15 Fourier Integrals and Fourier Transforms

CHAPTER

Fourier integrals and Fourier transforms extend the concept of Fourier series to non-periodic functions defined for all x. A non-periodic function which cannot be represented as Fourier series over the entire real line may be represented in an integral form. Fourier transforms are integral transforms similar to Laplace transforms. In fact, 'Fourier analysis', the term including various kinds of Fourier series, integrals and transforms find variety of applications in science and engineering.

15.1 FOURIER INTEGRAL

In the preceding chapter we have seen that if a function $f(x)$ is defined on $-\infty < x < \infty$ and is periodic over an interval $-l < x < l$ (and satisfies the other conditions), then it can be represented by a Fourier series. In many practical problems we come across functions defined on $-\infty < x < \infty$ that are not periodic, e.g. $f(x) = e^{-x^2}$, the graph of which is shown in Fig. 15.1.

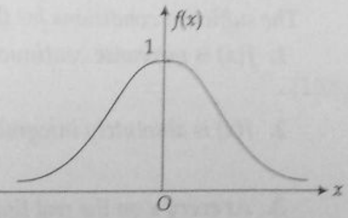

Fig. 15.1

We cannot expand such functions in Fourier series since they are not periodic, however we can consider such functions to be periodic but with an infinite period. *The Fourier integral can be regarded as an extension of the concept of Fourier series to non-periodic (or aperiodic) functions by letting $l \to \infty$.*

Consider any periodic function $f(x)$ of period $2l$ that can be represented by a Fourier series, then

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right), \qquad ...(15.1)$$

where $a_0 = \dfrac{1}{2l}\displaystyle\int_{-l}^{l} f(x)dx$, $a_n = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$ and $b_n = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$.

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Substituting the values for a_0, a_n and b_n, (15.1) becomes

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(u)du + \frac{1}{l}\sum_{n=1}^{\infty}\left[\cos\frac{n\pi x}{l}\int_{-l}^{l} f(u)\cos\frac{n\pi u}{l}du + \sin\frac{n\pi x}{l}\int_{-l}^{l} f(u)\sin\frac{n\pi u}{l}du\right]. \quad ...(15.2)$$

Set $w_n = \frac{n\pi}{l}$ and $\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{l} - \frac{n\pi}{l} = \frac{\pi}{l}$, (15.2) becomes

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(u)du + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[(\cos w_n x)\Delta w\int_{-l}^{l} f(u)\cos w_n u\,du + (\sin w_n x)\,\Delta w\int_{-l}^{l} f(u)\sin w_n u\,du\right] \quad ...(15.3)$$

The Eq. (15.3) is valid for any fixed finite l, arbitrary large.

We now let $l \to \infty$, and assume $f(x)$ to be absolutely integrable over the interval $(-\infty, \infty)$, that is, $\int_{-\infty}^{\infty} |f(x)|dx$ converges, then the value of the integral $\frac{1}{2l}\int_{-l}^{l} f(u)du$ tends to zero as $l \to \infty$; also $\Delta w = \pi/l \to 0$ and the infinite series in (15.3) becomes an integral from 0 to ∞, which represents $f(x)$ as

$$f(x) = \int_{0}^{\infty} [A(w)\cos wx + B(w)\sin wx]dw, \quad ...(15.4)$$

where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos wu\,du$ and $B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin wu\,du$, ...(15.5)

are called the *Fourier coefficients* and (15.4) is called the *Fourier integral representation of* $f(x)$.

The sufficient conditions for the validity of (15.4) are

1. *$f(x)$ is piecewise continuous on every interval $[-l, l]$.*
2. *$f(x)$ is absolutely integrable on the real axis, that is, $\int_{-\infty}^{\infty} |f(x)|dx$ converges.*
3. *At every x on the real line $f(x)$ has left and right hand derivatives.*

We state without proof the following convergence theorem for the Fourier integral, called the *Fourier Integral Theorem.*

Theorem 15.1 (Fourier integral theorem): *If $f(x)$ satisfies the conditions 1 to 3 stated above, then the Fourier integral of f converges to $f(x)$ at every point x at which f is continuous, and to the mean value $[f(x+0)+f(x-0)]/2$ at every point x at which f is discontinuous, where $f(x+)$ and $f(x-)$ are the right and left hand limits respectively.*

Example 15.1: Find the fourier integral representation of $f(x) = \begin{cases} 1, & \text{for } -1 \le x \le 1 \\ 0, & \text{for } |x| > 1 \end{cases}$ and hence prove that $\int_0^\infty \frac{\sin w}{w} dw = \frac{\pi}{2}$.

Solution: The graph of $f(x)$ is shown in Fig. 15.2. Clearly $f(x)$ is piecewise smooth and is absolutely integrable over $(-\infty, \infty)$. Thus $f(x)$ has a Fourier integral representation. The Fourier coefficients of $f(x)$ are

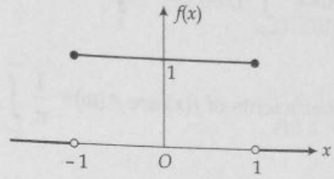

Fig. 15.2

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos wu \, du = \frac{1}{\pi} \int_{-1}^{1} \cos wu \, du = \left[\frac{\sin wu}{\pi w} \right]_{-1}^{1} = \frac{2 \sin w}{\pi w}, \text{ and,}$$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \sin wu \, du = \frac{1}{\pi} \int_{-1}^{1} \sin wu \, du = 0.$$

Hence the Fourier integral of $f(x)$ is $f(x) = \frac{2}{\pi} \int_0^\infty \frac{\cos wx \sin w}{w} dw$.

Since $x = \pm 1$ are the points of discontinuity of $f(x)$, thus at $x = \pm 1$

$$\frac{2}{\pi} \int_0^\infty \frac{\cos wx \sin w}{w} dw = \frac{1}{2} [f(x+0) + f(x-0)] = 1/2, \text{ for } \quad x = \pm 1.$$

Thus,
$$\int_0^\infty \frac{\cos wx \sin w}{w} dw = \begin{cases} \frac{\pi}{2}, & \text{for } -1 < x < 1 \\ \frac{\pi}{4}, & \text{for } x = \pm 1 \\ 0, & \text{for } |x| > 1 \end{cases} \quad \text{...(15.6)}$$

Set $x = 0$ in (15.6), we have $\int_0^\infty \frac{\sin w}{w} dw = \pi/2$.

Example 15.2: Find the Fourier integral representation of $f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x \le 0 \end{cases}$ and find the value of the resulting integral when, (a) $x < 0$, (b) $x = 0$, (c) $x > 0$. Also derive that $\int_0^\infty \frac{dw}{1+w^2} = \pi/2$.

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Solution: The given function $f(x)$ is piecewise smooth and is absolutely integrable over $(-\infty, \infty)$ since $\int_{-\infty}^{\infty} |f(x)|\,dx = \int_0^{\infty} e^{-x} dx = 1$. Thus $f(x)$ has a Fourier integral representation. The Fourier coefficients of $f(x)$ are $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos wu\,du = \frac{1}{\pi}\int_0^{\infty} e^{-u}\cos wu\,du = \frac{1}{\pi(1+w^2)}$, and,

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin wu\,du = \frac{1}{\pi}\int_0^{\infty} e^{-u}\sin wu\,du = \frac{w}{\pi(1+w^2)}.$$

Thus the Fourier integral repesentation of $f(x)$ is $f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{\cos wx + w\sin wx}{1+w^2}\,dw.$

At the point of discontinuity, $x = 0$,

$$\frac{1}{\pi}\int_0^{\infty} \frac{\cos wx + w\sin wx}{1+w^2}\,dw = \frac{1}{2}[f(x+0)+f(x-0)] = \frac{1}{2}[1+0] = \frac{1}{2}.$$

Thus,

$$\int_0^{\infty} \frac{\cos wx + w\sin wx}{1+w^2}\,dw = \begin{cases} 0, & x<0 \\ \pi/2, & x=0 \\ \pi e^{-x}, & x>0 \end{cases} \qquad \text{...(15.7)}$$

Set $x = 0$ in (15.7), we have $\int_0^{\infty} \frac{dw}{1+w^2} = \pi/2.$

15.2 FOURIER COSINE AND FOURIER SINE INTEGRALS

For an even or odd function the Fourier integral becomes simpler, analogous to the Fourier series expansion for the even or odd function. When $f(x)$ is an even function, then $f(u)\sin wu$ is an odd function of u, so from (15.5) we have $B(w) = 0$ and

$$A(w) = \frac{2}{\pi}\int_0^{\infty} f(u)\cos wu\,du. \qquad \text{...(15.8)}$$

Thus (15.4) simplifies to, $f(x) = \int_0^{\infty} A(w)\cos wx\,dw, \qquad \text{...(15.9)}$

called the *Fourier cosine integral representation* of $f(x)$.

Similarly, when $f(x)$ is an odd function, then $f(u)\cos wu$ is an odd function of u, so from (15.5) we have $A(w) = 0$ and

$$B(w) = \frac{2}{\pi}\int_0^\infty f(u)\sin wu\,du. \qquad ...(15.10)$$

Thus (15.4) simplifies to, $f(x) = \int_0^\infty B(w)\sin wx\,dw$, ...(15.11)

called the *Fourier sine integral representation* of $f(x)$.

The convergence result for the integral representations of even and odd functions is as follows.

Theorem 15.2: *If $f(x)$ is (i) piecewise continuous on each interval $[0, b]$, (ii) absolutely integrable on the real axis, and (iii) at every $x \in [0, \infty]$, $f(x)$ has left and right hand derivatives then the Fourier cosine and sine integrals of f converge to $f(x)$ at every point x at which f is continuous, and to the mean value $[f(x+0) + f(x-0)]/2$ at every point x at which f is discontinuous.*

Also similar to Fourier cosine and sine series defined on half period $[0, l]$, we can define Fourier cosine and Fourier sine integral representations of functions defined on the real half line $[0, \infty]$ by using respectively even or odd expansion of $f(x)$ to the whole real line.

Example 15.3: Find the Fourier cosine and sine integrals of $f(x) = e^{-kx}$, $x > 0$, $k > 0$.

Solution: Clearly $f(x)$ is differentiable and is absolutely integrable over $(0, \infty)$.

To obtain Fourier cosine representation, we have

$$A(w) = \frac{2}{\pi}\int_0^\infty f(u)\cos wu\,du = \frac{2}{\pi}\int_0^\infty e^{-ku}\cos wu\,du.$$

Consider $I = \int_0^\infty e^{-ku}\cos wu\,du = \left[\frac{e^{-ku}}{w^2+k^2}(w\sin wu - k\cos wu)\right]_0^\infty$.

