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ASSIGNMENT-1

UNIT 1: INFINITE SERIES

(B. Tech, Semester-1)

Q1. A geometric sequence is of the form $a, ar, ar^2, \dots$ partial sum of first n terms of the series can be written as.

$$S_n = \frac{a(1-r^n)}{(1-r)}, \text{ for } r \neq 1$$

We have the following cases:

CASE I ($|r|<1$)

As $n \to \infty$ $r^n \to 0$ $1-r^n \to 1$

$\therefore \lim_{n\to\infty} S_n = \frac{a}{1-r}$ (which is finite and unique) as

$\Rightarrow$ the series is convergent in the given conditions.

CASE II ($|r|>1$)

As $n \to \infty$ $r^n \to \infty$ $r^n - 1 \to \infty$ [when $r>1$]

$$S_n = a\frac{(r^n-1)}{(r-1)}$$

$\lim_{n\to\infty} S_n = \infty$ $\therefore$ series is divergent.

when $(r<-1)$ as $n\to\infty$ $r^n \to -\infty$ or ∞. Hence the $\lim_{n\to\infty} S_n$ oscillates b/w ∞ and $-\infty$.

CASE III ($|r|=1$) for $r=1$

$S_n = a + a + \dots + a + \dots$ (n times)

$= na$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} na = \infty$

$\therefore$ series is divergent

for $r=-1$

$S_n = a + (-a) + a + (-a) + \dots$ n terms.

$= \begin{cases} 0 & \text{when } n \text{ is even} \\ a & \text{when } n \text{ is odd} \end{cases}$

$\therefore$ series oscillates b/w 0 and a.

A geometric series is $\therefore$ (i) divergent for $r \geq 1$

(ii) oscillatory for $r \leq -1$

(iii) converges for $-1<r<1$.

Q-2 $\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \cdots$

$u_n = \frac{\sqrt{n+1}-1}{(n+2)^3-1}$

$v_n = \frac{1}{n^{5/2}}$ be an auxillary

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{n+1}-1}{(n+2)^3-1} (n^{5/2})$

$= \lim_{n\to\infty} \frac{(n+1)^{1/2}(n^{5/2}) - n^{5/2}}{(n+2)^3 - 1}$

$\lim_{n\to\infty} \frac{(n^6+n^5)^{1/2} - n^{5/2}}{(n+2)^3 - 1}$

$\lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^{1/2} - n^{-1/2}}{(1+2/n)^3 - 1/n^3}$

$= 1$ (finite and unique)

$\Rightarrow$ both the series converge or diverge together

v_n is a harmonic series with $p = 5/2$ which is known to converge.

$\therefore$ u_n also converges.

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T_1

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p series is of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$

e have 3 case & (i) $p>1$ (ii) $p<1$ (iii) $p=1$.

→ $p>1$

∵ grouping of series does not change the behaviour of the

, so we group them such that first group will contain 1, 2^{nd} gp.

contain 2 and 3rd group will contain 4 terms and so on.

$$\underbrace{\frac{1}{1^p}}_{T_1} + \underbrace{\left(\frac{1}{2^p} + \frac{1}{3^p}\right)}_{T_2} + \overbrace{\left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right)}^{T_3} + \dots$$

$$3^p > 2^p \quad (\because p>1)$$

$$\frac{1}{3^p} < \frac{1}{2^p}$$

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p}$$

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^{p-1}} \quad -(1)$$

$$4^p < 5^p < 6^p < 7^p$$

$$\frac{1}{4^p} > \frac{1}{5^p} > \frac{1}{6^p} > \frac{1}{7^p}$$

$$4 \times \frac{1}{4^p} > \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} \quad -(2)$$

$$\frac{1}{2^{2(p-1)}} > T_3$$

and ~~hence~~ so on and we deduce

$$T_1 + T_2 + T_3 + \dots < 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2(p-1)}} + \dots$$

