



I denote  $f(x+h)$  as  $f_h$  and  $f(x-h)$  as  $f_{-h}$ . The function and its derivatives are evaluated at  $x_0$  and is denoted by 0 on subscript. Taylor expansion gives,

$$f_h = f_0 + f'_0 h + \sum_{j=2}^{\infty} \frac{f_0^{(j)}}{j!} h^j$$

$$\Rightarrow \frac{f_h - f_0}{h} = f'_0 + \sum_{j=2}^{\infty} \frac{f_0^{(j)}}{j!} h^{j-1}$$

So the associated Truncation error is  $\epsilon_{tr}(h) = \frac{1}{2} f_0^{(2)} h$ . However as  $h$  becomes small, there is a loss of precision due to round off error in the evaluation of  $f_h - f_0$ . Round off error goes as,

$$\left| \frac{f_h - f_0}{h} \right| \leq \frac{\epsilon_M}{h}$$

where  $\epsilon_M$  is machine precision and equals to  $10^{-6}$  for single precision and  $10^{-15}$  for double precision.

So the total error,

$$\epsilon_{tot} = \frac{1}{2} f_0^{(2)} h + \frac{\epsilon_M}{h}$$

So the error is minimum at

$$h = \sqrt{\frac{2\epsilon_M}{f_0^{(2)}}}$$

For  $f(x) = x(x - 1)$  and for double precision at  $x = 1$ ,  $h \approx 4.47 \times 10^{-8}$   
This is evident from the plot above.