Special Theory of Relativity

An Introduction

DJM 8/01

Background

- Most basic concept of "relativity" is as old as Galilean and Newtonian Mechanics. Crudely speaking, it is:
  - The Laws of Physics look the same in different frames of reference
  - In Newtonian Relativity: This applies to Mechanics
  - In Einstein's Relativity: This applies to **all** Laws of Physics
- When applied to all laws, difficult issues arise.
  - Implications seem to violate intuition
  - masses increase, lengths are shortened, time expands
- Has generated fascination, has brought glamour...
  - Bottom line: STR is corroborated by experiment
- Until 20th century, Newton's Laws ($\mathbf{F} = d\mathbf{p}/dt$, $\mathbf{F}_{ab} = -\mathbf{F}_{ba}$, conservation of energy and momentum) were held true (and still are) and absolutely believed in (not any more, as written originally)

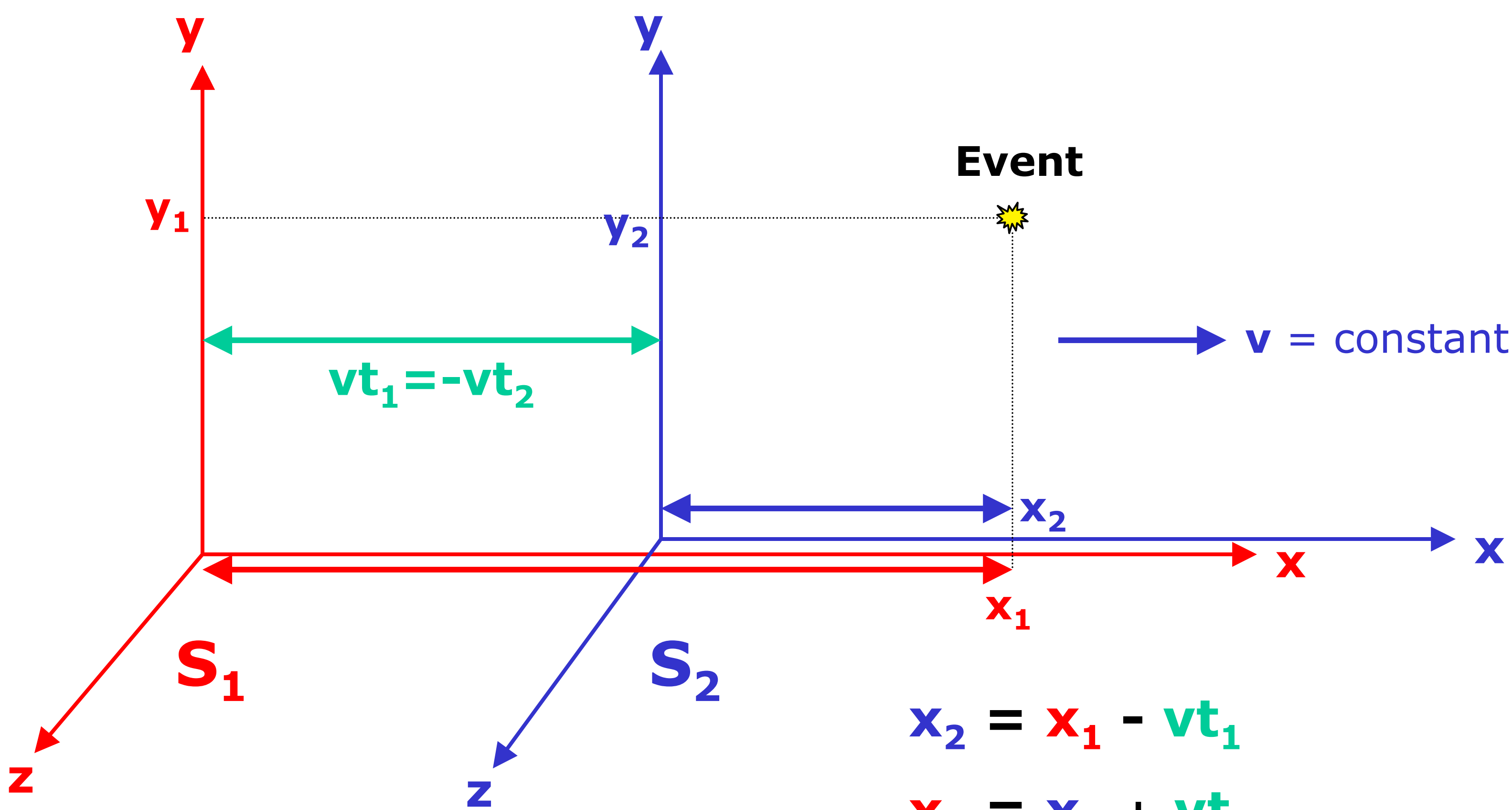
Background

- Recall: Newton's Laws are valid only in an "inertial" frame
  - An **inertial frame of reference** is one in which Newton's 1st Law holds; it exists only in empty space; it is at rest w.r.to stars)
- Basic Newtonian mechanics issue: A state of a system is specified at some time t_o by giving position coordinates (x_o,y_o,z_o) and velocity (v_{ox},v_{oy},v_{oz}) at t_o. Can calculate position (x,y,z) and velocity (v_x,v_y,v_z) at a later time t by knowing all forces acting on the system and applying Newton's laws.
- Often desirable to specify such a state in terms of a new set of coordinate axes, which is moving with respect to the first
- Three important questions arise:
  - How do we transform the state from the old to the new frame? i.e. How do we convert $x_1,y_1,z_1,t_1 \rightarrow x_2,y_2,z_2,t_2$?
  - What happens to Newton's Laws under this "transformation"?
  - What happens to E&M theory (Maxwell's Equations)?
- Theory of Relativity concerns itself with these questions

Background

- Albert Einstein formulated the modern Theory of Relativity. He proposed **two** such theories:
  - **The Special Theory of Relativity**
    - 1905 - Deals with the case of an inertial frame of reference moving with constant velocity with respect to another inertial frame
      - Its consequences are most important as $v \to c$
  - **The General Theory of Relativity**
    - 1915 - Deals with the case of an inertial frame of reference accelerating with respect to another inertial frame.
    - A relationship results between accelerated motion and gravitational effects
    - It is the current theory describing the gravitational interaction
    - Not based on quantum mechanics
    - Requires knowledge of tensor calculus

The S_1 and S_2 Inertial Frames



$$x_2 = x_1 - vt_1$$

$$x_1 = x_2 + vt_2$$

$$t_1 = t_2$$

