Formal Language and Automata Theory (CS21004)

Soumyajit Dey
CSE, IIT Kharagpur

Announcements

- The slide is just a short summary
- Follow the discussion and the boardwork
- Solve problems (apart from those we dish out in class)

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$$
\begin{array}{rcl}
\langle \text{stmt} \rangle & \models & \langle \text{if-stmt} \rangle \mid \langle \text{while-stmt} \rangle \mid \langle \text{begin-stmt} \rangle \\
\langle \ldots \rangle & \models & \mid \langle \text{assg-stmt} \rangle \\
\langle \text{if-stmt} \rangle & \models & \textit{if} \ \langle \text{bool-expr} \rangle \ \textit{then} \ \langle \text{stmt} \rangle \ \textit{else} \ \langle \text{stmt} \rangle \\
\langle \text{while-stmt} \rangle & \models & \textit{while} \ \langle \text{bool-expr} \rangle \ \textit{do} \ \langle \text{stmt} \rangle \\
\langle \text{begin-stmt} \rangle & \models & \textit{begin} \ \langle \text{stmt-list} \rangle \ \textit{end} \\
\langle \text{stmt-list} \rangle & \models & \langle \text{stmt} \rangle \mid \langle \text{stmt} \rangle \ ; \langle \text{stmt-list} \rangle \\
\langle \text{assg-stmt} \rangle & \models & \langle \text{var} \rangle \\
\langle \text{var} \rangle & \models & \langle \text{arith-expr} \rangle \\
\langle \text{bool-expr} \rangle & \models & \langle \text{arith-expr} \rangle \langle \text{compare-op} \rangle \langle \text{arith-expr} \rangle \\
\langle \text{compare-op} \rangle & \models & < \mid > \mid \leq \mid \geq = \mid \neq \\
\langle \text{arith-expr} \rangle & \models & \langle \text{var} \rangle \mid \langle \text{const} \rangle \mid \langle \text{arith-expr} \rangle \langle \text{arith-op} \rangle \langle \text{arith-expr} \rangle \\
\langle \text{arith-op} \rangle & \models & + \mid - \mid * \mid / \\
\langle \text{const} \rangle & \models & 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9 \\
\langle \text{const} \rangle & \models & a \mid b \mid c \mid \cdots \mid x \mid y \mid z
\end{array}
$$

For the string
while $x \leq y$ do begin $x = x + 1; y = y - 1$ end
The first few sentential forms in its derivation are
$\langle \text{stmt} \rangle$
$\langle \text{while-stmt} \rangle$
while $\langle \text{bool-stmt} \rangle$ do $\langle \text{stmt} \rangle$
while $\langle \text{arith-expr} \rangle \langle \text{compare-op} \rangle \langle \text{arith-expr} \rangle$ do $\langle \text{stmt} \rangle$
while $\langle \text{var} \rangle \langle \text{compare-op} \rangle \langle \text{arith-expr} \rangle$ do $\langle \text{stmt} \rangle$
while $\langle \text{var} \rangle \leq \langle \text{arith-expr} \rangle$ do $\langle \text{stmt} \rangle$
while $\langle \text{var} \rangle \leq \langle \text{var} \rangle$ do $\langle \text{stmt} \rangle$
while $\langle x \rangle \leq \langle \text{var} \rangle$ do $\langle \text{stmt} \rangle$
while $\langle x \rangle \leq \langle y \rangle$ do $\langle \text{stmt} \rangle$
while $\langle x \rangle \leq \langle y \rangle$ do $\langle \text{begin-stmt} \rangle$

Formally, a *context-free grammar* (CFG) is a quadruple
$G = (N, \Sigma, P, S)$
where

- N is a finite set (the *non-terminal symbols*),
- Σ is a finite set (the *terminal symbols*) disjoint from N,
- P is a finite subset of $N \times (N \cup \Sigma)^*$ (the *productions*), and
- $S \in N$ (the *start symbol*)

- $\{a^n b^n \mid n \geq 0\}$ is a CFL with CFG $S \rightarrow aSb \mid \epsilon$
- Palindromes over $\{a, b\}$ is a CFL with CFG $S \rightarrow aSa \mid bSb \mid a \mid b \mid \epsilon$
- Balanced parentheses over $\{(,)\}$ is a CFL with CFG $S \rightarrow (S) \mid SS \mid \epsilon$

How do you prove that each grammar generates the corresponding language ??

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It helps to restrict formal grammars to 'standard forms'. You can write algorithms that work on grammars assuming this standard

- A CFG is in *Chomsky Normal Form* (CNF) if all productions are of the form
  1. $A \to BC$ or
  2. $A \to a$

  where $A, B, C \in N$ and $a \in \Sigma$. Note that CNF form grammars cannot generate ϵ
- For any CFG G, there is a CFG G' in Chomsky Normal Form such that

$$L(G') = L(G) - \{\epsilon\}$$

Create grammar for ϵ free language

- For any CFG $G = (N, \Sigma, P, S)$, there is a CFG G' with no ϵ or unit productions $(A \rightarrow B)$ such that

$$L(G') = L(G) - \{\epsilon\}$$

.

- Proof: Let $\hat{P}$ be the smallest set of productions containing P and closed under the rules
  1. if $A \rightarrow \alpha B \beta$ and $B \rightarrow \epsilon$ are in $\hat{P}$; then $A \rightarrow \alpha\beta$ is in $\hat{P}$
  2. if $A \rightarrow B$ and $B \rightarrow \gamma$ are in $\hat{P}$, then $A \rightarrow \gamma$ is in $\hat{P}$
- Point to note : $\alpha, \beta, \gamma \in (N \cup \Sigma)^*$

Create grammar for ϵ free language

Note that for the language $\hat{G} = (N, \Sigma, \hat{P}, S)$, $L(G) = L(\hat{G})$

- $L(G) \subseteq L(\hat{G})$ since $P \subseteq \hat{P}$
- $L(G) = L(\hat{G})$: $\hat{P}$ only contains those rules as extra over P which are simulatable in two steps by existing rules of P

Create grammar for ϵ free language

Can prove that, for any $x \neq \epsilon$, $\exists$ a derivation $S \underset{\hat{G}}{\Rightarrow} x$ without ϵ and *unit* productions

- Remove ϵ and *unit* productions from $\hat{G}$

Create $G' = (N, \Sigma, P', S)$ where P' is same as $\hat{P}$ but w/o ϵ and *unit* productions

G' to CNF

- For each terminal $a \in \Sigma$ introduce a new nonterminal A_a and production $A_a \to a$ and replace all occurrences of a on the right hand side of old productions (except productions of the form $B \to a$) with A_a
- Now all productions are of the form
  1. $A \to a$ or
  2. $A \to B_1 B_2 \cdots B_k, k \geq 2$

  where the B_i are nonterminals.