When u tends to infinity, it becomes zero, and when u tends to zero, it becomes $-k/(w^2+k^2)$, since $k > 0$. Thus $I = k/(w^2+k^2)$ and hence

$$A(w) = \frac{2}{\pi}\int_0^\infty e^{-ku}\cos wu\,du = \frac{2k}{\pi(w^2+k^2)}. \qquad ...(15.12)$$

Thus the Fourier cosine integral $f(x) = \int_0^\infty A(w)\cos wx\,dx$ becomes

$$e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2+w^2}\,dx,\ k > 0. \qquad ...(15.13)$$

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On the similar lines, to obtain the Fourier sine integral representation of $f(x)$, we have

$$B(w) = \frac{2}{\pi}\int_0^\infty e^{-ku}\sin wu\,du = \frac{2w}{\pi(k^2+w^2)}, \quad \text{...(15.14)}$$

and thus $$f(x) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{w\sin wx}{k^2+w^2}\,dw,\ w>0. \quad \text{...(15.15)}$$

The integral representations (15.12) and (15.14) are called *Laplace integrals* because $A(w)$ is $2/\pi$ times the Laplace transform of $\cos wx$ and $B(w)$ is $2/\pi$ times the Laplace transform of $\sin wx$.

Example 15.4: Let $f(x) = \begin{cases} 1, & 0<x<1 \\ 0, & x\ge 1 \end{cases}$. Using the Fourier cosine integral representation of f, show that $\int_0^\infty \frac{\sin t}{t}\,dt = \pi/2$.

Solution: The function $f(x)$ is piecewise smooth and is also absolutely integrable over $(0,\infty)$. To obtain Fourier cosine representation, we have

$$A(w) = \frac{2}{\pi}\int_0^\infty f(u)\cos wu\,du = \frac{2}{\pi}\int_0^1 \cos wu\,du = \frac{2}{\pi}\left[\frac{\sin wu}{w}\right]_0^1 = \frac{2\sin w}{\pi w}.$$

Thus the Fourier cosine integral representation of $f(x)$ is

$$f(x) = \int_0^\infty A(w)\cos wx\,dw = \frac{2}{\pi}\int_0^\infty \frac{\sin w}{w}\cos wx\,dw.$$

The representation converges to $f(x)$ for every x in $(0,\infty)$ except at the point $x=1$ which is a point of discontinuity of $f(x)$. At $x=1$, the representation converges to

$$\frac{f(x+0)+f(x-0)}{2} = \frac{f(1+0)+f(1-0)}{2} = \frac{1}{2}.$$

Therefore $$\int_0^\infty \frac{\sin w}{w}\cos wx\,dw = \begin{cases} \pi/2, & 0<x<1 \\ \pi/4, & x=1 \\ 0, & x>1 \end{cases} \quad \text{...(15.16)}$$

and, hence for $x=1$, (15.16) gives $\int_0^\infty \frac{\sin w\cos w}{w}\,dw = \frac{\pi}{4}$, or $\int_0^\infty \frac{\sin 2w}{2w}\,dw = \frac{\pi}{4}$.

Setting $2w=t$ in it, we obtain $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$.

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Example 15.5: Solve the integral equation $\int_0^\infty f(x) \sin ax\, dx = e^{-a}$, where a is constant.

Solution: The given integral is Fourier sine integral represention.

Let
$$f(x) = \int_0^\infty A(w) \sin wx\, dw, \quad \text{...(15.17)}$$

where
$$A(w) = \frac{2}{\pi} \int_0^\infty f(u) \sin wu\, du. \quad \text{...(15.18)}$$

Comparing (15.18) with the given equation, we get

$$w = a \quad \text{and} \quad \frac{\pi A(w)}{2} = e^{-a}, \text{ thus } A(w) = \frac{2}{\pi} e^{-w},$$

and hence from (15.17) we have

$$f(x) = \frac{2}{\pi} \int_0^\infty e^{-w} \sin wx\, dw. \quad \text{...(15.19)}$$

Consider
$$I = \int_0^\infty e^{-w} \sin wx\, dw = \left[-\frac{e^{-w}}{1+x^2}(x \cos wx + \sin wx) \right]_0^\infty$$

When w tends to infinity, this becomes zero and at $w = 0$, it is $-x/(1 + x^2)$ and thus $I = x/(1 + x^2)$.

Using in (15.19), we get $f(x) = \dfrac{2x}{\pi(1+x^2)}, \quad x > 0.$

15.3 THE COMPLEX FOURIER INTEGRAL REPRESENTATION

Analogous to the complex form of the Fourier series discussed in Section 14.6, the Fourier integral can also be expressed in the equivalent complex form. This complex form provides the necessary plateform to develop the Fourier transform, (refer to Section 15.5), which are highly developed as a methodology like the Laplace transform.

Substituting the expressions for $A(w)$ and $B(w)$ from (15.5) into (15.4), we obtain

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty f(u)\{\cos wu \cos wx + \sin wu \sin wx\} du \right] dw$$

$$= \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty f(u) \cos w(u - x) du \right] dw. \quad \text{...(15.20)}$$

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Inserting, $\cos w(u-x) = \frac{1}{2}\left(e^{iw(u-x)} + e^{-iw(u-x)}\right)$, it becomes

$$f(x) = \frac{1}{2\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(u)\{e^{iw(u-x)} + e^{-iw(u-x)}\}du\right]dw$$

$$= \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(u)e^{iw(u-x)}du\,dw + \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(u)e^{-iw(u-x)}du\,dw. \qquad \text{...(15.21)}$$

In the first integral on the right side of (15.21), replace w by $-w$, we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{0}\int_{-\infty}^{\infty} f(u)e^{-iw(u-x)}du\,dw + \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(u)e^{-iw(u-x)}du\,dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{-iw(u-x)}du\,dw. \qquad \text{...(15.22)}$$

This is the *complex Fourier integral representation* of f on the real line. If we put

$$c(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)e^{-iwu}du, \qquad \text{...(15.23)}$$

then the integral (15.22) becomes

$$f(x) = \int_{-\infty}^{\infty} c(w)e^{iwx}dw. \qquad \text{...(15.24)}$$

The $c(w)$ as given in (15.23) is called the *complex Fourier integral coefficient of f.*

Example 15.6: If $f(x) = e^{-a|x|}$ for all real x and with $a > 0$, a positive constant, then find the complex Fourier integral representation of f.

Solution: The function is $f(x) = \begin{cases} e^{-ax}, & \text{for } x \geq 0 \\ e^{ax}, & \text{for } x < 0 \end{cases}$ $a > 0$ being a constant.

Obvious $f(x)$ is piecewise smooth and is absolutely integrable over the interval $(-\infty, \infty)$. The complex Fourier integral coefficient of f is given by

$$c(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)e^{-iwu}du = \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{au}e^{-iwu}du + \int_0^{\infty} e^{-au}e^{-iwu}du\right]$$

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$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{(a-iw)u}du + \int_{0}^{\infty} e^{-(a+iw)u}du\right] = \frac{1}{2\pi}\left[\left[\frac{e^{(a-iw)u}}{a-iw}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+iw)u}}{-(a+iw)}\right]_{0}^{\infty}\right]$$

$$= \frac{1}{2\pi}\left(\frac{1}{a+iw} + \frac{1}{a-iw}\right) = \frac{a}{\pi(a^2+w^2)}.$$

Thus, the complex Fourier integral representation, $f(x) = \int_{-\infty}^{\infty} c(w)e^{iwx}dw$, becomes

$$e^{-a|x|} = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{1}{a^2+w^2}e^{iwx}dw.$$

EXERCISE 15.1

1. Show that $f(x) = 1$, $(0 < x < \infty)$, cannot be represented by a Fourier integral.

Derive the Fourier integral representations of the following functions (Problems 2-5). At which points, if any, does the Fourier integral fail to converge to $f(x)$? To what value does the integral converge at those points?

2. $f(x) = \begin{cases} 100, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$

3. $f(x) = \begin{cases} bx/a, & |x| \le a \quad a, b > 0 \\ 0, & |x| > a \end{cases}$

4. $f(x) = \begin{cases} (\pi/2)\cos x, & |x| \le \pi/2 \\ 0, & |x| > \pi/2 \end{cases}$

5. $f(x) = \begin{cases} 0, & x < 0 \\ \cos x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$

In the following problems, find the integral representation as mentioned

6. $f(x) = e^{-2x} + e^{-3x}$, $x > 0$; cosine representation.

7. $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$; cosine representation.

8. $f(x) = \begin{cases} \sinh x, & 0 \le x \le 3 \\ 0, & x > 3 \end{cases}$; sine representation.

In the following problems, find the complex Fourier integral of the function and determine what this integral converges to

9. $f(x) = xe^{|x|}$, for all real x.

10. $f(x) = \begin{cases} \sin \pi x, & |x| \le 5 \\ 0, & |x| > 5 \end{cases}$ 11. $f(x) = \begin{cases} \cos x, & 0 \le x \le \pi/2 \\ \sin x, & -\pi/2 < x < 0 \\ 0, & |x| > \pi/2 \end{cases}$

12. Define a suitable function $f(x)$ and use the Fourier integral representation to show that

$$\int_0^\infty \frac{\sin ax}{x} dx = \pi/2, \ (a > 0).$$

13. If $\int_0^\infty f(x) \sin ax \, dx = \begin{cases} 1, & 0 < a < 1 \\ 0, & a > 1 \end{cases}$, then find $f(x)$.

14. Using the Fourier integral representation, show that

$$\int_0^\infty \frac{1 - \cos \pi w}{w} \sin (xw) dw = \begin{cases} \pi/2, & 0 < x \le \pi \\ 0, & x > \pi \end{cases}$$

15. Show that $\int_0^\infty \frac{\sin \pi w \sin wx}{1 - w^2} dw = \begin{cases} \frac{1}{2} \pi \sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$

15.4 FOURIER TRANSFORM AND ITS PROPERTIES

An *integral transform* is a transformation that produces from a given function, a new function which depends on a different variable and appears in the form of an integral. These transformations are mainly employed as a tool to solve certain initial and boundary value ordinary and partial differential equations arising in many areas of science and engineering. The Laplace transform as discussed in Chapter 13, is one such transform which has wide applications. Fourier transforms are the next other integral transforms which are of vital importance from the applications viewpoint in solving initial and boundary value problems.

We will discuss three transforms: *The Fourier transform, the Fourier cosine transform* and the *Fourier sine transform;* the first being complex and the latter two real. These transforms are obtained from the corresponding Fourier integral representations.

15.4.1 The Fourier Transform

The complex Fourier integral representation of function $f(x)$ on real line, refer to Eq. (15.22) is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{-iw(u-x)} du \, dw = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-iwu} du \right] e^{iwx} dw. \quad ...(15.25)$$

The expression in bracket, a function of w denoted by $F(w)$, is called the *Fourier transform* of $f(x)$ and since u is a dummy variable, we replace u by x and have

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$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx}dx, \qquad ...(15.26)$$

so that (15.25) becomes $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(w)e^{iwx}dw, \qquad ...(15.27)$

and is called the *inverse Fourier transform of F(w)*.

The function $f(x)$ and the associated Fourier transform $F(w)$ are called a *Fourier transform pair.*

Other common notations used for the Fourier transform of $f(x)$ are $\hat{f}(w)$ or, $\mathcal{F}(f(x))$ and the inverse Fourier transform is denoted by $\mathcal{F}^{-1}(f(x))$. Further, the choice of the normalizing factors $1/\sqrt{2\pi}$ in integrals (15.26) and (15.27) is optional and it is chosen here so, to make the two integrals as symmetric as possible. All that is required for the normalizing factors is that their product be $1/2\pi$. In fact we can write the normalizing factors in the general form as $k/2\pi$ and $1/k$, where k is an arbitrary scale factor.