R.H.S. of inequality is an ∞ G.P. with $a = 1$ & $r = \frac{1}{2^{p-1}}$

$$0 < \frac{1}{2^{p-1}} < 1 \quad (p>1)$$

∴ G.P. will give a finite value. (converge) K(say)

$$T_1 + T_2 + T_3 + \dots T_\infty < k.$$

∴ p series converges with $p>1$

CASE II : when $p=1$ similar grouping gives,

$$\underbrace{1}_{T_1} + \left(\frac{1}{2}\right) + \underbrace{\left(\frac{1}{3}\right) + \left\{\frac{1}{4}\right)}_{T_2} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \frac{1}{8}\right)}_{T_3}$$

$$1 + T_1 + T_2 + T_3 + \dots = \lim_{n\to\infty} S_n.$$

$3 < 4$

$\frac{1}{3} > \frac{1}{4}$ $\qquad \frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4}$ $\qquad \frac{1}{3} + \frac{1}{4} > \frac{1}{2}$

$5 < 6 < 7 < 8$

$\frac{1}{5} > \frac{1}{6} > \frac{1}{7} > \frac{1}{8}$ $\qquad \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$

$T_3 > \frac{1}{2}$

$S_2 = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$ (n-1) times

$S_2 = 1 + \frac{n-1}{2} = \frac{n+1}{2}$. — (3)

$\sum_{n=1}^{\infty} \frac{1}{n} > \frac{n+1}{2}$.

$\lim_{n\to\infty} \frac{n+1}{2} = \infty$ $\quad$ ∴ series will diverge.

CASE III : $p < 1$

$\frac{1}{1^p} = 1$, $\frac{1}{2^p} > \frac{1}{2}$, $\frac{1}{3^p} > \frac{1}{3}$ & $\frac{1}{4^p} > \frac{1}{4}$ . . . .

$S_n > \sum_{n=1}^{\infty} \frac{1}{n}$ → this series diverges. (shown above).

∴ p series will diverge.

Result: 1. p series is convergent if $p > 1$
2. p series is divergent if $p < 1$

(ii). the integral to be calculated is $\int_1^\infty \frac{dx}{x^p}$ if it is finite as.

For $p \neq 1$. $\int_1^\infty \frac{dx}{x^p} = \lim_{n\to\infty} \int_1^n \frac{dx}{x^p} = \lim_{n\to\infty} \left[\frac{n^{1-p} - 1}{1-p} \right]$

which is finite and equal to $\frac{1}{p-1}$ when $p > 1$ and tends to ∞ for $p < 1$

when $p = 1$ $\int_1^\infty \frac{dx}{x} = \ln x \Big|_1^\infty = \cancel{\ln(\infty)}$ $\lim_{n\to\infty} \ln(n) - \ln(1) = \infty$

∴ the p series is convergent for $p > 1$ and divergent for $p < 1$.

$Q = W + \Delta E \rightarrow U$ (internal energy)

$dq = du + dw$

$dq = du + pdv$

4

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \cdots$$ check for convergent or divergent.

$$u_n = \frac{2n-1}{(n)(n+1)(n+2)}$$

$v_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is an auxillary series.

v_n is a p series with $p = 2$ which is known to be convergent.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{(2n-1)\,\cancel{(n)}(n)}{\cancel{(n)}(n+1)(n+2)} = \lim_{n\to\infty} \frac{(2-1/n)(1)}{(1+1/n)(1+2/n)}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 2$$ (which is finite and non-zero)

$\therefore$ both u_n and v_n converge together

$\Rightarrow$ given series is convergent

5

$$u_n = \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n^p}$$

$v_n = \frac{1}{n^{p-1/2}}$ will be an auxillary series for u_n

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n^p} \cdot n^{p-1/2}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \sqrt{1 + 1/n} - 1$$

$$= \lim_{n\to\infty} 0 \quad \text{(finite)}.$$

$\therefore$ the nature of u_n depends upon the auxillary series v_n, which will ~~converge~~ diverge for $p - 1/2 \le 1$ or $p \le 3/2$ and converge for $p > 3/2$.

