Using Selective Memoization to Defeat Regular Expression Denial of Service (ReDoS)

James C. Davis
Computer Science
Virginia Tech
Blacksburg, VA, USA
davisjam@vt.edu

Francisco Servant
Computer Science
Virginia Tech
Blacksburg, VA, USA
fservant@vt.edu

Dongyoon Lee
Computer Science
Stony Brook University
Stony Brook, NY, USA
dongyoon@cs.stonybrook.edu

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***Abstract*—Regular expressions (regexes) are a denial of service vector in most mainstream programming languages. Recent empirical work has demonstrated that up to 10% of regexes have super-linear worst-case behavior in typical regex engines. It is therefore not surprising that many web services are reportedly vulnerable to regex denial of service (ReDoS).**

If the time complexity of a regex engine can be reduced transparently, ReDoS vulnerabilities can be eliminated at no cost to application developers. Unfortunately, existing ReDoS defenses — replacing the regex engine, optimizing it, or replacing regexes piecemeal — struggle with soundness and compatibility. Full memoization is sound and compatible, but its space costs are too high. No effective ReDoS defense has been adopted in practice.

We present techniques to provably eliminate super-linear regex behavior with low space costs for typical regexes. We propose *selective memoization* schemes with varying space/time tradeoffs. We then describe an *encoding scheme* that leverages insights about regex engine semantics to further reduce the space cost of memoization. We also consider how to safely handle *extended regex features*. We implemented our proposals and evaluated them on a corpus of real-world regexes. We found that selective memoization lowers the space cost of memoization by an order of magnitude for the median regex, and that run-length encoding further lowers the space cost to *constant* for 90% of regexes.

"Those who cannot remember the past are condemned to repeat it."
–George Santayana

***Index Terms*—Regular expressions, denial of service, ReDoS, algorithmic complexity attacks, memoization**

I. Introduction

Regular expressions (regexes) are a fundamental building block of computing systems [1]. It is unfortunate that such a widely used tool is a denial of service vector. For the sake of expressiveness and flexibility [2], most regex engines follow a backtracking framework with worst-case super-linear behavior in the length of the input string w (*e.g.,* $\mathcal{O}(|w|^2)$ or $\mathcal{O}(2^{|w|})$) [3]. Meanwhile, 30-40% of software projects use regexes to solve string matching problems [4], [5], and up to 10% of those regexes exhibit super-linear worst-case behavior [6], [7]. These trends expose regex-reliant services to an algorithmic complexity attack [8] known as Regex-based Denial of Service (ReDoS) [9]–[11].

The threat of ReDoS is well understood. Empiricists have demonstrated that software engineers commonly compose ReDoS-vulnerable regexes [5], [6] and that thousands of web services are exploitable [12]. For example, root cause analysis implicated super-linear regex evaluation in high-profile outages at Stack Overflow and Cloudflare. At Stack Overflow, a regex designed to format posts exhibited quadratic behavior, causing an outage for thousands of users [13]. At Cloudflare, a regex deployed to scan for malicious network traffic exhibited quartic behavior, delaying traffic through their firewall and causing outages for thousands of businesses [14], [15]. We know the risks — now we need a ReDoS defense.

ReDoS attacks require three ingredients: a slow (*i.e., backtracking*) regex engine, a slow regex, and reachability by user input. Assuming reachability, ReDoS defenses thus speed up the regex engine or change the regex. A proper ReDoS defense should be (1) *sound* (works for all regexes), (2) *backwards-compatible* (regex engines are "legacy systems" whose stability is critical), and (3) *low-cost*. Existing ReDoS defenses suffer from unsoundness or incompatibility. For example, moving regexes to a faster regex engine [3], [16] risks semantic differences [7], while refactoring slow regexes is an error-prone and piecemeal solution [6].

In light of these design goals, we propose memoization to speed up backtracking regex engines, addressing ReDoS *soundly* in a *backwards-compatible* manner with *small runtime costs* for typical regexes. For soundness, we prove theorems guaranteeing worst-case time complexity that is linear in $|w|$. For compatibility, our approach can be introduced within existing backtracking frameworks, just like other common regex optimizations (*e.g.,* [17]–[19]). To make our approach low-cost, we employ *selective memoization* and an *efficient data representation* to reduce asymptotic space complexity and typical space costs. Finally, we extend our techniques to two *extended regex features* (zero-width assertions, backreferences), achieving exponential time complexity reductions for typical implementations.

To evaluate the practical space and time costs of our approach, we measured its behavior on the largest available corpus of super-linear regexes [7]. On 42,766 (100%) of regexes that exhibit slow behavior in mainstream regex engines, our prototype achieves linear-in-$|w|$ time costs. Our selective memoization reduced the space cost for the median regex from 26 to 3 copies of w using a standard memo table. Using our efficient representation, the space cost falls from linear to *constant* for 38,489 (90%) of the super-linear regexes.

Our contributions are:

- We propose a novel memoized NFA to enable the analysis of memoization schemes (§VI).
- To reduce the problematic space complexity of memoization, we present two novel *selective memoization schemes* that reduce the space complexity of memoization (§VII).
- To further reduce space costs, we propose a novel space-efficient *memo function representation* (§VIII).
- We extend these techniques from regular expressions to two commonly-used extended regex features: backreferences and lookaround assertions. We provide novel bounds on the associated match problems within a backtracking framework, for current regex engines and after memoization (§IX).

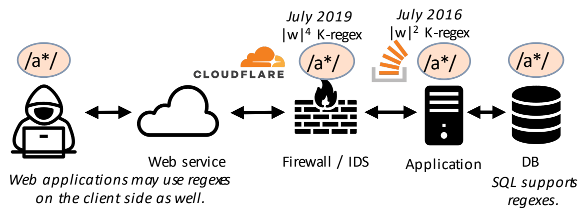


Fig. 1: Regexes across the system stack. ReDoS may occur when a slow regex meets unsanitized input on a slow regex engine. We discuss ReDoS outages at Cloudflare and Stack Overflow (§II).

- We evaluate our proposals on a large-scale corpus and report substantial benefits for a wide range of regexes (§X).

We have demonstrated techniques by which ReDoS can be defeated — soundly, for all commonly-used regex features, and with minimal changes to legacy regex engines.

Outline We begin with background material on ReDoS and regexes. Next we show the limitations of existing ReDoS defenses. Then we present our approach, evaluate, and conclude.

II. Regex Denial of Service (ReDoS)

Regexes are used in latency-sensitive contexts on the critical path. They validate untrusted input throughout the system stack (Figure 1), *e.g.,* to process HTML headers [12], [20] or detect cross-site scripting (XSS) attacks [21], [22]. Ironically, regexes used as a defensive filter may themselves be exploited.

A. ReDoS attacks

Crosby and Wallach observed that super-linear (*i.e.,* polynomial or exponential in $|w|$) regex engine behavior could be exploited in an algorithmic complexity attack [8], [9]. ReDoS attacks require three *ReDoS Conditions*:

1) The victim uses a regex engine with super-linear matching behavior in $|w|$ (*i.e.,* a backtracking implementation).[1]
2) The victim uses a super-linear regex (§III-B).
3) The regex is reachable by untrusted input.

If these conditions are met, then an attacker can submit input designed to trigger the super-linear regex behavior. The costly regex evaluation will divert computational resources (*e.g.,* CPUs, threads) and reduce or deny service to legitimate clients. Such attacks are applicable to most web services; Davis *et al.* have demonstrated super-linear regex behavior in every major programming language but Rust and Go [7]. ReDoS exploits are particularly problematic for services that use multiplexed architectures like Node.js [12], [24], [25].

B. Threat model

We suppose a realistic threat model: the attacker can specify the string w to which the victim's regex is applied (ReDoS Condition 3). This is in keeping with a primary use of regexes, namely to sanitize and process untrusted input [4], [26], [27].

[1]Some authors restrict ReDoS to exponential in w [23]. However, real-world outages have involved polynomial worst-case behavior [13], [14].

C. ReDoS in the wild: Two case studies

Thousands of ReDoS vulnerabilities have recently been identified [6], [12]. We illustrate these vulnerabilities through two case studies. These studies show the diverse usage of regexes and the implications of super-linear behavior.

The Q&A forum **Stack Overflow** had a 34-minute ReDoS outage in July 2016 [13]. They used the quadratic regex `.*\s+` (simplified) to trim trailing whitespace from each post as part of response generation, to improve rendering and reduce network traffic. A post with 20,507 tab characters reached the front page, triggering the worst-case quadratic behavior of the regex. Multiplied by all traffic to the website, the resulting computational load brought down their service.

ReDoS outages also occur due to dependencies on other services. The web infrastructure company **Cloudflare** had a 27-minute ReDoS outage in July 2019 [14], affecting the availability of thousands of their customers [15]. As part of an XSS detector, Cloudflare used the quartic regex `a*b?c?a*a*a*` (simplified) to detect JavaScript tokens within web traffic. Some typical traffic triggered the worst-case behavior of this regex, exhausting Cloudflare's computational resources and affecting their customers' web services.

III. Background on regular expressions

We explain ReDoS in algorithmic and engineering terms.

A. Regular expressions

Kleene introduced regular expressions (*K-regexes*) to describe strings constructed using a finite number of concatenations, repetitions, and disjunctions [28], [29]. Rabin and Miller showed that they were equivalent in expressive power to non-deterministic finite automata (NFAs) [30]. We will denote NFAs using the standard 5-tuple $\langle$states Q, start q_0, accepting F, alphabet Σ, transitions $\delta\rangle$ [31].[2] The typical regular expression notation is given in Figure 2, with equivalent NFAs according to the construction of Thompson [32] and Glushkov [33].

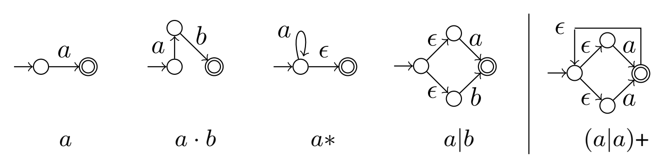


Fig. 2: K-regex operators and NFA equivalents for character, concatenation, repetition, and disjunction. NFAs for K-regexes have in-degree and out-degree ≤ 2, although this is higher under ϵ-closure. The final figure shows an exponentially ambiguous K-regex.

B. Regex languages and ambiguity

Regexes describe a *language*, a set of strings that meets the criteria embodied in the expression [31]. As with other pattern languages like context-free grammars [34], [35], regexes can be *ambiguous*: there may be multiple ways for a string to be in the language of the regex [36]. For example, the last expression

[2]Readers unfamiliar with this notation should refer to Table II.