The sufficient conditions for the existence of the Fourier transform are:

1. *f(x) is piecewise continuous on every finite interval; and*
2. *f(x) is absolutely integrable on the real axis.*

Similarly, for the existence of inverse Fourier transform of F(w), F(w) must be absolutely integrable over $(-\infty, \infty)$, *and thus* $\lim_{|w|\to\infty} F(w) = 0$.

Example 15.7: Find the Fourier transforms of

(a) $f(x) = \begin{cases} k, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$ (b) $f(x) = \begin{cases} a, & -l < x < 0 \\ b, & 0 < x < l \\ 0, & \text{otherwise} \end{cases} \quad a, b > 0$

(c) $f(x) = u(x+1) - u(x-1)$, where $u(x)$ is the unit-step function.

(d) $f(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$

Solution: (a) By definition

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx}dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{a} ke^{-iwx}dx = \frac{k}{\sqrt{2\pi}}\left[\frac{e^{-iwx}}{-iw}\right]_0^a = \frac{k}{iw\sqrt{2\pi}}(1 - e^{-iwa}).$$

(b) $$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx}dx = \frac{1}{\sqrt{2\pi}} \int_{-l}^{0} ae^{-iwx}dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{l} be^{-iwx}dx$$

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$$= \frac{a}{\sqrt{2\pi}}\left[\frac{e^{-iwx}}{-iw}\right]_{-l}^{0} + \frac{b}{\sqrt{2\pi}}\left[\frac{e^{-iwx}}{-iw}\right]_{0}^{l} = \frac{1}{iw\sqrt{2\pi}}[(b-a) + ae^{-iwl} - be^{-iwl}].$$

(c) The graph of $f(x) = u(x+1) - u(x-1) = \begin{cases} 1, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

is shown in Fig. 15.3.

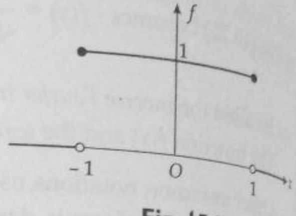

Fig. 15.3

By definition

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-iwx}}{-iw}\right)_{-1}^{1}$$

$$= \frac{e^{iw} - e^{-iw}}{\sqrt{2\pi}\, iw} = \sqrt{\frac{2}{\pi}}\,\frac{\sin w}{w}.$$

(d) By definition

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-iwx}}{-iw}\right]_{-a}^{a}$$

$$= \frac{1}{w\sqrt{2\pi}}\left[\frac{e^{iwa} - e^{-iwa}}{i}\right] = \sqrt{\frac{2}{\pi}}\,\frac{\sin wa}{w}.$$

Example 15.8: Find the Fourier transform of $f(x) = e^{-ax^2}$, $a > 0$.

Solution: By definition

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2}e^{-iwx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(ax^2+iwx)}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\left[-\left(\sqrt{a}x + \frac{iw}{2\sqrt{a}}\right)^2 + \left(\frac{iw}{2\sqrt{a}}\right)^2\right]}dx$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{4a}}\int_{-\infty}^{\infty} e^{-\left[\sqrt{a}x + \frac{iw}{2\sqrt{a}}\right]^2}dx = \frac{1}{\sqrt{2\pi a}}e^{-\frac{w^2}{4a}}\int_{-\infty}^{\infty} e^{-t^2}dt, \text{ setting } \sqrt{a}x + \frac{iw}{2\sqrt{a}} = t$$

$$= \frac{1}{\sqrt{2\pi a}}e^{-\frac{w^2}{4a}}.\sqrt{\pi}, \text{ since } \int_{-\infty}^{\infty} e^{-t^2}dt = 2\int_{0}^{\infty} e^{-t^2}dt = \Gamma(1/2) = \sqrt{\pi}$$

$$= \frac{1}{\sqrt{2a}}e^{-\frac{w^2}{4a}}.$$

Example 15.9: Find th

(a) $f(x) = e^{-|x|}$

Solution: (a) The funct

By definition

$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}$

$= \frac{1}{\sqrt{2\pi}}$

$= \frac{1}{\sqrt{2\pi}}$

(b) The function is $u($

Thus $\mathcal{F}(\delta(x)) = \lim_{k\to 0}$

$=$

Remark. A graph of $|F$ *amplitude spectrum* of $f(x)$.

For example, if $f(x)$

$F(w) = \sqrt{\frac{2}{\pi}}\,\frac{\sin w}{w}$, refer to

$\left(w, \sqrt{\frac{2}{\pi}}\left|\frac{\sin w}{w}\right|\right)$ is as show

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Example 15.9: Find the Fourier transform of the following functions

(a) $f(x) = e^{-|x|}$ (b) $\delta(x) = \lim\limits_{k\to 0}\frac{1}{k}[u(x) - u(u-k)]$; $u(x)$ being the unit-step function.

Solution: (a) The function is $f(x) = \begin{cases} e^{x} & -\infty < x \le 0 \\ e^{-x} & 0 < x < \infty \end{cases}$

By definition

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{(1-iw)x}dx + \int_{0}^{\infty} e^{-(1+iw)x}dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left[\frac{e^{(1-iw)x}}{(1-iw)}\right]_{-\infty}^{0} - \left[\frac{e^{-(1+iw)x}}{(1+iw)}\right]_{0}^{\infty}\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{(1-iw)} + \frac{1}{(1+iw)}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\frac{2}{1+w^2} = \sqrt{\frac{2}{\pi}}\frac{1}{1+w^2}.$$

(b) The function is $u(x) - u(x-k) = \begin{cases} 0, & x < 0 \\ 1, & 0 \le x < k \\ 0, & x \ge k \end{cases}$

Thus $\mathcal{F}(\delta(x)) = \lim\limits_{k\to 0}\left[\frac{1}{k}\mathcal{F}[u(x) - u(x-k)]\right] = \lim\limits_{k\to 0}\left[\frac{1}{k\sqrt{2\pi}}\int_{0}^{k} e^{-iwx}dx\right]$

$$= \frac{1}{\sqrt{2\pi}}\lim_{k\to 0}\left[\frac{1-e^{-iwk}}{iwk}\right] = \frac{1}{\sqrt{2\pi}}.$$

Remark. A graph of $|F(w)|$ versus w is called the *amplitude spectrum* of $f(x)$.

For example, if $f(x) = u(x+1) - u(x-1)$, then $F(w) = \sqrt{\frac{2}{\pi}}\frac{\sin w}{w}$, refer to Example 15.7c. The graph of $\left(w, \sqrt{\frac{2}{\pi}}\left|\frac{\sin w}{w}\right|\right)$ is as shown in Fig. 15.4 for $w \ge 0$.

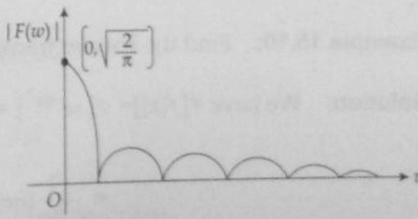

Fig. 15.4

15.4.2 Properties of Fourier Transform

The properties of Fourier transform help to simplify the calculations involving Fourier transf… and to obtain some results which are otherwise difficult to obtain.

1. *Linearity:* We state the following result.

Theorem 15.3 (Linearity theorem): *For any functions $f(x)$ and $g(x)$ whose Fourier transforms exist and for any constants a, b*

$$\mathcal{F}[af(x) + bg(x)] = a\mathcal{F}(f(x)) + b\mathcal{F}(g(x)),$$

where $\mathcal{F}(f(x))$ denotes the Fourier transform of $f(x)$.

The proof follows directly from the definition of Fourier transform.

2. *Fourier transform of derivatives:* It is stated as follows:

Theorem 15.4 (Transform of derivatives): *If $f(x)$ is a continuous function of x with $f(x) \to 0$ as $|x| \to \infty$, and $f'(x)$ is absolutely integrable over $(-\infty, \infty)$, then*

(a) $$\mathcal{F}[f'(x)] = iw\mathcal{F}[f(x)],$$

(b) $$\mathcal{F}[f^{(n)}(x)] = (iw)^n \mathcal{F}[f(x)],$$

and holds for all n such that the derivatives $f^{(r)}(x)$, $r = 1, 2, \ldots, n$ satisfy the sufficient conditions for the existence of the Fourier transforms.

Proof. (a) By definition, $\mathcal{F}[f'(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)e^{-iwx}dx$. Integrating by parts, we obtain

$$\mathcal{F}[f'(x)] = \frac{1}{\sqrt{2\pi}}\left[\left(f(x)e^{-iwx}\right)_{-\infty}^{\infty} - (-iw)\int_{-\infty}^{\infty} f(x)e^{-iwx}dx\right]$$

Since $f(x) \to 0$ as $|x| \to \infty$, therefore, $\mathcal{F}[f'(x)] = (iw)\mathcal{F}[f(x)]$

(b) The repeated application of result (a) gives result (b) provided the desired conditions are satisfied at each step.

Example 15.10: Find the Fourier transform of $f(x) = x\,e^{-ax^2}$, $a > 0$.

Solution: We have $\mathcal{F}[f(x)] = \mathcal{F}[xe^{-ax^2}] = \mathcal{F}\left[-\frac{1}{2a}(e^{-ax^2})'\right] = -\frac{1}{2a}\mathcal{F}[(e^{-ax^2})']$

$$= -\frac{1}{2a}(iw)\mathcal{F}[e^{-ax^2}], \quad \text{using differentiability}$$

$$= -\frac{iw}{2a}\left(\frac{1}{\sqrt{2a}}e^{-\frac{w^2}{4a}}\right), \quad \text{refer to Example 15.8}$$

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$$= \frac{-iw}{2a\sqrt{2a}} e^{-\frac{w^2}{4a}}.$$

Example 15.11: Show that

(a) $\mathcal{F}[x^n f(x)] = i^n \dfrac{d^n}{dw^n}[F(w)]$ ...(15.31)

(b) $\mathcal{F}[x^m f^{(n)}(x)] = i^{m+n} \dfrac{d^m}{dw^m}[w^n F(w)],$ ...(15.32)

where $F(w) = \mathcal{F}[f(x)]$ is the Fourier transform of $f(x)$.

Solution: (a) By definition of Fourier transform $F(w) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{-iwx} dx.$

Differentiating w.r.t. w and using Leibnitz rule to differentiate under integral sign, we have

$$\frac{d}{dw}[F(w)] = \frac{1}{\sqrt{2\pi}} \frac{d}{dw} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx = -\frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, f(x) e^{-iwx} dx$$

or,

$$i \frac{d}{dw}[F(w)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x f(x) e^{-iwx} dx = \mathcal{F}[x f(x)].$$

The repeated applications of the differentiation w.r.t. w leads to the desired result

$$\mathcal{F}[x^n f(x)] = i^n \frac{d^n}{dw^n}[F(w)].$$

(b) Consider $\mathcal{F}[x^m f^{(n)}(x)] = i^m \dfrac{d^m}{dw^m} \mathcal{F}[f^{(n)}(x)],$ using (15.31)

$$= i^m \frac{d^m}{dw^m}[(iw)^n F(w)], \quad \text{using (15.30)}$$

$$= i^{m+n} \frac{d^m}{dw^m}[w^n F(w)],$$

provided $f(x)$ and its successive derivatives satisfy the requisite conditions.

