MA-102: Assignment 3
Special Functions

Q1) a) $(1-x^2)y'' + 2xy' + n(n+1)y = 0$

$$\Rightarrow y'' + \underbrace{\left(\frac{2x}{1-x^2}\right)}_{p(x)} y' + \underbrace{\left(\frac{n(n+1)}{1-x^2}\right)}_{q(x)} y = 0$$

$1-x_0^2 = 0 \Rightarrow x_0^2 = 1 \Rightarrow x_0 = \pm 1$

At $x_0 = +1, -1$, $p(x)$ and $q(x)$ are not differentiable
$\Rightarrow$ these are singular points

$$\lim_{x \to x_0} (x-x_0) \times \frac{2x}{1-x^2}$$

$$= \lim_{x \to x_0} \frac{(x-x_0) \times 2x}{(1-x)(1+x)}$$

For $x_0 = 1$, $= \lim_{x \to 1} \frac{\overset{-1}{\cancel{(x-1)}} \times 2x}{\cancel{(1-x)}(1+x)}$

$= \frac{-2}{2}$

$= -1$, which is finite

For $x_0 = -1$, $= \lim_{x \to -1} \frac{\cancel{(x+1)} \times 2x}{(1-x)\cancel{(1+x)}}$

$= \frac{-2}{2}$

$= -1$, which is finite

$$\lim_{x\to x_0} (x-x_0)^2 \frac{n(n+1)}{\{1-x^2}$$

$$= \lim_{x\to x_0} (x-x_0)^2 \frac{n(n+1)}{(1-x)(1+x)}$$

At $x_0 = 1$, $= \lim_{x\to 1} \frac{(x-1)^{\cancel{2}}\, n(n+1)}{-\cancel{(x-1)}(x+1)}$

$= 0$, which is finite

At $x_0 = -1$, $\lim_{x\to -1} \frac{(x+1)^{\cancel{2}}\, n(n+1)}{(1-x)\cancel{(1+x)}}$

$= 0$ which is finite

$\therefore$ ~~the~~ $x_0 = \pm 1$ are regular singular points

b) $x^3(x-2)y'' + x^3 y' + 6y = 0$

$$\Rightarrow y'' + \frac{x^3 y'}{x^3(x-2)} + \frac{6y}{x^3(x-2)} = 0$$

~~$p(x) = \frac{1}{x-2} = \Rightarrow x_0 = 2$~~

$$p(x) = \frac{x^3}{x^3(x-2)} = \frac{1}{x-2} \quad \Rightarrow x_0 = 0, 2$$

$$q(x) = \frac{6}{x^3(x-2)}$$

$x_0 = 0, 2$ are the ~~ordinary points~~ singular points

At $x_0 = 0$

$$\lim_{x\to 0} x p(x) = \lim_{x\to 0} \frac{x}{x-2}$$

$$= 0$$

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{6\cancel{x^2}}{x^{\cancel{3}}(x-2)}$$

which is undefined

~~Irregula~~

At $x_0 = 2$

$$\lim_{x\to 2} \cancel{(x-2)} \times \frac{1}{\cancel{(x-2)}} = 1$$

$$\lim_{x\to 2} (x-2)^{\cancel{2}} \times \frac{6}{x^3 \cancel{(x-2)}} = 0$$

$\therefore x_0 = 0$ is an irregular ~~single~~ singular point

$x_0 = 2$ is a regular singular point

—x—

Q3) $2x^2 y'' + xy' - (x^2+1) y = 0$

$$\Rightarrow y'' + \left(\frac{x}{2x^2}\right) y' - \left(\frac{x^2+1}{2x^2}\right) y = 0$$

$$p(x) = \frac{1}{2x} \quad , \quad q(x) = -\left(\frac{x^2+1}{2x^2}\right)$$

$x_0 = 0$

$$\lim_{x\to 0} \frac{\cancel{x}}{2\cancel{x}} = \frac{1}{2}$$

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$$\lim_{x\to 0} \cancel{x^2} \times \frac{-(x^2+1)}{2\cancel{x^2}} = \frac{-1}{2}$$

