Exercise 10: A random point on the surface of the Earth

Part A

What are the ranges of the variables θ and ϕ?

θ is the angle between the z-axis and the point on the surface of the Earth. θ ranges from 0 to π. ϕ is the angle between the x-axis and the projection of the point on the surface of the Earth onto the x-y plane. ϕ ranges from 0 to 2π.

Verify that the two distributions p(θ) and p(ϕ) are correctly normalized—they integrate to 1 over the appropriate ranges.

$$p(\theta) = \frac{\sin\theta\, d\theta}{2}$$

$$\int_0^{\pi} p(\theta)\, d\theta = \int_0^{\pi} \frac{\sin\theta\, d\theta}{2} = \frac{\cos 0 - \cos\pi}{2} = 1$$

$$p(\phi) = \frac{d\phi}{2\pi}$$

$$\int_0^{2\pi} p(\phi)\, d\phi = \int_0^{2\pi} \frac{d\phi}{2\pi} = \frac{2\pi - 0}{2\pi} = 1$$

Part B

Let z be a uniform random number between 0 and 1 (i.e., $q(z) = 1$ between 0 and 1 and 0 otherwise).

Formulas for generating angles θ and ϕ

$$\int_0^{\theta'} p(\theta)\, d\theta = \int_0^{\theta'} \frac{\sin\theta\, d\theta}{2} = \frac{\cos 0 - \cos(\theta'(z))}{2} = z$$

Therefore,

$$\theta'(z) = \cos^{-1}(1 - 2z)$$

$$\int_0^{\phi'} p(\phi)\, d\phi = \int_0^{\phi'} \frac{d\phi}{2\pi} = \frac{\phi'(z) - 0}{2\pi} = z$$

Therefore,

$$\phi'(z) = 2\pi z$$