Example 15.12: Using the property of the Fourier transform of derivatives, find the Fourier transform of $f(x) = e^{-ax^2}$, $a > 0$.

Solution: Clearly $f(x)$ satisfies the requisite conditions of continuity and absolute integrability over the real axis for the existence of Fourier transform.

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It is easy to see that $f(x)$ satisfies the differential equation

$$f'(x) + 2a\,xf(x) = 0. \qquad \text{...(15.33)}$$

Taking the Fourier transform of (15.33), we have

$$\mathcal{F}[f'(x)] + 2a\,\mathcal{F}[xf(x)] = 0. \text{ It gives } iwF(w) + 2a\,(i\,F'(w)) = 0$$

or,

$$2a\,F'(w) + wF(w) = 0, \qquad \text{...(15.34)}$$

where $F(w)$ is the Fourier transform of $f(x)$. Rewriting (15.34) as $\dfrac{F'(w)}{F(w)} = -\dfrac{1}{2a}w$.

Integrating it w.r.t. w, we get $F(w) = A\exp\left[-\dfrac{w^2}{4a}\right]$, where A is an arbitrary constant.

To determine A we have $F(0) = A$ and also at $w = 0$,

$$F(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{\pi}}{\sqrt{a}} = \frac{1}{\sqrt{2a}}.$$

Thus,

$$F(w) = \frac{1}{\sqrt{2a}}e^{-w^2/4a}, \quad a > 0.$$

3. Shifting x by x_0 (the time-shifting); Scaling x by a; and Shifting w by w_0 (the frequency-shifting): The result is stated as follows.

Theorem 15.5 (Shifting and scaling): *If $f(x)$ has Fourier transform $F(w)$, then*

(a) $\mathcal{F}[f(x - x_0)] = e^{-iwx_0}\,F(w)$; *shifting on x-axis by* x_0.

(b) $\mathcal{F}[f(ax)] = \dfrac{1}{a}F(w/a)$, $a > 0$; *scalling x by a,*

(c) $\mathcal{F}[e^{iw_0x}f(x)] = F(w - w_0)$; *shifting w by* w_0.

The results follows immediately from the definition of the Fourier transform of $f(x)$.

Example 15.13: Find the Fourier transform of $f(x) = e^{-a(x-5)^2}$, $a > 0$, using shifting property.

Solution: By shifting property, we have

$$\mathcal{F}[e^{-a(x-5)^2}] = e^{-i5w}\mathcal{F}[e^{-ax^2}] = e^{-i5w}.\frac{1}{\sqrt{2a}}e^{-\frac{w^2}{4a}}, \quad \text{refer to Example 15.8}$$

$$= \frac{1}{\sqrt{2a}}e^{-\left(\frac{w^2}{4a} + i5w\right)}.$$

Example 15.14: Find the Fourier transform of $f(x) = 4e^{-|x|} - 5e^{-3|x+2|}$.

Solution: Using the linearity property

$$\mathcal{F}[f(x)] = 4\mathcal{F}(e^{-|x|}) - 5\mathcal{F}(e^{-3|x+2|}) = 4\mathcal{F}(e^{-|x|}) - 5e^{2iw}\mathcal{F}(e^{-|3x|}), \text{ using } x\text{-shifting}$$

$$= 4\mathcal{F}(e^{-|x|}) - \frac{5}{3}e^{2iw}\,\mathcal{F}(e^{-|x|})_{w\to w/3}, \text{ using scaling}$$

$$= 4.\frac{1}{\sqrt{2\pi}}\frac{2}{1+w^2} - \frac{5}{3}e^{2iw}.\frac{1}{\sqrt{2\pi}}\frac{2}{1+\left(\frac{w}{3}\right)^2}, \text{ refer to Example 15.9a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{8}{1+w^2} - \frac{30e^{2iw}}{9+w^2}\right].$$

4. **Fourier transform of integrals:** It is stated as follows.

Theorem 15.6 (Transforms of integrals): *If $\mathcal{F}[f(x)] = F(w)$, then*

$$\mathcal{F}\left[\int_{-\infty}^{x} f(t)\,dt\right] = \frac{1}{iw}F(w), \qquad \text{...(15.35)}$$

provided $F(w)$ satisfies $F(0) = 0$.

Proof. Let $g(x) = \int_{-\infty}^{x} f(t)dt$, then $g'(x) = f(x)$, since $\lim_{x\to-\infty} f(x) = 0$.

Also, $\mathcal{F}[g'(x)] = iw\mathcal{F}[g(x)]$. Substituting for $g(x)$ and $g'(x)$, it becomes

$$\mathcal{F}[f(x)] = iw\mathcal{F}\left[\int_{-\infty}^{x} f(t)\,dt\right],$$

which gives (15.35).

Example 15.15: Using the transform of integrals, find the Fourier transform of $f(x) = e^{-ax^2}$.

Solution: We have,

$$\mathcal{F}(e^{-ax^2}) = \mathcal{F}\left[-\ 2\ a \int_{-\ \infty}^{\infty}\ \right] = -2a\mathcal{F}\left[\int_{-\infty}^{x} xe^{-ax^2}\,dx\right]$$

$$= -2a\frac{1}{iw}\,\mathcal{F}(xe^{-ax^2}) = -2a.\frac{1}{iw}\left(\frac{-iw}{2a\sqrt{2a}}e^{-\frac{w^2}{4a}}\right), \text{ refer to Example 15.10}$$

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$$= \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}.$$

5. Fourier transform of convolutions: The *convolution* $f*g$ of two functions f and g is defined by

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\,dt.$$

We have the following result.

Theorem 15.7 (Convolution theorem): *Let $f(x)$ and $g(x)$ be two piecewise continuous, bounded and absolutely integrable functions on the x-axis, then the Fourier transform of $f*g$, the convolution of f and g, is*

$$\mathcal{F}(f*g) = \sqrt{2\pi}\, \mathcal{F}(f)\, \mathcal{F}(g). \quad \text{...(15.36)}$$

Proof. By definition of Fourier transform

$$\mathcal{F}[(f*g)(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t)\, g(x-t)\, e^{-iwx} \right) dx\, dt$$

Put $x - t = s$, then $x = (t + s)$, this becomes

$$\mathcal{F}[(f*g)(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, g(s) e^{-iw(t+s)} ds\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt} dt \int_{-\infty}^{\infty} g(s) e^{-iws} ds = \sqrt{2\pi}\, \mathcal{F}(f(x))\, \mathcal{F}(g(x)).$$

We observe that result in case of Fourier transform of convolution is the same as that of Laplace transform of convolution except for the factor $\sqrt{2\pi}$.

Taking inverse Fourier transform of (15.36), we obtain

$$(f*g)(x) = \sqrt{2\pi} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(f(x))\, \mathcal{F}(g(x)) e^{iwx} dw$$

or,
$$(f*g)(x) = \int_{-\infty}^{\infty} F(w)\, G(w) e^{iwx} dw, \quad \text{...(15.37)}$$

where $F(w) = \mathcal{F}(f(x))$ and $G(w) = \mathcal{F}(g(x))$.

The result (15.37) is particularly useful while solving partial differential equations using Fourier transforms.

Example 15.16: Find th

Solution: Lel $h(x)$ be the

$h(x)$ =

where $f(x) = \mathcal{F}^{-1}\left(\frac{1}{4+w^2}\right)$,

Using Example 15.9a a

$f(x)$ =

and, $g(x)$ =

Hence, $h(x)$ =

For $x > 0$, $\int_{-\infty}^{\infty} e^{-2|x-t|} e^{-3}$

$= \int_{-\infty}^{0} e^{-2(x-t)} e^{3t} dt + \int_{0}^{x} e^{-2(x}$

Similarly, for $x < 0$, $\int_{-\infty}^{\infty} e^{-2(x-t}$

and, for $x = 0$ $\int_{-\infty}^{\infty} e^{-2|x-t|} e^{-3|}$

Therefore, $h(x)$ =

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Example 15.16: Find the inverse Fourier transform of $F(w) = \dfrac{1}{(4+w^2)(9+w^2)}$

Solution: Let $h(x)$ be the inverse Fourier transform, then

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{(4+w^2)(9+w^2)} e^{iwx}\, dw = \frac{1}{\sqrt{2\pi}} (f * g)(x), \quad \text{using (15.37)}$$

where $f(x) = \mathcal{F}^{-1}\left(\dfrac{1}{4+w^2}\right)$, $g(x) = \mathcal{F}^{-1}\left(\dfrac{1}{9+w^2}\right)$ and $(f*g)$ is the convolution of f and g.

Using Example 15.9a and s aling property, we have

$$f(x) = \mathcal{F}^{-1}\left(\frac{1}{4+w^2}\right) = \frac{1}{4}\sqrt{2\pi}\, e^{-2|x|},$$

and,

$$g(x) = \mathcal{F}^{-1}\left(\frac{1}{9+w^2}\right) = \frac{1}{6}\sqrt{2\pi}\, e^{-3|x|}.$$

Hence,

$$h(x) = \frac{1}{\sqrt{2\pi}} (f * g)(x) = \frac{\sqrt{2\pi}}{24} e^{-2|x|} * e^{-3|x|} = \frac{\sqrt{2\pi}}{24} \int_{-\infty}^{\infty} e^{-2|x-t|}\, e^{-3|t|}\, dt$$

For $x > 0$, $$\int_{-\infty}^{\infty} e^{-2|x-t|}\, e^{-3|t|}\, dt = \int_{-\infty}^{0} e^{-2|x-t|} e^{-3|t|}\, dt + \int_{0}^{x} e^{-2|x-t|} e^{-3|t|}\, dt + \int_{x}^{\infty} e^{-2|x-t|} e^{-3|t|}\, dt$$

$$= \int_{-\infty}^{0} e^{-2(x-t)} e^{3t}\, dt + \int_{0}^{x} e^{-2(x-t)}\, e^{-3t}\, dt + \int_{x}^{\infty} e^{-2(t-x)} e^{-3t}\, dt = \frac{6e^{-2x}}{5} - \frac{4e^{-3x}}{5}.$$

Similarly . r $x < 0$, $$\int_{-\infty}^{\infty} e^{-2(x-t)} e^{-3|t|}\, dt = \frac{6e^{2x}}{5} - \frac{4e^{3x}}{5}.$$

and, for $x = 0$ $$\int_{-\infty}^{\infty} e^{-2|x-t|} e^{-3|t|}\, dt = \frac{2}{5}.$$

Therefore,

$$h(x) = \begin{cases} \sqrt{2\pi}\left(\dfrac{1}{20} e^{2x} - \dfrac{1}{30} e^{3x}\right) & x < 0 \\ \sqrt{2\pi}/60 & x = 0 \\ \sqrt{2\pi}\left(\dfrac{1}{20} e^{-2x} - \dfrac{1}{30} e^{-3x}\right) & x > \infty \end{cases}$$