6 a. $$1 + \left(1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \cdots\right. \quad (p>0).$$

$$u_n = \frac{n^p}{n!} \qquad u_{n+1} = \frac{(n+1)^p}{(n+1)!}$$

for Ratio test

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n^p \, \cancel{(n+1)!}(n+1)}{n! \, (n+1)^p} = \infty$$

$\infty > 1$

$\therefore$ through the ratio test we find that the ∞ series converges.

b. $\sum_{n=1}^{\infty} \frac{n!}{5^n}$

$u_n = \frac{n!}{5^n}$ $\quad u_{n+1} = \frac{(n+1)!}{5^{n+1}}$

Ratio test

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n!}{5^n} \frac{5^{n+1}}{(n+1)!}$

$= \lim_{n\to\infty} \frac{5}{n+1} = 0$

$0 < 1$

$\therefore$ through ratio test the given series diverges.

c. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$u_n = \frac{n!}{n^n}$ $\quad u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

Ratio test

$\lim_{n\to\infty} \frac{n!}{n^n} \frac{(n+1)^{n+1}}{(n+1)!} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$\Rightarrow$ ~~Ratio test has failed we now apply Raabe's test~~

~~Raabe's test~~

~~$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right)$~~

$e > 1$

$\therefore$ through the ratio test the ∞ series is convergent

7.

$$\sum_{n=1}^{\infty} \frac{3.6.9 \ldots 3n}{7.10.13 \ldots (3n+4)} x^n$$

$$u_n = \frac{3.6.9 \ldots 3n}{7.10.13 \ldots (3n+4)} x^n \qquad u_{n+1} = \frac{3.6.9 \ldots (3n+3)}{7.10.13 \ldots (3n+7)} x^{n+1}$$

Ratio test

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left(\frac{3n+7}{3n+3}\right)\frac{1}{x} = \lim_{n\to\infty}\left(\frac{3+7/n}{3+3/n}\right)\frac{1}{x} = \lim_{n\to\infty} \frac{1}{x} = \frac{1}{x}$$

$\therefore$ ~~the ratio test con~~

$\therefore$ the series converges for $\frac{1}{x} > 1 \Rightarrow x < 1$
and diverges for $\frac{1}{x} < 1 \Rightarrow x > 1$
and the ratio test fails for $\frac{1}{x} = 1 \Rightarrow x = 1$

we apply Raabe's test at $x = 1$.

$$= \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{3n+7}{3n+3} - 1\right)$$

$$= \lim_{n\to\infty} \frac{4n}{3n+3}$$

$$= \lim_{n\to\infty} \frac{4}{3+3/n} = \frac{4}{3}$$

$4/3 > 1 \Rightarrow$ through Raabe's test the series converges at $x = 1$.

$\therefore$ the series converges for $x < 1$ and diverges for $x > 1$.

8.

$$\sum \frac{(n!)^n}{(2n)!} x^n \quad , (x > 0)$$

$$u_n = \frac{(n!)^n}{(2n)!} x^n \qquad u_{n+1} = \frac{(n+1)!^{n+1}}{(2n+2)!} x^{n+1}$$

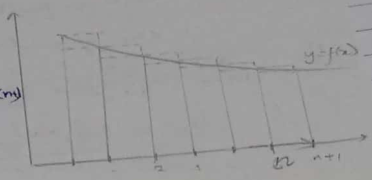
$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{(n!)^n \cancel{(2n+2)!}\,(2n+2)^2 (2n+1)}{\cancel{(2n)!}\,(n+1)^{n+1}(x)}$$

$$= \lim_{n\to\infty} \frac{2(2n+1)}{(n+1)!\,(n+1)^{n-1}(x)}$$

$Q = W + \Delta E \rightarrow 0, \ldots$

9 Let $f(x)$ be function where $f(x)$ decreases as x increases.

From the figure it is clear that



$f(1) + f(2) + \ldots + f(n) \geq \int_1^{n+1} f(x)\,dx \geq f(2) + f(3) + \ldots + f(n+1)$