Newtonian Relativity (Some History)

- **Galilean Transformation of space-time**
  - Two observers (1,2) are in their own, separate, inertial frame of reference S_1 and S_2
    - S_2 moves with respect to S_1 with v = constant along x-axis
  - Each observes the same "event" giving position and time as measured in their own frame of reference (x_1, y_1, z_1, t_1; x_2, y_2, z_2, t_2)
    - Each has own meter stick and clock
    - When temporarily at rest, sticks same length, clocks synchronized i.e. at the instant $t_1 = 0$, then $t_2 = 0$ and x in S_1 = x in S_2
  - When S_2 is moving with respect to S_1, the state (x and t) of the event as seen by the two observers is related by:

$x_2 = x_1 - vt_1$	$\mathbf{v} \rightarrow -\mathbf{v}$ gives	$x_1 = x_2 + vt_2$
$y_2 = y_1$	$(S_1 \rightarrow S_2)$	$y_1 = y_2$
$z_2 = z_1$		$z_1 = z_2$
$t_2 = t_1$		$t_1 = t_2$

Newtonian Relativity (Some History)

- Transformation of velocities and accelerations follow directly by taking derivatives

$$v_{2x} = v_{1x} - v \qquad \text{and} \qquad a_{2x} = a_{1x}$$
$$v_{2y} = v_{1y} \qquad\qquad a_{2y} = a_{1y}$$
$$v_{2z} = v_{1z} \qquad\qquad a_{2z} = a_{1z}$$

  - **This is the Newtonian answer to Question #1** (intuitive)
  - Important implicit assumptions: **space is absolute, time is absolute**
- Note (definition): $v_x = \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$ etc.
  - Implies that, for meaningful v_x, measurement of Δx and Δt must be made with respect to a single (the same) reference frame
  - When using Galilean Transformations this is not important because time is absolute and, therefore, $t_1 = t_2$

Newtonian Relativity (Some History)

- **All laws of mechanics are the same in all inertial frames**
  - Example: Newton's 2nd Law under Galilean transformation
  - When a mass accelerates, let the force acting on it be measured by S_1 and S_2. They find, respectively:

    $\mathbf{F}_1 = m\mathbf{a}_1$ and $\mathbf{F}_2 = m\mathbf{a}_2$ But $\mathbf{a}_1 = \mathbf{a}_2$

    Therefore $\mathbf{F}_1 = \mathbf{F}_2$

    And, therefore, Newton's 2nd Law retains its mathematical form in both inertial frames (is the same, "invariant"). All other laws of mechanics are also invariant, as they are derivable from 2nd Law
  - **Note assumption**: mass **m is the same** in both S_1 and S_2

    i.e. $m_1 = m_2$
- **This is Newtonian answer to Question #2**
- **Corollary:** Impossible to prove by mechanical tests that one frame is at absolute rest rather than in motion with v = const. w.r. to another one. ("Mechanics" looks the same in both)
  - Corollary: All inertial frames are equivalent; Length, time and mass are independent of relative motion of the observer

Failure of Newtonian Relativity
(Some More History)