G' to CNF

- For any production

$$A \rightarrow B_1 B_2 \cdots B_k$$

with $k \geq 3$, introduce a new nonterminal C and replace this production with the two productions

1. $A \rightarrow B_1 C$ and
2. $C \rightarrow B_2 B_3 \cdots B_k$

until all right hand sides are of length at most 2.

- Derive a CNF grammar for

$$\{a^n b^n | n \geq 0\} - \{\epsilon\} = \{a^n b^n | n \geq 1\}$$

- Starting with the grammar

$$S \rightarrow aSb|\epsilon$$

1. Remove ϵ productions to get $S \rightarrow aSB|ab$
2. Add non terminals A,B and replace the productions with $S \rightarrow ASB|AB, A \rightarrow a, B \rightarrow b$.
3. Add a nonterminal C and replace $S \rightarrow ASB$ with $S \rightarrow AC$ and $C \rightarrow SB$.

- The final grammar in CNF is $S \rightarrow AB|AC, C \rightarrow SB, A \rightarrow a, B \rightarrow b$.

- Derive a CNF grammar for the set of non-null strings of balanced parenthesis [ ]
- Starting with the grammar

$$S \rightarrow [S] | SS | \epsilon$$

1. Remove ϵ productions to get $S \rightarrow [S] | SS | [\,]$
2. Add non terminals A,B and replace the productions with $S \rightarrow SS | ASB | AB, A \rightarrow [, B \rightarrow]$.
3. Add a nonterminal C and replace $S \rightarrow ASB$ with $S \rightarrow AC$ and $C \rightarrow SB$.
4. The final grammar in CNF is $S \rightarrow SS | AB | AC, C \rightarrow SB, A \rightarrow [, B \rightarrow]$.

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- Consider the grammar $G = (\{S, A, B, C\}, \{a, b, c\}, S, P)$ where $P = \{S \rightarrow ABC, A \rightarrow aA, A \rightarrow \epsilon, B \rightarrow bB, B \rightarrow \epsilon, C \rightarrow cC, C \rightarrow \epsilon\}$.
- With this grammar, there is a choice of variables to expand. If we always expanded the leftmost variable first, we would have a *leftmost derivation*: $S \rightarrow ABC \rightarrow aABC \rightarrow aBC \rightarrow abBC \rightarrow abbBC \rightarrow abbC \rightarrow abbcC \rightarrow abbc$
- Conversely, if we always expanded the rightmost variable first, we would have a *rightmost derivation*: $S \rightarrow ABC \rightarrow ABcC \rightarrow ABc \rightarrow AbBc \rightarrow AbbBc \rightarrow Abbc \rightarrow aAbbc \rightarrow abbc$

- A grammar G is ambiguous if $\exists w \in L(G)$ for which
  - there are two or more distinct derivation/parse trees, or
  - there are two or more distinct leftmost derivations, or
  - there are two or more distinct rightmost derivations.
- Ambiguity is a property of a grammar, and it is usually (but not always) possible to find an equivalent unambiguous grammar.
- An *inherently ambiguous language* is a language for which no unambiguous grammar exists.

- Consider $S \rightarrow AS|\epsilon, A \rightarrow A1|0A1|01$. The string 00111 has the following two leftmost derivations from S:
  1. $S \rightarrow AS \rightarrow 0A1S \rightarrow 0A11S \rightarrow 00111S \rightarrow 00111$
  2. $S \rightarrow AS \rightarrow A1S \rightarrow 0A11S \rightarrow 00111S \rightarrow 00111$
- Intuitively, we can use $A \rightarrow A1$ first or second to generate the extra 1. The language of our example grammar is not inherently ambiguous, even though the grammar is ambiguous.
- Change the grammar to force the extra 1's to be generated last. $S \rightarrow AS|\epsilon, A \rightarrow 0A1|B, B \rightarrow B1|01$
- $\{a^i b^j c^k | i = j \text{ or } j = k\}$ is an inherently ambiguous language

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For every CFL A, $\exists k \geq 0$ such that $\forall z \in A$ with $|z| \geq k$ $\exists u, v, w, x, y$ such that

- $z = uvwxy$
- $vx \neq \epsilon$
- $|vwx| \leq k$
- $\forall i \geq 0,\ uv^iwx^iy \in A$

Intuitive (informal) idea :

- The set of a regular language grows in *one direction* per production : standard form is strictly right/left linear
- A Context Free language grows in *two directions* per production : Consider as standard form the CNF representation
- For sufficiently long strings, non terminals will repeat
- The subtree under the higher nonterminal instance can be copied for the lower instance
- This creates two separate possibilities of pumping

CNF grammar for $\{a^n b^n | n \geq 1\}$:
$S \rightarrow AB|AC, C \rightarrow SB, A \rightarrow a, B \rightarrow b.$

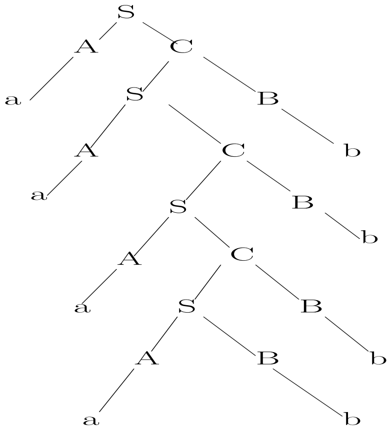


Figure: derivation tree for $a^4 b^4$:

Map the tree to uniform depth in all branches

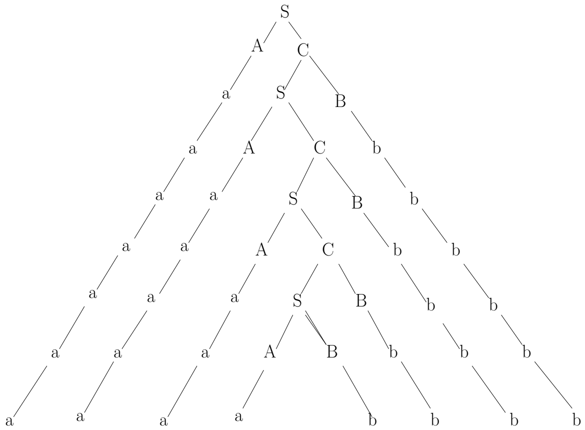
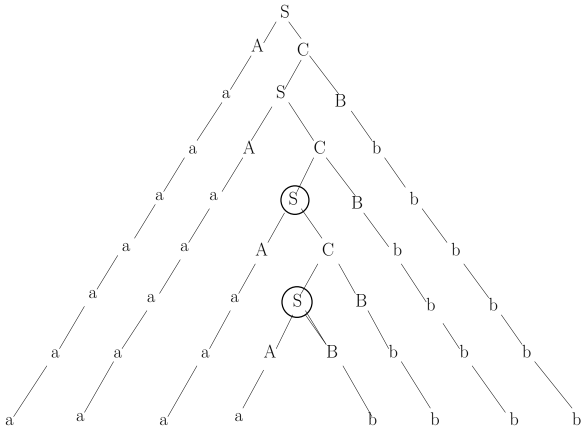
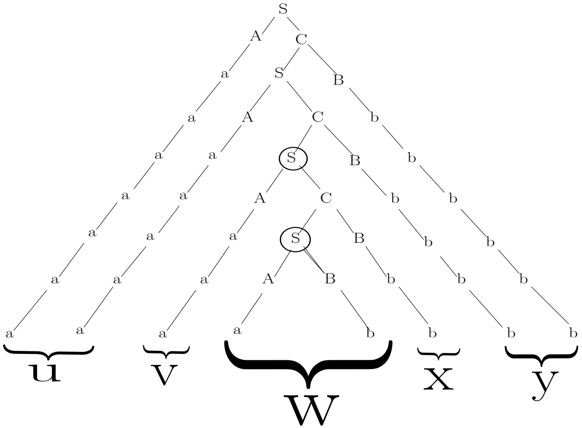


Figure: derivation tree for a^4b^4 :

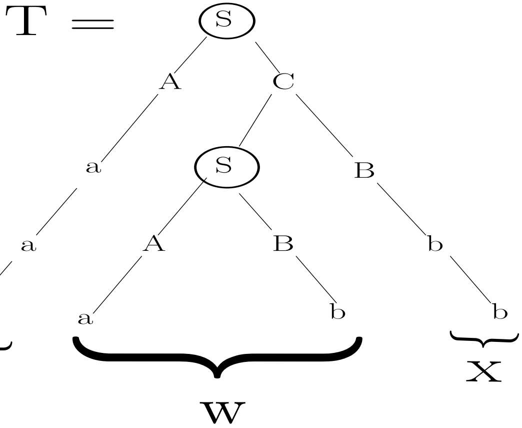
Consider a path with a repeating non-terminal (S here)



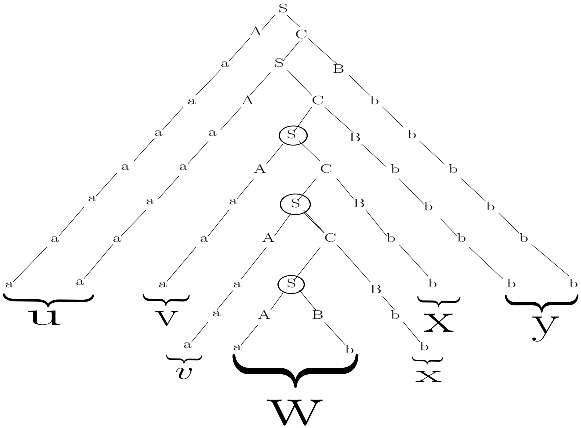
Note the decomposition of the string



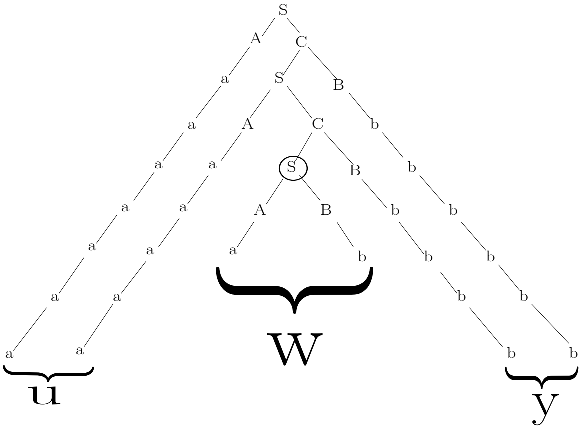
Check the subtree with two occurrences of S



Pumping up



Pumping down



Proof

- Let G be a grammar for A in CNF.
- $k = 2^{n+1}$, where n is the number of nonterminals of G
- Suppose $z \in A$ and $|z| \geq k$
- Any parse tree in G for z must be of depth $n + 1$
- The longest path in the tree is of length at least $n + 1$
- It must contain at least $n + 1$ occurrences of nonterminals.
- By pigeonhole principle, some nonterminal must occur more than once

- Say X is the nonterminal that appears more than once
- Break z up into substrings $uvwxy$ such that
-
  1. w is the string generated by the lower occurrence of X
  2. vwx is the string generated by the upper occurrence
- Let T and t be the subtree rooted at the upper and lower ocurrence of X respectively
- Replacing t with T once we get a valid subtree of uv^2wx^2y
- Repeat it to produce $uv^iwx^iy, i \geq 1$
- Replace T with t to get uv^0wx^0y

- Note that $vx \neq \epsilon$; otherwise $T = t$
- Also $|vwx| \leq k$ as we chose the first repeated occurrence of a nonterminal from the bottom.
- In the longest path, the depth of the subtree under the upper occurrence of X is at most $n + 1$
- So it cannot have more than $2^{n+1} = k$ terminals.

$A = \{a^n b^n a^n | n \geq 0\}$ is not context free

- Adversary picks k in step 1
- You pick $z = a^k b^k a^k$ such that $z \in A$ and $|z| = 3k \geq k$
- Adversary picks $z = uvwxy$ such that $vx \neq \epsilon$, and $|vwx| \leq k$
- You pick $i = 2$ and in all cases you can ensure $uv^2wx^2y \notin A$
  1. either v or x contain at least one a and at least one b
  2. v and x contains only a's or only b's
  3. one of v or x contains only a's and the other contains only b's

$A = \{ww | w \in \{a, b\}^*\}$ is not context free

- As CFLs are closed under intersection with regular sets, it suffices to show that

$$A' = A \cap L(a^* b^* a^* b^*) = \{a^n b^m a^n b^m | m, n \geq 0\}$$

  is not context-free
- Adversary picks k in step 1
- You pick $z = a^k b^k a^k b^k$ such that $z \in A'$ and $|z| \geq k$.
- call each of the four substrings of the form a^k or b^k as blocks
- Adversary picks $z = uvwxy$ such that $vx \neq \epsilon$, and $|vwx| \leq k$
- You pick $i = 2$ and in all cases you can win

- If one of v or x contains both a's and b's then $uv^2wx^2y \notin A'$
- If v and x are from the same block, then uv^2wx^2y has one block longer than the other three
- If v and x are on different blocks, then the blocks must be adjacent; otherwise $|vwx|$ would be greater than k. Thus one of the blocks containing v or x must be a block of a's and the other a block of b's. Then uv^2wx^2y either has two blocks of a's of different size (if vx contains an a) or two blocks of b's of different size (if vx contains a b) or both. Either way $uv^2wx^2y \notin A'$.

