ASSIGNMENT 4

LAPLACE TRANSFORMATION

1. Find the Laplace transform of

(a) $\frac{\cos\sqrt{t}}{\sqrt{t}}$ Answer: $\sqrt{\frac{\pi}{S}}\, e^{-1/4S}$

(b) $\sinh^2 t$ Answer: $\frac{2}{S(S^2-4)}, S > 2$

(c) $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ \sin t, & t \ge \pi \end{cases}$ Answer: $\frac{-e^{-S\pi}}{(S^2+1)}$

(d) $\sin^3 2t$ Answer: $\frac{48}{(S^2+4)(S^2+36)}, S > 0$

(e) $\mathcal{L}[t \sin at]$ Answer: $\frac{2aS}{(S^2+a^2)^2}$

(f) $\mathcal{L}[t^n e^{at}]$ Answer: $\frac{n!}{(S-a)^{n+1}}, S > a$

(g) Find the Laplace transform of $\frac{\sin at}{t}$. Does the Laplace transform of $\frac{\cos at}{t}$ exist?

Answer: $\mathcal{L}\left[\frac{\sin at}{t}\right] = \cot^{-1}\frac{S}{a}$; $\mathcal{L}\left[\frac{\cos at}{t}\right]$ does not exist.

2. Find $\mathcal{L}[G(t)]$, where $G(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$

Answer: $\frac{Se^{-2\pi S/3}}{(S^2+1)}, S > 0$

3. If $\mathcal{L}[f(t)] = \frac{(S^2-S+1)}{(2S+1)^2(S-2)}$ then prove that $\mathcal{L}[f(2t)] = \frac{(S^2-2S+4)}{4(S+1)^2(S-2)}$

4. If $\mathcal{L}[\sin\sqrt{t}] = \frac{\sqrt{\pi}}{2S^{3/2}} e^{-1/4S}$ then show that $\mathcal{L}\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{S}}\, e^{-1/4S}$

5. Find:

(a) $\mathcal{L}^{-1}\left[\frac{3S-2}{S^{5/2}} - \frac{7}{2S+2}\right]$ Answer: $6\left(\frac{t}{\pi}\right)^{1/2} - \frac{8t}{3}\left(\frac{t}{\pi}\right)^{\frac{1}{2}} - \frac{7}{3}e^{-2t/3}$

(b) $\mathcal{L}^{-1}\left[\frac{2S^2-6S+5}{S^3-6S^2+11S-6}\right]$ Answer: $\frac{e^t}{2} - e^{2t} + \frac{5e^{3t}}{3}$

(c) $\mathcal{L}^{-1}\left[\frac{S^2}{S^4+4a^2}\right]$ Answer: $\frac{1}{2a}(\cosh at \sin at + \sinh at \cos at)$

(d) $\mathcal{L}^{-1}\left[\frac{e^{-4S}}{(S-3)^4}\right]$ Answer: $\frac{1}{6}(t-4)^3 e^{3(t-4)} H(t-4)$

(e) $\mathcal{L}^{-1}\left[\log\frac{1+S}{S}\right]$ Answer: $\frac{1-e^{-t}}{t}$

6. State and prove Convolution theorem and hence solve the following:

(a) $\mathcal{L}^{-1}\left[\frac{S}{(S^2+a^2)^2}\right]$ Answer: $\frac{t \sin at}{2a}$

(b) Find $f(t)$ as the solution of integral equation

$$f(t) = t + e^{-2t} + \int_0^t f(\tau) e^{2(t-\tau)} d\tau$$

Answer: $f(t) = \frac{1}{45}(14e^{3t} - 5 + 30t + 36e^{-2t})$

7. Find the solution of initial value problem

$$y'' + 8y' + 17y = f(t), y(0) = 0, y'(0) = 0$$

Where $f(t)$ is a periodic function $f(t) = \begin{cases} 1, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$ and $f(t + 2\pi) = f(t)$

Answer: $y(t) = \frac{1}{34}[(-1)^n + 1] - \frac{A^{n+1}-1}{A-1} g(t), n = 0,1,2, \dots, A = e^{4\pi}$ and $g(t) = e^{-4t}(4 \sin t + \cos t)/17$.

8. Solve $(D + 1)^2 y = t, if\ y(0) = -3, y'(1) = -1$.

Answer: $y(t) = t - 2 + (t - 1)e^{-t}$.

9. Solve $(tD^2 + (1 - 2t)D - 2)y = 0\ if\ y(0) = 1, y'(0) = 2$.

Answer: $y(t) = e^{-2t}$.

10. Solve $\frac{dx}{dt} = 2x - 3y, \frac{dy}{dt} = y - 2x\ if\ x(0) = 8\ and\ y(0) = 3$.

Answer: $x(t) = 3e^{4t} + 5e^{-t}, y(t) = 5e^{-t} - 2e^{4t}$.

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