Exercise 6: Importance Sampling

Part A:

General derivation:

Formula for integration:

For any general function $g(x)$, we can define a weighted average over the interval from a to b thus:

$$\langle g \rangle_w = \frac{\int_a^b w(x) g(x)\, dx}{\int_a^b w(x)\, dx}$$

where $w(x)$ is the weight function. Now consider again the general one-dimensional integral:

$$I = \int_a^b f(x)\, dx$$

Setting $g(x) = \frac{f(x)}{w(x)}$,

$$\left\langle \frac{f(x)}{w(x)} \right\rangle_w = \frac{\int_a^b w(x) \frac{f(x)}{w(x)}\, dx}{\int_a^b w(x)\, dx} = \frac{\int_a^b f(x)\, dx}{\int_a^b w(x)\, dx} = \frac{I}{\int_a^b w(x)\, dx}$$

$$I = \left\langle \frac{f(x)}{w(x)} \right\rangle_w \int_a^b w(x)\, dx$$

For purposes of normalization, let us define the probability density function $p(x)$ as:

$$p(x) = \frac{w(x)}{\int_a^b w(x)\, dx}$$

Let us sample N random points x_i with this probability density (i.e., $p(x)$). That is, the probability of generating a value in the interval between x and $x + dx$ will be $p(x)\, dx$. Then the average number of samples that fall in this interval is $Np(x)\, dx$, and so for any function $g(x)$ and a large N,

$$\sum_{i=1}^{N} g(x_i) \approx \int_a^b N p(x) g(x)\, dx$$

Therefore,

$$\langle g \rangle_w = \int_a^b p(x) g(x)\, dx \approx \frac{1}{N} \sum_{i=1}^{N} g(x_i)$$

$$I \approx \frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i)}{w(x_i)}\int_a^b w(x)\,dx$$

Transformation method to generate random numbers from an arbitrary distribution:

1. Suppose we have a source of random floating-point numbers z drawn

from a distribution with probability density $q(z)$ (i.e., the probability of generating a number in the interval z to $z+dz$ is $q(z)\,dz$). 2. Suppose you have a function $x = x(z)$. When z is a random number, $x(z)$ is also a random number with probability distribution $p(x)$. 3. Our goal is to choose the function $x(z)$ so that x has the distribution we want. The probability of generating a value of x between x and $x+dx$ is by definition equal to the probability of generating a value of z in the corresponding z interval (i.e., z and $z+dz$). That is, 1. $p(x)\,dx = q(z)\,dz$ 2. We will have z be a uniform random number between 0 and 1 (i.e., $q(z) = 1$ between 0 and 1 and 0 otherwise).

$$\int_{-\infty}^{x(z)} p(x^{'})\,dx^{'} = \int_0^z q(z)\,dz^{'} = \int_0^z dz^{'} = z$$

For our specific case:

Probability density function:

$$p(x) = \frac{x^{-1/2}}{\int_0^1 x^{-1/2}\,dx} = \frac{1}{2\sqrt{x}}$$

$$\int_0^{x(z)} \frac{1}{2\sqrt{x}^{'}}\,dx^{'} = \frac{1}{2}\left(2\sqrt{x}\right) = z$$

$$x(z) = z^2$$