$S_n \geq$ $\quad$ $\therefore \geq S_{n+1} - f(1)$

from inequality 2 $\quad \lim_{n \to \infty} \frac{2(2+1/n)}{n!\,(1+1/n)\,(n+1)^{n-1} x}$

S_n

Taki

$= \lim_{n \to \infty} \frac{4}{n!\,(n+1)^{n-1} x}$

He $\quad = 0 \quad (x > 0)$

Si $\quad \therefore$ the given series is divergent.

is $\lim_{n \to \infty}$

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ii) By i

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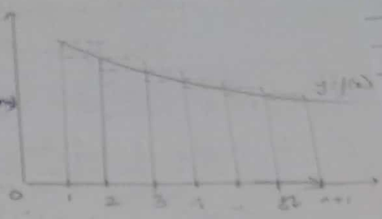
9 Let $f(x)$ be function where $f(x)$ decreases as x increases.



From the figure it is clear that

$$f(1)+f(2)+\dots+f(n) \geq \int_1^{n+1} f(x)\,dx \geq f(2)+f(3)+\dots+f(n+1)$$

from inequality 2

$$S_n \geq \int_1^{n+1} f(x)\,dx \geq S_{n+1} - f(1)$$

$$S_{n+1} \leq \int_1^{n+1} f(x)\,dx + f(1)$$

Taking limit as $n \to \infty$, we obtain.

$$\lim_{n\to\infty} S_{n+1} \leq \int_1^{\infty} f(x)\,dx + f(1)$$

Hence if $\int_1^{\infty} f(x)\,dx$ is finite, then so is $\lim_{n\to\infty} S_{n+1}$.

Similarly from inequality 1 we observe that

$S_n \geq \int_1^{n+1} f(x)\,dx$ if $\int_1^{\infty} f(x)\,dx$ is ∞, then so is $\lim_{n\to\infty} S_n$. & $\because$ the series is +ve it either converges or diverges depending upon the value of $\int_1^{\infty} f(x)\,dx$.

ii) By integral test we will check

$$\int_1^{\infty} e^{-px}\,dx = \int_1^{\infty} \frac{e^{-px}}{-p}\,dx(-p) = \frac{e^{-px}}{-p}\Big|_1^{\infty}$$

$$= \frac{e^{-p(\infty)} - e^{-p}}{-p}$$

which will be finite and unique and equal to $\frac{e^{-p}}{p}$ for $p>0$

and ∞ for $p \leq 0$.

$\therefore$ the integral is finite for $p>0$.

$\therefore$ the series converges for $p>0$.

10. $1 + \frac{1}{2\sqrt[3]{2}} + \frac{1}{3\sqrt[3]{3}} + \frac{1}{4\sqrt[3]{4}} + \dots$

$$\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$$

By integral we will check.

$$\int_1^{\infty} \frac{dx}{x^{5/4}} = \lim_{n\to\infty} \int_1^{n} \frac{dx}{x^{5/4}} = \lim_{n\to\infty} \frac{n^{-1/4} - 1^{-1/4}}{-1/4}$$

$= 1$ which is finite and unique.

$\therefore$ the given harmonic series is convergent.

11

$$\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \dots \quad (0<x<1)$$

$$u_n = \frac{(-1)^{n+1} x^n}{1+x^n}$$

$$u_{n+1} = \frac{(-1)^{n+2} x^{n+1}}{1+x^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(-1)^{n+1}(x^n)(1+x^{n+1})}{(1+x^n)(-1)^{n+2}x^{n+1}}$$

$$= \lim_{n\to\infty} \frac{-1\,(1+x^{n+1})}{(x)\,(1+x^n)} = \lim_{n\to\infty} \frac{(-1)(1+x^{n+1})}{x+x^{n+1}}$$

as $0<x<1$

$\Rightarrow n\to\infty \Rightarrow x^n \to 0$

$$\lim_{n\to\infty} = \left(\frac{-1}{x}\right)$$

which is finite and unique.

and hence the series is convergent.