TABLE I: Worst-case time and space complexities for typical regex recognition algorithms, for a K-regex R and candidate string w, and omitting automaton construction costs. Although Thompson's algorithm has better theoretical guarantees, Spencer's algorithm is widely used (*e.g.,* JavaScript-V8, Java, PHP, Python, Ruby, Perl, .NET). The last row gives representative complexity of our approach.

Algorithm	$\lvert Q\rvert$	**Time cxty.**	**Space cxty.**
Spencer	$\mathcal{O}(\lvert R\rvert)$	$\mathcal{O}(\lvert Q\rvert^{2+\lvert w\rvert}\times\lvert w\rvert)$	$\mathcal{O}(\lvert Q\rvert\times\lvert w\rvert)$
Thompson	$\mathcal{O}(\lvert R\rvert)$	$\mathcal{O}(\lvert Q\rvert^{2}\times\lvert w\rvert)$	$\mathcal{O}(\lvert Q\rvert)$
Memo-Spencer	$\mathcal{O}(\lvert R\rvert)$	$\mathcal{O}(\lvert Q\rvert^{2}\times\lvert w\rvert)$	$\mathcal{O}(\lvert Q\rvert\times\lvert w\rvert)$

in Figure 2 is ambiguous, because it can parse the string 'a' in two different ways. If there is a maximum number of distinct parses for any string in the language, a regular expression or NFA is called *finitely ambiguous*, otherwise it is *infinitely ambiguous*. Finite ambiguity results from disjunctions, *e.g.,* the 2-ambiguous `a|a`, while infinite ambiguity requires the use of a quantifier, *e.g.,* `(a|a)*`. Infinite ambiguity is one of the necessary conditions for typical super-linear regex behavior [26], [37].

C. *String problems and regex engines*

Software engineers use regular expressions to answer two string problems [31]. The *recognition problem* tests whether or not a candidate string w is in the language of a regular expression R (*i.e.,* the regex *matches* the string). This permits engineers to determine appropriate control flow in a software application. The *parse problem* returns the matching substring and any sub-captures ("capture groups"). This supports activities like scraping web pages or log files.

Programming languages use a *regex engine* to solve these problems. These engines follow two general algorithms that can be modeled as NFA simulation (with extensions for parsing [38] and irregular features [39], [40]). Under this model, the algorithms search for a path to an accept state, either depth-first (Spencer's) or breadth-first (Thompson's). The semantics of regular expression membership follow the NFA membership problem, with extensions and deviations discussed by Câmpeanu [41], [42] and Berglund [43]–[46].

Table I summarizes these algorithms and our approach.

1) The Spencer algorithm: This resolves non-determinism using a backtracking [47] NFA simulation (Listing 1). Given a choice of edges, it tries one and saves the others for later. Its state consists of its current *simulation position* $\pi = \langle q \in Q, i \in \mathbb{N}^{|w|}\rangle$ (an automaton state and an index i into the string w), and a *backtracking stack* that records the un-tested alternative branches, for $\mathcal{O}(|Q|\times|w|)$ space complexity.

Spencer's search policy can exhibit exponential time complexity [23], [26], [37]. To see this, examine the last illustration of Figure 2. Consider a backtracking simulation of this NFA on the string $w = a^k b$, and count the number of paths to the accept state for the prefix $a^k = a \overset{k}{\ldots} a$. There are two paths to the accept state on the first a; twice that many for a^2; and twice again for a^3. The geometric recurrence yields 2^k paths for the prefix a^k. When the simulation encounters the suffix b, each of these paths results in failure. Note that many of these paths are redundant. Nevertheless, the backtracking algorithm may attempt exponentially many paths before returning a mismatch, with time complexity $\mathcal{O}(|Q|^{2+|w|}\times|w|)$.

Listing 1 Spencer-style backtracking K-regex recognition. Regex engines often use an explicit stack, not recursion.

```python
# Invoke as recognize(NFA, w, NFA.q0, 0)
# Returns True if the NFA recognizes w
def recognize(NFA, w, currQ, i):
  if i == len(w): # Max recursion depth
    return True if currQ in NFA.F else False
  for nextQ in NFA.delta(currQ, w[i], i): # O(Q)
    if recognize(NFA, w, nextQ, i+1): # DFS
      return True # Bubble up a success
  return False # Backtrack
```

2) The Thompson algorithm: This resolves non-determinism using a lockstep NFA simulation. When there is a choice of edges, it simulates taking all of them together. Its state tracks the current offset i and the vertex-set of the currently activated NFA states: $\langle i, \Phi \subseteq Q\rangle$, for space complexity $\mathcal{O}(|Q|)$. As for time complexity, for each character in the input string the algorithm queries the transition function δ for each of the current NFA states Φ. Thus the time complexity is $\mathcal{O}(|Q|^2\times|w|)$.

D. *The persistence of ReDoS: Regexes in practice*

In principle, applying Thompson's algorithm is a straightforward remedy for ReDoS. There are obstacles in practice.

Dialects Perl introduced regexes as a first-class programming language construct using a derivative of Spencer's library [48]. All subsequent programming languages support regexes and generally follow Perl-Compatible Regular Expressions (PCRE) notation and semantics [1], [49]. However, they also maintain independent (and inconsistent) regex engines. This practice leads to dueling regex specifications [49], [50] and an embarrassment of dialects [1], [7], [51].

Extended features PCRE's extended regexes (*E-regexes*) are more expressive and more powerful than Kleene's K-regexes. They offer "syntax sugar" regular operators (*K-compatible*) including character classes (`[a-z]`), cuts `(?>a)` [1], and limited repetition (`a{3,5}`), as well as irregular operators like zero-width assertions `(?=a)` and backreferences (`(a)\1`) [52].

Engine implementations In practice, regex features dictate regex engine algorithms. Spencer's backtracking algorithm is used by all "PCRE" regex engines, including those of Java, JavaScript-V8, PHP, Python, Ruby, Perl, and .NET [1], [7]. This is partly historical accident, as many descend from Spencer's library. But it is also an engineering decision; Spencer was aware of Thompson's approach but chose backtracking for its greater flexibility (E-regexes) and simplicity of implementation [2]. Thompson's algorithm is used only by Rust [53] and Go [54], as well as Google's standalone RE2 regex engine [16]. These engines do not support E-regexes.

Maintainer priorities The maintainers of Spencer-style Regex engines are aware of the threat of ReDoS. Several engines have longstanding bug reports describing problematic time costs [55], [56]. Historically, maintainers have viewed super-linear regexes as aberrant usage that should be addressed at the application level. New research has demonstrated the extent of the ReDoS problem [6], [12], and regex engine maintainers are beginning to take ReDoS seriously [57]. Their current approaches, discussed next, are unsound or incompatible.

IV. Limitations of Existing ReDoS defenses

Our threat model assumes that developers will continue to use regexes to handle user input (ReDoS Condition 3). Defenders should therefore address Condition 1 (slow engine) or 2 (slow regex). All existing defenses have serious shortcomings.

A. Slow engines: Remove super-linear matching behavior

Addressing Condition 1 would be a fundamental solution. If a regex engine guaranteed linear-in-$|w|$ match times, then ReDoS would be addressed once-for-all.

1) Use a faster matching algorithm: This comes via application-level substitution or engine-level overhaul.

Applications can adopt a third-party linear-time regex engine. For example, after their outage Cloudflare moved from Lua's backtracking regex engine to RE2 [14]. Unfortunately, substituting one regex engine for another is fraught. E-regexes cannot be ported, because the current generation of linear-time regex engines only support K-regexes. Most K- and K-compatible regexes can be ported, but this is complicated by the abundance of regex dialects. Some dialectal differences are syntactic and amenable to standard transpilation techniques [58], but others are not [7]. Regexes are under-tested [59] so finding portability problems may be difficult.

At the regex engine level, the engine maintainers could overhaul their regex engine to use a faster algorithm. Although regex engine maintainers know about ReDoS [55], [56], the engineering cost of leaving the backtracking framework appears unpalatable. Maintainers regularly improve common-case performance (*e.g.,* V8 [60], [61] and .NET [57]), but do not undertake an algorithmic overhaul to address ReDoS. Some newer regex engines are designed around Thompson's algorithm (RE2, Rust, Go), but legacy engines may have too much technical inertia to follow suit.

2) Backtracking optimizations: Although no regex engine has changed its algorithmic framework to address ReDoS, maintainers have incorporated "inline" optimizations that fit into the backtracking framework. Some optimizations find good starting points for an NFA simulation [17]–[19], and are orthogonal to our approach. More apropos to our approach, other optimizations remove some of the redundant paths in the backtracking search. For example, .NET [57] and Perl use *prefix factoring* to unite overlapping paths, while *caching* is used to accelerate backtracking in Perl and a rare path in RE2. We evaluate the unsoundness of these caches in §X using machinery introduced in §VI and §VII.

3) Capping super-linear behavior: Three backtracking regex engines defend against ReDoS by limiting the resources consumed by a regex match. The .NET regex engine offers a wall clock *time-based* cap on matches [62]. The PHP and Perl regex engines use *counter-based* caps [63], [64], throwing an exception if a match exceeds their cost measures. Resource caps cannot be enforced in a backwards-compatible manner. Resource caps are a perennial "Goldilocks" problem [25], [65]: too tight rejects valid input, too loose permits ReDoS. Moreover, users want a fast regex *answer*, not an exception.

B. Slow regexes: Remove super-linear regexes

If a super-linear regex engine must be used, application developers may instead refactor their use of super-linear regexes. This defense is problematic in two ways. *First*, it is *ad hoc*. The maintainers of individual computing systems must determine that they are vulnerable, choose a refactoring strategy, and apply it correctly. This process is error prone [6]).[3] *Second*, it is indirect, putting the burden for a solution on application developers rather than addressing the root cause. Developers might prefer a regex with super-linear structure, *e.g.,* to facilitate maintainability or comprehension [68]. Our approach accommodates such preferences in linear time.

V. Our approach: Context and Overview

We propose a memoization approach to guarantee linear-time matching within the backtracking algorithmic framework. Our approach improves on the state of practice (Spencer's algorithm). We emphasize that we do not attempt to improve on the theoretical state of the art for regex matching (Thompson's algorithm). In the 60 years since Thompson proposed his algorithm, practitioners have shown no inclination to abandon the backtracking framework in their legacy regex engines (§IV). Therefore, we propose an approach that can be adopted within a backtracking framework with minimal changes.

Our approach builds on Michie's general function memoization technique [69], an optimization that spends space to save on time. Memoization records a function's known input-output pairs to avoid evaluating it more than once per input. Many algorithms benefit from the time savings [70]–[72]. Space costs depend on the input-output domain.