$\therefore$ $x_0 = 0$ is a regular singular point

Let $y = \sum_{n=0}^{\infty} a_n (x)^{n+r}$ $\quad \{a_0 \neq 0,\ r \in R\}$

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2}$$

$$2x^2 y'' + xy' - x^2 y - y = 0$$

Substituting

$$\Rightarrow 2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r}$$

$$-\sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$n+2 = m \quad \Rightarrow n+2 \longrightarrow n$

$\Rightarrow n = m-2$

$$\cancel{\Rightarrow 2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}}$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^{n+r}\left[2(n+r)(n+r-1) + (n+r) - \cancel{a_n} 1\right] - \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$\Rightarrow a_0 x^r (r-1)(2r+1) + a_1 x^{r+1} r(2r+3) \cancel{+ \sum_{n=2}^{\infty} x^{n+r}}$$

$$+ \sum_{n=2}^{\infty} x^{n+r}\left[a_n\{2(n+r)(n+r-1) + (n+r) - 1\} - a_{n-2}\right] = 0$$

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$$(\quad) x^{r} = 0$$

$$\Rightarrow a_0 (r-1)(2r+1) = 0$$

$$\Rightarrow r = 1, \frac{-1}{2}$$

$r_1 = 1,\ r_2 = \frac{-1}{2}$, $(r_1 - r_2) = \frac{3}{2}$ which is not an integer

$$(\quad) x^{r+1} = 0$$

$$\Rightarrow a_1 r(2r+3) = 0$$

$$\Rightarrow \boxed{a_1 = 0}$$

$$(\quad) x^{n+r} = 0$$

$$\Rightarrow a_n \left[2(n+r)(n+r-1) + (n+r-1)\right] - a_{n-2} = 0$$

$$\Rightarrow a_n (n+r-1)(2n+2r+1) = a_{n-2}$$

$$\Rightarrow a_n = \frac{a_{n-2}}{(n+r-1)(2n+2r+1)}$$

For $r=1$

$$a_n = \frac{a_{n-2}}{n(2n+3)}$$

$n=2$ $\quad a_2 = \frac{a_0}{14}$

$n=3$ $\quad a_3 = 0$

$n=4$ $\quad a_4 = \frac{a_2}{44} = \frac{a_0}{616}$

$n=5$ $\quad a_5 = 0$

$$\therefore y_1(x) = x[a_0 + a_1x + a_2x^2 + \dots]$$
$$= x\left[a_0 + \frac{a_0x^2}{14} + \frac{a_0}{616}x^4 + \dots\right]$$
$$= a_0x\left(1 + \frac{x^2}{14} + \frac{x^4}{616} + \dots\right)$$

For $r = \frac{-1}{2}$

$$a_n = \frac{a_{n-2}}{n(2n-3)}$$

$n=2$ $\quad a_2 = \frac{a_0}{2}$

$n=3$ $\quad a_3 = 0$

$n=4$ $\quad a_4 = \frac{a_2}{20} = \frac{a_0}{40}.$

$n=5$ $\quad a_5 = 0$

$$y_2(x) = x^{-1/2}(a_0 + a_1x + a_2x^2 + \dots)$$
$$= \frac{1}{\sqrt{x}}\left(a_0 + \frac{a_0x^2}{2} + \frac{a_0x^4}{40} + \dots\right)$$
$$= \frac{a_0}{\sqrt{x}}\left(1 + \frac{x^2}{2} + \frac{x^4}{40} + \dots\right)$$

$\therefore$ the two LI solutions are

$$y_1(x) = a_0x\left(1 + \frac{x^2}{14} + \frac{x^4}{616} + \dots\right)$$

and $y_2(x) = \dfrac{a_0}{\sqrt{x}}\left(1 + \dfrac{x^2}{2} + \dfrac{x^4}{40} + \dots\right)$

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Q4) $xy'' + y' - xy = 0$

Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Substituting

$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$

$n-1 = m \Rightarrow n = m+1 \Rightarrow n-1 \to n$ in the first & 2 term

In the third term, $n+1 = m \Rightarrow n = m-1 \Rightarrow n+1 \to n$

$\Rightarrow \sum_{n=1}^{\infty} n(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$

$\Rightarrow a_1 + \sum_{n=1}^{\infty} [n(n+1)a_{n+1} + (n+1)a_{n+1} - a_{n-1}] x^n = 0$

