinant that is 0. So, we want R_2 and R_3 to be linear combinations of one another.

$$\frac{-1}{2+c} = \frac{c}{-15}$$

$$c(2+c) = 15$$

$$c^2 + 2c - 15 = 0$$

$$(c+5)(c-3) = 0$$

$$c = -5 \ or \ c = 3$$

3.2.63

Solve each system using Cramer's rule.

$$x_1 - 2x_3 = 6$$

$$x_1 + x_2 + 3x_3 = -5$$

$$2x_2 + x_3 = 4$$

$$D = \begin{vmatrix} 1 & 0 & -2 \\ 1 & 2 & 3 \\ 0 & 2 & 1 \end{vmatrix}$$

$$-R_1 + R_2 \to R_2$$

$$\begin{vmatrix} 1 & 0 & -2 \\ 0 & 2 & 5 \\ 0 & 2 & 1 \end{vmatrix}$$

$$-R_2 + R_3 \to R_3$$

$$\begin{vmatrix} 1 & 0 & -2 \\ 0 & 2 & 5 \\ 0 & 0 & -4 \end{vmatrix}$$

$$D = 1 * 2 * -4 = -8$$

$$D_{x_1} = \begin{vmatrix} 6 & 0 & -2 \\ -5 & 2 & 3 \\ 4 & 2 & 1 \end{vmatrix}$$

$$= 6\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} -5 & 2 \\ 4 & 2 \end{vmatrix}$$

$$= 6(2-6) - 2(10-8) = -24 - 4$$

$$D_{x_1} = -28$$

$$D_{x_2} = \begin{vmatrix} 1 & 6 & -2 \\ 1 & -5 & 3 \\ 0 & 4 & -4 \end{vmatrix}$$

$$= 1\begin{vmatrix} -5 & 3 \\ 4 & -4 \end{vmatrix} - 6\begin{vmatrix} 1 & 3 \\ 0 & -4 \end{vmatrix} - 2\begin{vmatrix} 1 & -5 \\ 0 & 4 \end{vmatrix}$$

$$D_{x_2} = 1(20-12) - 6(-4) - 2(4) = 2 + 24 - 8 = 18$$

$$D_{x_3} = \begin{vmatrix} 1 & 0 & 6 \\ 1 & 2 & -5 \\ 0 & 2 & 4 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & -5 \\ 2 & 4 \end{vmatrix} 6 \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$$

$$D_{x_3} = 1(8+10) + 6(1) = 18 + 6 = 24$$

$$x_1 = \frac{D_{x_1}}{D} = \frac{-28}{-8} = \frac{7}{2}$$

$$x_2 = \frac{D_{x_2}}{D} = \frac{18}{-8} = \frac{-9}{4}$$

$$x_3 = \frac{D_{x_3}}{D} = \frac{24}{-8} = -3$$