PH 354: hw 2, problem 11

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The set of equations whose root is to be found is,

$$f(x, y) = 0$$

$$g(x, y) = 0$$

Jacobian is defined as $J = \begin{bmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{bmatrix}$
According to Newton's method, $\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = -J^{-1} \begin{bmatrix} f \\ g \end{bmatrix}$
This gives

$$\Delta x = -\frac{f\partial_y g - g\partial_y f}{D}$$

$$\Delta y = -\frac{-f\partial_x g + g\partial_x f}{D}$$

where $D = \partial_x f \partial_y g - \partial_y f \partial_x g$
Defining a complex function $\phi(z = x + iy) = f(x, y) + ig(x, y)$
For the function $\phi(z) = 0$, Newton's method gives,

$$\Delta z = \Delta x + i\Delta y = -\frac{\phi(z)}{\phi'(z)}$$

Now $f(x, y)$ and $g(x, y)$ follows the following Cauchy-Riemann (CR) criteria,
$\partial_x f = \partial_y g$ and $\partial_x g = -\partial_y f$
D defined above becomes $(\partial_x g)^2 + (\partial_y g)^2$

$$\therefore \Delta z = -\frac{f + ig}{\partial_x f + i\partial_x g}$$

Using the CR criteria this can be rewritten to,

$$\Delta z = \Delta x + i\Delta y = -\frac{f\partial_y g - g\partial_y f}{D} - i\frac{-f\partial_x g + g\partial_x f}{D}$$

This is exactly the same expression obtained in the two dimensional version of Newton's method using the Jacobian.