Memoization techniques have previously been applied to parsing problems, notably "Packrat Parsers" for context-free grammars (CFGs) and parsing expression grammars (PEGs) [47], [73]–[76]. Memoization can also be used for regex matching. If the input-output pairs in Spencer's algorithm were recorded, then redundancy can be eliminated. As an example, consider Listing 1 when applied to the final illustration from Figure 2. The many paths to the accepting NFA vertex correspond to many (redundant) recursive queries to `recognize` for the same simulation positions. Memoizing the results of `recognize` would eliminate this redundancy.

Full memoization, *i.e.,* recording every input-output pair for `recognize`, has never been applied for practical regex matching due to its space complexity [3], [43], [77], [78]. Our case studies corroborated this conclusion (§II); large regexes are used on long input and so the $\mathcal{O}(|Q| \times |w|)$ space complexity of full memoization is too costly. To employ memoization for regex matching in practice, its space complexity must be reduced. Some regex engines have done so unsoundly using heuristics (§X). Our memoization techniques offer linear-time

[3] The regex refactoring process might be automated. Existing techniques are unsound [57], [66] or entail exponential space complexity [67].

K-regex matching *soundly and with low space costs*. For E-regexes, we obtain linear-time matching for lookaround assertions and parameterized costs for backreferences.

Here are the key ingredients of our approach. First, we prove that full memoization records more data than necessary for linear-in-$|w|$ K-regex time complexity. This goal can be achieved by selectively memoizing visits to only a subset of the automaton vertices (decreasing space *complexity*). Second, we consider the embodiment of the memo function M, and observe that for many regexes its state will be low-entropy and compressible via run-length encoding (decreasing space *cost*). Third, we extend these techniques to two E-regex features (lookaround assertions and backreferences). For lookaround assertions the benefits are notable, reducing the time complexity of typical implementations from exponential to linear.

We divide our presentation into four parts: a formalization of Memoized Non-Deterministic Finite Automata (M-NFA) to reason about our approach (§VI); analyses of the behavior of M-NFA for K-regex recognition under three memoization schemes (§VII); a space-efficient representation of the memo function (§VIII); and extensions for E-regexes (§IX).

VI. Memoized non-deterministic finite automata

Using memoization, a regex engine can record the areas of the search space that it has explored. If it revisits those regions, it can short-circuit a redundant exploration. Although a *full memoization* approach is a standard technique from Packrat Parsing, we introduce two novel *selective memoization* schemes in §VII whose properties are more subtle.

To provide a framework for the analysis of our selective memoization schemes, we introduce a novel extension of Rabin-Miller NFAs. We define this entity as a *Memoized Non-Deterministic Finite Automaton* (M-NFA), with components described in Table II.[4] This model enables us to reason about the behavior of an NFA simulation algorithm when applied to an M-NFA. The additional components are:

M The memo function M of an M-NFA is updated during the backtracking simulation. It initially returns 0 to all queries. After a simulation position π is marked, the memo function returns 1 for subsequent queries to that position.

δ_M An M-NFA's memoized transition function δ_M accepts the typical arguments to δ, plus a candidate string index i:

$$\delta_M(q, \sigma, i) = \{r \in Q \mid r \in \delta(q, \sigma) \wedge M(r, i+1) = 0\}$$

In other words, δ_M uses the memo function M to dynamically eliminate redundant transitions during the simulation.

Simulation An M-NFA can be simulated on a string w beginning from q_0, by repeated application of the memoized transition function δ_M (see Listing 2). If the simulation ends in a state $q \in F$, the M-NFA accepts the candidate string. Note that for K-regexes, the outcome of a match starting from a given position π is determined solely by the current position, and not on previous decisions. Thus, the memo function tracks at most the $|Q| \times |w|$ possible positions.

[4] Strictly speaking, an M-NFA is not finite because it uses memory based on w. The components M and δ_M can be viewed as maintained by a memoized simulation of the NFA. We feel the M-NFA conceit is simpler.

TABLE II: Components of a Memoized Non-Deterministic Finite Automaton (M-NFA) derived from an NFA $A = \langle Q, q_0 \in Q, F \subseteq Q, \Sigma, \delta \rangle$. The components of A are listed above the mid-rule (cf. §III-A). The components of the M-NFA for A include those, plus the additional components below the mid-rule: M and δ_M.

Component	Meaning
Q	Automaton states
$q_0 \in Q$	Initial state
$F \subseteq Q$	Accepting states
Σ	String alphabet: $w \in \Sigma*$
$\delta : Q \times \Sigma \to \mathbb{P}(Q)$	Transition function
$M : Q \times \mathbb{N}^{\|w\|} \to \{0, 1\}$	Memo function ("memo table")
$\delta_M : Q \times \Sigma \times \mathbb{N}^{\|w\|} \to \mathbb{P}(Q)$	Memoized transition function

Listing 2 Memoized backtracking K-regex recognition. Differences from Listing 1 are highlighted.

```
# Invoke as memoRecognize(MNFA, w, MNFA.q0, 0)
def memoRecognize(MNFA, w, currQ, i):
  if i == len(w): # Max recursion depth
    return True if currQ in MNFA.F else False
  for nextQ in MNFA.deltaM(currQ, w[i], i) :
    if memoRecognize(MNFA, w, nextQ, i+1):
      return True # Bubble up a success
  MNFA.M.mark(currQ, i)  # This position failed
  return False # Backtrack
```

Memoization scheme During M-NFA simulation, the choice of which simulation positions π to memoize is determined by a *memoization scheme*. Our schemes memoize all simulation positions associated with a selected subset $Q_{sel.}$ of the automaton vertices, *i.e.,* all $\pi = \langle q \in Q_{sel.}, i \in \mathbb{N}^{|w|} \rangle$.

Ambiguity We define an ambiguous M-NFA analogous to an ambiguous NFA (§III-A). An M-NFA is *ambiguous* if there exists a string w such that when it is simulated from q_0, there are multiple paths to an accepting state. Note that the memo function eliminates possible paths in an M-NFA, so an ambiguous NFA does not always have an ambiguous M-NFA.

Space complexity Based on our M-NFA model, a memoization scheme incurs additional space complexity $\mathcal{O}(|Q_{sel.}| \times |w|)$ to store the memo function.

Time complexity Based on our M-NFA model, the time cost of an M-NFA simulation can be calculated as:

$$(\textit{\# sim. pos.}) \times (\textit{max visits per pos.}) \times (\textit{cost per visit}). \quad (1)$$

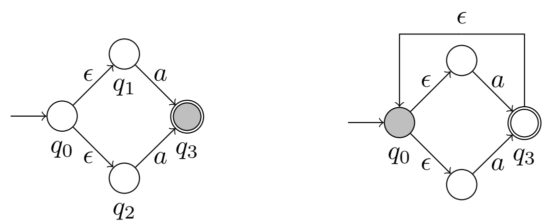
For K-regexes there are $|Q| \times |w|$ simulation positions. We assume that visits cost $\mathcal{O}(|Q|)$ per the loop in Listing 2, with $\mathcal{O}(1)$ updates to M and queries to δ_M [79]. If each position is visited once (Table III), the time complexity is $\mathcal{O}(|Q|^2 \times |w|)$.

VII. Selective memoization for K-regex recognition

In this section we present three selective memoization schemes for backtracking M-NFA simulations that follow Listing 2. Table III summarizes their properties. For each scheme we bound the number of visits per simulation position. We sketch intuition here using Figure 3. See appendix for proofs.

TABLE III: Time and space complexity of K-regex matching using the selective memoization schemes. Each scheme adds space complexity $|Q_{sel.}|\times|w|$, and they are ordered from largest to smallest vertex set size. The time complexity for $Q_{ancestor}$ has a factor $f(Q)$ that depends on the NFA's ambiguous structure (see appendix).

Memo scheme	Visits per pos.	Time cxty.	Add'l. space cxty.
None (Spencer)	$\mathcal{O}(\lvert Q\rvert^{\lvert w\rvert})$	$\mathcal{O}(\lvert Q\rvert^{2+\lvert w\rvert}\times\lvert w\rvert)$	—
$Q_{all} = Q$	≤ 1	$\mathcal{O}(\lvert Q\rvert^2\times\lvert w\rvert)$	$\lvert Q\rvert\times\lvert w\rvert$
$Q_{in-deg>1}$	≤ 1	$\mathcal{O}(\lvert Q\rvert^2\times\lvert w\rvert)$	$\lvert Q_{in-deg}\rvert\times\lvert w\rvert$
$Q_{ancestor}$	$\leq f(Q)$	$\mathcal{O}(\lvert Q\rvert^2\times\lvert w\rvert\times f)$	$\lvert Q_{anc.}\rvert\times\lvert w\rvert$



(a) Q_{in-deg}: M-NFA for $(a|a)$ (b) $Q_{anc.}$: M-NFA for $(a|a)+$

Fig. 3: Memoized automata used to illustrate the selective memoization schemes. Shading indicates the vertices associated with memoized simulation positions for (a) $Q_{in-deg>1}$ and (b) $Q_{ancestor}$.

A. Select all vertices: Q_{all}

In this scheme, we record the outcome of the simulation positions associated with every vertex of the M-NFA, $Q_{all} = Q$. From the definition of δ_M, each simulation position can be reached at most once. Theorem 1 follows directly.

> **Theorem 1.** *Let the memo function track simulation positions involving all M-NFA vertices Q. Then the M-NFA is unambiguous — every simulation position $\pi = \langle q, i\rangle$ is visited at most once.*

B. Select vertices with in-degree > 1: $Q_{in-deg>1}$

This scheme memoizes only the vertices $Q_{in-deg>1}$ with in-degree greater than one. Like Q_{all}, memoizing $Q_{in-deg>1}$ eliminates redundant visits; the M-NFA is unambiguous.

> **Theorem 2.** *Let the memo function track simulation positions involving the M-NFA vertices with in-degree greater than one, $Q_{in-deg>1}$. Then the M-NFA is unambiguous — every simulation position π is visited at most once.*

Proof sketch Figure 3 (a) illustrates this scheme. The shaded vertex q_3 has in-degree 2. By memoizing it, on the string a it can only be visited from q_1 and not q_2. Intuitively, in order for there to be redundancy (ambiguity) in the M-NFA simulation, we must reach the same simulation position π along two different paths. To reach this position twice: (1) There must have been a choice; and (2) Both branches must have led to π along edges with the same labels. At the fork, the two paths diverged; later, they converged. For two paths to converge, they must share some vertex with in-degree > 1. *Memoizing visits to this vertex prevents multiple visits to π.*

Observations This scheme eliminates non-determinism (outgoing edges with the same label) that results in ambiguity (branches converge). Per Figure 2, these conditions may arise in the Thompson-Glushkov construction from the use of disjunctions or Kleene stars (*e.g.,* `a|a` or `a*a*`). The $Q_{in-deg>1}$ scheme memoizes visits to the convergence points.

