HW # 3

Q1)

Part A

$x = (-b \pm \sqrt{b^2 - 4ac})/(2a)$

$$x = \frac{(-b \pm \sqrt{b^2-4ac})}{2a} \cdot \frac{(-b \mp \sqrt{b^2-4ac})}{(-b \mp \sqrt{b^2-4ac})} = \frac{b^2 - (b^2-4ac)}{2a(-b \mp \sqrt{b^2-4ac})}$$

$$= \frac{\cancel{4ac}\ 2c}{\cancel{2a}(-b \mp \sqrt{b^2-4ac})} = \frac{2c}{-b \mp \sqrt{b^2-4ac}}$$

Q4)

A)

$$\frac{dx}{dt} = -x + ay + x^2y \quad ; \quad \frac{dy}{dt} = b - ay - x^2y$$

Finding the stationary point(s)

$$\therefore \left.\frac{dx}{dt}\right|_{x_*} = 0 = -x_* + ay_* + x_*^2 y_* \Rightarrow x_* = y_*(x_*^2 + a) \quad — \text{①}$$

$$\left.\frac{dy}{dt}\right|_{y_*} = 0 = b - ay_* - x_*^2 y_* \Rightarrow y_* = b/(a + x_*^2) \quad — \text{②}$$

Using ① & ②, $x_* = (x_*^2 + a) \cdot \dfrac{b}{(x_*^2 + a)} = b$ — ③

Using ② & ③, $y_* = \dfrac{b}{a + b^2}$

B) ① & ② are the rearrangements

The method failed to converge

C) $x = f(x) = y(a + x^2)$; $y = f(y) = b/(a + x^2)$

Q5>

$$I(\lambda) = \frac{2\pi hc^2 \lambda^{-5}}{e^{hc/\lambda k_B T} - 1}$$

$$\frac{dI}{d\lambda} = -\frac{2\pi hc^2\left((5\lambda k_B T - hc)\, e^{hc/\lambda k_B T} - 5\lambda k_B T\right)}{\lambda^7 k_B T \left(e^{hc/\lambda k_B T} - 1\right)^2}$$

For λ corresponding to I_{max}, set $dI/d\lambda = 0$

$$(5\lambda k_B T - hc)\, e^{hc/\lambda k_B T} = 5\lambda k_B T$$

$$e^{-hc/\lambda k_B T} = 1 - \frac{1}{5}\frac{hc}{\lambda k_B T}$$

Let $hc/\lambda k_B T = x$

$\therefore\ e^{-x} = 1 - x/5$

i.e, $5e^{-x} + x - 5 = 0$ (Call it $f(x)$)

$\therefore\ b = hc/k_B x$

$f(0) = 0$; $f(1) = 5/e - 4 < 0$

$\lim_{x\to\infty} f(x) = \infty$; $f(2) = 5/e^2 - 3 < 0$

$f(3) = 5/e^3 - 2 < 0$

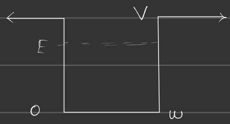
$f(4) = 5/e^4 - 1 < 0$

$f(5) = 5/e^5 > 0$

$\therefore$ A root lies between 4 & 5

$\therefore$ [4, 5] are the bounds of the bisection method

Q6>



Trapped implies bound $\therefore$ $0 < E < V$

$$E\psi(x) = \begin{cases} \dfrac{-\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + V\psi & ; \ x \notin (0,w) \\[2ex] \dfrac{-\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} & ; \ x \in (0,w) \end{cases}$$

$$\therefore \frac{d^2\psi}{dx^2} = \frac{-2m(E-V)\psi}{\hbar^2} = +k_1^2\psi \quad ; \ x \notin (0,w) \qquad \text{i.e, } k_1 = \frac{\sqrt{2m(V-E)}}{\hbar}$$

