CS229 Problem Set 2

Solution to Problem 4

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June 16, 2025

4.

(a)

Yes, K_1 and K_2 are both PSD, so $K_1 + K_2$ is PSD.

$$z^T K z = z^T (K_1 + K_2) z = z^T K_1 z + z^T K_2 z \geq 0$$

(b)

No, although K_1 and K_2 are both PSD, $K_1 - K_2$ may not be PSD.
For example, let $K_2 = 2K_1$:

$$z^T K z = z^T (K_1 - K_2) z = z^T (K_1 - 2K_1) z = -z^T K_1 z \leq 0$$

(c)

Yes, K_1 is PSD, so aK_1 (for $a \in \mathbb{R}^+$) is PSD.

$$z^T K z = z^T (aK_1) z = a \cdot z^T K_1 z \geq 0$$

(d)

No, K_1 is PSD, so $-aK_1$ (for $a \in \mathbb{R}^+$) is not PSD.

$$z^T K z = z^T (-aK_1) z = -a \cdot z^T K_1 z \leq 0$$

(e)

Yes, $K_1 K_2$ is PSD.

$$
\begin{aligned}
z^T K z &= \sum_i \sum_j z_i K_{ij} z_j \\
&= \sum_i \sum_j z_i K_1(x^{(i)}, x^{(j)}) K_2(x^{(i)}, x^{(j)}) z_j \\
&= \sum_i \sum_j z_i \phi_1(x^{(i)})^T \phi_1(x^{(j)}) \phi_2(x^{(i)})^T \phi_2(x^{(j)}) z_j \\
&= \sum_i \sum_j z_i \sum_a \phi_{1a}(x^{(i)}) \phi_{1a}(x^{(j)}) \sum_b \phi_{2b}(x^{(i)}) \phi_{2b}(x^{(j)}) z_j \\
&= \sum_a \sum_b \left(\sum_i \sum_j z_i \phi_{1a}(x^{(i)}) \phi_{1a}(x^{(j)}) \phi_{2b}(x^{(i)}) \phi_{2b}(x^{(j)}) z_j \right) \\
&= \sum_a \sum_b \left(\sum_i z_i \phi_{1a}(x^{(i)}) \phi_{2b}(x^{(i)}) \right)^2 \geq 0
\end{aligned}
$$

(f)

Yes, K is PSD.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function:

$$
\begin{aligned}
z^T K z &= \sum_i \sum_j z_i K_{ij} z_j \\
&= \sum_i \sum_j z_i f(x^{(i)}) f(x^{(j)}) z_j \\
&= \left(\sum_i z_i f(x^{(i)}) \right)^2 \geq 0
\end{aligned}
$$

(g)

Yes, $K_3(\phi(x), \phi(z))$ is a valid kernel, no matter what the inputs are.

(h)

Yes, $p(K_1)$ is a valid kernel.

Let $p(x)$ be a polynomial with coefficients $c_k > 0$, $k = 0, 1, \ldots, n$:

$$p(x) = \sum_{k=0}^{n} c_k x^k$$

Then,

$$K(x, z) = p(K_1(x, z)) = \sum_{k=0}^{n} c_k \left(K_1(x, z)\right)^k$$

From (e), we know $K(x, z) = K_1(x, z) K_2(x, z)$ is valid, so $K(x, z) = (K_1(x, z))^k$ is valid.

From (a) and (c), $K(x, z) = K_1(x, z) + K_2(x, z)$ and $K(x, z) = aK_1(x, z)$ are valid for $a \in \mathbb{R}^+$.

So,

$$K(x, z) = \sum_{k=0}^{n} c_k \left(K_1(x, z)\right)^k$$

is a valid kernel.