C. Select cycle ancestors: $Q_{ancestor}$

This scheme memoizes the vertices $Q_{ancestor}$ that are cycle ancestors according to a topological sort from q_0. The effect of this memoization scheme is to prevent the compounding of ambiguity; it eliminates infinite ambiguity, but still permits f redundant visits. The M-NFA is thus finitely ambiguous. In the appendix we provide a precise bound on f.

> **Theorem 3.** *Let the memo function track simulation positions involving the M-NFA vertices that are "cycle ancestors", $Q_{ancestor}$, i.e., automaton vertices to which back-edges are directed in a topological sort from q_0. Then the M-NFA is* f*-ambiguous. Every simulation position will be visited at most a finite number of times* f.

Proof sketch Figure 3 (b) illustrates this scheme. The shaded vertex q_0 is an ancestral node that can compound the two ambiguous paths to q_3. By memoizing q_0, only one of the two simple paths to q_3 can form a cycle with q_0, and the M-NFA is 2-ambiguous just as `a|a` is. Intuitively, infinite ambiguity arises when finite ambiguity is compounded by the presence of cycles in an NFA. For example, `a?a?` and `a|a` are finitely ambiguous, while `a*a*` and `(a|a)*` are infinitely so. These infinitely ambiguous variations add cycles to the NFA. *Memoizing the cycle ancestors permits ambiguity, but this ambiguity is bounded because all but one of the possible cycles are eliminated by δ_M.* This yields the result in Theorem 3.

Observations This memoization scheme permits redundant visits that result from finite ambiguity. Although Cox observes that regexes can be constructed with arbitrarily large finite ambiguity [3], we lack analysis tools to determine whether such regexes are practical or pathological. Regardless, the degree of redundancy permitted by this memoization scheme is not super-linear — it is limited as a function of the automaton, not of the length of w. Beyond a certain point, increasing $|w|$ will not increase the worst-case behavior.

In terms of K-regex features, Figure 2 shows that ancestral back-edges only occur from the use of Kleene stars. The $Q_{ancestor}$ selection scheme memoizes the vertices to which these back-edges are directed. If there is finite ambiguity in the sub-pattern to which the Kleene star is applied, that ambiguity remains after $Q_{ancestor}$-memoization.

This selection scheme has lower space complexity than the preceding one because $Q_{ancestor} \subseteq Q_{in-deg>1}$. Under standard regex-to-NFA constructions, all vertices in an NFA graph are reachable from q_0 and thus have in-degree ≥ 1. The vertices in $Q_{ancestor}$ have an additional in-degree due to the back-edge, and hence are also in $Q_{in-deg>1}$.

D. Time and space complexity

Time In §VI we stated that the time complexity of an M-NFA simulation depends on the maximum number of visits to each simulation position. Our theorems provide upper bounds on visits, resulting in the time complexities given in Table III.

Space Memoization schemes incur additional space complexity $\mathcal{O}(|Q_{sel.}|\times|w|)$. This space complexity decreases monotonically as: $Q = Q_{all} \supseteq Q_{in-deg>1} \supseteq Q_{ancestor}$.

E. Discussion

1) Semantics: The use of memoization eliminates redundant path exploration, but does not otherwise affect the behavior of the regex engine. Thus, the existing semantics of a regex engine will be unchanged by the introduction of memoization, whether that engine follows PCRE, POSIX, or another dialect.

2) Parsing with memoization: These selective memoization schemes improve the worst-case performance of the K-regex *recognition* problem (regex match) for Spencer-style backtracking regex engines. They similarly improve the worst-case performance of the other regular string problem, *parsing* (capture groups). A regex engine is free to populate capture groups during a memoized backtracking simulation just as it currently does [38], enabling it to answer parse queries as well.

3) Costs of selective memoization: The automaton vertices selected by our selective memoization schemes can be identified during a parsing pass. These vertices are associated with disjunction and Kleene star operators in a 1-to-1 manner. The space cost of the selective memoization schemes depends on $|Q_{sel.}|$, *e.g.,* on the number of Kleene stars in typical patterns.

4) Unnecessary memoization: Our formal selection approaches will eliminate all super-linear behavior. It is instructive to consider unsound variations, cf. §X-A.

Our selective memoization schemes may involve unnecessary memoization. Our schemes select vertices according to analysis on an "unlabeled skeleton" of the NFA. Only vertices that meet a stronger condition actually need be memoized, namely that they be reachable along multiple ambiguous paths. For example, the regex `a|b` is unambiguous, so the vertex with in-degree 2 need not be memoized. However, our approximations have the advantage that the vertices involved can be identified in $\mathcal{O}(|Q|)$ steps during the NFA construction. For example, the vertices in $Q_{ancestor}$ are a superset of the "pivot nodes" necessary for infinitely ambiguous behavior; identifying those pivot nodes requires $\mathcal{O}(|Q|^6)$ to $\mathcal{O}(2^{|Q|})$ time complexity [26], [37], [80]. In our evaluation we find that $Q_{in-deg>1}$ and $Q_{ancestor}$ are typically small, so further refinement may not be worthwhile. Faster ambiguity analyses for typical regexes would enable further space reduction.

VIII. Representations of the Memo Function

Implementing an M-NFA requires an embodiment of the memo function indicated in Table II (the implementation of `mark` in Listing 2). Our selective memoization schemes will decrease the amount of state tracked by this memo function. If this state can be efficiently represented, the total space cost of memoization can be made lower still. We discuss three implementations of the memo function, two conventional and one novel. Their properties are summarized in Table IV.

TABLE IV: Properties of the memo function representations, ordered by their best-case space complexity. The first two are standard techniques. Based on regex engine semantics, we propose the application of run-length encoding to the memo function.

Representation	Access time	Space complexity
Memo table	$\mathcal{O}(1)$	$\Theta(\lvert Q_{sel.}\rvert\times\lvert w\rvert)$
Positive entries	$\mathcal{O}(1)$	$\Omega(\lvert w\rvert)$; $\mathcal{O}(\lvert Q_{sel.}\rvert\times\lvert w\rvert)$
Run-length encoding	$\mathcal{O}(\log k)$	$\Omega(\lvert Q_{sel.}\rvert)$; $\mathcal{O}(\lvert Q_{sel.}\rvert\times\lvert w\rvert)$

A. Memo table

One implementation of the memo function is a *memo table*, an array with a cell indicating each input-output pair. For K-regex memoization, this memo table would be a two-dimensional array whose cells are 0-1 valued. This array offers optimal access times and requires $|Q_{sel.}|\times|w|$ space.

B. Positive entries

Because the entries in the memo table can contain only two values, only the cells with one of the values need be tracked. In our context, we might track only the visited positions. A missing entry means no visit.

A data structure with efficient random access and update times (*e.g.,* a hash table) can be used to store only the visited cells, as is common for memoization in functional programming [71], [72], [81], This approach offers the same asymptotic access times as an array, although with larger constants. However, its space complexity is input-dependent and may be superior to an array. On some pathological inputs, the space cost is $\Omega(|w|)$, when only one of the memoized vertices is visited along w. For example, this would occur if the input exploits only some of the potential ambiguity in the regular expression. In the worst case, all of the memoized vertices are visited repeatedly, for cost $\mathcal{O}(|Q_{sel.}|\times|w|)$.

C. Run-length encoding

We propose to further decrease the space cost by compressing the information in the memo table. We interpret the memo table as an array of $|Q_{sel.}|$ *visit vectors*, one per memoized NFA vertex, each of length $|w|$. In the context of regular expressions these visit vectors may be compressible because the engine's search regime is ordered.

The ordered search regimes necessitated by regex engine match semantics (*e.g.,* PCRE's leftmost-greedy behavior) cause the memo function to be updated in an orderly manner. Updates will tend to accrete to adjacent indices as characters are processed. Because the alphabet of the memo table is small, this property results in visit vectors with consistently low entropy, and hence a high compression ratio. This observation has separately been described in terms of the locality of NFA traversals [82].

As a result, it is reasonable to expect that the memo tables in the K-regex context will be compressible. Many states

TABLE V: Time complexity for E-regexes within a backtracking (BT) framework, before and after memoization. See text for full treatment.

E-regex feature	Time: BT w/o memo	Time: BT w/ memo
REWZWA	$\mathcal{O}(\lvert Q\rvert^{3+2*\lvert w\rvert}\times\lvert w\rvert^2)$	$\mathcal{O}(\lvert Q\rvert^2\times\lvert w\rvert)$
REWBR	$\mathcal{O}(\lvert Q\rvert^{2+\lvert w\rvert})\times\lvert w\rvert^2)$	$\mathcal{O}(\lvert Q\rvert^2\times\lvert w\rvert^{2+2*\lvert\mathbf{CG}_{BR}\rvert})$

may be visited at only a few distinct indices of the input string, resulting in intervening compressible runs of 0's due to unvisited states. Other states, *e.g.,* quantifier destinations, may be visited at many indices. Some of these visits will be especially compressible. Consider, for example, the behavior of a regex engine on a monadic quantifier like `/.*/`. The NFA vertex corresponding to the quantifier would be memoized under all of our selective memoization schemes. Once an NFA simulation reaches this structure, it will recursively check for regex matches after consuming each of the *adjacent* characters in an ordered manner, accreting a compressible sequence of memo table entries.[5] Such quantifiers are common. In Davis *et al.*'s regex corpus [7], we found that among the 253,216 regexes that use an unbounded quantifier, fully 103,664 (41%) include the catch-all quantifier `/.*/` or `/.+/`.

As a compression strategy, we propose to use run-length encoding (RLE) [83] because it supports in-place updates. When a visit vector is implemented as a binary tree, with elements the runs keyed by their offsets [84], this scheme offers $\mathcal{O}(\log k)$ access time for a vertex with k runs. If the encoding scheme is effective, k will be small and the time cost will be competitive with the two other memo function representations we have discussed. If it is ineffective, k may be as large as $|w|$, substantially increasing the time cost of this scheme relative to the others. By the same token, if the encoding scheme is effective, the space cost of this scheme is *constant* for each visit vector. It remains $\mathcal{O}(|w|)$ per vector in the worst case.