12

$$\frac{1}{a1^2+b} + \frac{1}{a2^2+b} + \frac{1}{a3^2+b} + \dots$$

$$\sum_{n=1}^{\infty} \frac{1}{an^2+b}$$

$$u_n = \frac{1}{n^2\left(\frac{a}{} + \frac{b}{n^2}\right)}$$

$v_n = \frac{1}{n^2}$ becomes the auxillary series.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{a+\frac{b}{n^2}} = \frac{1}{a}.$$ (which is finite and unique)

and both the series either converge or diverge together

the series $v_n = \frac{1}{n^2}$ is known to converge

$\therefore$ the given series is convergent.

16 (a)

13

$$\frac{1}{1^p}+\frac{1}{3^p}+\frac{1}{5^p}+\cdots$$

$$=\left(\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\cdots\right)-\left(\frac{1}{2^p}+\frac{1}{4^p}+\frac{1}{6^p}+\cdots\right)$$

$$=\left(\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\cdots\right)-\frac{1}{2^p}\left(\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots\right)$$

$$u_n=\left(1-\frac{1}{2^p}\right)\left(\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots\right)$$

$v_n = \sum_{k=1}^{\infty}\frac{1}{n^p}$ be an auxillary series

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\left(1-\frac{1}{2^p}\right)\cancel{\left(\sum_{n=1}^{\infty}\frac{1}{n^p}\right)}}{\cancel{\left(\sum_{n=1}^{\infty}\frac{1}{n^p}\right)}}$$

$= 1-\frac{1}{2^p}$ which is finite and unique $\in \forall p \in R$.

$\therefore$ v_n and u_n either converge or diverge together.

the series v_n is known to converge for $p<1$ and diverge for $p\geq 1$

$\therefore$ u_n converges for $p\geq 1$ and diverges for $p\leq 1$.

14

$$\sum_{n=1}^{\infty}\frac{1}{(an+b)^p}$$

$$u_n=\cancel{\sum_{k=1}^{\infty}}\frac{1}{n^p(a+b/n)^p}$$

$v_n = \frac{1}{n^p}$ is the auxillary series.

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{(a+b/n)^p}=\frac{1}{a^p}$$ which is finite and unique.

$\therefore$ the series u_n and v_n either converge or diverge together.

series v_n converges for $p>1$ and diverges for $p<1$

$\therefore$ the series u_n converges for $p>1$ and diverges for $p<1$.

16 (a) $\frac{1^2\ 2^2}{1}$ 15

$\sum_{n=1}^{\infty} \frac{x^{n-1}}{1+x^n}$; $x>0$

$\cancel{u_n = \frac{x^{n-1}}{1+x^n}}$

$\cancel{v_n = \frac{1}{x}}$

$u_n = \frac{x^{n-1}}{1+x^n}$ $\quad u_{n+1} = \frac{x^n}{1+x^{n+1}}$

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^{n-1}(1+x^{n+1})}{(1+x^n)\,x^n}$

$= \lim_{n\to\infty} \frac{1+x^{n+1}}{x+x^{n+1}}$

Case I : when $0<x<1$ and $n\to\infty$ $x^n, x^{n+1} \to 0$

$\Rightarrow \lim_{n\to\infty} \frac{1}{x}$ (which is finite and unique)

Case II : when $x>1$ and $n\to\infty$ $x^n, x^{n+1} \to \infty$

$\lim_{n\to\infty} \frac{1+x^{n+1}}{x+x^{n+1}}$ ~~$\to\infty$~~ $\frac{\frac{1}{x^{n+1}}+1}{\frac{1}{x^n}+1} = 1$

(b)

Case III : when $x=1$ and $n\to\infty$.

