PH354 HW # 2 (Note : Q13 is at the end)

Q5)

$$I_2 - I_1 = ch_1^2 - ch_2^2$$

$$h_i = \frac{b-a}{n_i - 1} \quad \forall\, i \in 1,2$$

$n_i$ is the number of points i.e. # slices + 1

$$\therefore \frac{h_1^2}{h_2^2} = \left(\frac{n_2-1}{n_1-1}\right)^2$$

$$\therefore ch_1^2 = \left(\frac{n_2-1}{n_1-1}\right)^2 ch_2^2$$

$$\Rightarrow \quad ch_2^2 = \frac{I_2 - I_1}{\left(\frac{n_2-1}{n_1-1}\right)^2 - 1}$$

Q6)

$$I_2 - I_1 = ch_1^4 - ch_2^4$$

$$\therefore ch_2^4 = \frac{I_2 - I_1}{\left(\frac{n_2-1}{n_1-1}\right)^4 - 1}$$

Q7) $$R_{i,m} - R_{i-1,m} = C_m h_{i-1}^{2m} - C_m h_i^{2m}$$

$$\therefore C_m h_i^{2m} = \frac{R_{i,m} - R_{i-1,m}}{\left(\frac{h_{i-1}}{h_i}\right)^{2m} - 1}$$

**Q10)** $E = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + V(x)$

$$\therefore \sqrt{\frac{2(E - V(x))}{m}} = \frac{dx}{dt}$$

$$\int_0^{T/4} dt = \int_0^a \left(\frac{2(V(a) - V(x))}{m}\right)^{-1/2} dx$$

$$T = \sqrt{8m} \int_0^a \frac{dx}{\sqrt{a^4 - x^4}}$$

Consider $V(x) = k|x|^n$

$$T = \sqrt{8m} \int_0^a \frac{dx}{\sqrt{ka^n - kx^n}} = \sqrt{\frac{8m}{k}} \int_0^a \frac{dx}{\sqrt{a^n - x^n}}$$

Let $y = x/a \quad \therefore a\,dy = dx$

$$\therefore T = \sqrt{\frac{8m}{k}} \frac{a}{a^{n/2}} \int_0^1 \frac{dy}{\sqrt{1-y^n}} = \sqrt{\frac{8m}{ka^{n-2}}} \int_0^1 \frac{dy}{\sqrt{1-y^n}}$$

i.e. $T \propto a^{1-n/2}$

**Q12)** The Stefan-Boltzmann constant

$$I(\omega) = \frac{\hbar}{4\pi^2 c^2} \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1}$$

let $\beta = (k_B T)^{-1}$

$$\therefore I(\omega) = \frac{1}{4\pi^2 c^2 \hbar^2 \beta^3} \frac{(\beta\hbar\omega)^3}{e^{\beta\hbar\omega} - 1}$$

$$W = \int_0^\infty I(\omega)\,d\omega = \frac{1}{4\pi^2 c^2 \hbar^2 \beta^3} \int_0^\infty \frac{(\beta\hbar\omega)^3}{e^{\beta\hbar\omega} - 1}\,d\omega$$

Let $\beta\hbar\omega = x$

$$\therefore d\omega = \frac{dx}{\beta\hbar}$$

$$\therefore W = \frac{1}{4\pi^2 c^2 \hbar^3 \beta^4} \int_0^\infty \frac{x^3}{e^x - 1}\,dx$$

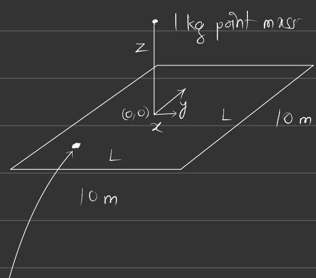
$$W = \frac{(k_B T)^4}{4\pi^2 c^2 \hbar^3} \int_0^\infty \frac{x^3}{e^x - 1}\,dx$$

It is known that $\int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx = \zeta(s)\Gamma(s)$

Riemann-Zeta function | Gamma function

$$\therefore \int_0^\infty \frac{x^3}{e^x - 1} = \zeta(4)\Gamma(4) = 3!\,\frac{\pi^4}{90} = \frac{\pi^4}{15}$$

**Q14)**




Consider a piece of sheet of mass $dm = \sigma\,dx\,dy$

$$\therefore d\bar{F} = \frac{G\sigma\,dx\,dy}{|r_{12}|^2}\left((\hat{r}_{12}\cdot\hat{x})\hat{x} + (\hat{r}_{12}\cdot\hat{y})\hat{y} + (\hat{r}_{12}\cdot\hat{z})\hat{z}\right)$$

