Causal Inference

Assignment

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Q1. $Y = \beta_0 + \beta_x X + \beta_2 Z + \varepsilon$

$$ATE = E[Y \mid do(X=1)] - E[Y \mid do(X=0)]$$

$$= E[\beta_0 + \beta_2 Z + \beta_x X + \varepsilon \mid do(x=1)]$$
$$- E[\beta_0 + \beta_2 Z + \beta_x X + \varepsilon \mid do(X=0)]$$

$$= \beta_0 - \beta_0 + \beta_2 (E[Z \mid do(X=1)] - E[Z \mid do(X=0)])$$
$$+ \beta_x (E[X \mid do(X=1)] - E[X \mid do(X=0)])$$

$$= \beta_2 (E[Z \mid do(X=1)] - E[Z \mid do(X=0)])$$
$$+ \beta_x (1 - 0)$$

$\because$ Z satisfies backdoor criterion wrt X & Z $\therefore E[Z \mid do(X=x)] = E[Z]$

$\therefore \boxed{ATE = \beta_x}$

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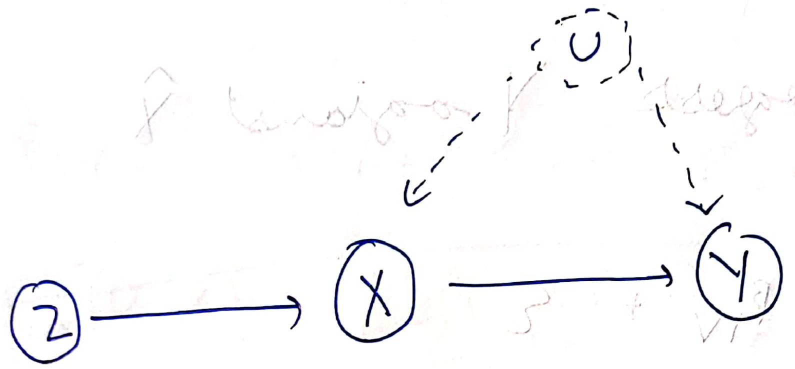
Q2. Consider the following SCM for instrumental variable formulation.




So for finding $\hat{\beta}_{IV}$ we first regress X against the IV Z.

Let

$$X = \cancel{\beta_{XZ} Z} + \varepsilon$$
$$\beta_{XZ} Z$$

$\therefore$ By OLS Regression formula:

$$\hat{\beta}_{XZ} = (Z^T Z)^{-1} Z^T X$$

$$\therefore \hat{X} = \cancel{\hat{\beta}_{XZ} Z} \; Z \hat{\beta}_{XZ}$$

$$\cancel{\therefore \hat{\beta}_{XZ} =}$$

$$\therefore \boxed{\hat{X} = Z (Z^T Z)^{-1} Z^T X}$$

Which is

$$\cancel{\hat{X} = P_2 Z X}$$

$$\boxed{\hat{X} = P_2 X}$$

Now we regress $Y$ against $\hat{X}$

$$\therefore Y = \hat{X} \beta_{IV} + \varepsilon$$

$$\therefore \hat{\beta}_{IV} = (\hat{X}^T \hat{X})^{-1} \hat{X}^T Y$$

Substituting the value $\hat{X} = P_2 X$

$$\hat{\beta}_{IV} = (X^T P_2^T P_2 X)^{-1} X^T P_2 Y$$

Consider $P_2^T P_2 = \left(Z(Z^T Z)^{-1} Z^T\right)^T \left(Z(Z^T Z)^{-1} Z^T\right)$

$$= Z\cancel{(Z^T Z)^{-1} Z^T} Z (Z^T Z)^{-1} Z^T$$

$$= Z (Z^T Z)^{-1} Z^T = P_2$$

$$\therefore \boxed{\hat{\beta}_{IV} = (X^T P_2 X)^{-1} X^T P_2 Y}$$

None if $X$ & $Z$ have same dims

$$\hat{\beta}_{IV} = (X^T Z (Z^T Z)^{-1} Z^T X)^{-1} X^T Z (Z^T Z)^{-1} Z^T Y$$

$X^T Z$ & $Z^T X$ are now sq inv matrices

$$\Rightarrow \hat{\beta}_{IV} = (Z^T X)^{-1} \cancel{Z^T Z} \cancel{(X^T Z)^{-1}} \cancel{X^T Z} \cancel{(Z^T Z)^{-1}} Z^T$$

$$\therefore \boxed{\hat{\beta}_{IV} = (Z^T X)^{-1} Z^T Y}$$