Hw2

SECTION 2.1 - 2.b

```python
ITERATION: 0  a=-1.25 | b=2.5     | p_0 = 0.625    | F(p_0) = -0.228515625
ITERATION: 1  a=0.625 | b=2.5     | p_1 = 1.5625   | F(p_1) = 4.594482421875
ITERATION: 2  a=0.625 | b=1.5625  | p_2 = 1.09375  | F(p_2) = 0.3496398925781
ITERATION: 3  a=0.625 | b=1.09375 | p_3 = 0.859375 | F(p_3) = -0.281902313232
```

SECTION 2.1 - 18

$$\begin{aligned} \mathrm{TOL} &= 10^{-3} \\ f(x) &= x^3 + x - 4 \\ [a, b] &= [1, 4] \end{aligned}$$

We know the upper bound for error:

$$|p - p_0| < \frac{b-a}{2^n}$$

Upper bound required $\mathrm{TOL} = 10^{-3} = \frac{4-1}{2^n}$

$$\frac{3}{2^n} \leq 10^{-3}$$

$$\frac{2^n}{3} \geq 1000$$

$$2^n \geq 3000$$

$$n \geq \log_2 3000 \approx 11.55$$

$$\implies n \geq 12$$

Theoretically the bisection method will require 12 iterations to conform 10^{-3} units within the actual root. The real iteration count may be less.

2.2.10

$$g(x) = 2^{-x}$$
$$g'(x) = -\ln 2 \cdot 2^{-x}$$
$$g''(x) = (\ln 2)^2 \cdot 2^{-x}$$

Observe that $\forall x \in \mathbb{R}$ $g''(x) > 0$, thus $g'(x)$ contains no inflection points.

Observe that $\forall x \in \mathbb{R}$ $g'(x) < 0$.

$g'(\frac{1}{3}) = -.550\ldots$	$g(\frac{1}{3}) = .793\ldots$
$g'(1) = -0.3466\ldots$	$g(1) = 0.5$

This means that for $x \in [\frac{1}{3}, 1]$, $|g'(x)| \leq g'(\frac{1}{3}) < 1$

Since g is monotonic decreasing $\forall x \in [\frac{1}{3}, 1]$, $g(1) \leq g(x) \leq g(1/3)$

$\Rightarrow$ $g : [\frac{1}{3}, 1] \to [1/3, 1]$

This combined with the fact that $2^{-x} \in C^{\infty}(\mathbb{R})$ satisfies theorem 2.3.

2^{-x} has unique fixed point in $[\frac{1}{3}, 1]$

2.2.10.b

$$g(x) = 2^{-x}$$
$$p_0 = 1$$
$$p_1 = g(p_0) = 1/2$$

we require $|p_n - p| \le 10^{-4} \Rightarrow \frac{k^n}{1-k} |p_1 - p_0| \le 10^{-4}$

$$= \frac{1}{2} \cdot \frac{1}{1-k} k^n$$

$k = 0.5508$

$$1.11308 k^n \le 10^{-4}$$

$$k^n \le .00009$$

$$.5508^n \le .00009$$

$$n \ln .5508 \le \ln(.00009)$$

$$n = \frac{\ln(.00009)}{\ln(.5508)}$$

$$\lceil n \rceil = 16$$

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$A \in \mathbb{R}^+ \rightarrow f(x) = 2x - Ax^2$

Since $x = f(x)$ exists:

$$x = 2x - Ax^2$$
$$0 = x - Ax^2$$
$$0 = x(1 - Ax)$$

Zero is a trivial fixed point.

However $1 - Ax = 0$ is also a fixed point:

$$1 - Ax = 0$$
$$\left(\tfrac{1}{A}\right) Ax = 1 \left(\tfrac{1}{A}\right)$$
$$x = 1/A$$

Since $\frac{1}{A}$ is a fixed point if $A \in \mathbb{R}^+$, $\lim_{n\to\infty} p_n = p = 1/A$

20 B

$$g(x) = 2x - Ax^2$$

$$g'(x) = 2 - 2Ax$$

if $\exists K \in (0,1)$ s.t $\forall x \in I, |g'(x)| < K$ then $\forall p \in I$, the sequence $p_n = g(p_{n-1})$ will converge.

$$|g'(x)| < 1$$

$$|2 - 2Ax| < 1$$

$2 - 2Ax < 1$
$-2Ax < -1$
$Ax > \frac{1}{2}$
$x > \frac{1}{2A}$

$2Ax - 2 < 1$
$2Ax < 3$
$Ax < \frac{3}{2}$
$x < 3/2A$

Interval $I = \left[\frac{1}{2A}, \frac{3}{2A}\right]$

Additional 1

$$\cos^2 x + 6 = x$$

$$\cos^2 x = x - 6$$

Since $0 \leq \cos^2(x) \leq 1$, $6 \leq \cos^2 x + 6 \leq 7$

The interval $[6, 7]$ contains the fixed point for $\cos^2 x + 6$.

Additional 2

if we suppose that $\sqrt{2} \in [1,2]$ we can employ the bisection algorithm for:

$$f(x) = x^4 - 4$$

See that $f(\sqrt{2}) = (2^{1/2})^4 - 4 = 4 - 4 = 0$

Additional 3

$* - \forall x \in \mathbb{R} \ x^2 + 2 > 0$

$$g(x) = x$$

$$\frac{8+2x}{2+x^2} = x \overset{*}{\rightarrow} 8 + 2x = 2x + x^3$$

$$8 = x^3$$

$$\boxed{2 = x}$$

Additional 4

1. $x^3 + e^x = x$

2. $x^3 - x + e^x = 0$

$$x - e^x = x^3$$

$$\sqrt[3]{x - e^x} = x$$