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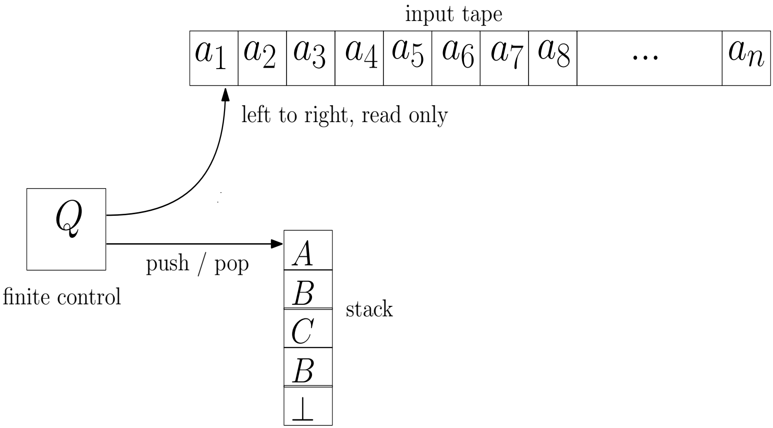

Figure: Non deterministic Pushdown Automata

A non deterministic PDA is a 7 tuple
$M = (Q, \Sigma, \Gamma, \delta, s, \perp, F)$,

- Q is a finite set (the states),
- Σ is a finite set (the input alphabet),
- Γ is a finite set (the stack alphabets),
- $s \in Q$ (the start state),
- $\perp \in \Gamma$ (the initial stack symbol), and
- $F \subseteq Q$(the final or accept states),
- $\delta \subseteq (Q \times (\Sigma \cup \varepsilon) \times \Gamma) \times (Q \times \Gamma^*), \delta$: NPDA has ϵ transitions (can move w/o input)

$((p, a, A), (q, B_1 \cdots B_k)) \in \delta$: from a state p while reading some input symbol a with some stack top element A, the PDA moves to another state q, pops A, pushes $B_1 \cdots B_k$ (B_k first)

- $((p, a, A), (q, B_1 \cdots B_k)) \in \delta$: from a state p while reading some input symbol a with some stack top element A, the PDA moves to another state q, pops A, pushes $B_1 \cdots B_k$ (B_k first)
- $((p, \epsilon, A), (q, B_1 \cdots B_k)) \in \delta$: from a state p with some stack top element A, the PDA moves to another state q, pops A, pushes $B_1 \cdots B_k$ (B_k first) w/o reading any input symbol

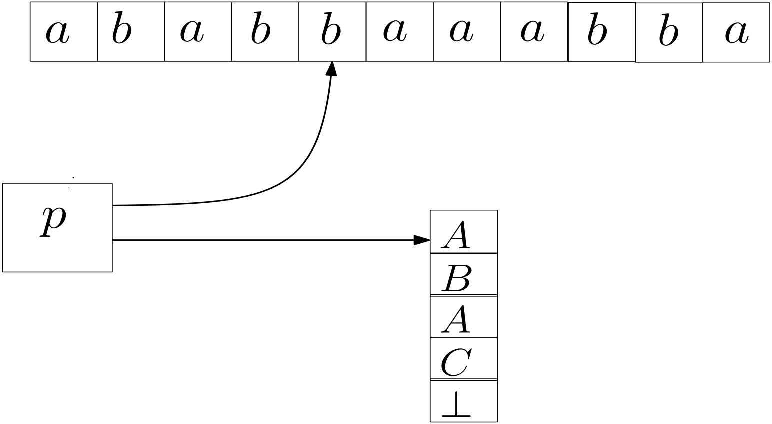

Figure: Example of Non deterministic Pushdown Automata

Config : $(p, baaabba, ABAC\bot)$: (state, unread tape, stack content)

- The start configuration on input x is $(s, x, \perp)$. That is, the machine always starts in its start state s with its read head pointing to the leftmost input symbol and the stack containing only the symbol $\perp$.
- The next configuration relation $\xrightarrow[M]{1}$ describes how the machine can move from one configuration to another in one step.
- If $((p, a, A), (q, \gamma)) \in \delta$, then for any $y \in \Sigma^*$ and $\beta \in \Gamma^*$, $(p, ay, A\beta) \xrightarrow[M]{1} (q, y, \gamma\beta)$;
- and if $((p, \varepsilon, A), (q, \gamma)) \in \delta$,then for any $y \in \Sigma^*$ and $\beta \in \Gamma^*$, $(p, y, A\beta) \xrightarrow[M]{1} (q, y, \gamma\beta)$

Acceptance

- by **final state** : $(s, x, \perp) \xrightarrow[M]{*} (q, \epsilon, \gamma)$: get to some $q \in F$ when string exhausted
- by **empty stack** : $(s, x, \perp) \xrightarrow[M]{*} (q, \epsilon, \epsilon)$: pop stack bottom element when string exhausted

Both kinds of M/Cs can be converted to an equivalent NPDA M that has a single final state t and M can empty its stack after it enters state t.

PDA for balanced parenthesis

- $Q = \{q\}$,
- $\Sigma = \{[,]\}$,
- $\Gamma = \{\perp, [\}$,
- start state $= q$
- initial stack symbol $= \perp$,
- and let δ consist of the following transitions:
  1. $((q, [, \perp), (q, [\perp))$;
  2. ((q,[,[), (q,[[));
  3. $((q,], [), (q, \varepsilon))$;
  4. $((q, \varepsilon, \perp), (q, \varepsilon))$.