$\boxed{a_1 = 0}$

$(\quad) x^n = 0$

$\Rightarrow n(n+1)a_{n+1} + (n+1)a_{n+1} = a_{n-1}$

$\Rightarrow a_{n+1} = \frac{a_{n-1}}{(n+1)^2}$

$n=1$ $a_2 = \frac{a_0}{4}$

$n=2$ $a_3 = 0$

$n=3 \quad a_4 = \frac{a_2}{16} = \frac{a_0}{64}$

$\therefore \; y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$
$= a_0 + \frac{a_0}{4} x^2 + \frac{a_0}{64} x^4 + \ldots$

$\Rightarrow y(x) = a_0 \left(1 + \frac{x^2}{4} + \frac{x^4}{64} + \ldots\right)$

Q2) a) Let $y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} y^n(x_0)(x-x_0)^n$

b) Substituting $x=0$ in the main equation,
$y''(0) + y(0) = 0$
$\Rightarrow y''(0) = -1$

Differentiating given equation,
$y'''(1-x^2) - 2xy'' + 2y' + 2xy'' + y' = 0$
$\Rightarrow y'''(1-x^2) + 3y' = 0$
$\Rightarrow y'''(0) + 3y'(0) = 0$
$\Rightarrow y'''(0) = -3$

$\therefore \; y(x) = 1 + x - \frac{1}{2!} x^2 - \frac{3}{3!} x^3 + \frac{3}{4!} x^4 + \ldots$
$= 1 + x - \frac{x^2}{2} - \frac{x^3}{2} + \frac{x^4}{8} + \ldots$

c) $y''(0) - y(0) = 0$
$\Rightarrow y''(0) = 2$

$y'''(0) = y'(0) = 0$
$y''''(0) = y''(0) = 2$

$\therefore\ y(x) = 2 + x^2 + \frac{2}{4!}x^4 + \frac{2}{6!}x^6 + \dots$

$= 2 + x^2 + \frac{x^4}{12} + \frac{x^6}{72} + \dots$

Q5) $x^2 y'' + x^3 y' + (x^2-2) = 0$

$\Rightarrow y'' + xy' + \left(\frac{x^2-2}{x^2}\right) = 0$

$p(x) = x$

$q(x) = \frac{x^2-2}{x^2} = 1 - \frac{2}{x^2}$

At $x_0 = 0$, which is a singular point,

$\lim_{x\to 0} x^2 = 0$

$\lim_{x\to 0} \cancel{x^2}\left(\frac{x^2-2}{\cancel{x^2}}\right) = -2$

$\therefore\ x_0 = 0$ is an RSP. We will use Frobenius Method.

Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ $\{a_0 \neq 0, r \in R\}$

$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$

$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

Substituting,

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r+2} + x^2 - 2 = 0$$

Let $m = n+2 \Rightarrow n = m-2$

$n+2 \to n$

$$\cancel{\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} + x^2 - 2 = 0}$$

$$\Rightarrow \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}(n+r-2)x^{n+r} + x^2 - 2 = 0$$

$$\Rightarrow r(r-1)a_0 x^r + x^2 - 2 + r(r+1)a_1 x^{r+1} + \cancel{\sum_{n=2}^{\infty} x^{n+r}}$$

$$+ \sum_{n=2}^{\infty} x^{n+r}\left[(n+r)(n+r-1)a_n + (n+r-2)a_{n-2}\right] = 0$$

$$(\quad)x^r = 0$$

Q7) Let $u = (x^2-1)^n$

$$\Rightarrow \frac{du}{dx} = u_1 = 2nx(x^2-1)^{n-1} = \frac{2nxu}{x^2-1}$$

$$\Rightarrow (1-x^2)u_1 + 2nxu = 0$$

Differentiating it $(n+1)$ times using Leibnitz theorem

$$(1-x^2)u_{n+2} + (n+1)(-2x)u_{n+1} + \frac{1}{2!}(n+1)(n)(-2u_n) +$$

$$2n[xu_{n+1} + (n+1)u_n] = 0$$

$$\Rightarrow (1-x^2)u_n'' - 2xu_n' + n(n+1)u_n = 0 \quad \text{— (1)}$$

where $u_n' = \dfrac{du_n}{dx}$ and $u_n'' = \dfrac{d^2u_n}{dx^2}$

Eq. ① is Legendre DE in $y = cu_n$, where c is an arbitrary constant. Since $P_n(x)$ is the finite series solution of the Legendre equation,

