PH 354: hw 2, problem 12

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For a squence $x_1, x_2, x_3, ...$ converges to r
For $\lambda > 0$ and $\alpha \geq 1$

$$\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|^{\alpha}} = \lambda$$

Here α is defined as the rate of convergence.
For any Fixed Point Iteration method, $x_{n+1} = g(x_n)$
$g(x) = x - \frac{f(x)}{f'(x)}$ is the form of $g(x)$ for Newton-Raphson (NR) method
where $f(x) = 0$ is the equation whose root is r
Taylor expansion of $g(x)$ about r gives

$$g(x) = g(r) + g'(r)(x - r) + \frac{1}{2}g"(r)(x - r)^2 + ...$$

Using $x_{n+1} = g(x_n)$ and $g(r) = r$

$$\frac{x_{n+1} - r}{x_n - r} = g'(r) + \frac{1}{2}g"(r)(x_n - r) + ...$$

In NR method, $g'(x) = \frac{f"(x)}{[f'(x)]^2}f(x)$ and $\therefore g'(r) = 0$ as $f(r) = 0$
This gives $\frac{|x_{n+1}-r|}{|x_n-r|^2} = \left|\frac{1}{2}g"(r)\right|$
Hence $\alpha = 2$ and the convergence is Quadratic

For any general method,

$$\frac{x_{n+1} - r}{x_n - r} = g'(r) + \frac{1}{2}g"(r)(x_n - r) + ...$$

$$\implies \lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|} = |g'(r)|$$

Using Taylor expansion,

$$x_{n+1} = r + g'(r)(x_n - r) + \frac{1}{2}g"(r)(x_n - r)^2 + ...$$

$$x_n = r + g'(r)(x_{n-1} - r) + \frac{1}{2}g"(r)(x_{n-1} - r)^2 + ...$$

Subtracting one equation from the other and rearranging,

$$\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|} = |g'(r)| = \lim_{n\to\infty} \frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|}$$

For large enough n, $\frac{|x_{n+1}-r|}{|x_n-r|} \approx \frac{|x_{n+1}-x_n|}{|x_n-x_{n-1}|}$
Defining $e_n = x_n - r$,
$\frac{|e_{n+1}|}{|e_n|^\alpha} = \lambda$ and also $\frac{|e_n|}{|e_{n-1}|^\alpha} = \lambda$

$$\therefore \left|\frac{e_{n+1}}{e_n}\right| = \left|\frac{e_n}{e_{n-1}}\right|^\alpha$$

$$\implies \alpha = \frac{\log\left(\left|\frac{e_{n+1}}{e_n}\right|\right)}{\log\left(\left|\frac{e_n}{e_{n-1}}\right|\right)}$$

For large enough n, this can be approximated to

$$\alpha \approx \frac{\log\left(\left|\frac{x_{n+1}-x_n}{x_n-x_{n-1}}\right|\right)}{\log\left(\left|\frac{x_n-x_{n-1}}{x_{n-1}-x_{n-2}}\right|\right)}$$

This equation can be solved numerically to find the rate of convergence of any generic algorithm.