IX. Memoization for E-regexes

Most regex engines support extensions beyond K-regexes. They should offer E-regex implementations that do not (unnecessarily) risk ReDoS. Up to 5% of general-purpose regexes use E-regex operators, most commonly involving zero-width (lookaround) assertions (REWZWA) and backreferences (REWBR) [4], [5]. In this section we study the worst-case performance of backtracking implementations for these features. Table V gives time complexity improvements.

We give the first description of how REWZWA are implemented within typical regex engines, the first analysis of the time complexity of REWZWA within the backtracking framework, and the first application of memoization to reduce REWZWA complexity to linear in $|w|$. REWBR has been studied more carefully because of its use in IDSs (*e.g.,* Snort rulesets). The general REWBR recognition problem is NP-hard [44], [52], [85], and the time complexity for an E-regex can be parameterized in terms of the number of backreferences [40], [86] (cf. §XII). As we did for K-regexes, we show how to use memoization to match the best known time complexity within a backtracking framework, and apply selective memoization to improve space complexity. Our time complexity analyses use the cost formula from §VI, Eqt. (1).

A. Regexes with zero-width assertions (REWZWA)

In typical backtracking implementations, the time complexity of REWZWA is exponential. When REWZWA are implemented with memoized backtracking, the cost can be lowered to linear in w — from $\mathcal{O}(|Q|^{2*|w|+1})$ to $\mathcal{O}(|Q|^2\times|w|)$.

1) Semantics: Many E-regexes dialects support *zero-width assertions*. In PCRE, a zero-width assertion permits a condition about w (*assertion*) to be tested without advancing the simulation state (*zero-width*). Some assertions examine a fixed number of characters, *e.g.,* the anchor $ or the word boundary `\b`. Others may examine the entire string w. These latter are commonly called *lookaround assertions*.

We examined the expressiveness of lookaround assertions in various programming languages, and opted for a middle-of-the-road semantic: our lookaround assertion model permits the user to test whether, beginning at index i, a substring of w matches a K-regex either rightward (*lookahead*) or leftward (*lookbehind*).[6] For example, the REWZWA `(?=a+)\w+(?<=z+)` matches all strings of "words" that begin with one or more a's and end with one or more z's.

2) Backtracking implementations: We examined the regex engines of Perl, PHP, Python, and JavaScript-V8. These regex engines expand the asserted pattern into the automaton, match it within the backtracking framework, but omit advancing the offset into w because the assertions are zero-width. In terms of standard NFA representations, this type of implementation can be modeled as introducing α-edges that describe a regular string pattern via sub-automata. Like ϵ-edges, these α-edges consume no characters.

3) Current time and space complexity: The space complexity of REWZWA in these implementations is the same as for K-regexes. The time complexity is worse. For REWZWA with K-regex assertions, because the assertions do not consume characters, *each step* of the simulation may traverse α-edges at an exponential cost. A coarse bound for the time complexity is thus $\mathcal{O}(|Q|^{|w|}\times|Q|^{|w|}) = \mathcal{O}(|Q|^{2*|w|})$. Following Eqt. (1) more carefully, we have $|Q|\times|w|$ simulation positions, up to $|Q|^{|w|}$ visits per position, and any α-edges bear the full cost of a backtracking sub-simulation over w before processing the $|Q|$ edges from δ:

$$\mathcal{O}\big((|Q|\times|w|)\times(|Q|^{|w|})\times(|Q|+|Q|^{2+|w|}\times|w|)\ \big)$$
$$=\mathcal{O}\big(|Q|^{3+2*|w|}\times|w|^2\big)$$

Note that the second and third terms are both exponential for the same reason: redundancy during backtracking.

[5]Non-greedy quantifiers do not change this locality, but merely reverse the order in which the indices are explored. The exploration entropy remains low.

[6]Rust and Go do not support lookaround assertions. For lookahead, JavaScript, Java, Python, PHP, & Ruby support K-regexes and some E-regexes features (*e.g.,* backreferences). For lookbehind, JavaScript and Java support K-regexes. Perl, PHP, Ruby, & Python support only fixed-width assertions.

4) Application of memoization: By applying memoization, we can reduce the time complexity for REWZWA to linear in $|w|$. This reduction in time complexity follows from a simple observation: for each α-edge, the simulations of the corresponding sub-automaton will all operate on the same sub-automaton and on some substring of w. Rather than treating these simulations independently, we can retain what we learn in one for use in the next.

Extension of M-NFA To operationalize this observation, we must extend the REWZWA model in two ways. First, convert the top-level automaton and each sub-automaton to M-NFAs. Second, preserve all memo functions across the simulation of the higher-level M-NFA, remembering the results from sub-simulations that begin at different indices i of w.

Time complexity Because REWZWA implementations use a single large automaton (flat, with no levels), in practice this modeling requires minimal modification to an existing memoized implementation. The time complexity analysis is easy to follow when expressed in terms of this single automaton. Supposing a $|Q_{all}|$ or $|Q_{in-deg>1}|$ memoization scheme for simplicity, we obtain the familiar:

$$\mathcal{O}\big(\ (|Q|{\times}|w|)\ \times\ 1\ \times\ |Q|\ \big) = \mathcal{O}\big(|Q|^2{\times}|w|\big)$$

We assume as before that the top- and sub-automata are of similar sizes. Memoization removes redundancy in the second and third terms. This result can be obtained in our hierarchical α model by dividing the cost across the hierarchy of automata, using the insight that each sub-automaton simulation cost is *amortized* $|Q|^2{\times}|w|$ because the memo function is preserved from one sub-simulation to the next.

Space complexity The space complexity of such a simulation is the standard $\mathcal{O}(|Q|{\times}|w|)$. This is the same as would be required to simulate a similarly-sized K-regex regex, one for which the zero-width sub-automata were instead concatenated into the outer automaton, *e.g.,* `a(?=b*)` $\rightarrow$ `ab*`.

Selective memoization Our proposal for REWZWA is essentially recursive. The selective memoization schemes proposed earlier are directly applicable.

B. Regexes with backreferences (REWBR)

The general REWBR matching problem is NP-complete and all known solutions are exponential in some combination of $|Q|$ and $|w|$. Compared to typical backtracking implementations, memoization reduces the time complexity from exponential in $|w|$ to exponential in $|Q|$ — roughly speaking, from $\mathcal{O}(|Q|^{|w|})$ to $\mathcal{O}(|w|^{|Q|})$. Since we typically expect $|w| \gg |Q|$, the reduction can be substantial.

1) Semantics: Many E-regex dialects support a *backreference operator*. In PCRE, this operator permits an intra-pattern reference to the substring of w matched by an earlier labeled sub-pattern (cf. capture groups, §III-C). For example, the E-regex `(a|b)\1` matches “aa” and “bb” but not “ab” or “ba”. There are dialectal variations of REWBR semantics [41], [44], [52], [87], but the details are unimportant for our analysis.

2) Backtracking implementations: Spencer-style regex engines support parsing by tracking the substring of w matched by each labeled sub-pattern using a pair of indices $\langle j, k\rangle$. To evaluate a backreference at offset i, they compare the subsequent characters from *w[i:]* to the current contents of the appropriate sub-pattern. This can be modeled by introducing β-edges that perform string (not character) comparison.

3) Current time & space complexity: REWBR cannot be evaluated in linear time in $|w|$, nor by finite automata (which can only encode memory via the current vertex). Whereas K-regex recognition is a context-free task (§V), REWBR are context-sensitive. A REWBR regex can be crafted that requires the exploration of all the (exponentially many) paths through the automaton [44], [52], [85]. Of course, in the worst case, existing Spencer-style engines *already* explore all these paths (hence ReDoS). But note that it is more expensive to test a β-edge than a regular one, because it entails an $\mathcal{O}(|w|)$ string comparison. The worst-case time complexity for a REWBR match within an un-memoized backtracking framework is thus:

$$\begin{aligned}\mathcal{O}\big(\ (|Q|{\times}|w|)\ \times\ |Q|^{|w|}\ \times\ (|Q|+|w|)\ \big)\\ = \mathcal{O}\big(|Q|^{2+|w|}{\times}|w|^2\big)\end{aligned}$$

The simplified form uses the property $|Q| + |w| \leq |Q|{\times}|w|$.

4) Application of memoization:

Extension of M-NFA Because the contents of a capture group depend on the path taken through the automaton, REWBR disrupts our path-independent memoization scheme. The outcome of the extended δ_M function (for the β-edges) is no longer determined solely by q and $w[i]$, but also requires knowledge of the appropriate capture group. If we reach this edge in a different state, we must re-evaluate it. But *if reached in the same state, this redundant visit can still be eliminated.*

For example, the regex `<(\w+)>(a|a)+</\1>` uses a backreference to match HTML tags. It contains the exponential K-sub-regex `(a|a)+`. This sub-pattern may result in exponentially many visits to the β-edge, all of which share the same capture group vector $\mathbf{CG}$ and vary only in their simulation positions. Incorporating $\mathbf{CG}$ into the memo functions permits these redundant paths to be eliminated. To accomplish this, we extend the M-NFA described in Table II:

Capture groups The simulation must track the capture group vector $\mathbf{CG}$, where CG_i denotes the i^{th} capture group.

Memo functions The domains of the REWBR memo function M' and transition function δ'_M must consider the simulation position π as well as the capture group vector $\mathbf{CG}$. Each CG_i can be represented with a pair (a, b) of indices into w.

The memo functions now depend on the current simulation position $\pi = \langle q, i\rangle$ as well as the *path state* $\langle CG_1, CG_2, \ldots, CG_{|Q|}\rangle$ of the $\mathbf{CG}$ configuration.

Time complexity The change in memo functions inflates the first term in Eqt. (1), but is offset by the decrease in the second term. In the first term, each CG_i can take on $|w|^2$ distinct values. Again supposing a Q_{all} or $Q_{in-deg>1}$ memoization scheme, we have time complexity of:

$$\begin{aligned}\mathcal{O}\big(\ (|Q|{\times}|w|{\times}(|w|^2)^{|\mathbf{CG}|})\ \times\ 1\ \times\ (|Q|+|w|)\ \big)\\ = \mathcal{O}\big(|Q|^2{\times}|w|^{2+2*|\mathbf{CG}|}\big)\end{aligned}$$

Note that the exponents have changed places when compared to the un-memoized version: from $\mathcal{O}(|Q|^{|w|})$ by counting (potentially-overlapping) paths to $\mathcal{O}(|w|^{2*|Q|})$ by counting distinct path configurations. This bound remains problematic if more than a few backreferences are used. In §X we report that typical regexes use backreferences sparingly.