$$= \frac{-2mE\psi}{\hbar^2} = -k_2^2\psi \quad ; \ x \in (0,w) \qquad k_2 = \frac{\sqrt{2mE}}{\hbar}$$

$$\therefore \psi(x) = Ae^{-k_1 x} + Be^{k_1 x} \quad ; \ x \notin (0,w)$$

$$= C\sin k_2 x + D\cos k_2 x \quad ; \ x \in (0,w)$$

• Constraints

1> Physicality: The wavefunction must not blow up

$\therefore$ for $x \geq w$, $B = 0$, & for $x \leq 0$, $A = 0$

2> Continuity of $\psi(x)$

At $x = 0$: $B = D$ — (1)

At $x = w$: $Ae^{-k_1 w} = C\sin k_2 w + D\cos k_2 w$ — (2)

3> Continuity of $\psi'(x)$

At $x = 0$: $k_1 B = k_2 C$ — (3)

At $x = w$: $-k_1 A e^{-k_1 w} = k_2 C\cos k_2 w - k_2 D\sin k_2 w$ — (4)

Using (1) & (3) in (2) & (4)

$$-k_1 \cdot \frac{k_1 B}{k_2}\sin k_2 w - k_1 B\cos k_2 w = k_1 B\cos k_2 w - k_2 B\sin k_2 w$$

$$\Rightarrow \left(\frac{k_2}{k_1} - \frac{k_1}{k_2}\right)\sin k_2 w = 2\cos k_2 w$$

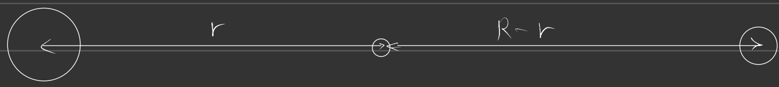
$$\tan(k_2 w) = \frac{2}{\frac{k_2}{k_1} - \frac{k_1}{k_2}}$$

Conversion from J to eV

$$\tan(\alpha\sqrt{E}) = 2\left(\sqrt{\frac{E}{V-E}} - \sqrt{\frac{V-E}{E}}\right) \quad ; \quad \alpha = \frac{w\sqrt{2m}}{\hbar}$$

in eV

Q 8)



For the satellite:

$\dfrac{G M_e m_s (-\hat{x})}{r^2}$ $\qquad$ $\dfrac{G M_m m_s}{(R-r)^2}$

$a = \omega^2 r$

$\therefore$ For equilibrium: $\quad -\dfrac{GM_e}{r^2} + \dfrac{GM_m}{(R-r)^2} - \omega^2 r = 0$

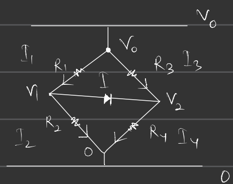
$$\frac{-1}{r^2} + \frac{\alpha}{(R-r)^2} - r\cdot\beta = 0 \quad ; \quad \alpha = M_m/M_e \quad ; \quad \beta = \omega^2/GM_e$$

$$\frac{-(R-r)^2 + \alpha r^2 - r^3(R-r)^2\beta}{r^2(R-r)^2} = 0 \Rightarrow \alpha r^2 - (R-r)^2 - \beta r^3 (R-r)^2 = 0$$

Let r/R be x $\quad \therefore \quad \alpha x^2 - (1-x)^2 - \beta x^3 R^3 (1-x)^2 = 0$

i.e., $\alpha x^2 - (1-x)^2 - \gamma x^3 (1-x)^2 = 0 \quad ; \quad \gamma = \beta R^3$

Q9)