	Configuration		Transition
	(q,	[[[ ]] [ ]] [ ], $\perp$)	start configuration
$\rightarrow$	(q,	[[ ]] [ ]] [ ], [$\perp$)	transition (1)
$\rightarrow$	(q,	[ ]] [ ]] [ ], [[$\perp$)	transition (2)
$\rightarrow$	(q,	]] [ ]] [ ], [[[$\perp$)	transition (2)
$\rightarrow$	(q,	] [ ]] [ ], [[$\perp$)	transition (3)
$\rightarrow$	(q,	[ ]] [ ], [$\perp$)	transition (3)
$\rightarrow$	(q,	]] [ ], [[$\perp$)	transition (2)
$\rightarrow$	(q,	] [ ], [$\perp$)	transition (3)
$\rightarrow$	(q,	[ ], $\perp$)	transition (3)
$\rightarrow$	(q,	], [$\perp$)	transition (1)
$\rightarrow$	(q,	ϵ, $\perp$)	transition (2)
$\rightarrow$	(q,	ϵ, ϵ)	transition (4)

Design an NPDA for $\{a, b\}^* - \{ww \mid w \in \{a, b\}^*\}$

CFG to NPDA

All productions of $G = (\Sigma, N, P, S)$ are of the form: $A \rightarrow cB_1B_2.....B_k$, Where $c \in \Sigma \cup \{\epsilon\}$ and $k \geq 0$. An equivalent NPDA M with only one state that accepts by empty stack is $M = (\{q\}, \Sigma, N, \delta, q, S, \emptyset)$ where

- q is the only state.
- Σ is the input alphabet of M
- N is the set of stack alphabet of M
- δ is the transition relation
- q is the start state
- S, is the start symbol of G, is the initial stack symbol of M
- $\emptyset$, the null set, is the set of final states

Conversion Rule

- For each production $A \rightarrow cB_1B_2.....B_k$ in P
- The transition relation δ is defined as $((q, c, A), (q, B_1B_2....B_k))$
  1. δ has one transition of each production of G
  2. When in state q scanning input symbol c with A on top of the stack, pop A off the stack, push $B_1B_2.....B_k$ onto the stack (B_k first) and enter in q state
  3. For $c = \epsilon$, when in state q with A on top of the stack, without scanning an input symbol, pop A off the stack, push $B_1B_2.....B_k$ onto the stack (B_k first) and enter in q state

Example Conversion

Consider the set of nonnull balanced strings of parentheses [ ]. The list of production rules of the grammar and corresponding transition of NPDA for this set are as follows:

1. $S \rightarrow [BS$ $((q, [, S), (q, BS))$
2. $S \rightarrow [B$ $((q, [, S), (q, B))$
3. $S \rightarrow [SB$ $((q, [, S), (q, SB))$
4. $S \rightarrow [SBS$ $((q, [, S), (q, SBS))$
5. $B \rightarrow]$ $((q,], B), (q, \epsilon))$

Example Derivation for the i/p 'x = [[[ ]] [ ]]'

Rule applied	*Sentential forms in a leftmost* derivation of x in G	*Configuration of M in an accepting computation of M on input x*
	S	(q, [[[ ]] [ ]], S)
(3)	[SB	(q, [[ ]] [ ]], SB)
(4)	[[SBSB	(q, [ ]] [ ]], SBSB)
(2)	[[[BBSB	(q,]] [ ]], BBSB)
(5)	[[[ ]BSB	(q,] [ ]], BSB)
(5)	[[[ ]]SB	(q, [ ]], SB)
(2)	[[[ ]] [BB	(q,]], BB)
(5)	[[[ ]] [ ]B	(q,], B)
(5)	[[[ ]] [ ]]	(q, ϵ, ϵ)

Lemma: For any $z, y \in \Sigma^*, \gamma \in N^*$, and $A \in N, A \xrightarrow[G]{n} z\gamma$ via a leftmost derivation iff $(q, zy, A) \xrightarrow[M]{n} (q, y, \gamma)$

- For example, in the fourth row of the table above, we would have z = [[[, y =]] [ ]], γ = BBSB, A = S and n=3
- *Proof:* Try Yourself

CFG G converted to NPDA M

Language Equivalence : $L(M) = L(G)$
Proof:

$$x \in L(G)$$

$\Leftrightarrow S \xrightarrow[G]{*} x$ by a leftmost derivation. [definition of L(G)]

$\Leftrightarrow (q, x, S) \xrightarrow[M]{*} (q, \epsilon, \epsilon)$ using the last Lemma

$\Leftrightarrow x \in L(M)$ definition of L(M)

NPDA to CFG

- Every NPDA can be simulated by an NPDA with one state.
- Every NPDA with one state has an equivalent CFG

Conclusion : With NPDA $\rightarrow$ CFG and CFG $\rightarrow$ NPDA we have established that 'NPDAs and CFGs are equivalent in expressive power'

Single state NPDA to CFG

- Consider a NPDA M with one state that accepts by empty stack
  $M = (\{q\}, \Sigma, \Gamma, \delta, q, \perp, \emptyset)$
- The equivalent CFG be $G = (\Gamma, \Sigma, P, \perp)$, where P contains a production $A \rightarrow cB_1B_2...B_k$ for every transition $((q, c, A), (q, B_1B_2...B_k)) \in \delta$

(the conversion we discussed is invertible).

Arbitrary NPDA to single state NPDA

Consider a NPDA $M = (Q, \Sigma, \Gamma, \delta, s, \perp, \{t\})$

- M has a single final state t and M can empty its stack after it enters state t.
- Let $\Gamma' \stackrel{def}{=} Q \times \Gamma \times Q$. Elements of Γ' are written $\langle pAq \rangle$, where $p, q \in Q$ and $A \in \Gamma$

Equivalent NPDA $M' = (\{*\}, \Sigma, \Gamma', \delta', *, \langle s \perp t \rangle, \emptyset)$

- M' has one state * and accepts by empty stack.
- $(*, x, pAq) \xrightarrow[M']{*} (*, \epsilon, \epsilon)$ iff $(p, x, A) \xrightarrow[M]{*} (q, \epsilon, \epsilon)$

For each transition $((p, c, A), (q_0, B_1 B_2 ... B_k)) \in \delta$, where $c \in \Sigma \cup \{\epsilon\}$

- $((*, c, \langle pAq_k \rangle), (*, \langle q_0 B_1 q_1 \rangle \langle q_1 B_2 q_2 \rangle .... \langle q_{k-1} B_k q_k \rangle)) \in \delta'$. For all possible choice of $q_1, q_2, ..., q_k$.
- For k=0, $((*, c, \langle pAq_0 \rangle), (*, \epsilon)) \in \delta'$ iff $((p, c, A), (q_0, \epsilon)) \in \delta$

M' nondeterministically guesses a computation sequence, saves the guessed sequence in stack
verifies later in case of an actual run.