Space complexity We will suppose a full memoization scheme for simplicity. There are $|Q|\times|w|$ simulation positions, and $|\mathbf{CG}|$ capture groups. The memo function's domain is thus

$$\mathcal{O}\big(|Q|\times|w|\times(|w|^2)^{|\mathbf{CG}|}\big) = \mathcal{O}\big(|Q|\times|w|^{1+2*|\mathbf{CG}|}\big).$$

Selective memoization Selective memoization can be applied to reduce the cost of REWBR. First, only the indices of the *backreferenced* capture groups affect the simulation result, so we need only memoize the sub-vector $\mathbf{CG}_{BR} \subseteq \mathbf{CG}$. This reduces the corresponding exponent for both time and space complexity. Second, as with REWZWA, the $Q_{in-deg>1}$ and $Q_{ancestor}$ can be applied to REWBR to reduce the space complexity's vertex factor $|Q|$.

X. Evaluation

The memoization schemes described in §VII provably eliminate ReDoS by guaranteeing K-regex matching times that are linear in $|w|$. We extended this guarantee to REWZWA within the backtracking framework, and parameterized the super-linearity for REWBR. Our selective memoization schemes incur varying additional space complexity, which may be offset by efficient representations of the memo function (§VIII).

Here we analyze comparable defenses, experimentally confirm our security guarantees (time complexity), and assess the practicality for typical regexes (space complexity and costs).

A. Security analysis: Existing memoization-like defenses

We surveyed regex engine implementations and behavior described in the literature [3], [7], [16], and identified memoization-like ReDoS defenses in Perl and RE2. These defenses unsoundly target only exponential worst-case behavior. To avoid the $|Q|\times|w|$ cost of full memoization, Perl unsoundly and incompletely memoizes certain vertices. while RE2 unsoundly caches a constant number of simulation results. Both schemes exhibit exponential and polynomial behavior.

Perl Since 1999, the Perl regex engine has employed an unsound semi-selective memoization scheme called the "super-linear cache". It is undocumented and its workings have not previously been described in the scientific literature.[7] Perl memoizes visits to the first k=16 NFA vertices associated with repetitions `A*` if the language of `A` has strings of varying lengths, *e.g.,* `(a|aa)*`. Perl omits other repetitious states, *e.g.,* bounded repetition (`a{3}`) or repetition with a fixed-length pattern (`.*.*`). The memo table is erased when a backreference β-edge is tested.

Perl's approach is similar in spirit to our $Q_{ancestor}$ scheme, but it is restricted to k=16 cycle ancestors and only considers those whose cycles can be of varying lengths. This scheme is not sound — *e.g.,* it protects `(a*)*` (otherwise exponential), but not `(a|a)*` (exponential) nor `a*a*` (quadratic). We conjecture that this scheme was designed to eliminate exponential worst-case K-regex behavior due to nested quantifiers [6], but did not consider other forms of exponential behavior nor weaker polynomial behavior. Davis *et al.* described many regexes this defense failed to protect [7].

RE2 Although the RE2 regex engine generally uses Thompson's lockstep algorithm, in rare cases it uses backtracking [16]. An unsound *memoization cache* records visits to C=32 KB's worth of simulation positions (full memoization). This scheme is effective if $|Q|\times|w| \le C$, *e.g.,* for small regexes and small inputs. It is not useful for large regexes or large inputs, the common case for polynomial worst-case behavior (*e.g.,* Stack Overflow's outage). Our space-efficient memoization proposal is more suited for those cases.

B. Dataset and measurement instruments for our approach

Regex corpus Researchers have collected several regex corpuses [4], [6], [7], [59]. We use Davis *et al.*'s regex corpus [7], which is the largest and most representative of typical regex practices [5]. It contains 537,806 regexes from 193,524 software projects. This corpus includes 51,224 super-linear K-regexes and inputs to trigger worst-case behavior.

Prototype We implemented the selective memoization and encoding schemes within a Spencer-style backtracking regex engine published by Cox [88]. The baseline regex engine supported K-regexes, plus capture groups, the optional (`?`) and non-greedy (`*?`) quantifiers, and the catch-all character class (`.`). Unmodified, it could evaluate approximately 17% of the regex corpus and 15% of the super-linear K-regexes.

To enable measurement on a wider variety of regexes, we added support for commonly-used features including anchors, escape sequences, and character classes. We also implemented REWZWA (lookahead) and REWBR ($\le$ 9 groups) following the algorithms described in §IX. Our prototype supports 357,013 (66%) of the corpus regexes. We will refer to these as the *supported regexes*. Among these are 43,171 (84%) of the super-linear K-regexes.

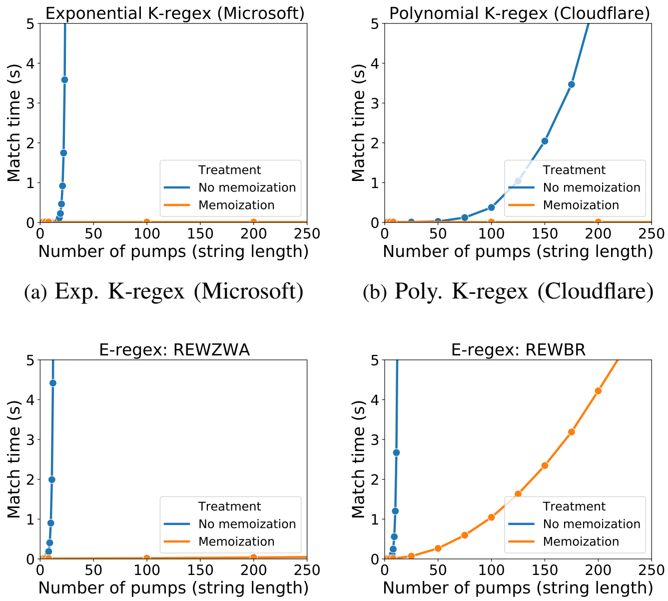
We used appropriate data structures for the memo function representations. We implemented the memo table representation using a bitmap; the positive entry representation using Hanson's `uthash` hash table [89]; and the RLE-based representation using Biggers's `avl_tree` balanced binary tree [90]. We parameterized RLE with runs of length 1 for the common case of monadic quantifiers. Discounting vendored files, our modifications added 5,408 lines (prototype, tests, and measurement tools). We will open-source it.

C. Security analysis: Time complexity

This experiment is to confirm our theoretical guarantees.

1) Methodology: Our theorems predict the maximum number of visits to each simulation position, with the effect that the worst-case behavior grows linearly with $|w|$. As shown in §VII (Table III), the selection schemes vary in the bound they offer on the number of visits to each simulation position.

[7] We describe it as of February 2020. See https://github.com/Perl/perl5 commit 34667d08d.



(a) Exp. K-regex (Microsoft) (b) Poly. K-regex (Cloudflare)

(c) Sample REWZWA (d) Sample REWBR

Fig. 4: Case studies of applying our techniques to K-regexes and E-regexes. The regexes are: (a) Exponential Microsoft username regex (responsibly disclosed); (b) Cloudflare regex (§II); (c,d) Hand-crafted examples. All K-regexes and REWZWA can be handled in linear-in-$|w|$. For REWBR, memoization reduces the degree of the exponent.

For simplicity of presentation, for each selection scheme we therefore measured *the total number of simulation position visits* as we increased $|w|$ (it should grow linearly).

We tested each supported super-linear regex with problematic inputs of varying lengths. Each regex has an *input signature* consisting of a constant prefix, a list of consecutive "pump strings", and a constant suffix. The more times each pump string is repeated, the more times certain simulation positions will be visited by an un-memoized backtracking search. We varied the length of each input, from 10,000 pumps to 100,000 pumps at intervals of 10,000. Since our measurements are of algorithmic quantities, we measured each configuration only once, with full memoization.

2) Results: On these regex-input pairs, the prototype behaved as predicted. The total number of simulation position visits grew linearly with $|w|$ in every case. We illustrate the corresponding reduction in matching time in Figure 4.

D. Practicality analysis: Space complexity and costs

This experiment assesses the practicality of our approach. We measure the typical space complexities of the supported regexes (*i.e.,* values for $|Q_{sel.}|$) and the actual space costs of memoization under different memo function representations.

1) Methodology: The **space complexity** of a selective memoization scheme depends on the size of the selected vertex set Q_{sel}. We measured typical sizes of Q_{all}, $Q_{in-deg>1}$, and $Q_{ancestor}$ for the supported regexes. We used a non-minimized Thompson construction for the NFAs.

The **space costs** of our implementations depend on properties of the regexes, the inputs, and the memo function representations. For all of the supported super-linear regexes, we measured the space used under each of the nine combinations of selection scheme and memo function representation. For this experiment we used 100,000 pumps (resulting in strings of length $\approx 10^6 - 10^7$ characters). The space costs we report correspond to the asymptotic costs of each combination:

Memo table : $|Q_{sel.}| \times |w|$ for the selected vertex set $Q_{sel.}$.

Positive entry : The number of distinct simulation positions visited during simulation (maximized at the end).

RLE : Across all $q \in Q_{sel.}$, we summed the maximum number of runs observed at q during the simulation. These values may be maximized mid-simulation, *e.g.,* if distinct runs arise but are eventually merged.

As in the previous experiment, the results are deterministic, so it was only necessary to measure each configuration once.

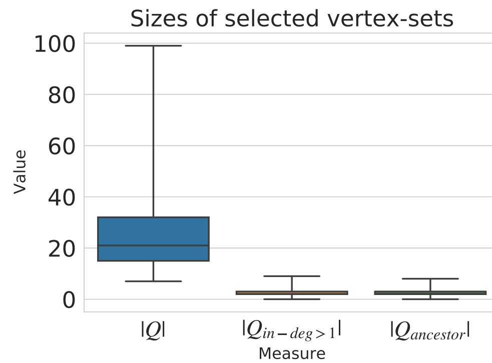


Fig. 5: Sizes of the vertex sets for the selective memoization schemes. Whiskers indicate the (1, 99)th percentiles. Outliers are not shown. Among the supported regexes, the results are similar for all regexes (shown) and the super-linear subset (not shown).

2) Results: **Space complexity** The distribution of $Q_{sel.}$ sizes under various selection schemes is shown in Figure 5. We note two aspects of the data. First, as has been conjectured, our data show that full memoization would indeed be costly. The 75th percentile of $|Q| = |Q_{all}|$ is 32. On long inputs, the $|Q|\times|w|$ cost of full memoization would therefore be significant for many regexes. Second, our data show that $|Q| \gg |Q_{in-deg>1}| \approx |Q_{ancestor}|$. Our selective memoization schemes will thus exhibit space complexities an order of magnitude lower than the full memoization scheme.