$I_1 = I + I_2$

$I + I_3 = I_4$

$I_1 - I_2 = I_4 - I_3$

$I_1 - I_2 + I_3 - I_4 = 0$

$V_0 - V_1 = I_1 R_1$

$V_0 - V_2 = I_3 R_3$

$V_1 = I_2 R_2$

$V_2 = I_4 R_4$

$I = I_0 \left(e^{(V_1 - V_2)/V_T} - 1\right)$

$V = V_1 - V_2 = I_2 R_2 - I_4 R_4$

$\therefore$ 4 unknowns (I_1, I_2, I_3, I_4)

$$\Rightarrow F \left\{ \begin{array}{l} V_0 - I_1 R_1 - I_2 R_2 = 0 \\ V_0 - I_3 R_3 - I_4 R_4 = 0 \\ I_1 - I_2 - I_0 \left(e^{(I_3 R_3 - I_1 R_1)/V_T} - 1\right) = 0 \\ I_1 - I_2 + I_0 \left(- \quad '' \quad \right) = 0 \end{array} \right.$$

$I_2 R_2 - I_4 R_4 = I_3 R_3 - I_1 R_1$

i.e., $I_1 R_1 + I_2 R_2 = I_3 R_3 + I_4 R_4$

We want $I = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix}$; $J = \begin{pmatrix} -R_1 & -R_2 & 0 & 0 \\ 0 & 0 & -R_3 & -R_4 \\ \alpha & -1 & \beta & 0 \\ 1-\alpha & 0 & 1-\beta & -1 \end{pmatrix}$

$$\alpha = 1 + \frac{R_1 I_0}{V_T} e^{(I_3 R_3 - I_1 R_1)/V_T}$$

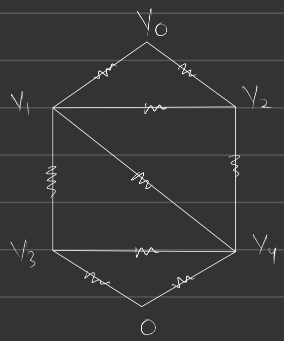
$$\beta = -\frac{I_0 R_3}{V_T} e^{(I_3 R_3 - I_1 R_1)/V_T}$$

$$I_{n+1} = I_n - J^{-1} F$$

B) The results V_D matched with the thumb rule

i.e, $V_D \approx 0.6\,V$

Q10)



$$\frac{1}{R}\Big((V_4-V_3)+(V_4-V_1)+(V_4-V_2)+V_4\Big)=0$$

$$4V_4-(V_1+V_2+V_3)=0 \qquad \text{for } V_4$$

$$\frac{1}{R}\Big((V_3-V_4)+(V_3-V_1)+V_3\Big)=0$$

$$3V_3-(V_1+V_4)=0 \qquad \text{for } V_3$$

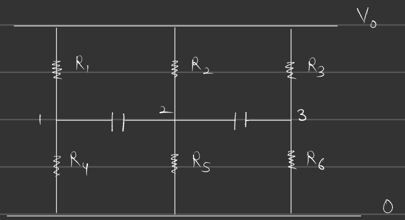
$$V=\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix}$$

$$\frac{1}{R}\Big((V_2-V_0)+(V_2-V_4)+(V_2-V_1)\Big)=0$$

$$3V_2-V_0-V_1-V_4=0$$

$$\therefore \begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & -1 & -1 & 4 \\ -1 & 0 & 3 & -1 \\ -1 & 3 & 0 & -1 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \begin{pmatrix} V_0 \\ 0 \\ 0 \\ V_0 \end{pmatrix}$$

Q13>



$$V_k = x_k e^{i\omega t} \quad ; \forall k = 0,1,2,3$$

$$\therefore \frac{dV_k}{dt} = i\omega \cdot V_k$$

At 1 : $\frac{(V_1 - V_0)}{R_1} + \frac{(V_1 - 0)}{R_4} + C_1 \cdot \frac{d(V_1 - V_2)}{dt} = 0$

$$\frac{(x_1 - x_0)}{R_1} + \frac{(x_1)}{R_4} + C_1 \cdot i\omega(x_1 - x_2) = 0$$

$$\therefore \left(\frac{1}{R_1} + \frac{1}{R_4} + i\omega C_1\right) x_1 - i\omega C_1 x_2 = \frac{x_0}{R_1}$$