NPDA to CFG - another option

Consider a PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, \perp, \{q_F\})$ with the restriction that every transition either pushes a symbol or pops a symbol from the stack, i.e. $\delta(q, a, X)$ contains either $(q', \gamma X)$, $\gamma \in \Gamma$ or (q', ϵ). Consider the equivalent grammar $G_P = (V, T, P, S)$ such that $V = \{A_{p,q} : p, q \in Q\}$, $T = \Sigma$, $S = A_{q_0,q_F}$ and P has transitions of the following form:

- $\forall q \in Q \quad (A_{q,q} \rightarrow \epsilon) \in P$
- $\forall p, q, r \in Q \quad (A_{p,q} \rightarrow A_{p,r} A_{r,q}) \in P$
- $(A_{p,q} \rightarrow a A_{r,s} b) \in P$ if $\delta(p, a, \epsilon)$ contains (r, X) and $\delta(s, b, X)$ contains (q, ϵ)

NPDA to CFG

Theorem: *If $A_{p,q} \stackrel{*}{\Rightarrow} x$ then x can bring the PDA P from state p on empty stack to state q on empty stack.*

Proof:
We can prove this theorem by induction on the number of steps in the derivation of x from $A_{p,q}$.

NPDA to CFG

<u>*Base case*</u>: If $A_{p,q} \stackrel{*}{\Rightarrow} x$ in one step, then the only rule to generate a terminal string in one step is $A_{p,p} \rightarrow \epsilon$

<u>*Inductive step:*</u> If $A_{p,q} \stackrel{*}{\Rightarrow} x$ in n + 1 steps. The first step in the derivation must be $A_{p,q} \rightarrow A_{p,r}A_{r,q}$ or $A_{p,q} \rightarrow aA_{r,s}b$.

- If it is $A_{p,q} \rightarrow A_{p,r}A_{r,q}$, then the string x can be broken into two parts x_1x_2 such that $A_{p,r} \stackrel{*}{\Rightarrow} x_1$ and $A_{r,q} \stackrel{*}{\Rightarrow} x_2$ in at most n steps. The theorem easily follows in this case.
- If it is $A_{p,q} \rightarrow aA_{r,s}b$, then the string x can be broken as ayb such that $A_{r,s} \stackrel{*}{\Rightarrow} y$ in n steps. Notice that from state p, on reading a the PDA pushes a symbol X to stack, while it pops X in state s on reading b and goes to state q.

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The following grammar generates the well-parenthesized propositional expressions

$$
\begin{aligned}
E &\implies (EBE)|(UE)|C|V, \\
B &\implies \vee|\wedge|\rightarrow|\leftrightarrow, \\
U &\implies \neg, \\
C &\implies \top|\bot, \\
V &\implies P|Q|R|\cdots .
\end{aligned}
$$

The words well-parenthesized means that there must be parentheses around any compound expression. E.g.

$$(((P \vee Q) \wedge R) \vee (Q \wedge (\neg P)))$$

A parser

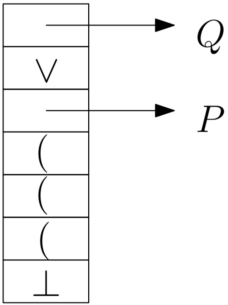
Start with the initial stack symbol $\perp$ and scan the expression from left to right.

1. If the symbol is a (, push it in the stack.
2. If the symbol is an operator, push it in the stack.
3. If the symbol is a constant, push it in the stack.
4. If the symbol is a variable, push it in the stack.
5. If the symbol is a), do a reduce
    1. Create a new node.
    2. Pop the top. It should be a constant, variable or another node.
    3. Pop the top. It should be an operand.
    4. Pop the top. It should be a constant, variable or another node.
    5. Pop the top. It should be a (.
    6. Push the node in the stack.

Parsing Example

$$(((P \vee Q) \wedge R) \vee (Q \wedge (\neg P)))$$

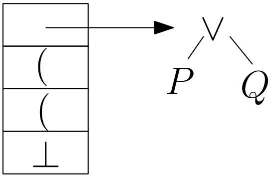
- Push the 3 ('s in the stack.(i)
- Push P in the stack.(iv)
- Push $\vee$ in the stack.(ii)
- Push Q in the stack.(iv)



Parsing Example contd.

$$(((P \vee Q) \wedge R) \vee (Q \wedge (\neg P)))$$

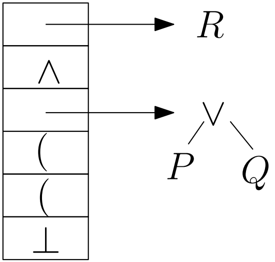
- Scan) and reduce.
- Pop the stack until (and create a new node.



Parsing Example contd.

$$(((P \lor Q) \land R) \lor (Q \land (\neg P)))$$

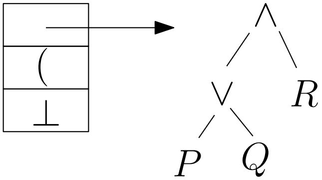
- Push $\land$ and R in the stack



Parsing Example contd.

$$(((P \vee Q) \wedge R) \vee (Q \wedge (\neg P)))$$

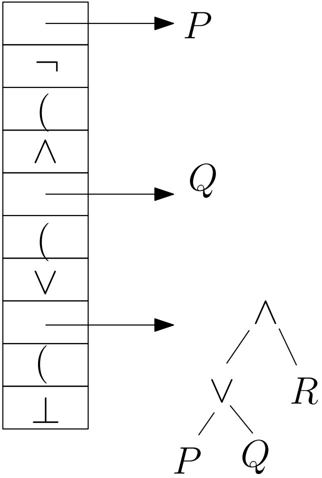
- Scan) and reduce.



Parsing Example contd.

$$(((P \vee Q) \wedge R) \vee (Q \wedge (\neg P)))$$

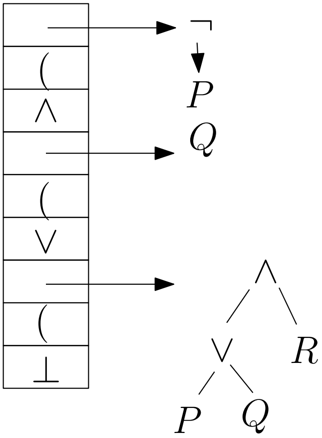
- Push everything until the next).