$$P_n(x) = cu_n = c\frac{d^n}{dx^n}\left[(x^2-1)^n\right] \quad -②$$

Setting $P_n(1) = 1$,

$$\Rightarrow P_n(1) = c\frac{d^n}{dx^n}\left[(x^2-1)^n\right]_{x=1}$$

$$= c\frac{d^n}{dx^n}\left[(x-1)^n(x+1)^n\right]_{x=1}$$

$= c\left[n!(x+1)^n + \text{terms containing } (x-1) \text{ and its higher powers}\right]_{x=1}$

$$= cn!\,2^n$$

$$\Rightarrow c = \frac{1}{n!\,2^n} \quad -③$$

Substituting ③ in ②, we obtain

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

Hence proven.

Q9) We have $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\left[(x^2-1)^n\right]$

$$\Rightarrow P_n'(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[n(x^2-1)^{n-1} \times 2x\right]$$

$$= \frac{1}{2^{n-1}(n-1)!}\frac{d^n}{dx^n}\left[x(x^2-1)^{n-1}\right]$$

$$= \frac{1}{2^{n-1}(n-1)!}\left[x\frac{d^n}{dx^n}(x^2-1)^{n-1} + n\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\right]$$

$$= x\frac{d}{dx}\left[\frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}[(x^2-1)]^{n-1}\right] + \frac{n}{2^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}[(x^2-1)^{n-1}]$$

$$= xP'_{n-1}(x) + nP_{n-1}(x)$$

Hence proven

Q11) b) We have $\frac{d}{dx}[x^v J_v(x)] = x^v J_{v-1}(x)$

$$\Rightarrow vx^{v-1}J_v(x) + x^v J'_v(x) = x^v J_{v-1}(x) \quad -①$$

Also $\frac{d}{dx}[x^{-v}J_v(x)] = -x^{-v}J_{v+1}(x)$

$$\Rightarrow -vx^{-v-1}J_v(x) + x^{-v}J'_{v}(x) = -x^{-v}J_{v+1}(x)$$

$$\Rightarrow -vx^{v-1}J_v(x) + x^v J'_v(x) = -x^v J_{v+1}(x) \quad -②$$

Subtracting ② from ① and dividing by x^v,

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x}J_v(x)$$

$$\Rightarrow 2vJ_v(x) = x[J_{v-1}(x) + J_{v+1}(x)]$$

Hence proven

Q10) a) This is the orthogonality of Legendre ~~Polynomiamls~~ Polynomials on $[-1,1]$

$P_n(x)$ is a solution of the Legendre equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0 \quad \text{for } \alpha = n$$

For non-negative integers m and n,

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \quad \text{--- (1)}$$

$$(1-x^2)P_m''(x) - 2xP_m'(x) + m(m+1)P_m(x) = 0 \quad \text{--- (2)}$$

Multiplying (1) by $P_m(x)$ and (2) by $P_n(x)$ and subtracting the resultant equations, we obtain

$$(1-x^2)(P_n''P_m - P_m''P_n) - 2x(P_n'P_m - P_m'P_n) + [n(n+1) - m(m+1)]P_nP_m = 0$$

$$\Rightarrow \frac{d}{dx}\left[(1-x^2)(P_n'P_m - P_m'P_n)\right] + [n(n+1) - m(m+1)]P_nP_m = 0$$

Integrating over the interval $[-1,1]$, we obtain

$$\left[(1-x^2)(P_n'P_m - P_m'P_n)\right]_{-1}^{1} + [n(n+1) - m(m+1)]\int_{-1}^{1} P_nP_m\,dx = 0$$

~~For $m \neq n$, we get~~

The 1st term vanishes at $x = \pm 1$

$$\Rightarrow \int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \quad , \quad \underline{m \neq n}$$

Case $m = n$

Using the generating function given by

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$$

Squaring both sides and integrating wrt x over $[-1,1]$,

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \int_{-1}^{1} \left[\sum_{n=0}^{\infty} P_n(x)t^n\right]^2 dx \quad \text{--- (3)}$$