At 3 : $\frac{(V_3 - V_0)}{R_1} + \frac{(V_3 - 0)}{R_4} + C_2 \cdot \frac{d(V_3 - V_2)}{dt} = 0$

$$\frac{(x_3 - x_0)}{R_3} + \frac{(x_3)}{R_6} + C_2 \cdot i\omega(x_3 - x_2) = 0$$

$$\therefore \left(\frac{1}{R_3} + \frac{1}{R_6} + i\omega C_2\right) x_3 - i\omega C_2 x_2 = \frac{x_0}{R_3}$$

At 2 : $\frac{(V_2 - V_0)}{R_2} + \frac{(V_2 - 0)}{R_5} + C_1 \cdot \frac{d(V_2 - V_1)}{dt} + C_2 \cdot \frac{d(V_2 - V_3)}{dt} = 0$

$$\therefore \frac{(x_2 - x_0)}{R_2} + \frac{x_2}{R_5} + i\omega C_1 (x_2 - x_1) + i\omega C_2 (x_2 - x_3) = 0$$

$$x_2\left(\frac{1}{R_2} + \frac{1}{R_5} + i\omega C_1 + i\omega C_2\right) - i\omega C_2 x_3 - i m C_1 x_1 = 0$$

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Q14)

For V_1 : $\frac{(V_1 - V_0)}{R} + \frac{(V_1 - V_2)}{R} + \frac{(V_1 - V_3)}{R} = 0$

$\therefore \quad 3V_1 - V_0 - V_2 - V_3 = 0$

for V_2 : $\frac{(V_2 - V_0)}{R} + \frac{(V_2 - V_1)}{R} + \frac{(V_2 - V_3)}{R} + \frac{(V_2 - V_4)}{R} = 0$

$\therefore \quad 4V_2 - V_0 - V_1 - V_3 - V_4 = 0$

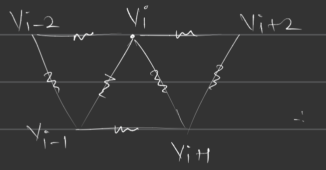
for V_N : $\frac{(V_N - 0)}{R} + \frac{(V_N - V_{N-1})}{R} + \frac{(V_N - V_{N-2})}{R} = 0$

$\therefore \quad 3V_N - V_{N-1} - V_{N-2} = 0$

for V_{N-1} : $\frac{(V_{N-1} - 0)}{R} + \frac{(V_{N-1} - V_N)}{R} + \frac{(V_{N-1} - V_{N-3})}{R} + \frac{(V_{N-1} - V_{N-2})}{R} = 0$

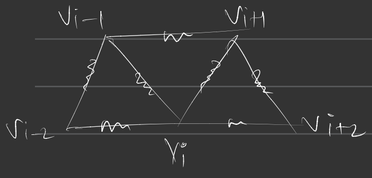
$\therefore \quad 4V_{N-1} - V_N - V_{N-2} - V_{N-3} = 0$

for V_i $\quad i \neq 1, 2, N, N-1$



$\frac{(V_i - V_{i-2})}{R} + \frac{(V_i - V_{i-1})}{R} + \frac{(V_i - V_{i+1})}{R} + \frac{(V_i - V_{i+2})}{R} = 0$

$\therefore \quad 4V_i - V_{i-2} - V_{i-1} - V_{i+1} - V_{i+2} = 0$



i.e. By symmetry, this arrangement leads to the same expression

Q16)

A) $$\int_0^L \sin\left(\frac{\pi m x}{L}\right)\sin\left(\frac{\pi n x}{L}\right) = \frac{L}{2}\delta_{mn}$$

S.T. $\hat{H}\psi = E\psi$ implies $\sum_{n=1}^{\infty} \psi_n \int_0^L \sin\frac{\pi m x}{L}\,\hat{H}\sin\frac{\pi n x}{L}\,dx = \frac{1}{2}LE\psi_m$