$\lim_{n\to\infty} \frac{1+x^{n+1}}{x+x^{n+1}} = \frac{1+1}{1+1} = 1$

$\Rightarrow$ ratio test has failed for $x\geq 1$ we'll proceed to Raabe's test

$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{x^{n+1}+1}{x+x^{n+1}} - 1\right)$

$= \lim_{n\to\infty} n\left(\frac{\cancel{x^{n+1}}+1-x-\cancel{x^{n+1}}}{x+x^{n+1}}\right)$

$= \lim_{n\to\infty} n\left(\frac{1-x}{x+x^{n+1}}\right)$

16 (a) $\frac{1^2 2^2}{1!} + \frac{2^2 3^2}{2!} + \frac{3^2 4^2}{3!} + \cdots$

$$u_n = \frac{n^2 (n+1)^2}{n!}$$

&

$$u_{n+1} = \frac{(n+1)^2 (n+2)^2}{(n+1)!}$$

Ratio test

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n^2 \cancel{(n+1)^2}\,(n+1)\cancel{!}}{\cancel{n!}\,\cancel{(n+1)^2}(n+2)^2} = \lim_{n\to\infty} \left(\frac{n}{n+2}\right)^2 (n+1)$$

$$= \lim_{n\to\infty} \frac{(n+1)}{\left(\frac{2}{n}+1\right)^2} = \infty$$

$\infty > 1$

$\therefore u_n$ is an ∞ series which is convergent.

(b) $\frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \frac{1.2.3.4}{3.5.7.9} + \cdots$

$$u_n = \frac{n!}{\cancel{(2n+1)}\,(2n+3)\cdot(5)\,(7)\ldots(2n+3)}$$

$$u_{n+1} = \frac{(n+1)!}{(3)(5)(7)\ldots(2n+3)(2n+5)}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\cancel{n!}\,\cancel{(3)(5)(7)\ldots(2n+3)}\,(2n+5)}{\cancel{(3)(5)(7)\ldots(2n+3)}\,(n+1)\cancel{!}}$$

$$= \lim_{n\to\infty} \frac{2n+5}{n+1} = \lim_{n\to\infty} \frac{2+5/n}{1+1/n} = 2$$

$2 > 1$

$\Rightarrow u_n$ is an ∞ series which is convergent.

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17. $\frac{x^2}{1} + \left(\frac{2^2 x^4}{3.4} + \frac{2^2 . 4^2 x^6}{3.4.5.6} + \cdots\right)$

we ~~f~~ will ignore the first given term as it won't alter the nature of ∞ series.

Ratio test

$$u_n = \frac{x^{2(n+1)}\; 2^2\, 4^2 \ldots (2n)^2\; 2}{(2n+2)!}$$

$$u_{n+1} = \frac{x^{2(n+2)}\; 2^2\, 4^2 \ldots (2n+2)^2\; 2}{(2n+4)!}$$

– (13)

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^{2(n+1)}\, 2^2\, 4^2 \cdots (2n)^2\, x}{(2n+2)!} \cdot \frac{(2n+4)!\,(2n+4)(2n+3)}{x^{2(n+2)}\, 2^2\, 4^2 \cdots (2n)^2 (2n+2)^2 (2)\, x^2}$$

$$\lim_{n\to\infty} \frac{(2n+4)(2n+3)}{(2n+2)^2\, x^2}$$

$$\lim_{n\to\infty} \frac{(2+4/n)(2+3/n)}{(2+2/n)^2\, x^2}$$

$$\lim_{n\to\infty} \frac{1}{x^2}$$

u_n converges for $\frac{1}{x^2} > 1 \Rightarrow x^2 < 1$, u_n diverges for $\frac{1}{x^2} < 1 \Rightarrow x^2 > 1$

for $x = 1$ the ratio test fails.