Here $r_{12}$ is the radial vector from point mass to dm

Now, $(\hat{r}_{12}\cdot\hat{i}) = \frac{i}{|r_{12}|}$ $\quad \forall i \in \langle x, y, z\rangle$

$$\therefore d\bar{F}_z = -\frac{G\sigma\,dx\,dy}{|\bar{r}_{12}|^2}\cdot\frac{z}{|r_{12}|}$$

$$= -\frac{G\sigma z\,dx\,dy}{(x^2+y^2+z^2)^{3/2}}$$

since $|\bar{r}_{12}| = \sqrt{x^2+y^2+z^2}$

$$\therefore |F_z| = G\sigma z \int_{-L/2}^{L/2}\int_{-L/2}^{L/2} \frac{dx\,dy}{(x^2+y^2+z^2)^{3/2}}$$

## Q15>

Part B: Assume $a > 1$

$$f(x) = x^{a-1}e^{-x}$$

$$f'(x) = (a-1)x^{a-2}e^{-x} - x^{a-1}e^{-x}$$

For extrema, $f'(x) = 0$

ie, $(a-1)x_0^{a-2} = x_0^{a-1}$

$\Rightarrow x_0 = a-1$

$$f''(x_0) = x_0^{a-3}\left(x_0^2 + 2(1-a)x_0 + a^2 - 3a + 2\right)e^{-x_0}$$

$$= (1-a)\,e^{a-1}(a-1)^{a-3}$$

$\therefore f''(x_0) < 0 \Rightarrow$ Maximum at $x_0$

Part C:

Change of variable

$$z = \frac{x}{c+x} \rightarrow dz = \frac{dx}{(c+x)} - \frac{x\,dx}{(c+x)^2} = \frac{dx}{(c+x)^2}$$

$$cz + xz = x \Rightarrow x - xz = cz \Rightarrow x = cz/1-z$$

$$dx = \frac{c\,dz}{(1-z)^2} \quad \therefore \int_0^\infty f(x,a)\,dx = \int_0^1 \frac{c}{(1-z)^2} f\left(\frac{cz}{1-z}, a\right)dz$$

$$\int_0^\infty x^{a-1}e^{-x}\,dx = \int_0^1 \frac{(cz)^{a-1}}{(1-z)^{a-1}} e^{-cz/(1-z)} \frac{dz\,c^2}{(1-z)^2}$$

$$= c^{a+1}\int_0^1 \frac{z^{a-1}e^{-cz/(1-z)}}{(1-z)^{a+1}}\,dz$$

Peak of the integrand is $a-1$

i.e. $z_0 = \frac{x_0}{c+x_0}$

$$\therefore \frac{1}{2} = \frac{a-1}{c_0+a-1} \Rightarrow c+a-1 = 2a-2$$

$\therefore \underline{c_0 = a-1}$

## Q16>

Part A> $g(u) = \sin^2 \alpha u$

Separation of the slits $= d$ should be the difference between 2 consecutive minima of $g(u)$

$\sin^2(\alpha u - \pi) = \sin^2(\alpha u)$, $\pi$ is the period

$\therefore u_1 = \pi/\alpha$

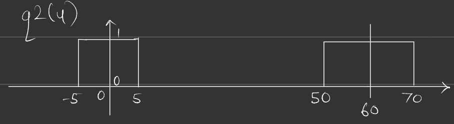
ie, $d = \pi/\alpha$

Part E II>

slit 1 $\quad$ $d = 60\,\mu m$ $\quad$ slit 2

$a_1 = 10\,\mu m$ $\quad$ $a_2 = 20\,\mu m$

Since we don't know where the central maximum is apriori, we will assume the origin (ie, $u = 0$) to be at the centre of slit 1




The oscillatory behaviour of $I(x)$ is determined by that of $g(u)$ & $e^{2\pi i \cdot xy/\lambda f}$

In the interval of integration, there are 10 slits, i.e. $\sim 10$ oscillations due to $g(u)$

$\frac{xy}{\lambda f}$ for fixed $x$ $\in \left( \pm \frac{x \times 11 \times 20 \times \cancel{10^{-6}}}{0.5 \times \cancel{10^{-6}} \times 2} \right)$

i.e. $\in \left( \pm x \times 220 \right)$

Maximum value of $x$ considered is 5 cm

(Screen is 10 cm wide)

i.e. $\frac{xy}{\lambda f} \leq 0.05 \times 220 = 11$

This is as expected (Oscillations depend only on number of slits & are of that order $\sim 10$)

$\therefore$ Having $n = 100$ points for gaussian quadrature should be safe

Q13>

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\psi_n(x)|^2 dx$$

$$= \langle 5 | x^2 | 5 \rangle$$

$$x^2 = \frac{\hbar}{2m\omega} (a + a^{+})^2$$

$$\therefore \langle x^2 \rangle_5 = \frac{\hbar}{2m\omega} \langle 5 | aa + aa^{+} + a^{+}a + a^{+}a^{+} | 5 \rangle$$

$$= \frac{\hbar}{2m\omega} \langle 5 | aa^{+} + a^{+}a | 5 \rangle$$

$$= (2n+1) \frac{\hbar}{2m\omega}$$

For all constants $= 1$

$$\langle x^2 \rangle_5 = 11/2$$

$$\therefore \sigma_{x_5} = \sqrt{5.5} \approx 2.345$$