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$$= \sqrt{2\pi}\left(\frac{1}{20}e^{-2|x|} - \frac{1}{30}e^{-3|x|}\right), -\infty < x < \infty.$$

The table below gives some functions $f(x)$ and their Fourier transforms $F(\omega)$.

f(x)	$F(\omega) = \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx$
1. $\begin{cases}1 & \lvert x\rvert < a \\ 0 & \lvert x\rvert > a\end{cases}$, (a > 0), or	$\sqrt{\frac{2}{\pi}}\left(\frac{\sin a\omega}{w}\right)$
2. $\frac{1}{x}$	$\begin{cases}\frac{i}{\sqrt{2\pi}}, & \omega > 0 \\ 0 & \omega = 0 \\ \frac{-i}{\sqrt{2\pi}}, & \omega < 0\end{cases}$
3. $\begin{cases}1, & a < x < b \\ 0, & \text{otherwise}\end{cases}$ (0 < a < b)	$\frac{1}{\sqrt{2\pi}}\left(\frac{e^{-ia\omega} - e^{-ib\omega}}{i\omega}\right)$
4. $\begin{cases}a - \lvert x\rvert, & \lvert x\rvert < a \\ 0, & \lvert x\rvert > a\end{cases}$	$\sqrt{\frac{2}{\sqrt{\pi}}}\left(\frac{1-\cos aw}{\omega^2}\right)$
5. $\frac{\sin ax}{x}$, a > 0	$\begin{cases}\sqrt{\pi/2} & \lvert\omega\rvert < a \\ 0, & \lvert\omega\rvert > a\end{cases}$
6. $\begin{cases}e^{-ax}, & x > 0 \\ 0, & x < 0\end{cases}$ (a > 0)	$\frac{1}{\sqrt{2\pi}}\frac{1}{a+i\omega}$
7. $\begin{cases}e^{ax}, & b < x > c \\ 0, & \text{otherwise}\end{cases}$ (a > 0)	$\frac{1}{\sqrt{2\pi}}\left[\frac{e^{(a-i\omega)c} - e^{(a-i\omega)b}}{a - i\omega}\right]$
8. $e^{-a\lvert x\rvert}$, (a > 0)	$\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+\omega^2}\right)$
9. $xe^{-a\lvert x\rvert}$(a > 0)	$-\sqrt{\frac{2}{\pi}}\frac{2ia\omega}{(a^2+\omega^2)^2}$
10. $\lvert x\rvert e^{-a\lvert x\rvert}$, (a > 0)	$\sqrt{\frac{2}{\pi}}\frac{(a^2-\omega^2)}{(a^2+\omega^2)}$
11. e^{-ax^2}, (a > 0)	$\frac{1}{\sqrt{2a}}e^{-\omega^2/4a}$

Contd.

12. $\frac{1}{a^2+x^2}$, (a > 0)	
13. $\frac{x}{a^2+x^2}$, (a > 0)	
14. $\begin{cases}e^{-x}x^a & x > 0 \\ 0, & x \le 0\end{cases}$	
15. $\delta(x)$	
16. $J_0(ax)$, (a > 0)	

1. Find the Fou

(a) $f(x) = \begin{cases}1, \\ 0,\end{cases}$

(c) $f(x) = \begin{cases}e^x \\ 0,\end{cases}$

2. Find the Fou

Hence show

3. Find the Fou

(a) $f(x) = e^{-}$

(c) $f(x) = \begin{cases}x \\ 0\end{cases}$

4. Show that i

12. $\dfrac{1}{a^2+x^2}, (a>0)$	$\dfrac{1}{a}\sqrt{\dfrac{\pi}{2}}\, e^{-a\lvert w\rvert}$
13. $\dfrac{x}{a^2+x^2}, (a>0)$	$\dfrac{-i}{2a}\sqrt{\dfrac{\pi}{2}}\, we^{-a\lvert w\rvert}$
14. $\begin{cases} e^{-x}x^{a} & x>0 \\ 0, & x\le 0 \end{cases}$	$\dfrac{\Gamma(a)}{\sqrt{2\pi}(1+iw)^a}$
15. $\delta(x)$	$\dfrac{1}{\sqrt{2\pi}}$
16. $J_0(ax), (a>0)$	$\sqrt{\dfrac{2}{\pi}}\dfrac{u(a-\lvert w\rvert)}{(a^2-w^2)^{1/2}}$

EXERCISE 15.2

1. Find the Fourier transforms of

(a) $f(x) = \begin{cases} 1, & 0<x<a \\ 0, & \text{otherwise} \end{cases}$ (b) $f(x) = \begin{cases} e^{iwx}, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$

(c) $f(x) = \begin{cases} e^{x}, & \lvert x\rvert<a \\ 0, & \text{otherwise} \end{cases}$ (d) $f(x) = \begin{cases} x, & 0<x<a \\ 0, & \text{otherwise} \end{cases}$

2. Find the Fourier transform of $f(x) = \begin{cases} 1-x^2, & \lvert x\rvert<1 \\ 0, & \lvert x\rvert>1 \end{cases}$

Hence show that $\displaystyle\int_0^{\infty} \frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}\,dx = -\frac{3\pi}{16}$

3. Find the Fourier transform of

(a) $f(x) = e^{-x^2/2}$ (b) $f(x) = \dfrac{\sin ax}{x}, a>0$

(c) $f(x) = \begin{cases} x^a e^{-x}, & x>0 \\ 0, & x\le 0 \end{cases}$ (d) $f(x) = \begin{cases} x^2, & \lvert x\rvert<x_0 \\ 0, & \text{otherwise} \end{cases}$

4. Show that if $f(x)$ has a finite jump discontinuity at $x = a$, then

$$\mathcal{F}[f'(x)] = iwF(w) - \frac{1}{2\pi}[f(a+)-f(a-)]e^{-iwa}$$

and hence find the Fourier transform of $f''(x)$ when $f(x) = \begin{cases} x, & 0 \le x < a \\ 0, & \text{otherwise.} \end{cases}$

5. Using Fourier transform solve $y'(x) - 4y(x) = u(x)e^{-4x}$, where $u(x)$ is the unit step function.
6. Using the fact that the Bessel function $J_0(x)$ satisfies the differential equation $xf''(x) + f'(x) + xf(x) = 0$, find the Fourier transform of $J_0(x)$, use $\int_0^{\infty} J_0(x)dx = 1$.
7. Use convolution to find the inverse Fourier transform of the functions

   (a) $\dfrac{1}{(1+iw)^2}$ (b) $\dfrac{\sin 3w}{w(2+iw)}$

8. Using convolution find the Fourier transform of $f(x) = \begin{cases} xe^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$
9. Show that Fourier transform of $f(x) = \begin{cases} 1, & b > x c \\ 0, & \text{otherwise} \end{cases}$ is $F(w) = \dfrac{e^{-ibw} - e^{-icw}}{iw\sqrt{2\pi}}$.

   Using this result find the inverse Fourier transform of $\dfrac{i}{\sqrt{2\pi}} \dfrac{e^{-ib(a-w)} - e^{-ic(a-w)}}{(a-w)}$.

10. Evaluate $\int_{-\infty}^{\infty} \delta(x-3)\, u(x-3)e^{-5x}dx$, where $\delta(x)$ is the Dirac-Delta function and $u(x)$ is the unit-step function.

15.5 FOURIER COSINE AND FOURIER SINE TRANSFORMS AND THEIR PROPERTIES

The Fourier cosine and sine transforms can be considered as special cases of the Fourier transform of $f(x)$ when $f(x)$ is even or odd function of x over the real axis.

15.5.1 The Fourier Cosine Transform

Consider $f(x)$ to be a piecewise continuous and absolutely integrable function of x over the real axis and so its Fourier transform $F(w)$ exists, refer to 15.26, and is given by

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx}dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)[\cos wx - i \sin wx]dx \quad (15.38)$$

Further if $f(x)$ is an even function of x, then $f(x)\cos wx$ is even function of x and $f(x) \sin wx$ is odd function of x and so the right side of (15.38) simplifies to

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$$F_c(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos wx \, dx \qquad \text{...(15.39)}$$

is called the *Fourier cosine transform of* $f(x)$, (also denoted by $\mathcal{F}_c$ or $\hat{f}_c$).

The *inverse Fourier cosine transform of* $F_c(w)$ corresponding to the inverse Fourier transform (15.27) can be obtained as follows. Consider

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_c(w) e^{iwx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_c(w)[\cos wx + i \sin wx] dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(w) \cos wx \, dx,$$

for we note from (15.39) that $F_c(w)$ and, hence $\mathcal{F}_c(w) \cos wx$ is an even function of w, and $\mathcal{F}_c(w) \sin wx$ is an odd function of w.

The integral denoted by $f(x) = \mathcal{F}_c^{-1}(w)$ and defined as

$$f(x) = \mathcal{F}_c^{-1}(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(w) \cos wx \, dx \qquad \text{...(15.40)}$$

is called *the inverse Fourier cosine transform* of $\mathcal{F}_c(w)$.

15.5.2 The Fourier Sine Transform

Similarly, considering $f(x)$ to be an odd function of x, piecewise continuous and absolutely integrable over the real axis, we arrive at integrals

$$F_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin wx \, dx \qquad \text{...(15.41)}$$

(also denoted by $\mathcal{F}_s$, or $\hat{f}_s$), defined as the *Fourier sine transform of* $f(x)$, and its inverse

$$f(x) = \mathcal{F}_s^{-1}(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(w) \sin wx \, dw \qquad \text{...(15.42)}$$

defined as *the inverse Fourier sine transform of* $F_s(\omega)$.

The sufficient conditions for the existence of Fourier cosine and sine transforms are:

1. $f(x)$ *is piecewise continuous on each finite interval* $[0, l]$; *and*
2. $f(x)$ *is absolutely integrable on the positive real axis.*

Similarly, as in case of inverse Fourier transform, *for the existence of inverse Fourier cosine and sine transforms,* $F_c(w)$ *and* $F_s(w)$ *must be absolute integrable over* $(0, \infty)$.

Remarks: 1. Whenever $f(x)$ is discontinuous, then expression on the left of (15.40) and (15.42) replaced by $[f(x+0) + f(x-0)]/2$ because the Fourier cosine and sine transforms satisfy the same convergence properties as the Fourier transform.

2. We have derived Fourier cosine and sine transforms as special cases of the Fourier transform, when $f(x)$ being even or odd, however, as in case of Fourier cosine and Fourier sine integrals, these two transforms respectively can be defined when $f(x)$ given on semi-infinite interval say, $0 < x < \infty$, and is extended to the domain $-\infty < x < \infty$ as even or odd function.

Example 15.17: Find Fourier cosine and sine transforms of $f(x) = \begin{cases} 1, & 0 \le x \le a \\ 0, & x > a \end{cases}$

Solution: By definition

$$F_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx = \sqrt{\frac{2}{\pi}} \int_0^a \cos wx \, dx = \sqrt{\frac{2}{\pi}} \left(\frac{\sin wx}{w} \right)_0^a = \sqrt{\frac{2}{\pi}} \frac{\sin aw}{w},$$

$$F_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx = \sqrt{\frac{2}{\pi}} \int_0^a \sin wx \, dx = \sqrt{\frac{2}{\pi}} \left(\frac{-\cos wx}{w} \right)_0^a = \sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos aw}{w} \right)$$

15.5.3 Properties of Fourier Cosine and Sine Transforms

Like Fourier transform the Fourier cosine and sine transforms also satisfy certain properties which are useful from applications point of view.