- Maxwell's (4) equations describe electromagnetism successfully
- Maxwell's equations predict the existence of e.m. waves propagating through free space with speed $c = 3.00 \times 10^8$ m/s
  - Question #1: With respect to what frame is c to be measured?
  - Question #2: Through what medium do e.m. waves propagate?
  - 19th century answer: "Ether" hypothesis - existence of a massless but elastic substance permeating all space
    - Electromagnetic waves propagate through the ether
    - c is to be measured with respect to the ether
  - An experiment is needed to test the presence of ether!
- **Michelson-Morley Experiment (1881)**
  - **Premise:** If one believes in the ether (and that c is $3\text{x}10^8$ w.r. to it), and in Maxwell's Eqns, and in Galilean transformations for E&M, then if one measures the speed of light in a frame moving w.r. to ether, one should be able to measure v of this moving frame.
  - **Plan:** Measure earth's speed as it moves through ether
  - **Result:** Null; can not detect any motion of the earth through ether

Failure of Newtonian Relativity
(Some More History)

- **Implications:**
  - **Question (remember #3?):** Do Maxwell's equations obey Newtonian Relativity? (i.e. do they stay "invariant" under Galilean transformations, under the ether hypothesis?)
  - **Short Answer!:** No!
  - If they did, equations would produce c+v for the speed of light (and in a moving space ship, electrical or optical phenomena would be different than in a stationary space ship; then, some one on the spaceship could measure the speed of the space ship)
    - Contradicted by Michelson-Morley experiment.
  - Another problem: Maxwell's equations seem to predict the same value for c for all inertial frames. Therefore:
    - If the source of the e.m. waves is moving, the e.m. wave is predicted to move through space with the **same c.** But:
    - This is contradicted by Galilean transformations.
    - Experiments support $c = 3.00 \times 10^8$ m/s in all frames

What to Do?

- Are Maxwell’s equations not the same in all inertial frames?
  - Maxwell’s equations could be wrong (only 20 years old!)
  - If so, Galilean transformations would be retained and
  - Maxwell’s equations should be fixed to obey Galilean transformation and
  - A preferred inertial frame must exist which should produce different values for c as one moves through it. Keep looking!
- Are Galilean transformations wrong?
  - Then, keep Maxwell’s equations as they are; experiments OK
  - Either hypothesis would have to be abandoned
  - Absolute space and absolute time would have to be abandoned
- What about c?
  - All moving frames seem to give same value: 3.00×10^8 m/s
  - Poincare: “A total conspiracy of nature is itself a law of nature”
- Einstein to the rescue:
  - Fruitless to continue to show experimentally that a special inertial frame exists. Proposes Two Postulates.

Special Theory of Relativity Einstein's Two Postulates (1905)

- Postulate I (Principle of Relativity)

  "All laws of physics must be the same ("invariant") in all inertial reference frames"

  (Can not detect absolute uniform motion)

- Postulate II (Constancy of c)

  "The speed of light in vacuum is constant (same value, $c = 3.00 \times 10^8$ m/s) in all inertial reference frames, regardless of motion of source or observer"

Some Observations

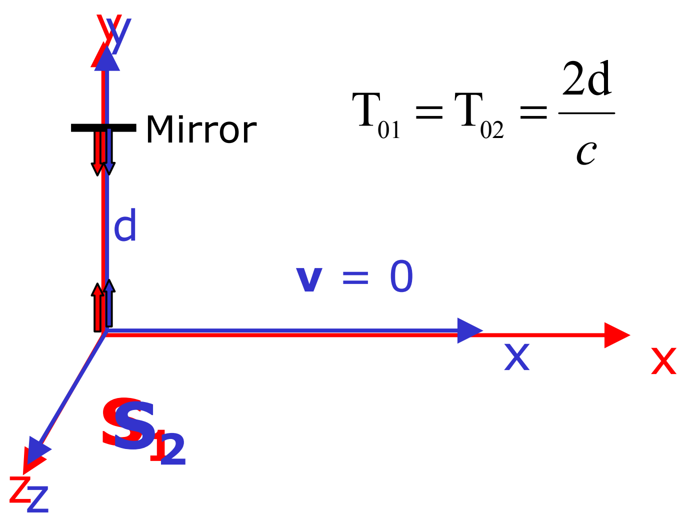
- Problems solved by STR:
  - Keep Maxwell's equations
  - Accept measurements of c, all producing 3.00×10^8 m/s
  - No need for either
    - build a theory on experimental results, not the other way around!
- New issues arise:
  - Galilean transformations are abandoned
    - need new transformation equations
  - Newton's laws don't obey Postulate I any longer
    - have to be changed
    - Absolute space and absolute time must be abandoned
    - Constancy of mass (and conservation of mass) must be abandoned
    - Laws are OK the way they are as long as $v \ll c$
  - New and non-intuitive predictions appear
  - Experimental verification needed
  - Acceptance by scientific community is slow

