PH 354: hw 3, problem 15

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$$\Gamma(a) = \int_0^\infty x^{a-1} \exp{(-x)} dx$$

So the Integrand $I = x^{a-1} \exp{(-x)}$

$$\frac{dI}{dx} = I \left(\frac{a-1}{x} - 1 \right)$$

and

$$\frac{d^2I}{dx^2} = \frac{dI}{dx} \left(\frac{a-1}{x} - 1 \right) - \frac{I}{x^2}(a-1)$$

The turning point of I is at $\frac{dI}{dx}(x = x_0) = 0 \implies x_0 = a - 1$
So at $x_0 = a - 1$, $\frac{d^2I}{dx^2}(x = x_0) = -(a-1)^{a-2} \exp{[-(a-1)]} < 0$
So the turning point is a maxima.
Changing variables to $z = \frac{x}{c+x}$, the limits of integration change from 0 and ∞ to 0 and 1 respectively.
If the peak of the integrand is desired to be placed in the midway ($z = \frac{1}{2}$) in the new variable z, then $z = \frac{1}{2}$ should correspond to $x = a - 1$

$$\therefore \frac{a-1}{c+a-1} = \frac{1}{2} \implies c = a - 1$$

In the new variable, dx and dz are related as $dx = \frac{c}{(1-z)^2} dz$
Also x^{a-1} and $\exp{(-x)}$ can cause overflow and underflow respectively in the integration domain. Therefore a better alternative but equivalent expression of the integral is $I(x) = \exp{[(a-1)\ln{(x)} - x]}$
The expression $(a-1)\ln{(x)} - x$ is well behaved and doesn't involve any term that quickly becomes very large or very small and can cause overflow or underflow issues respectively.