1. Linearity: For any two functions $f(x)$ and $g(x)$ whose Fourier cosine and sine transforms exist and for any constants a and b

$$\hat{f}_c\,[af(x) + bg(x)] = a\,\hat{f}_c\,[f(x)] + b\,\hat{f}_c\,[g(x)] \text{ and, } \quad \hat{f}_s\,[af(x) + bg(x)] = a\,\hat{f}_s\,[f(x)] + b\,\hat{f}_s\,[g(x)]$$

The proofs follow directly from the definition of Fourier cosine and sine transforms.

2. Shifting w by w_0 and scaling x by a: If $F_c(w)$ and $F_s(w)$ are the Fourier cosine and sine transforms of $f(x)$, then

(a) $\hat{f}_c\,[\cos(w_0 x) f(x)] = \dfrac{1}{2}\,[F_c(w + w_0) + F_c(w - w_0)]$

(b) $\hat{f}_c\,[\sin(w_0 x) f(x)] = \dfrac{1}{2}\,[F_s(w + w_0) - F_s(w - w_0)]$

(c) $\hat{f}_s\,[\cos(w_0 x) f(x)] = \dfrac{1}{2}\,[F_s(w + w_0) - F_s(w - w_0)]$

(d) $\hat{f}_s\,[\sin(w_0 x) f(x)] = \dfrac{1}{2}\,[F_c(w - w_0) - F_c(w + w_0)]$

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(e) $\hat{f}_c\,[f(ax)] = \dfrac{1}{a}\,F_c(w/a),\ a > 0$

(f) $\hat{f}_s\,[f(ax)] = \dfrac{1}{a}\,F_s(w/a),\ a > 0.$

These results follow directly from the definitions of the Fourier cosine and sine transforms. For example, to prove (b) we have

$$\sin w_0 x \cos wx = \frac{1}{2}\,[\sin (w_0 + w)x + \sin (w_0 - w)x] = \frac{1}{2}\,[\sin (w + w_0)x - \sin(w - w_0)x].$$

Thus,
$$\hat{f}_c\,[\sin (w_0 x) f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(w_o x)\cos(wx)\, f(x)\,dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty \sin(w + w_0)x\, f(x)\,dx - \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(w - w_0)x\, f(x)\,dx\right]$$

$$= \frac{1}{2}\,[F_s(w + w_0) - F_s(w - w_0)].$$

To prove (e), we have $\hat{f}_c\,[f(ax)] = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty f(ax) \cos wx\, dx = \frac{1}{a}\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \frac{w}{a}\, x\, dx = \frac{1}{a}\, F_c(w/a)$

3. Fourier cosine and sine transforms of derivatives

Let $f(x)$ and $f'(x)$ be continuous and absolutely integrable on the interval $[0, \infty)$ and $f''(x)$ be piecewise continuous on every subinterval $[0, l]$. Then

(a) $\hat{f}_c\,[f'(x)] = w\,F_s(w) - \sqrt{\dfrac{2}{\pi}}\, f(0)$ (b) $\hat{f}_s\,[f'(x)] = -wF_c(w)$

(c) $\hat{f}_c\,[f''(x)] = -w^2 F_c(w) - \sqrt{\dfrac{2}{\pi}}\, f'(0)$ (d) $\hat{f}_s\,[f''(x)] = -w^2 F_s(w) + w\sqrt{\dfrac{2}{\pi}}\, f(0)$

Proof. (a) By definition

$$\hat{f}_c\,[f'(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos wx\, dx = \sqrt{\frac{2}{\pi}}\left[[f(x) \cos wx]_0^\infty + w \int_0^\infty f(x) \sin wx\, dx\right]$$

$$= wF_s(w) - \sqrt{\frac{2}{\pi}}\, f(0),\ \text{ assuming that } f(x) \to 0 \text{ as } x \to \infty.$$

The result (b) can be proved on the similar lines as in (a).

To prove (c), by definition

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$$\hat{f}_c[f''(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f''(x)\cos wx\,dx = \sqrt{\frac{2}{\pi}}\left[[f'(x)\cos wx + wf(x)\sin wx]_0^\infty + w^2\int_0^\infty f(x)\cos wx\,dx\right]$$

$$= -w^2F_c(w) - \sqrt{\frac{2}{\pi}}\,f'(0), \text{ assuming } f(x), f'(x) \to 0 \text{ as } x \to \infty.$$

The result (d) can be proved on the similar lines as in (c).

Example 15.18: Find the Fourier cosine and sine transforms of $f(x) = e^{-ax}$, $x \geq 0, a > 0$, by using Fourier cosine and sine transforms of derivatives.

Solution: Here $f(x) = e^{-ax}$, this gives $f'(x) = -ae^{-ax}$ and $f''(x) = a^2e^{-ax}$.

Thus, $$\hat{f}_c[f''(x)] = \hat{f}_c[a^2e^{-ax}] = a^2\hat{f}_c[e^{-ax}] = a^2F_c(w), \quad \text{...(15.43)}$$

where $F_c(w)$ denotes the Fourier cosine transform of $f(x) = e^{-ax}$.

Also, $$\hat{f}_c[f''(x)] = -w^2F_c(w) - \sqrt{\frac{2}{\pi}}\,f'(0) = -w^2F_c(w) + a\sqrt{\frac{2}{\pi}}, \quad \text{...(15.44)}$$

since $f'(0) = -a$

From (15.43) and (15.44), we have

$$a^2F_c(w) = -w^2F_c(w) + a\sqrt{\frac{2}{\pi}} \text{ or, } F_c(w) = \sqrt{\frac{2}{\pi}}\,\frac{a}{w^2+a^2}$$

To find Fourier sine transform, consider

$$\hat{f}_s[f''(x)] = a^2\hat{f}_s[e^{-ax}] = a^2F_s(w), \quad \text{...(15.45)}$$

where $F_s(w)$ denotes the Fourier sine transform of $f(x) = e^{-ax}$.

Also, $$\hat{f}_s[f''(x)] = -w^2F_s(w) + w\sqrt{\frac{2}{\pi}}\,f(0) = -w^2F_s(w) + w\sqrt{\frac{2}{\pi}} \quad \text{...(15.46)}$$

From (15.45) and (15.46), we have

$$a^2F_s(w) = -w^2F_s(w) + w\sqrt{\frac{2}{\pi}} \text{ or, } F_s(w) = \sqrt{\frac{2}{\pi}}\,\frac{w}{w^2+a^2}$$

Remark: While solving second order differential equations using integral transforms when the domain is semi-infinite real line, $(0 < x < \infty)$, we need to choose among the Laplace, Fourier cosine and Fourier sine transforms. The Laplace transform will possibly be the best in case of the initial value problems. In case of boundary-value type, the choice will be between Fourier cosine and sine transforms. To use Fourier cosine transform we need to know $f'(0)$; and to use sine transform we require $f(0)$. Thus, the choice may be made accordingly between the two transforms on the basis of

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the conditions prescribed. We illustrate this in the example to follow next. However, a few specific applications of Fourier transforms to the solutions of partial differential equations are considered in Section 17.15.

Example 15.19: By applying an integral transform, solve the boundary value problem

$$f''(x) - f(x) = 3e^{-2x}, \quad (0 < x < \infty), \quad f(0) = x_0, \quad f(\infty) \text{ bounded} \qquad \text{...(15.47)}$$

Solution: The domain of definition $0 < x < \infty$ is semi-infinite and the problem is boundary-value problem with $f(0) = x_0$; so the clear choice is to apply Fourier sine transform. Applying Fourier sine transform to (15.47) and using the linearity property, we have

$$\hat{f}_s\,[f''(x)] - \hat{f}_s\,[f(x)] = 3\,\hat{f}_s\,[e^{-2x}]$$

or,

$$-w^2F_s(w) + w\sqrt{\frac{2}{\pi}}f(0) - F_s(w) = 3\sqrt{\frac{2}{\pi}}\,\frac{w}{w^2+4} \qquad \text{...(15.48)}$$

(refer to Example 15.18 for $a = 2$)

Using $f(0) = x_0$ and solving (15.48) for $F_s(w)$, we have

$$F_s(w) = \sqrt{\frac{2}{\pi}}\,\frac{wx_0}{w^2+1} - 3\sqrt{\frac{2}{\pi}}\,\frac{w}{(w^2+4)\,(w^2+1)} = \sqrt{\frac{2}{\pi}}\left[(x_0-1)\,\frac{w}{w^2+1} + \frac{w}{w^2+4}\right] \qquad \text{...(15.49)}$$

Taking inverse Fourier sine transform of (15.49) and using the linearity property of the inverse transform we have, refer to Example 15.18 for $a = 1, 2$,

$$f(x) = (x_0 - 1)e^{-x} + e^{-2x}, \qquad \text{...(15.50)}$$

as the solution of (15.47).

Remark: While using the result of $\hat{f}_s\,[f''(x)]$ to obtain the solution of Eq. (15.47), we have implicitly used that $f(x)$ and $f'(x)$ both tends to zero as x tends to infinity and in fact we can verify that (15.50) satisfies these conditions.

15.6 PARSEVAL IDENTITIES FOR FOURIER TRANSFORMS

The Parseval identities for Fourier transform and Fourier cosine and sine transforms are given by

(a) $\displaystyle\int_{-\infty}^{\infty} f(x)\,\bar{g}\,(x)dx = \int_{-\infty}^{\infty} F(w)\,\bar{G}\,(w)dw$ (b) $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2dx = \int_{-\infty}^{\infty} |F(w)|^2dw$

(c) $\displaystyle\int_{0}^{\infty} f(x)\,g(x)dx = \int_{0}^{\infty} F_c(w)G_c(w)dw$ (d) $\displaystyle\int_{0}^{\infty} |f(x)|^2dx = \int_{0}^{\infty} |F_c(w)|^2dw$

(e) $\displaystyle\int_{0}^{\infty} f(x)g(x)dx = \int_{0}^{\infty} F_s(w)G_s(w)dw$ (f) $\displaystyle\int_{0}^{\infty} |f(x)|^2dx = \int_{0}^{\infty} |F_s(w)|^2dw,$

where $F(w)$, $F_c(w)$ and $F_s(w)$ are respectively the Fourier transform, Fourier sine and Fourier cosine transforms of $f(x)$ respectively and 'bar' denotes the complex conjugate.