Space costs The relative space costs of the memo function representations are given in Figure 6, calculated as proportions of $|Q|\times|w|$ (*i.e.,* the space complexity of full memoization). The proportional decrease for the Positive Entry representation follows the decrease in $|Q_{sel.}|$. In contrast, the RLE representation achieves *constant* space costs for most regexes. It has a 95th percentile of 12 for $Q_{in-deg>1}$ and 10 for $Q_{ancestor}$.

E. Extensions: Complexity of E-regexes

§IX presented novel algorithms for evaluating two E-regex features (REWZWA and REWBR) within a backtracking framework. In Figure 4 we illustrated the effect of memoization on these features. Unfortunately, we cannot systematically evaluate our approach because of limitations in the regex corpus. The state-of-the-art super-linear regex analyses [23], [26], [37], [91] cannot identify super-linear behavior arising

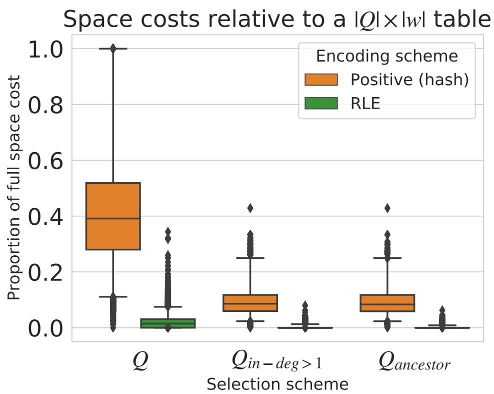


Fig. 6: Relative space costs to handle super-linear K-regexes with input pumped 100,000 times. Costs are relative to a baseline of full memoization with a memo table ($|Q|\times|w|$). Whiskers indicate the $(1, 99)^{\text{th}}$ percentiles. Outliers are shown.

from extended features (§IX).[8] A new regex analysis for E-regexes lies beyond the scope of this work.

However, we can provide measurements to indicate typical parameter values for REWBR. Although several authors have parameterized the worst-case time complexity of REWBR in general [40], [86], no regex corpus has been analyzed to understand typical REWBR use. Per our analysis in §IX-B, we are interested in $|\mathbf{CG}_{BR}|$ and $|Q_{BR}|$. To characterize these costs, we present novel measurements of the use of REWBR.

In the regex corpus, most regexes contain few distinct back-referenced capture groups and few backreference uses. Among the $\approx 3,500$ REWBR in the corpus, 98% contain at most three backreferences ($|BR| \leq 3$), and 98% contain backreferences to at most two distinct capture groups ($|\mathbf{CG}_{BR}| \leq 2$). For typical regexes the worst-case time and space complexity of REWBR is a (relatively) small polynomial of $|w|$.

An additional parameterization can further characterize the costs of typical backreference-using regexes [40]. If a backreferenced capture group has unbounded width (*i.e.,* contains a *), then it adds a factor of $|w|^2$ to the time and space complexities. If it has limited width, then it can take on only $|w|$ distinct indices, not $|w|^2$, removing a factor of 2 in the exponent. In the corpus, 78% of the REWBR regexes use only fixed-width capture groups, most commonly to find a matching single or double quote to a string, *e.g.,* a pattern `('|")\w+\1`.

F. Analysis

In §X-C we substantiated our theoretical guarantees. In §X-D we showed that if regex engines combine selective memoization with a good memo function representation, they can defeat ReDoS with lower space complexity, and achieve *constant* space costs for 90% of K-regexes. These benefits extend to REWLA. There is even hope for REWBR, as typical usage entails low-polynomial time and space complexity.

G. Limitations

The corpus of Davis *et al.* is limited to the regexes employed in general-purpose programming (*i.e.,* popular open-source software modules) [7]. It may not be representative of the regexes used in specialized application domains, *e.g.,* IDS [39]. We aim to support general-purpose regex engines, and do not propose to compete with specialized regex engines tailored to a particular domain [82], [92].

Our prototype does not support all K-compatible nor E-regex features. To maximize corpus support, we added feature support following popularity measurements by Chapman & Stolee and Davis *et al.* [4], [5].

[8] [91] supports E-regexes but only finds problematic K-regex sub-patterns.

XI. Discussion

A. How much space overhead is too much?

Super-linear regex behavior anywhere in the software stack can expose applications to ReDoS. We have demonstrated how to eliminate all such behavior through memoization, at the cost of some space overhead. Regexes are a programming primitive used in myriad ways, so we cannot offer a general answer to the question "How much space overhead is too much?" As one data point, Cox proposed 32 KB as a reasonable memory footprint for memoization overheads [16]. Server-class machines can likely afford the cost of full memoization, whether KB or GB. This cost may be too great for containerized or memory-capped application deployments, which might benefit from our selective memoization techniques. For components in cyber-physical systems, *e.g.,* IoT devices, memory is at a premium and further optimizations may be necessary (*e.g.,* our space-efficient memo function representation).

A study of the memory constraints across all ReDoS-vulnerable deployments is beyond the scope of this paper. Our theoretical bounds on space complexity can be used by practitioners to analyze and choose appropriate memory configurations. Our techniques accommodate a wide range of usage in a general-purpose regex engine, with small constant costs for a large fraction of regexes.

B. Incorporating into real-world Spencer regex engines

We believe our techniques can be applied to real-world backtracking regex engines with minimal disruption to the codebase. Conceptually, memoization is an "inline" change for these regex engines. For example, the memoization modifications for our prototype involve a single `if`-condition within the backtracking framework (Listing 2). In real-world regex engines the scope of changes could be slightly more involved. For example, Perl's unsound memoization scheme is updated at three points instead of one to accommodate local optimizations. However, in the many backtracking regex engines we have examined there is a search loop comparable to Listing 1 with clear points where memoization could be introduced. The difficulty has always lain not in introducing memoization, but rather in determining how little can be memoized while ensuring soundness, and in measuring the cost of memoization for typical regexes.

Based on our evaluation, we recommend that the maintainers of backtracking regex engines incorporate a $|Q_{in-deg>1}|$-based memoization scheme for K-regexes. Compared to $Q_{ancestor}$, the $Q_{in-deg>1}$ selection scheme has a stronger time complexity guarantee (Table III) and similar space costs for typical regexes. However, the outlier values of $Q_{in-degree>1}$ are an order of magnitude larger than those of $Q_{ancestor}$, so $Q_{ancestor}$ might be used as a back-up option in some cases. In either case, an RLE-based memo function representation is quite effective for the common case of the visit vectors associated with "catch-all" quantifications. A memo table may be more useful for other visit vectors.

XII. Other Related Work

We treated much of the related work earlier in the paper. Our work is also informed by other applications of memoization, and by research into domain-specific regex engines.

A. Other applications of memoization

Memoization has been widely applied in functional programming [71], logic programming [72], [93], [94], and dynamic programming [70], [95]. Most relevant to our work, memoization has also been applied to parsing problems, *e.g.,* "Packrat Parsers" for context-free grammars (CFGs) and parsing expression grammars (PEGs) [47], [73]–[76].

Any algorithm that relies on memoization must address the accompanying space cost. Researchers have explored strategies including caching policies [16], [81], [96], partial memoization to avoid part (but not all) of the repeated computation [97], [98], and selective memoization to record some (but not all) of the input-output pairs [99]. These techniques have occasionally been applied to string parsing [100], but in general the space costs of string parsing are not critical.[9] Context matters — when CFGs are used to parse software projects (not a latency-critical task), the input is trusted, and the modules being parsed are typically not of great length [74]. When researchers have considered security-sensitive parsing contexts, *e.g.,* XML [101]–[103] and JSON [104], they have focused on time rather than space. For regexes both attributes must be considered. Large regexes are deployed on latency-critical paths on long untrusted input. Time costs must be lowered to avoid ReDoS, but this must be done while minimizing space overheads.

B. Domain-specific regex engines

We have described optimization techniques for the general-purpose regex engines provided with programming languages, and evaluated them on measurements of regex usage in a large sample of software modules. Other researchers have tailored regex engine optimizations for specific application domains. For intrusion-detection systems (IDS), regex evaluations must run as close to line speed as possible, and researchers have focused on how to make these evaluations efficient and predictable without regard to algorithmic extensibility [39], [40], [92], [92]. For information retrieval and analysis contexts, *e.g.,* genomic analysis and text exploration, researchers have proposed "fuzzy" approximate matching semantics and algorithms [105]–[110]. Our techniques support general-purpose regex usage, without requiring substantial changes to existing algorithmic frameworks. Our work is orthogonal to these domain-specific optimizations.

[9]Packrat Parsers for CFGs have $\mathcal{O}(|G|\times|w|)$ space complexity, where G represents the rules in the grammar [74].

C. Other treatments of E-regexes

§IX presents the first analysis of the problematic current behavior of REWZWA, and the potential REWZWA performance within the backtracking framework. The time complexity we have achieved for REWZWA using memoization is unsurprising from an automata-theoretic perspective, because our REWZWA model does not add expressive power to a K-regex. The set of regular languages is closed under intersection [30], and lookaround assertions are intersection-like — the language of $R_1 = A(B)$ is similar to the language of $R_2 = A \cap B$. Our approach is a means of obtaining linear time complexity without pre-computing the intersection.

Other researchers have shown that REWBR problems have exponential worst-case time complexity (and are NP-hard) [44], [52], [85], and that their time complexity can be parameterized in terms of the number of backreferences [40], [86]. Becchi & Crowley proposed a Thompson-style algorithm for REWBR [39], and Namjoshi & Narlikar discussed the time complexity of such an approach [40]. We show how to obtain the same time complexity for legacy regex engines, by fitting our approach within the framework of backtracking. As noted in §V, our primary goal is not to exceed the theoretical state of the art, but to enable the advance of the state of practice.

XIII. Concluding remarks

Regular expressions are used for string processing in every layer of the software stack. Most programming languages have chosen to implement their regex engines within the algorithmic framework of backtracking. This simplifies the implementation and gives users expressiveness. But it also introduces exponential worst-case behavior that can lead to regex-based denial of service (ReDoS).

The ReDoS problem has been rumored since the 1960s [32] and defined carefully since 2002, but has thus far defied a practical solution. Although there are alternative algorithmic frameworks with better time complexity guarantees, they complicate implementations and limit extensibility and expressiveness. Meanwhile, a memoization-based optimization would fit nicely into the backtracking framework, but these have been rejected for their high space complexity.