Parsing Example contd.

$$(((P \vee Q) \wedge R) \vee (Q \wedge (\neg P)))$$

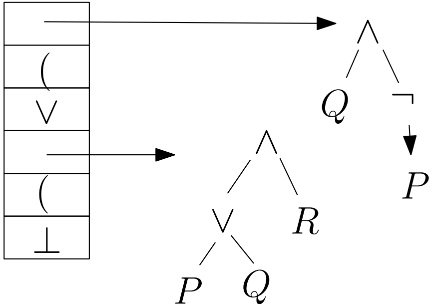
- Scan the first of the final 3) and reduce.



Parsing Example contd.

$$(((P \vee Q) \wedge R) \vee (Q \wedge (\neg P)))$$

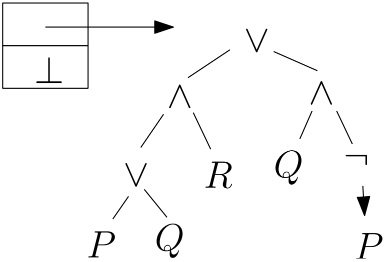
- Scan the next) and reduce.



Parsing Example contd.

$$(((P \vee Q) \wedge R) \vee (Q \wedge (\neg P)))$$

- Scan the final 3) and reduce.



Operator Precedence

Consider the grammar for arithmetic expressions.

$$E \to E + E|E - E|E \cdot E|E/E| - E|C|V|(E),$$
$$C \to 0|1,$$
$$V \to a|b|c.$$

This grammar is ambiguous. The equivalent unambiguous grammar is

$$E \to E + F|E - F|F,$$
$$F \to F \cdot G|F/G|G,$$
$$G \to -G|H,$$
$$H \to C|V|(E),$$
$$C \to 0|1,$$
$$V \to a|b|c.$$

Parsing Example

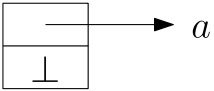
Given a precedence relation,on the operators, we modify the parsing rules as follows. When we scan a binary operator B

1. Check if top of the stack is an operand
2. Look at the stack symbol A immediately below
    - If A has lower precedence than B push B
    - Otherwise, reduce

Consider the following example

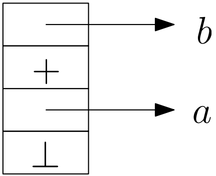
$$a + b.c + d$$

- Start with the stack containing $\perp$
- Scan and push a in the stack



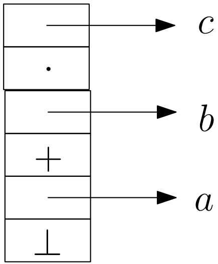
Parsing Example contd.

- Scan $+$. Check that $\perp$ has lower precedence so push $+$.
- Scan and push b



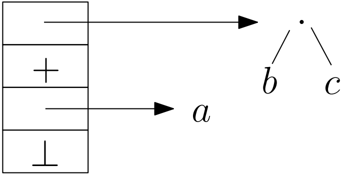
Parsing Example contd.

- Scan $\cdot$. Check that $+$ has lower precedence so push $\cdot$.
- Scan and push c

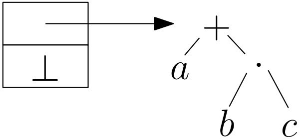


Parsing Example contd.

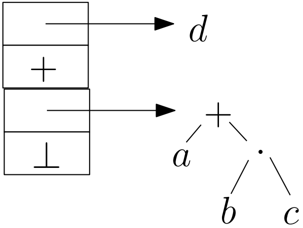
- Scan the second $+$. Check that $\cdot$ has higher precedence so reduce.



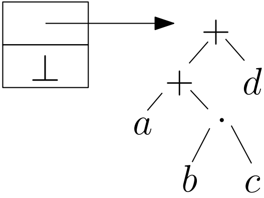
- Check that $+$ has equal precedence so reduce.



Parsing Example contd.



- Check that $\perp$ has lower precedence so push $+$.
- Scan and push d



- End of expression so reduce.

Parsing Example contd.

Why reduce in case of equal precedence ?

- Eventually you have to reduce everything
- No point keeping things in stack when it can be reduced
- Unnecessary Push n Pop operations in that case

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Closure Properties of CFL

Context-free languages are closed under the following operations:

1. Union
2. Concatenation
3. Kleene closure
4. Homomorphism
5. Substitution
6. Inverse-homomorphism
7. Reverse

Closure Properties of CFLs

- CFL are closed under intersection with regular sets.
- CFL are not closed under intersection.

Closure Properties of CFL

Context-free languages are **not closed** *under intersection and complementation.*

Proof:

Consider the languages

$L_1 = \{0^n 1^n 2^m : n, m \geq 0\}$ and

$L_2 = \{0^m 1^n 2^n : n, m \geq 0\}$

Both languages are CFLs.

What is $L_1 \bigcap L_2$?

Closure Properties of CFL

Proof: contd...

$$L = L_1 \bigcap L_2 = \{0^n 1^n 2^n : n \geq 0\} \text{ and it is not a CFL.}$$

Hence CFLs are not closed under **intersection**.

Use Demorgans law to prove non-closure under **complementation**.

Linear Grammars n languages

A linear grammar is a CFG that has at most one nonterminal in the R.H.S. of each of its productions.

- Consider $\{a^i b^i \mid i \geq 0\}$
- $P = \{S \to aSb, S \to \epsilon\}$

We call such languages as linear languages. Recall, the special cases

- **left-linear grammars** : a type of linear grammar with R.H.S. nonterminals strictly at the left ends;
- **right-linear grammars** : a type of linear grammar with R.H.S. nonterminals strictly at the right ends;

Linear Grammars n languages

- All linear languages are context-free
- There are CFLs that are non-linear : balanced parenthesis (the famous "Dyck language")
  $S \rightarrow (S)|SS|\epsilon$

Thus

- regular languages are a proper subset of linear languages
- linear languages are a proper subset of CFLs

Linear Grammars n languages

- Linear languages that are regular are deterministic (DPDA acceptable naturally)
- there exist linear languages that are nondeterministic

Ex : the language of even-length palindromes on $\{0, 1\}$ has the linear grammar $S \rightarrow 0S0|1S1|\epsilon$. Require NPDA

- linear languages are closed under intersection with regular sets
- linear languages are closed under homomorphism and inverse homomorphism.