Consequences - Simultaneity

- **Usual Meaning:** Two events occurring at the same time
- Simultaneity is not an absolute concept anymore
  - It depends on motion of the observer
  - Taken for granted in Newtonian physics
    - But it doesn't matter, since $v \ll c$ in Newtonian world
- **Precise new definition needed:**
  - Two events in an inertial frame of reference are simultaneous if light signals from the locations of these events reach an observer positioned halfway between them at the same time
- **Consequences:**
  - Two events simultaneous in one inertial frame are not simultaneous in another frame in motion w.r.to the first
  - $t(x)$; $t_1 \neq t_2$; t_1 and t_2 are intimately intertwined with x_1, x_2

Consequences - Time Dilation

- Clocks slow down when seen by observer moving w.r. to them!
  - Consider the duration (time interval T) of an "event"
  - Let "event" be: a light ray goes from the origin of S_2 to a mirror at $y_2 = d$, is reflected there, and returns to the origin.
  - If S_1 and S_2 are momentarily at rest, both agree, using their own clocks and meter sticks, the duration of the round trip ("proper time", i.e. measured in the rest frame of the event, at a fixed location) is:



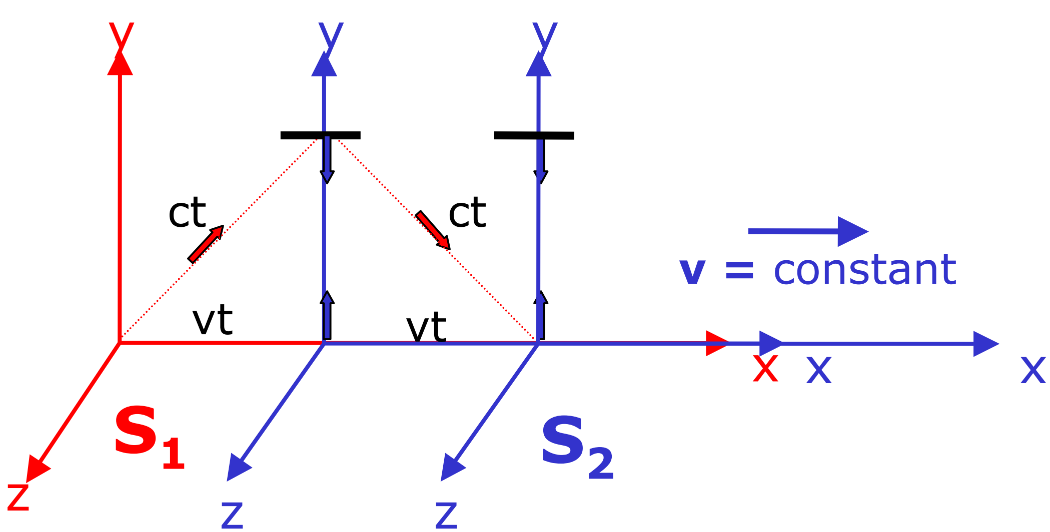
$$T_{01} = T_{02} = \frac{2d}{c}$$

Next: let S_2 move along x-axis of S_1 with v = constant.
S_1 will measure the time interval again. Call it T_1

Consequences - Time Dilation

- As seen by S_1, light sent by S_2 follows path shown, taking time **t** to get from the origin of S_2 to the mirror and back
- New time interval T_1, measured by S_1, is $T_1 = 2t$, where

$$(ct)^2 = d^2 + (vt)^2 \quad \text{solve for t:} \quad t = \frac{d}{\sqrt{c^2 - v^2}}, \ \therefore T_1 = \frac{2d}{c}\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{or}$$



$$T_1 = \gamma T_{02} \quad \text{where } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Since $\gamma \geq 1$ always,

$\Rightarrow$ $\mathbf{T_1 \geq T_{02}}$

(Relativistic T > Proper T)

Time Dilation Example: Muon (μ) Lifetime

- When at rest in the laboratory (i.e. in its "rest frame") the μ has (average) lifetime $\tau_o = 2.2 \times 10^{-6}$ s, before it decays.
  - At the speed of light it would travel ~600 m before decaying
- We know μ are produced high in the atmosphere, ~10 km up, from collisions of cosmic rays with the atmosphere
- How are we able to observe them on the earth?
  - As measured by an earth based observer, the life time of the μ is dilated ("non-proper" time interval): $\tau = \gamma\, \tau_o$
  - Since a high energy (relativistic) μ has speed $v \to c$

    then $\gamma >> 1$

    and $\tau >> \tau_o$
  - Therefore it has enough time to reach the earth and be observed