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We note that Fourier transform is defined for real and complex functions both, thus to prove (a) consider

$$\int_{-\infty}^{\infty} f(x)\,\bar{g}(x)dx = \int_{-\infty}^{\infty} f(x)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{G}(w)\,e^{-iwx}\,dw\right\}dx = \int_{-\infty}^{\infty}\bar{G}(w)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-iwx}\,dx\right\}dw$$

$$= \int_{-\infty}^{\infty}\bar{G}(w)\,F(w)dw = \int_{-\infty}^{\infty} F(w)\,\bar{G}(w)dw, \text{ which is (a).}$$

Put $g(x) = f(x)$ in (a), we obtain

$$\int_{-\infty}^{\infty} f(x)\,\bar{f}(x)dx = \int_{-\infty}^{\infty} F(w)\,\bar{F}(w)dw, \text{ or, } \int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|F(w)|^2dw, \text{ which is (b).}$$

Similarly to prove (c), consider

$$\int_{-\infty}^{\infty} f(x)g(x)dx = \int_{-\infty}^{\infty} f(x)\left\{\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} G_c(w)\cos wx\,dw\right\}dx = \int_{-\infty}^{\infty} G_c(w)\left\{\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos wx\,dx\right\}dw$$

$$= \int_{0}^{\infty} G_c(w)F_c(w)dw = \int_{0}^{\infty} F_c(w)G_c(w)dw, \text{ which is (c).}$$

The result (d) follows from (c). Similarly we can prove (e) and (f).

These results are useful in solving certain improper integrals.

Example 15.20: Find the Fourier cosine transform of $f(x) = xe^{-ax}$, $x > 0$, $a > 0$ and then evaluate

$$\int_{0}^{\infty}\frac{(a^2-x^2)\,(b^2-x^2)}{(a^2+x^2)^2\,(b^2+x^2)^2}\,dx$$

using the Parseval identity for the cosine transforms.

Solution: By definition of the Fourier cosine transform

$$F_c(w) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos wx\,dx = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} xe^{-ax}\cos wx\,dx = \text{Re}\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} xe^{-ax}\,e^{iwx}dx$$

$$= \text{Re}\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} xe^{-(a-iw)x}dx = \text{Re}\sqrt{\frac{2}{\pi}}\left[\left(\frac{xe^{-(a-iw)x}}{-(a-iw)}\right)_0^{\infty} + \int_{0}^{\infty}\frac{e^{-(a-iw)x}}{(a-iw)}\,dx\right]$$

$$= \text{Re}\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{e^{-(a-iw)x}}{(a-iw)}dx = \text{Re}\sqrt{\frac{2}{\pi}}\left(\frac{e^{-(a-iw)x}}{-(a-iw)^2}\right)_0^{\infty}$$

$$= \text{Re}\sqrt{\frac{2}{\pi}}\frac{1}{(a-iw)^2} = \text{Re}\sqrt{\frac{2}{\pi}}\frac{(a+iw)^2}{(a^2+w^2)^2} = \sqrt{\frac{2}{\pi}}\frac{a^2-w^2}{(a^2+w^2)^2}.$$

Thus, $F_c(w) = \sqrt{\frac{2}{\pi}}\frac{a^2-w^2}{(a^2+w^2)^2}$.

For the Fourier cosine transform the Parseval identity is

$$\int_0^\infty F_c(w)G_c(w)dw = \int_0^\infty f(x)g(x)dx \qquad \text{...(15.51)}$$

Set $f(x) = xe^{-ax}$, $a > 0$ and $g(x) = xe^{-bx}$, $b > 0$ and correspondingly

$F_c(w) = \sqrt{\frac{2}{\pi}}\frac{a^2-w^2}{(a^2+w^2)^2}$ and $G_c(w) = \sqrt{\frac{2}{\pi}}\frac{b^2-w^2}{(b^2+w^2)^2}$, (15.51) becomes

$$\frac{2}{\pi}\int_0^\infty \frac{(a^2-w^2)(b^2-w^2)}{(a^2+w^2)^2(b^2+w^2)^2}dw = \int_0^\infty x^2e^{-(a+b)x}dx$$

$$= \left[x^2\frac{e^{-(a+b)x}}{-(a+b)} - (2x)\frac{e^{-(a+b)x}}{(a+b)^2} + 2\frac{e^{-(a+b)x}}{-(a+b)^3}\right]_0^\infty = \frac{2}{(a+b)^3}$$

or, $$\int_0^\infty \frac{(a^2-w^2)(b^2-w^2)}{(a^2+w^2)^2(b^2+w^2)^2}dw = \frac{\pi}{(a+b)^3}$$

Changing w to x, we get the desired integral.

Example 15.21: Using Parseval identities show that

(a) $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$ (b) $\displaystyle\int_0^\infty \frac{x^2}{(x^2+a^2)^2}dx = \frac{\pi}{4a}$

Solution: Consider $f(x) = e^{-ax}$, $a > 0$.

The Fourier cosine and sine transform of $f(x)$, respectively are

$F_c(w) = \sqrt{\frac{2}{\pi}}\frac{a}{w^2+a^2}$ and $F_s(w) = \sqrt{\frac{2}{\pi}}\frac{a}{w^2+a^2}$, (refer to Example 15.18)

To prove (a) consider the Parseval identity for the Fourier cosine transform

$\int_0^\infty [F_c(w)]^2 = \int_0^\infty [f(x)]^2dx$. Set $f(x) = e^{-ax}$ and $F_c(w) = \sqrt{\frac{2}{\pi}}\frac{a}{w^2+a^2}$, we have

$$\frac{2}{\pi}\int_0^\infty \frac{a^2}{(w^2+a^2)^2}\,dw = \int_0^\infty e^{-2ax}dx = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{1}{2a}, \text{ or } \int_0^\infty \frac{1}{(w^2+a^2)^2}\,dw = \frac{\pi}{4a^3}.$$

Changing w to x we get the desired result.

To prove (b), consider the Parseval's identity for the Fourier sine transform,

$\int_0^\infty [F_s(w)]^2 = \int_0^\infty [f(x)]^2 dx$. Set $f(x) = e^{-ax}$ and $F_s(w) = \sqrt{\frac{2}{\pi}}\frac{w}{(w^2+a^2)}$, we have

$$\frac{2}{\pi}\int_0^\infty \frac{w^2}{(w^2+a^2)^2}\,dw = \int_0^\infty e^{-2ax}dx = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{1}{2a}, \text{ or } \int_0^\infty \frac{w^2}{(w^2+a^2)^2} = \frac{\pi}{4a}.$$

Changing w to x, we get the desired result.

Example 15.22: Using the Parseval identity prove that $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$.

Solution: Consider $f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

The Fourier transform of $f(x)$, refer to Example 15.7d for $a = 1$, is $F(w) = \sqrt{\frac{2}{\pi}}\left(\frac{\sin w}{w}\right)$.

The Parseval identity for the Fourier transform is $\int_{-\infty}^\infty |F(w)|^2 dw = \int_{-\infty}^\infty |f(x)|^2 dx$.

Substituting for $F(w)$ and $f(x)$, it gives

$$\frac{2}{\pi}\int_{-\infty}^\infty \frac{\sin^2 w}{w^2}\,dw = \int_{-1}^1 dx = 2, \quad \text{or} \quad \int_{-\infty}^\infty \frac{\sin^2 w}{w^2}\,dw = \pi$$

Changing w to x, we get, $\int_{-\infty}^\infty \frac{\sin^2 x}{x^2}\,dx = \pi$, or $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$.

15.7 THE FINITE FOURIER COSINE AND SINE TRANSFORMS

The Fourier cosine and sine transforms defined on $[0, \infty)$ are motivated by the respective integral representations of a function. In many applications we are to deal with problems defined on finite intervals, and hence we define *the finite Fourier cosine and sine transforms* using Fourier cosine and sine series instead of integrals.

Finite Fourier cosine transform: Suppose f is piecewise continuous on $[0, \pi]$, then *the finite Fourier cosine transform of f* denoted by $F_c(n)$ is defined as

$$F_c(n) = \int_0^{\pi} f(x) \cos nx \, dx \qquad \text{...(15.52)}$$

for $n = 0, 1, 2, \ldots$

Also the Fourier cosine series representation of $f(x)$ on the interval $[0, \pi]$, refer to Eq. (14.30) for $l = \pi$, is $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$, where

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}F_c(0), \text{ and } a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx = \frac{2}{\pi}F_c(n). \text{ Thus,}$$

$$f(x) = \frac{1}{\pi}F_c(0) + \frac{2}{\pi}\sum_{n=1}^{\infty} F_c(n) \cos nx \qquad \text{...(15.53)}$$

can be interpreted as the *inverse finite Fourier cosine transform.*

Finite Fourier sine transform: Suppose f is piecewise continuous on $[0, \pi]$, then *the finite Fourier sine transform* of $f(x)$ denoted by $F_s(n)$ is defined as

$$F_s(n) = \int_0^{\pi} f(x) \sin nx \, dx \qquad \text{...(15.54)}$$

for $n = 1, 2, \ldots$

Also the Fourier sine representation series of $f(x)$ on the interval $[0, \pi]$, refer to Eq. (14.31) for $l = \pi$, is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx = \frac{2}{\pi}F_s(n)$. Thus,

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} F_s(n) \sin nx \qquad \text{...(15.55)}$$

can be interpreted as the *inverse finite Fourier sine transform.*

In case the domain of definition for $f(x)$ is $[0, l]$, then (15.52), (15.53), (15.54) and (15.55) respectively become

$$F_c(n) = \int_0^{l} f(x) \cos \frac{n\pi x}{l} dx, \qquad \text{...(15.56)}$$

$$f(x) = \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} F_c(n) \cos \frac{n\pi x}{l}, \qquad \text{...(15.57)}$$

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$$F_s(n) = \int_0^l f(x)\sin\frac{n\pi x}{l}\,dx, \qquad ...(15.58)$$

and, $$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_s(n)\sin\frac{n\pi x}{l}. \qquad ...(15.59)$$

Example 15.23: If $f(x) = x^2, 0 \le x \le 4$, find finite Fourier cosine and sine transform of $f(x)$.