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Deterministic Pushdown Automata

A PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, \perp, F)$ is deterministic if

- $\delta(q, a, X)$ has at most one member for every $q \in Q$, $a \in \Sigma$ or $a = \epsilon$, and $X \in \Gamma$.
- If $\delta(q, a, X)$ is nonempty for some $a \in \Sigma$ then $\delta(q, \epsilon, X)$ must be empty.

Example: $L = \{0^n 1^n : n \geq 1\}$.

Deterministic Pushdown Automata

- Theorem:

  Every regular language can be accepted by a deterministic pushdown automata that accepts by final states.

- Theorem ($DPDA \neq PDA$):
  There are some CFLs, for instance $\{ww^R\}$ *that can not be accepted by a DPDA.*

Closure Properties of DCFLs

A deterministic context free language is a language accepted by a deterministic PDA(DPDA). Every DCFL is a CFL but not vice versa.

- DCFL are not closed under union
- DCFL are not closed under reversal
- DCFL are closed under complementation

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CYK Algorithm

Cubic-time Algorithm to check if a string x belongs to a grammar G in CNF.
Consider the following grammar for the set of all non-null strings with equal number of a's and b's.

$$
\begin{aligned}
S &\to AB|BA|SS|AC|BD, \\
A &\to a, \\
B &\to b, \\
C &\to SB, \\
D &\to SA.
\end{aligned}
$$

CYK Algorithm contd.

We'll run the algorithm on the string

	a		a		b		b		a		b	
0		1		2		3		4		5		6

For $0 \leq i < j \leq n$, x_{ij} denote the substring of x between lines i and j. Build a table T with $\binom{n}{2}$ entries, one for each pair i, j.

0						
–	1					
–	–	2				
–	–	–	3			
–	–	–	–	4		
–	–	–	–	–	5	
–	–	–	–	–	–	6

$T_{i,j}$ refers to substring $x_{i,j}$.

We will fill each entry $T_{i,j} \in T$ with the set of nonterminals of G that produce substring $x_{i,j}$. We start with substring of length 1. For each substring $c = x_{i,i+1}$, if there is a production $X \to c \in G$, we write X in $T_{i,j}$.

0						
A	1					
–	A	2				
–	–	B	3			
–	–	–	B	4		
–	–	–	–	A	5	
–	–	–	–	–	B	6

For each substring $c = x_{i,i+2}$, we break the substring into two non-null substrings $x_{i,i+1}$ and $x_{i+1,i+2}$ and check the table entries

0						
A	1					
ϕ	A	2				
–	S	B	3			
–	–	ϕ	B	4		
–	–	–	S	A	5	
–	–	–	–	S	B	6

Now proceed to strings of length 3. For each substring $c = x_{i,i+3}$, we have two ways to proceed: we break the substring into two non-null substrings $x_{i,i+1}$ and $x_{i+1,i+3}$ or $x_{i,i+2}$ and $x_{i+2,i+3}$. For example,$x_{0,3} = x_{0,1}x_{1,3} = x_{0,2}x_{2,3}$. For the first, $A \in T_{0,1}$ and $S \in T_{1,3}$ but AS is not present in the right side of any production. Similarly nothing produces $T_{0,2}$. So $T_{0,3}$ is ϕ

0						
A	1					
ϕ	A	2				
ϕ	S	B	3			
–	–	ϕ	B	4		
–	–	–	S	A	5	
–	–	–	–	S	B	6

For $x_{1,4} = x_{1,2}x_{2,4} = x_{1,3}x_{3,4}$, $A \in T_{1,2}$ and $\phi \in T_{2,4}$. But $S \in T_{1,3}$ and $B \in T_{3,4}$ and $C \to SB \in G$. So $T_{1,4}$ is labeled with C.

0						
A	1					
ϕ	A	2				
ϕ	S	B	3			
–	C	ϕ	B	4		
–	–	–	S	A	5	
–	–	–	–	S	B	6

Continue in this fashion for stings of length three,four etc. For strings of length four there are 3 ways to break them up and every one must be checked. the final result is

0						
A	1					
ϕ	A	2				
ϕ	S	B	3			
S	C	ϕ	B	4		
D	S	ϕ	S	A	5	
S	C	ϕ	C	S	B	6

We see that $T_{0,6}$ is labeled with S so we conclude that x is generated by G.

Algorithm 1: CYK Algorithm

```
for i := 0 to n − 1 do    /* 1 length strings first */
    T_{i,i+1} := Φ                    /* Initialize to Φ */
    for A → a ∈ G do
        if a = x_{i,i+1} then
            T_{i,i+1} := T_{i,i+1} ∪ {A}
for m := 2 to n do        /* for each length m ≥ 2 */
    for i := 0 to n − m do    /* for each substring */
        T_{i,i+m} := Φ                /* of length m */
        for j := i + 1 to i + m − 1 do  /* for all ways
         to breakup the string */
            for A → BC ∈ G do
                if B ∈ T_{i,j} ∧ C ∈ T_{j,i+m} then
                    T_{i,i+m} := T_{i,i+m} ∪ {A}
```

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- An unrestricted grammar is a 4-tuple $G = (V, \Sigma, S, P)$, where V and Σ are disjoint sets of variables and terminals, respectively. S is an element of V called the start symbol, and P is a set of productions of the form

$$\alpha \to \beta$$

  where $\alpha, \beta \in (V \cup \Sigma)^*$ and α contains at least one variable.
- A context-sensitive grammar (CSG) is an unrestricted grammar in which no production is length-decreasing. In other words, every production is of the form $\alpha \to \beta$, where $|\beta| \geq |\alpha|$.
- A language is a context-sensitive language (CSL) if it can be generated by a context-sensitive grammar.

A CSG Generating $L = \{a^n b^n c^n | n \geq 1\}$

$$
\begin{aligned}
S &\rightarrow SABC|\mathcal{A}BC \\
BA &\rightarrow AB \\
CA &\rightarrow AC \\
CB &\rightarrow BC \\
\mathcal{A} &\rightarrow a \\
aA &\rightarrow aa \\
aB &\rightarrow ab \\
bB &\rightarrow bb \\
bC &\rightarrow bc \\
cC &\rightarrow cc
\end{aligned}
$$

The derivation for the string *aabbcc* is

$$
\begin{aligned}
S &\rightarrow aSBC \\
&\rightarrow aaBCBC \\
&\rightarrow aabCBC \\
&\rightarrow aabBCC \\
&\rightarrow aabbCC \\
&\rightarrow aabbcC \\
&\rightarrow aabbcc
\end{aligned}
$$