Consequences - Length Contraction

- A length gets shortened when measured by observer in motion with respect to it!
  - Consider the length between the earth and a star.
  - A space ship (S_2) travels from the earth to the star moving with constant speed v along x-axis of an observer at rest on the earth (S_1)
  - **Earth observer** *measures* the distance to the star L_{01} ("proper length") and *computes* the time it takes for the space ship (the moving frame) to get there as $T_1 = L_{01}/v$; (T_1 is "non-proper" T).
  - **Space ship** observer, at rest in S_2, sees the star moving with -v; *measures* time T_{02} ("proper time", at fixed place) it takes to reach the star and *computes* distance ("non-proper" L) to be: $L_2 = vT_{02}$
  - The two time measurements are related by $T_1 = \gamma T_{02}$
  - Therefore, substituting T_{02} in L_2: $L_2 = vT_1/\gamma$, or:

$$\boxed{L_2 = \frac{L_{01}}{\gamma}}$$

∴ Since $\gamma > 1$ always, ⇒ $L_2 < L_{01}$ (Relativistic L < Proper L)

Length Contraction Example: Human Space Travel

- A 25 year-old astronaut wishes to travel from earth to Alpha Pegasus (distance L_0 = 100 lt yr away), using a fast spaceship
  - Will he/she get there in his/her lifetime?
- Use v = 0.95c for spaceship speed w.r. to earth (γ = 3.2)
- Astronaut measures $T_{02} = L_2/v = (L_0/\gamma)v = 100/(3.2*0.95)$ = 32.8 yr (measures proper time interval, computes contracted length)
- Therefore, astronaut will be and feel 25 + 32.8 = 57.8 years old upon arrival, will get there alive and well.
- Earth observer computes: $T_1 = L_{01}/v = 100\text{yr}/0.95\text{yr/ltyr} = 105$ yr for astronaut to get there, (measures proper length, computes dilated time interval), $T_1 = \gamma T_{02} = 3.2*32.8 = 105$ yr
  - Biological, (e.g. heart beat) and other processes on spaceship will "slow down" as seen by earth observer

Lorentz Transformations

- Recall, Galilean transformations are not valid when $v \to c$
- Need new equations to transform $(x_1,y_1,z_1,t_1) \to (x_2,y_2,z_2,t_2)$
- Assume relativistic transformation for $x_1 \to x_2$ is the same as the classical one except for an unknown constant γ, therefore,

$x_2 = \gamma(x_1 - vt_1)$ (1)	and	$x_1 = \gamma(x_2 + vt_2)$ (2)
$y_2 = y_1$		$y_1 = y_2$
$z_2 = z_1$		$z_1 = z_2$
$t_2 = ?$		$t_1 = ?$

- γ must be <u>the same</u> in both equations (Postulate I). Substitute x_2 from (1) in (2) and solve for t_2

$$t_2 = \frac{x_1 - \gamma^2(x_1 - vt_1)}{\gamma v} \text{ (3). Form } \frac{\Delta x_2}{\Delta t_2}; \text{ Set } \frac{\Delta x_2}{\Delta t_2} = \frac{\Delta x_1}{\Delta t_1} = c \text{ (Postulate II)}$$

- Solve for γ. $\boxed{\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}}$ Substitute γ in (1), (2), (3) and get final result:

Lorentz Transformations

$$x_2 = \frac{x_1 - vt_1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y_2 = y_1$$

$$z_2 = z_1$$

$$t_2 = \frac{t_1 - \frac{vx_1}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

or

$$x_1 = \frac{x_2 + vt_2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y_1 = y_2$$

$$z_1 = z_2$$

$$t_1 = \frac{t_2 + \frac{vx_2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

- Note: Lorentz Transformations are valid for any v.
  - As $v \to 0$, they reduce to the Galilean Transformations,
  - Therefore, G. T. are good enough for Newtonian Mechanics
- $v > c$ produces imaginary, non-physical numbers.
- Therefore c becomes a limiting speed for physical phenomena

Lorentz Transformation of Velocities

- By differentiation,

$$\boxed{\begin{aligned} v_{2x} &= \frac{v_{1x} - v}{1 - \frac{v}{c^2} v_{1x}} \\ v_{2y} &= \frac{v_{1y}}{\gamma\left(1 - \frac{v}{c^2} v_{1x}\right)} \\ v_{2z} &= \frac{v_{1z}}{\gamma\left(1 - \frac{v}{c^2} v_{1x}\right)} \end{aligned}}$$

or

$$\boxed{\begin{aligned} v_{1x} &= \frac{v_{2x} + v}{1 + \frac{v}{c^2} v_{2x}} \\ v_{1y} &= \frac{v_{2y}}{\gamma\left(1 + \frac{v}{c^2} v_{2x}\right)} \\ v_{1z} &= \frac{v_{2z}}{\gamma\left(1 + \frac{v}{c^2} v_{2x}\right)} \end{aligned}}$$