In this paper we revisited this longstanding problem from a data-driven perspective. We empirically confirmed claims that full memoization is overly expensive. Then, we proposed two selective memoization schemes that offer comparable time complexity guarantees with lower space complexity. Using a newly-available large-scale regex corpus, we evaluated these schemes and found that they offer an order of magnitude reduction in memoization space costs for typical regexes. Furthermore, leveraging insights into regex engine search semantics, we showed that memoization space costs can be further reduced to constant for typical regexes. In short, we have demonstrated a provably sound ReDoS defense that fits within a backtracking framework, and shown how to reduce its space cost from problematic to constant for the majority of super-linear regexes.

APPENDIX
PROOFS OF THE SELECTIVE MEMOIZATION THEOREMS

Here we present proofs of Theorems 1 to 3.

A. Definitions

These definitions are used in Theorems 2 and 3.

Definition 1 (Simulation position)**.** *For a regex engine following the backtracking algorithm given in Listing 1, we define a* simulation position $\pi = \langle q \in Q, i \in \mathbb{N}^{|w|} \rangle$ *as one of the possible simulation positions on which the* `recurse` *function is called. Two simulation positions are* different *if they differ in the automaton vertex q or the candidate string index i. If a simulation position is subscripted π_i, we may denote its automaton vertex as q_i.*

Definition 2 (Simulation path)**.** *We define a* simulation path *of simulation positions, denoted $\Pi = \pi_0\pi_1 \dots \pi_n$. This represents a valid sequence of positions visited by the backtracking algorithm. In a simulation path, π_0 is the position $\langle q_0, 0\rangle$, and each π_i is in $\delta(\pi_{i-1})$. Two simulation paths are* different *if they are of different lengths, or if at some index i they contain different simulation positions, i.e., are at different automaton vertices.*

We introduce the following concept to refine the statement of Theorem 3 from that given in §VII.

Definition 3 (Bounded ambiguity)**.** *Let A be an ϵ-free NFA. We define its* bounded ambiguity *as:*

$$boundedAmbiguity(A) = \max_{0 \le i \le |Q|} \Big(\max_{s,t \in Q} \Big(\text{\# distinct simulation paths } s \rightsquigarrow t \text{ of length } i \Big) \Big)$$

Note that $boundedAmbiguity(A)$ differs from the ambiguity of A. An automaton can be infinitely ambiguous, *i.e.,* increasingly-long candidate strings can be defined whose ambiguity is larger than any finite bound. In contrast, our definition of $boundedAmbiguity(A)$ captures the maximum possible ambiguity for strings of length no more than $|Q|$.

B. Assumptions (M-NFA pre-processing steps)

In our theorems and proofs, we assume that the M-NFAs involved have two additional properties: having one accepting state, and being ϵ-free. These properties are standard proof tactics for automata [31], [111].

First, we assume that the M-NFAs are modified to have a single accepting state q_F. This ensures that if a candidate string is ambiguous, then the ambiguous paths all terminate at the same vertex q_F. Any M-NFA can be converted with no change in its language: introduce q_F, direct ϵ-edges to it from the vertices in F, and update F to $F = \{q_F\}$.

Second, we assume that the M-NFAs are ϵ-free. This has the convenience of ensuring that the string index i increases for consecutive simulation positions in a simulation path, *i.e.,* every step consumes a character from w. Any M-NFA can be converted with no change in its language: δ must be defined as δ_ϵ, computing the ϵ-closure such that every transition consumes a character from w. However, for the standard Thompson NFA construction, our proofs hold with minor modifications.

C. Theorems and proofs

Theorem 1. *Let the memo function track simulation positions involving all M-NFA vertices, $Q_{all} = Q$. Then every simulation position $\langle q, i\rangle$ will be visited at most once, and the M-NFA is unambiguous.*

Proof. This result follows trivially from the definition of δ_M, the memoized transition function. See §VII-A. □

Theorem 2. *Let the memo function track simulation positions involving the M-NFA vertices with in-degree greater than one, $Q_{in-deg>1}$. Then every simulation position will be visited at most once, and the M-NFA is unambiguous.*

Put simply, this theorem states that all ambiguity is the result of joining paths. Without splits, only one path is possible; without joins, different paths cannot lead to ambiguity because of the q_F assumption.

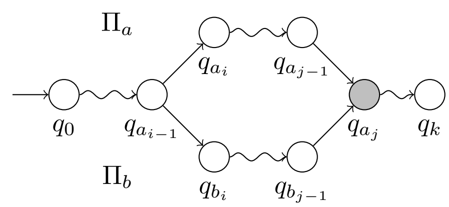


Fig. 7: Illustration for the proof of Theorem 2. Vertex $q_{a_j} = q_{b_j}$ must have in-degree > 1 because it is the first point of convergence after the split at vertex $q_{a_{i-1}} = q_{b_{i-1}}$.

Proof. The proof proceeds by contradiction. Suppose the $Q_{in-deg>1}$ memoization scheme is employed but some search position π_k is visited twice. This means an M-NFA simulation traverses different simulation paths Π_a and Π_b,

$$\Pi_a = \pi_{a_0} \dots \pi_{a_i} \dots \pi_{a_j} \dots \pi_{a_k} \dots, 0 \le k \le |w|$$
$$\Pi_b = \pi_{b_0} \dots \pi_{b_i} \dots \pi_{b_j} \dots \pi_{b_k} \dots, 0 \le k \le |w|,$$

such that (Figure 7):

1) The paths diverge. At some $i < k$, $\pi_{a_i} \neq \pi_{b_i}$, *e.g.,* at a point of non-determinism.
2) The paths converge. There is some smallest j, $i < j \le k$, such that $\pi_{a_j} = \pi_{b_j}$, and so $q_{a_j} = q_{b_j}$.

The paths must diverge, else they would not be different and π_k would be visited only once. They must converge, else $\pi_{a_k} \neq \pi_{b_k}$. Now, because the paths converge, the vertex in $\pi_{a_j} = \pi_{b_j}$ must have in-degree > 1. In more detail, since j was the earliest point of convergence after $i-1$ on the two paths, it must be that $q_{a_j} \in \delta(q_{a_{j-1}}, w_{j-1})$ and likewise $q_{b_j} \in \delta(q_{b_{j-1}}, w_{j-1})$. Since $\pi_{a_{j-1}} \neq \pi_{b_{j-1}}$, we have $q_{a_{j-1}} \neq q_{b_{j-1}}$, and so the in-degree of $q_{a_j} > 1$.

But if $q_{a_j} \in Q_{in-deg>1}$, the M-NFA simulation will not traverse both Π_a and Π_b. We are memoizing all visits to automaton vertices with in-degree > 1. Without loss of generality, suppose we first visit q_{a_j} via the simulation path Π_a, thus marking π_{a_j} in the memo function M. When we backtracked to $\pi_{a_{i-1}}$ and evaluated the alternative path Π_b, at $\pi_{b_{j-1}}$ we would have found the path eliminated by the memo function: $q_{a_j} \notin \delta_M(q_{b_{j-1}}, w_{j-1})$. So we cannot reach the search position $\pi_{a_j} = \pi_{b_j}$ more than once along these paths, because Π_b would terminate at $\pi_{b_{j-1}}$.

□

Next we re-state Theorem 3, refined using the definition of $boundedAmbiguity(A)$.

Theorem 3 (Refined)**.** *Let the memo function track simulation positions involving the M-NFA vertices that are "cycle ancestors", $Q_{ancestor}$, i.e., the automaton vertices to which any back-edges in a topological sort from q_0 are directed. Then the M-NFA is finitely ambiguous. A simulation position involving a vertex t will be visited at most (1) Once if $t \in Q_{ancestor}$; (2) $boundedAmbiguity(A)$ times if t is not reachable from a cycle ancestor; and (3) $|Q_{ancestor}|\times|Q|\times boundedAmbiguity(A)$ times otherwise.*

Put simply, this theorem states that when back-edges can be taken at most once from any simulation position, then ambiguity in the simulation cannot compound. The simulation will retain any ambiguity in its "cycle-free" analog (*i.e.,* a variant that has the back-edges removed). The ambiguity may increase as the result of back-edges taken at different offsets, but it remains bounded. Recall the illustration in Figure 3 (b): ambiguity remains possible but is limited. An example of the theorem calculations is given in Figure 8.

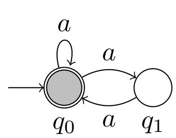


Fig. 8: Illustration of the calculations for Theorem 3. This is an ϵ-free M-NFA for the regex $(a|aa)*$. Here, $|Q| = 2$ and $|Q_{ancestor}| = 1$. There are two distinct $q_0 \rightsquigarrow q_0$ paths of length two, so $boundedAmbiguity(A) = 2$. The theorem gives a bound of $1 * 2 * 2 = 4$ visits to any simulation position, although this bound is not realized in this automaton because the two paths share a memoized vertex q_0.

Proof. Choose a target simulation position $\pi = \langle t \in Q, i\rangle$. We will show the visit bound for each case.

Case $t \in Q_{ancestor}$**:** If $t \in Q_{ancestor}$, then the memo function ensures that π is visited at most once.

Case $q \in Q_{ancestor} \not\rightsquigarrow t$: If there is no path from a cycle ancestor to t, then every simulation path reaching π must be cycle-free and thus contain $\leq |Q|$ positions. The bound then follows from the definition of $boundedAmbiguity(A)$. This result also covers the case when $Q_{ancestor} = \emptyset$.

Case $q \in Q_{ancestor} \rightsquigarrow t$: We partition the space on i.

Clearly, if $i \leq |Q|$ then π can be visited at most $boundedAmbiguity(A)$ times. So suppose $i > |Q|$. Consider two observations. First, any simulation path containing more than $|Q|$ positions must include a cycle. Second, for the same reason, after a simulation path makes its final visit to a simulation position involving some $q \in Q_{ancestor}$, that simulation path must terminate within $|Q|$ steps. This is because the back-edges to $Q_{ancestor}$ are the only means of introducing a cycle, and without further cycles a simulation path must come to an end.

Now then, as we have supposed that $i > |Q|$, so the distinct simulation paths to π must all include some "cycle" simulation position $\pi' = \langle q \in Q_{ancestor}, j\rangle$ at most $|Q|$ steps beforehand. Recall that the memo function assumed in this theorem will prevent more than one simulation path through each such π'. There are $|Q_{ancestor}| \times |Q|$ possible cycle positions π', so all of the distinct simulation paths to π must share at most $|Q_{ancestor}| \times |Q|$ distinct simulation path prefixes. From these π', each simulation prefix may diverge up to $boundedAmbiguity(A)$ times to reach π. Multiplying, we obtain an upper bound of $|Q_{ancestor}|\times|Q|\times boundedAmbiguity(A)$ distinct simulation paths that can reach π.

□

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