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n \sin\left(\frac{\pi n x}{L}\right)$$

$$\hat{H}\psi = \sum_{n=1}^{\infty}\psi_n \hat{H}\sin\left(\frac{\pi n x}{L}\right) = \sum_{n=1}^{\infty}\psi_n E\sin\frac{\pi n x}{L}$$

$$\therefore \sum_{n=1}^{\infty}\psi_n\int_0^L \sin\left(\frac{\pi m x}{L}\right)\hat{H}\sin\left(\frac{\pi n x}{L}\right)dx = \int_0^L \sin\left(\frac{\pi m x}{L}\right)\sum_{n=1}^{\infty}\psi_n E\sin\frac{\pi n x}{L}$$

$$= \sum_{n=1}^{\infty} E\int_0^L \sin\left(\frac{\pi m x}{L}\right)\sin\left(\frac{\pi n x}{L}\right)\psi_n\,dx$$

$$= \sum_{n=1}^{\infty}\frac{EL}{2}\psi_n\,\delta_{nm} = \frac{1}{2}LE\psi_m$$

Rearranging,

$$\sum_{n=1}^{\infty}\psi_n\frac{2}{L}\int_0^L \sin\left(\frac{\pi m x}{L}\right)\hat{H}\sin\left(\frac{\pi n x}{L}\right)dx = E\psi_m$$

$$= \hat{H}\psi_m$$

$$= \sum_{n=1}^{\infty} H_{mn}\psi_n$$

$$\therefore H_{mn} = \frac{2}{L}\int_0^L \sin\left(\frac{\pi m x}{L}\right)\hat{H}\sin\left(\frac{\pi n x}{L}\right)dx$$

B) $V(x) = ax/L$

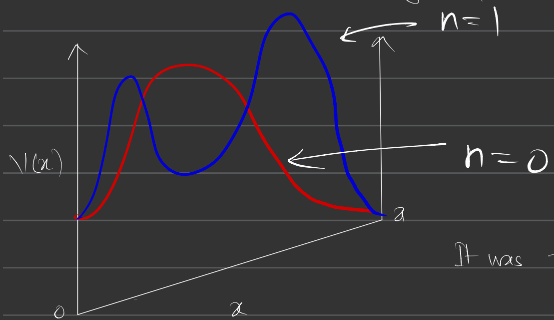
$$H_{mn} = -\frac{2}{L}\left(\frac{-\hbar^2}{2M}\right)\int_0^L \sin\frac{\pi m x}{L}\sin\frac{\pi n x}{L}\,dx \left(\frac{\pi n}{L}\right)^2 + \frac{2a}{L^2}\int_0^L x\sin\frac{\pi m x}{L}\sin\frac{\pi n x}{L}\,dx$$

$$= \frac{\pi^2\hbar^2 n^2}{2ML^2}\cdot\delta_{mn} + \frac{2a}{L^2}\cdot\alpha_{mn}$$

$$\alpha_{mn} = \begin{cases} 0 & m \neq n \quad \text{same parity} \\ -\left(\frac{2L}{\pi}\right)^2 \frac{mn}{(m^2-n^2)^2} & m \neq n \quad \text{different "} \\ \frac{L^2}{4} & m = n \end{cases}$$

$$\therefore H_{mn} = \begin{cases} \frac{(n\hbar\pi)^2}{2ML^2} + \frac{a}{2} & m = n \\ -\frac{8a}{\pi^2}\frac{mn}{(m^2-n^2)^2} & m \neq n \quad \text{different parity} \\ 0 & m \neq n \quad \text{same parity} \end{cases}$$

Upon graphing, we find that the ground state peaks nearer to the left side as expected.
The excited states have higher peaks near the right side as expected.



Ground & first excited shown here

It was found that $\int_0^L |\Psi(x)|^2\,dx = 5/2$