- If $v_{2x} = c$, then $v_{1x} = c$
  - i.e. if v = c in one frame, then v = c in any frame
- These equations refer to measurements done in <u>different</u> frames. Vector addition of velocities in the <u>same</u> frame remain the same

Lorentz Transformation of Velocities Example

- Two spaceships A and B approach the Earth from opposite directions
- Each ship moves with v = 0.80c with respect to the Earth
- How fast is each ship moving with respect to the other?
  - i.e. what is the speed of one spaceship measured in the rest frame of the other?
- Must establish S_1 and S_2: Let S_1 be the earth, let S_2 be spaceship A, moving w.r.to S_1 with speed v = +0.80c
- "Event" is spaceship B, moving with speed v_{B1} = -0.80c in S_1
- Speed v_{B2} of spaceship B (event) relative to A (S_2) is:

$$\boxed{v_{B2} = \frac{v_{B1} - v}{1 - \frac{v_{B1} v}{c^2}} = \frac{-0.80c - 0.80c}{1 - \frac{(-0.80c)(0.80c)}{c^2}} = -0.98c}$$

- Note: $v_{B2} \neq$ (Galilean) $v_{B1} - v$ = -0.80c – 0.80c = -1.6c

Relativistic Dynamics

- Newton's Laws are invariant under Galilean transformations but not under Lorentz transformations. They must be changed
- Experiment shows that Newton's Laws:
  - Force laws (with **p** = m**v**)
  - Conservation of energy
  - Conservation of momentum

  may retain their (mathematical) form IF the classical concept of mass is modified: m $\neq$ constant, independent of relative motion between object and observer. Conservation of mass is abandoned!
- In relativistic physics, let: m = m(v), that is:
  - A mass m moving with speed v with respect to an observer is different than the mass m_0 of the same object when at rest with respect to the observer ("proper mass" or "rest mass")
- Can show: $\boxed{m = \dfrac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}} \quad \text{or,} \;\; m = \gamma m}$

$\therefore$ Since $\gamma > 1$ always, $\Rightarrow$ $m > m_0$

(Relativistic m > Proper m)

Relativistic Dynamics

- Newton's Laws (now relativistically correct):

$\vec{p} = m\vec{v}$	New Definition of Momentum, where
$m = \gamma m_0$	Relativistic Mass, then :
$\vec{F} = \dfrac{d\vec{p}}{dt}$	Second Law of Motion
$K = W = \int_0^s \vec{F} \cdot d\vec{s}$	Definition of Work, K.E.
$\left[\Sigma \vec{p}_i\right]_{\text{initial}} = \left[\Sigma \vec{p}_i\right]_{\text{final}}$	Conservation of Momentum

- Note: As $v \to c$ and $m >> m_0$, even a small Δv requires a large F to produce it. Example:
  - Going from v = 0 to v = 0.01c (Δv = 0.01c), $\Delta p = 0.010 m_0 c$. Going from v = 0.98c to v = 0.99c (same Δv = 0.01c),

    $\Delta p = 2.10 m_0 c$ (or 210 times more).

    Therefore F is 210 times more in the second case
- What about Energy/Conservation of Energy?

Relativistic Energy

- Kinetic Energy (in one dimension):

$$K = \int_0^v F ds = \int_0^v \frac{dp}{dt} ds = \int_0^v v dp = \int_0^v v d\left(\frac{m_0 v}{\sqrt{1 - v^2/c^2}} \right) = \int_0^v \frac{m_0 v dv}{\sqrt[3]{1 - v^2/c^2}} = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} - m_0 c^2$$

- Or: $K = mc^2 - m_0c^2 = (m - m_0)c^2 = \gamma m_0c^2 - m_0c^2$ (1)
- Note: $mc^2 = m(v \neq 0)$; $m_0c^2 = m(v=0)$
- Interpret m(v=0) as "Rest Energy" E_0, $\therefore$ $\boxed{E_0 = m_0c^2}$
  - No classical equivalent, but not in conflict with Newtonian physics
    - Can add rest energy on both sides of any Newtonian energy balance without destroying it
- Interpret $K + E_0$ as "Total Energy" E, $\therefore$ $\boxed{E = mc^2}$ from (1).

  i.e. $E = K + E_0 = mc^2 = \gamma m_0c^2 = \gamma E_0$

Examples

- Find the velocity of a 6 GeV electron
  - K = 6 GeV
  - Find γ: $\gamma = \frac{E}{E_0} = \frac{K + m_0c^2}{m_0c^2} = 1 + \frac{K}{m_0c^2} = 1 + \frac{6x10^3\,\mathrm{MeV}}{0.511\,\mathrm{MeV}} = 1.174268x10^4$
  - Find v: $v = c\sqrt{1 - \frac{1}{\gamma^2}} = c\sqrt{1 - \frac{1}{(1.174268x10^4)^2}} = 0.999999996c$
- Find the rate at which the sun loses mass if it radiates energy at the rate of 3.85×10^{26} W
  - P = E/s = 3.85×10^{26} W
  - M/s = $(E/c^2)/s$ =
    = $[(3.85 \times 10^{26})/(3.00 \times 10^{8})^2]/s$= 4.28×10^{9} kg/s