From LHS, we obtain

$$\int_{-1}^{1} \frac{dx}{(1-2t+t^2)} = \left[\frac{\ln(1-2xt+t^2)}{-2t}\right]_{-1}^{1}$$

$$= \frac{-1}{2t}\left[\ln(1-2t+t^2) - \ln(1+2t+t^2)\right]$$

$$= \frac{1}{t}\left[\ln(1+t) - \ln(1-t)\right]$$

$$\Rightarrow \int_{-1}^{1} \frac{dx}{(1-2t+t^2)} = 2\left(1 + \frac{t^2}{3} + \frac{t^4}{5} + \dots \frac{t^{2n}}{2n+1} + \dots\right) \quad — (4)$$

RHS of (3), using the orthogonal property of the Legendre Polynomials for $m \neq n$ gives

$$\int_{-1}^{1}\left[\sum_{n=0}^{\infty} P_n(x)t^n\right]^2 dx = \sum_{n=0}^{\infty}\left(\int_{-1}^{1} P_n^2(x)\,dx\right)t^{2n} \quad — (5)$$

From (4) and (5), equating the coefficient of t^{2n} gives

$$\cancel{\int_{-1}^{1} P_n^2(x)(dx)} \quad \int_{-1}^{1} P_n^2(x)\cdot dx = \frac{2}{2n+1}, \quad \text{for } n \geq 0$$

Hence proven.

b) This is called the orthogonality of Bessel Functions.

We know that $u(x) = J_\nu(j_n \cancel{x} x)$ satisfies the Bessel eq^n

$$x^2 u'' + xu' + (j_n^2 x^2 - \nu^2)u = 0 \quad — (1)$$

and $v(x) = J_v(j_m x)$ satisfies the equation
$$x^2 v'' + x v' + (j_m^2 x^2 - v^2) v = 0 \qquad -②$$

Multiplying ① by v and ② by u and subtracting the resulting equations, we obtain

$$x^2(u''v - uv'') + x(u'v - uv') = (j_m^2 - j_n^2) x^2 uv$$
$$\Rightarrow [x(u'v - uv')]' = (j_m^2 - j_n^2) x uv$$

Integrating it over $[0,1]$,
$$\int_0^1 [x(u'v - uv')]' \, dx = \int_0^1 (j_m^2 - j_n^2) x uv \, dx \qquad -③$$

$$\text{LHS} = \int_0^1 [x(u'v - uv')]' \cdot dx = [x(u'v - uv)]_0^1$$
$$= [u'v - uv']_{x=1}$$

$u = J_v(j_n x)$ gives $u' = j_n J_v'(j_n x)$
Similarly, $v = J_v(j_m x)$ gives $v' = j_m J_v'(j_m x)$

Thus $\text{LHS} = [u'v - uv']_{x=1}$

$$= [j_n J_v'(j_n x) J_v(j_m x) - j_m J_v'(j_m x) J_v(j_n x)]_{x=1}$$

$$= j_n J_v'(j_n) J_v(j_m) - j_m J_v'(j_m) J_v(j_n)$$
$= 0$, since j_m and j_n are the zeros of $J_v(x)$

Next, the RHS of ③ $= (j_m^2 - j_n^2) \int_0^1 x uv \, dx$

$$= (j_m^2 - j_n^2) \int_0^1 x J_v(j_n x) J_v(j_m x) \, dx$$

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Comparing the two sides, we obtain $\cancel{\int_0^1 x J_\nu (j}$

$$\int_0^1 x J_\nu(j_n x) J_\nu(j_m x)\,dx = 0 \quad , \text{ for } j_n \neq j_m$$

In case $j_n = j_m$, then considering j_n as a root of $J_\nu(x) = 0$ and j_m as a variable approaching j_n, the ~~LHS = $j_n J_\nu'(j_n) J$~~ LHS $= j_n J_\nu'(j_n) J_\nu(j_m)$, thus we have

$$\lim_{j_m \to j_n} \int_0^1 x J_\nu(j_n x) J_\nu(j_m x)\,dx = \lim_{j_m \to j_n} \frac{j_n J_\nu'(j_n) J_\nu(j_m)}{j_m^2 - j_n^2}$$

$$= \lim_{j_m \to j_n} \frac{j_n J_\nu'(j_n) J_\nu'(j_m)}{2 j_m}$$

$$= \frac{1}{2} \{J_\nu'(j_n)\}^2$$

$$= \frac{1}{2} [J_{\nu+1}(j_n)]^2$$

$\therefore$ Hence proven.