we proceed for Raabe's test

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right)$$

$$\lim_{n\to\infty} n\left(\frac{(2+4/n)(2+3/n)}{(2+2/n)^2\, x^2} - 1\right)$$

$$\cancel{x = 1}$$

$$\cancel{\lim_{n\to\infty} n\,(1 - 1)}$$

$$\cancel{= 0}$$

$$\lim_{n\to\infty} n\left(\frac{(2n+4)(2n+3) - (2n+2)^2}{(2n+2)^2}\right)$$

$$\lim_{n\to\infty} n\left(\frac{\cancel{4n^2} + 14n + 12 - \cancel{4n^2} - 4 - 8n}{4n^2 + 4 + 8n}\right)$$

$$\lim_{n\to\infty} n\,\frac{(6n+8)}{4n^2+8n+4}$$

$$\lim_{n\to\infty} \frac{6n^2+8n}{4n^2+8n+4}$$

$$= \lim_{n\to\infty} \frac{6 + 8/n}{4 + 8/n + 4/n^2}$$

$$= 3/2 > 1$$

$3/2 > 1$

$\therefore$ the ~~∞~~ ∞ series converges for $x^2 \le 1$, diverges for $x^2 > 1$

18 Leibnitz test : An alternating series $u_1 - u_2 + u_3 - u_4 + \cdots$ is convergent if each term is numerically less than its preceding term that is if $u_1 > u_2 > u_3 > \cdots$ $\lim_{n\to\infty} u_n = 0$

To establish this consider S_{2n}.

① — $S_{2n} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2n-1} - u_{2n})$

② — $S_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - (u_6 - u_7) \cdots - (u_{2n-2} - u_{2n-1}) - u_{2n}$

Each term in brackets in ① is +ve since $u_1 > u_2 > u_3 \cdots$, hence S_{2n} is +ve and increases n increases.

Similarly each term in brackets in ② is +ve S_{2n} is always less than u_1

Thus the sequence S_{2n} is monotonically increasing and bounded

$\therefore \lim_{n\to\infty} S_{2n}$ exists and is finite.

Further,

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty}(S_{2n} + u_{2n+1}) = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} u_{2n+1}$$

$$= \lim_{n\to\infty} S_{2n},$$

$\because \lim_{n\to\infty} u_{2n+1} = 0$. Thus S_{2n} tends to the same finite limit whether n is odd or even.

When $\lim_{n\to\infty} u_n \neq 0$ then $\lim_{n\to\infty} S_{2n} \neq \lim_{n\to\infty} S_{2n+1}$

Thus in this case the series is oscillatory.

$$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$$

$$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \cdots$$

the terms of the given series are

$$\lim_{n\to\infty} \frac{\cos n\pi}{\sqrt{n}} = 0$$

Each term is numerically smaller from the preceding

$\frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}} > \frac{1}{\sqrt{4}} > \cdots$ $\therefore$ the series is convergent

18 a. If a series containing arbitrary terms $u_1+u_2+u_3+\dots u_n$ is such that the series $|u_1|+|u_2|+|u_3|+\dots+|u_n|$ is convergent, then the series Σu_n is said to be absolutely convergent.
In the case when $\Sigma|u_n|$ is ~~convergent~~ divergent but Σu_n is convergent series is said to be conditionally convergent.

$$|u_n| = \frac{\Sigma n}{2n^3} = \frac{(n)(n+1)}{2(n^3)}$$

$v_n = \frac{1}{n}$ (which is divergent)

~~either~~

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{n^2(n+1)}{2(n^3)n} = \frac{1}{2}$$ (finite and unique)

$\therefore$ the series $|u_n|$ diverges along with v_n.