Conservation of Energy

- The rest energy m_0c^2 is of more than casual interest
  - Processes exist where the total rest mass is not conserved
  - Experiment shows that a change ΔE_0 is compensated by a change ΔK so that the total energy change $\Delta E = 0$, therefore:
- The Total Energy **E is the conserved energy quantity** now.
  - **Law of Conservation of Energy applies to total energy E**
  - In relativistic physics there are no separate laws of conservation of mass and energy
- If there is Potential Energy involved, then $E = K + V + E_0$
- K is equal to the increase in the mass of the object when it moves with speed v
- As $v \to 0$, $K \to 1/2\ m_0v^2$, as it should, (binomial expansion)
  - Note: K can not be obtained from $K = 1/2\ mv^2$ by using $m = \gamma m_0$

Energy - Momentum Relationship

- In Newtonian physics: $K = p^2/2m$ relates K to p
- To find corresponding expression in relativistic physics, use:

  $E = \gamma m_0 c^2$ and $p = \gamma m_0 v$ with $\gamma = (1 - v^2/c^2)^{-1/2}$

  to derive: $\mathbf{E^2 = p^2c^2 + (m_0c^2)^2}$ (very important/useful)
- $E^2 - p^2c^2 = (m_0c^2)^2$ is "invariant" (same value in all frames)
- Interesting limiting cases:
  - If v << c: $$E = m_0c^2\sqrt{1+\frac{p^2}{m_0^2c^2}} = m_0c^2\left(1+\frac{1}{2}\frac{p^2}{m_0^2c^2}+\ldots\right) = m_0c^2 + \frac{p^2}{2m_0}$$
    - Get $K = p^2/2m$
  - If $v \approx c$: $E >> E_0$, therefore $E \approx pc$ and $K \approx E$
- If $m_0 = 0$, $E = pc$, $E = K$ and **must have v = c**

  otherwise $E = mc^2 = \gamma m_0 c^2 = 0$
- If $v = c$ (and $E \neq 0$), **must have $m_0 = 0$** (and $E = pc$)

  otherwise $m = \gamma m_0 \rightarrow \infty$ (0/0 is OK!)

Units: Mass, Energy, Momentum

$$E^2 = p^2c^2 + (m_0c^2)^2$$

- In nuclear/particle physics preferred unit of energy is MeV

  $$1 \text{ MeV} = 1.6 \times 10^{-13} \text{ J}$$

- Since mc^2 or m_0c^2 represent energy, their units are MeV
  - Therefore, mass m can be given in units of MeV/c^2
  - **Example:** Rest mass of a proton is $m_{p0} = 938 MeV/c^2$;

    This means: rest energy of a proton is $m_{p0}c^2 = 938$ MeV
- Similarly: pc has units of MeV
  - Therefore, p can be given in units of MeV/c
  - **Example:** A proton with (total) E = 2 GeV has momentum p:

$$p = \sqrt{\frac{E^2 - (m_0c^2)^2}{c^2}} = \sqrt{\frac{(2000\text{MeV})^2}{c^2} - \left(\frac{938\text{MeV}}{c^2}\right)^2} = \sqrt{(2000^2 - 938^2)\left(\frac{\text{MeV}}{c}\right)^2} = 1766\text{MeV/c}$$

Example

- Compute the momentum of a 500 MeV proton
  - From $E^2 = p^2c^2 + (m_0c^2)^2 = p^2c^2 + E_0^2$
  - And from $E = K + m_0c^2$
  - Compute:

$$p = \frac{1}{c}\sqrt{E^2 - E_0^{\,2}} = \frac{1}{c}\sqrt{(K + m_0c^2)^2 - (m_0c^2)} =$$

$$= \frac{1}{c}\sqrt{(500\,\mathrm{MeV} + 938\,\mathrm{MeV})^2 - (938\,\mathrm{MeV})^2} = 1090\,\mathrm{MeV/c}$$