Solution: By definition

$$F_c(n) = \int_0^4 f(x)\cos\frac{n\pi x}{l}\,dx = \int_0^4 x^2\cos\frac{n\pi x}{4}\,dx$$

$$= \left[\frac{4x^2}{n\pi}\sin\frac{n\pi x}{4}\right]_0^4 - \int_0^4 \frac{4}{n\pi}2x\sin\frac{n\pi x}{4}\,dx = -\frac{8}{n\pi}\int_0^4 x\sin\frac{n\pi x}{4}\,dx$$

$$= -\frac{8}{n\pi}\left[\frac{-4x}{n\pi}\cos\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4}\right]_0^4 = \frac{128}{n^2\pi^2}(-1)^n.$$

Similarly, $$F_s(n) = \int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \int_0^4 x^2\sin\frac{n\pi x}{4}\,dx$$

$$= \left[\frac{-4}{n\pi}x^2\cos\frac{n\pi x}{4}\right]_0^4 + \int_0^4 2x\frac{4}{n\pi}\cos\frac{n\pi x}{4}\,dx$$

$$= -\frac{64}{n\pi}(-1)^n + \frac{8}{n\pi}\left[\frac{4x}{n\pi}\sin\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4}\right]_0^4$$

$$= \frac{-(-1)^n 64}{n\pi} + \frac{128}{n^3\pi^3}((-1)^n - 1).$$

Example 15.24: Find $f(x), 0 < x < \pi$, if its finite Fourier sine transform is

$$F_s(n) = \frac{1-\cos n\pi}{n^2\pi^2}, n = 1, 2, \ldots$$

Solution: By definition

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_s(n)\sin\frac{n\pi x}{l} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n^2\pi^2}\sin nx$$

$$= \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\sin nx = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^2}.$$

1. Find the Fourier c
(a) $f(x) = e^{-x}, x > 0$
(c) $f(x) = x^{\alpha-1}, 0 <$
2. Find the Fourier c
(a) $f(x) = \begin{cases}\cos x, \\ 0,\end{cases}$
3. Explain why the
sine transform
(a) $f(x) = 1$
4. Find the Fourier s
Hence, obtain the
5. Find the Fourier c
6. Find the finite Fo
$[0, \pi]$
(a) $f(x) = \sin ax, a >$
7. Solve the integral
Hence show that
8. Solve the integral e
9. Using Parseval ide

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EXERCISE 15.3

1. Find the Fourier cosine and Fourier sine transforms of
   (a) $f(x) = e^{-x}, x > 0$ (b) $f(x) = xe^{-ax}$
   (c) $f(x) = x^{\alpha-1}, 0 < \alpha < 1$ (d) $h(x) = \int_0^\infty f(x)\, g(x)\, dx$
2. Find the Fourier cosine and Fourier sine transforms of
   (a) $f(x) = \begin{cases} \cos x, & 0 \le x \le a \\ 0, & x > a \end{cases}$ (b) $f(x) = e^{-x} \cos x, x > 0$
3. Explain why the following functions have neither Fourier cosine transform nor Fourier sine transform
   (a) $f(x) = 1$ (b) $f(x) = e^x$
4. Find the Fourier sine transform of e^{-ax}, $a > 0$ and prove that
   $$\int_0^\infty \frac{x \sin \alpha x}{a^2 + x^2}\, dx = \frac{\pi}{2} e^{-a\alpha}, \quad \alpha > 0.$$
   Hence, obtain the Fourier sine transform of $x/(a^2 + x^2)$.
5. Find the Fourier cosine transform of e^{-ax} and hence evaluate $\int_0^\infty \frac{\cos \alpha x}{x^2 + a^2}\, dx$
6. Find the finite Fourier cosine and sine transforms of the following functions defined on $[0, \pi]$
   (a) $f(x) = \sin ax, a > 0$ (b) $f(x) = \sinh ax, a > 0$
7. Solve the integral equation $\int_0^\infty f(x) \cos \alpha x\, dx = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$

   Hence show that $\int_0^\infty \frac{\sin^2 x}{x^2} dx = \pi/2.$
8. Solve the integral equation $\int_0^\infty f(x) \sin \alpha x\, dx = \begin{cases} 1, & 0 \le \alpha < 1 \\ 2, & 1 \le \alpha < 2 \\ 0, & \alpha \ge 2 \end{cases}$
9. Using Parseval identities for sine and cosine transforms of $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$,

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evaluate (a) $\int_0^\infty \frac{(1-\cos x)^2}{x^2}\,dx$ (b) $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$.

10. Solve using a cosine or sine transform the boundary value problem

$$f''(x) - 9f(x) = 50e^{-3x}, \quad 0 < x < \infty, \quad f(0) = 0, \quad f(\infty) \text{ bounded.}$$

ANSWERS

Exercise 15.1 (p. 859)

2. $\frac{100}{\pi}\int_0^\infty \frac{1}{w}[\sin 2w \cos wx + (1-\cos 2w)\sin wx]dw$ for all x, except at $x = 0$ and $x = 2$; and $f(0) = f(2) = 100$ but Fourier integral converges to the average value 50.

3. $\frac{2b}{aw}\int_0^\infty \frac{\sin wx\,(\sin wa - wa\cos wa)}{w^2}\,dw$, for all x except at $x = \pm a$; and $f(a) = b$, $f(-a) = -b$ but Fourier integral converges to the average value $b/2$ and $-b/2$.

4. $\int_0^\infty \frac{\cos\frac{\pi w}{2}\cos wx}{1-w^2}\,dw$, for all x.

5. $\frac{1}{\pi}\int_0^\infty \frac{w[\sin wx - \sin w(\pi + x)]}{w^2-1}\,dw$, for all x, except at $x = 0$ and $x = \pi$. At $x = 0$ Fourier integral converges to the average value $1/2$ and at $x = \pi$, converges to $-1/2$.

6. $\frac{2}{\pi}\int_0^\infty \left(\frac{2}{4+w^2}+\frac{3}{9+w^2}\right)\cos wx\,dw$ 7. $\frac{-2}{\pi}\int_0^\infty \frac{[1+\cos w\pi]}{w^2-1}\cos wx\,dw$

8. $\frac{2}{\pi}\int_0^\infty \frac{[\sin 3w\cosh 3 - w\cos 3w\sinh 3]}{1+w^2}\sin wx\,dw$, except at $x = 3$. At $x = 3$, converges to $\frac{1}{2}\sin h\,3$, the average of $f(3+0)$ and $f(3-0)$.

9. $\frac{i}{\pi}\int_{-\infty}^{\infty}\left[\frac{-2w(1-w^2)^2}{((1-w^2)^2+4w^2)^2} - \frac{8w^3}{((1-w^2)^2+4w^2)^2}\right]e^{-iwx}\,dw$; converges to $xe^{|x|}$ for all real x.

10. $i\int_{-\infty}^{\infty}\left(\frac{\sin 5w}{w^2-\pi^2}\right)e^{iwx}\,dw$; converges to $\sin \pi x$ for $|x| < 5$ and to zero for $|x| \geq 5$.

11. $\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[-\frac{\cos(\pi w/2)}{w^2-1}+\frac{i\sin(\pi w/2)-w}{w^2-1}+\frac{1-w\sin(\pi w/2)}{w^2-1}+i\frac{w}{w^2-1}\cos(\pi w/2)\right]e^{-iwx}\,dw$

converges to $\cos x$, for $0 < x < \pi/2$, to $\sin x$ for $-\pi/2 < x < 0$, to 0 for $|x| > \pi/2$, to $1/2$ at $x = 0$, to $-1/2$ at $x = -\pi/2$ and to 0 at $x = \pi/2$.

13. $f(x) = 2(1 - \cos x)/\pi$, $x > 0$.

Exercise 15.2 (p. 871)

1. (a) $\frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{iwa}}{iw}\right)$ (b) $\frac{i}{\sqrt{2\pi}}\left(\frac{1-e^{-i(w-a)}}{a-w}\right)$

(c) $\frac{i}{\sqrt{2\pi}}\frac{(e^{a-iaw}-e^{-a+iaw})}{(1-iw)}$ (d) $\frac{[(1+iaw)e^{-iaw}-1]}{\sqrt{2\pi}\,w^2}$

2. $-\frac{4}{\sqrt{2\pi}\,w^3}(w\cos w-\sin w)$

3. (a) $e^{-w^2/2}$ (b) $\sqrt{\pi/2}$, $|w| < a$; 0, otherwise

(c) $\frac{\sqrt{a}}{\sqrt{2\pi}(1+iw)^2}$ (d) $\sqrt{\frac{2}{\pi}}\,\frac{[(x_0^2w^2-2)\sin x_0w+2x_0w\cos x_0\,w]}{w^3}$

5. $-\frac{1}{8}e^{-4|x|}$. 6. $\sqrt{\frac{2}{\pi}}\,\frac{1}{(1-w^2)^{1/2}}$, $0 < w^2 < 1$.

7. (a) $u(x)\cdot xe^{-x}$ (b) $\frac{1}{4}[1-e^{-2(x+3)}]u(x+3)-\frac{1}{4}[1-e^{-2(x-3)}]u(x-3)$

8. $\frac{1}{\sqrt{2\pi}}(1+iw)^2$ 10. $4e^{-15}$.

Exercise 15.3 (p. 883)

1. (a) $F_c(w)=\sqrt{\frac{2}{\pi}}\,\frac{1}{1+w^2}$, $F_s(w)=\sqrt{\frac{2}{\pi}}\,\frac{w}{1+w^2}$

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(b) $F_c(w) = \sqrt{\dfrac{2}{\pi}}\,\dfrac{a^2 - w^2}{(a^2 + w^2)^2}$ $\quad F_s(w) = \sqrt{\dfrac{2}{\pi}}\,\dfrac{2aw}{(w^2 + a^2)^2}$

(c) $F_c(w) = \sqrt{\dfrac{2}{\pi}}\,\dfrac{\Gamma(\alpha)}{w\alpha} \cos\dfrac{\alpha\pi}{2}$, $\quad F_s(w) = \sqrt{\dfrac{2}{\pi}}\,\dfrac{\Gamma(\alpha)}{w\alpha} \sin\dfrac{\alpha\pi}{2}$

(d) $H_c(w) = \displaystyle\int_0^\infty F_c(w)\, G_c(w)dw$, $\quad H_s(w) = \displaystyle\int_0^\infty F_s(w) G_s(w)dw$

2. (a) $F_c(w) = \dfrac{1}{\sqrt{2\pi}}\left[\dfrac{\sin a(1-w)}{1-w} + \dfrac{\sin a(1+w)}{1+w}\right]$,

$$F_s(w) = \sqrt{\frac{2}{\pi}}\,\frac{w}{w^2-1} - \frac{1}{\sqrt{2\pi}}\left[\frac{\sin a(1+w)}{1+w} - \frac{\cos a(1-w)}{1-w}\right]$$

(c) $F_c(w) = \dfrac{1}{\sqrt{2\pi}}\left[\dfrac{1}{1+(1+w)^2} + \dfrac{1}{1+(1-w)^2}\right]$,

$$F_s(w) = \frac{1}{\sqrt{2\pi}}\left[\frac{1+w}{1+(1+w^2)} - \frac{1-w}{1+(1-w)^2}\right]$$

4. $\sqrt{\dfrac{2}{\pi}}\,\dfrac{w}{w^2+a^2}, \sqrt{\dfrac{\pi}{2}}\,e^{-aw}$ $\qquad$ 5. $\sqrt{\dfrac{2}{\pi}}\,\dfrac{a}{(w^2+a^2)}, \dfrac{\pi}{2a}\,e^{-\alpha a}$

6. (a) $c_o = (1 - \cos a\pi)/a$,

$$c_n = \frac{1}{2(n-a)}[\cos\{(n-a)\pi\} - 1] - \frac{1}{2(n+a)}[\{(n+a)\pi - 1\}]$$

$c_n = 0$, if $n = a$ an integer.

$s_n = \left[\dfrac{\sin(n-a)\pi}{2(n-a)} \;\; \dfrac{\sin(n+a)\pi}{2(n+a)}\right]$, $s_n = \pi/2$, if $n = a$, an integer.

7. $f(x) = \dfrac{2(1-\cos x)}{\pi x^2}$ $\qquad$ 8. $f(x) = (2 + 2\cos x - 4\cos 2x)/\pi x$.

9. (a) $\pi/2$ $\qquad$ (b) $\pi/2$

10. $f(x) = -\dfrac{25}{3}\,xe^{-3x}$

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