but the actual series is of the type $u_1-u_2+u_3-u_4+\dots$

$$u_n = \frac{(-1)^{n+1}(n)(n+1)}{2n^3}$$

$$u_n - u_{n-1} = \frac{\cancel{(-1)^{n+1}}(n)(n+1)}{2n^3 2} - \frac{(n-1)(n)}{2(n-1)^3}$$

$$\frac{1}{2}\left(\frac{n+1}{n^2} - \frac{n}{(n-1)^2}\right)$$

$$\frac{(n+1)(n-1)^2 - n^3}{2(n^2)(n-1)^2}$$

$$\frac{\cancel{n^3} - n^2 - n + 1}{2n^2(n-1)^2} < 0$$

Thus each term is numerically less than the preceding term.
$\Rightarrow$ the series is now convergent
$\therefore$ the series was conditionally convergent.

b. $|u_n| = \sum_{n=1}^{\infty} \sin 1/n$

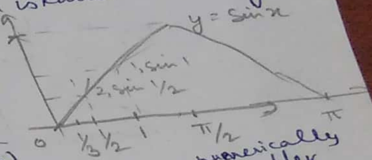
$v_n = \sum_{n=1}^{\infty} \frac{1}{n}$

$\lim_{n\to\infty} \frac{|u_n|}{v_n} = \lim_{n\to\infty} \frac{\sin 1/n}{1/n} = 1$ (finite and unique)

$\Rightarrow |u_n|$ and v_n converge/diverge together.

v_n is harmonic series of order 1 and is known to diverge.

$\therefore |u_n|$ diverges.



$u_n = (-1)^{n-1} \sin 1/n$. is an alternating series

where u_1 :: $\sin 1 > \sin 1/2 > \sin 1/3 \ldots$ (graphically)

$u_1 > u_2 > u_3 \ldots$

(each term is numerically smaller than preceding term)

$\Rightarrow$ it passes Leibnitz test

and so u_n is convergent.

$\therefore u_n$ is convergent and $|u_n|$ is divergent.

$\therefore u_n$ is conditionally convergent

$|u_n| = \frac{n+2}{2^n+5}$

$|u_{n+1}| = \frac{n+1+2}{2^{n+1}+5} = \frac{n+3}{2^{n+1}+5}$

Ratio test.

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{(n+2)(2^{n+1}+5)}{(2^n+5)(n+3)}$$

$$= \lim_{n\to\infty} \frac{(1+2/n)(2+5/2^n)\cancel{(2^n)}\cancel{(n)}}{(1+3/n)(1+5/2^n)\cancel{(2^n)}\cancel{(n)}}$$

$$= \lim_{n\to\infty} \frac{(1)(2)}{(1)(1)}$$

$= 2.$

$2 > 1$

$\Rightarrow |u_n|$ is convergent

$\because$ $u_1 + u_2 + u_3 + \dots \leq |u_1| + |u_2| + |u_3| + \dots$ — (1)

$\therefore$ ~~u_1~~ $\sum_{n=1}^{\infty} |u_n| =$ finite and unique ($\because |u_n|$ is convergent)

$\therefore$ $\sum_{n=1}^{\infty} u_n =$ finite and unique (using (1))

$\therefore$ u_n is also convergent.

$\Rightarrow$ the series absolutely convergent.

21

$$\sum_{n=1}^{\infty} \left(1 - \cos \frac{\pi}{n}\right)$$

~~1-cos2x~~

$(1 - \cos 2x = -2 \sin^2$

$$u_n = \sum_{n=1}^{\infty} -2 \sin^2 \frac{\pi}{2n}$$

$$v_n = \sum_{n}^{\infty} \frac{\pi^2}{(2n)^2}$$

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{-2 \sin^2 (\pi/2n)}{(\pi/2n)(\pi/2n)} = -2 \text{ (finite and unique)}$$

$\frac{\pi}{2n} \to 0$ $\quad \sin(\pi/2n) \to 0$ $\quad \frac{\sin(\pi/2n)}{(\pi/2n)} \to 1$

$\Rightarrow$ v_n and u_n both converge or diverge together

v_n is a harmonic series of order 2 which is known to converge

$\Rightarrow$ u_n is also convergent.

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