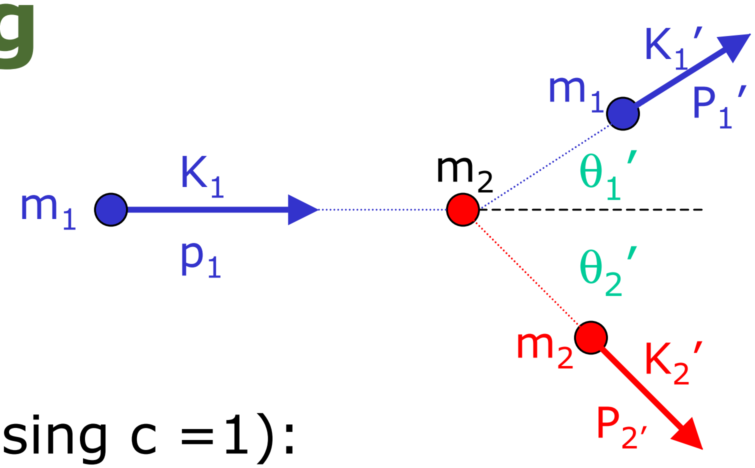
- Redo, using system of units where c = 1
  - Convenient since, now, p, E, m will all have units of energy
  - Here $E^2 = p^2 + m_0^2$ and $E = K + m$
  - Therefore:

$$p = \sqrt{(K + m_0)^2 - m_0^{\,2}} = \sqrt{(500\mathrm{MeV} + 938\mathrm{MeV})^2 - 938^2\,\mathrm{MeV}^2} = 1090\,\mathrm{MeV}$$

    - in the “regular” system of units, would write p = 1090 MeV/c

Relativistic Two-Body Kinematics
Elastic Scattering

Given: m_1, m_2, K_1, θ_1'
Find: K_1', K_2', θ_2'



- Conservation of Energy (using c =1):

$$E_1 + m_2 = E_1' + E_2' \qquad (1)$$

- Conservation of Momentum:

$$p_1 = p_1'\cos\theta_1' + p_2'\cos\theta_2' \qquad (2)$$

$$0 = p_1'\sin\theta_1' - p_2'\sin\theta_2' \qquad (3)$$

- Recall: $E_1 = K_1 + m_1$ and $E_1^2 = p_1^2 + m_1^2$
  $E_1' = K_1' + m_1$ and $E_1'^2 = p_1'^2 + m_1^2$
  $E_2' = K_2' + m_2$ and $E_2'^2 = p_2'^2 + m_2^2$
- Solve (1), (2), (3) for K_1', K_2', θ_2'

Relativistic Two-Body Kinematics Elastic Scattering – Results (1)

$$E_1' = \frac{A_1(E_1 + m_2) \pm p_1 \cos\theta_1' \sqrt{A_1^2 - 4m_1^2\left[(E_1 + m_2)^2 - p_1^2 \cos^2\theta_1'\right]}}{2\left[(E_1 + m_2)^2 - p_1^2 \cos^2\theta_1'\right]}$$

$$E_2' = \frac{A_2(E_1 + m_2) \pm p_1 \cos\theta_2' \sqrt{A_2^2 - 4m_2^2\left[(E_1 + m_2)^2 - p_1^2 \cos^2\theta_2'\right]}}{2\left[(E_1 + m_2)^2 - p_1^2 \cos^2\theta_2'\right]}$$

$$\cot\theta_2' = \frac{-(1 + \rho_1\rho_2)\cot\theta_1' \pm (\rho_1 + \rho_2)\sqrt{\gamma^2(1 - \rho_1^2) + \cot^2\theta_1'}}{1 - \rho_1^2}$$

Where A_1, A_2, ρ_1, ρ_2, γ are given by:

Relativistic Two-Body Kinematics Elastic Scattering – Results (2)

$$A_1 = 2\left(m_1^2 + m_1 m_2 + m_2 K_1\right)$$

$$A_2 = 2\left(m_2^2 + m_1 m_2 + m_2 K_1\right)$$

$$\rho_1 = \frac{A_1 p_1}{(E_1 + m_2)\sqrt{A_1^2 - 4m_1^2\left[(E_1 + m_2)^2 - p_1^2\right]}}$$

$$\rho_2 = \frac{A_2 p_1}{(E_1 + m_2)\sqrt{A_2^2 - 4m_2^2\left[(E_1 + m_2)^2 - p_1^2\right]}}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v_{cm}^2}{c^2}}} = \frac{E_1 + m_2}{\sqrt{m_1^2 + m_2(2E_1 + m_2)}}$$

Relativistic Two-Body Kinematics Elastic Scattering - Observations

- **Note:** Not any θ_1, θ_2 are possible in two-body kinematics
  - Square root in E_1, E_2 must be > 0

$$\sin^2\theta_1' \leq \frac{A_1^2 - 4m_1^2\left[(m_1 + m_2)^2 + 2m_2K_1\right]}{4m_1^2 p_1^2}$$

$$\sin^2\theta_2' \leq \frac{A_2^2 - 4m_2^2\left[(m_1 + m_2)^2 + 2m_2K_1\right]}{4m_2^2 p_1^2}$$

- **Note:** If $m_1 = m_2$ then $\theta_1' + \theta_2' < \pi/2$ (unlike classical mechanics)