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ROLL NO - DTU/2K17/A2/156

MATHS ASSIGNMENT - 2

classmate

Date

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Q1 Statement → If a function $f(x)$ is such that $f(x)$, $f'(x)$, $f''(x)$ ... $f^{n-1}(x)$ are continuous in the (closed interval) $[a, a+h]$ and $f^n(x)$ exists in the (open interval) $(a, a+h)$, then there exist atleast one number θ such that $f(a+h) = f(a) + hf'(a) \dots \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^n(a+\theta h)$ where $\theta \in (0,1)$

PROOF →

We define a function $\phi(x)$ involving $f(x)$ and its derivatives $f'(x)$, $f''(x)$, ... $f^n(x)$, designed so as to satisfy the conditions for Rolle's theorem. Let

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \dots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) + \frac{(a+h-x)^n A}{n!} \quad - (1)$$

where A is a constant to be determined such that.

$$\phi(a) = \phi(a+h) \quad - (2)$$

From (1) & (2) we obtain

$$f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n A}{n!} = f(a+h)$$

As rolle's conditions are satisfied $\phi'(a+\theta h)$ must be zero for atleast one $\theta \in (0,1)$

$$\phi'(x) = f'(x) - f'(x) + (a+h-x) f''(x) - (a+h-x) f''(x) \dots \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) A$$

$$= \frac{(a+h-x)^{n-1}}{(n-1)!} [f^{(n)}(x) - A]$$ other terms cancelling in pairs

Therefore $\phi'(a+\theta h) = 0$ gives,

$$\phi'(a+\theta h) = 0 = \frac{(h-\theta h)^{n-1}}{(n-1)!}\left[f^{(n)}(a+\theta h) - A\right]$$

$\Rightarrow \quad f^{(n)}(a+\theta h) = A \quad$ since $(1-\theta) \neq 0$ for $\theta \in (0,1)$

Substituting this value of A we get

$$\cancel{f(a)} = $$

$$f(a+h) = f(a) + f'(a) \times h + \frac{h^2}{2!} f''(a) + \cdots$$

$$\frac{h^n}{n!} f(a+\theta h).$$

Q2 $\quad \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Approximating the Taylor's polynomial upto degree three we get

$$\sin(x) = x - \frac{x^3}{3!} + \cancel{\frac{x^5}{5!}}$$

$$R_3(x) = \frac{x^5}{5!}$$

For $\frac{x^5}{5!} \leq 0.001$

$$x^5 \leq 0.12$$

$$x \leq 0.654$$

$$\Rightarrow \quad c \in [0, 0.654]$$

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Q3.

$$e^{2x} = 1 + 2x\,(\cancel{x}) + \frac{4x^2}{2!}(\cancel{x})^2 + \frac{8x^3}{3!}(\cancel{x})^3 + \frac{16x^4}{4!}(\cancel{x})^4 + \dots$$

Approximating till fourth degree

$$e^{2x} \approx 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!}$$

Error term $R_5(x) = \frac{32x^5}{5!} e^{(2\theta x)}$

$x^5 e^{(2\theta x)}$ is an increasing function for $x \in (0, \infty$

$\Rightarrow$ Error is max when $x = 0.5$

Max error $= \frac{32 \times (0.5)^5}{5!} e^{2\theta(0.5)}$

For max error $\theta = 1 \Rightarrow \frac{32 \times (0.5)^5}{5!} e^1 = \underline{\underline{\frac{e}{120}}}$

Q4.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f^2(0) + \frac{x^3}{3!} f^3(0) + \dots$$

(a) For $f(x) = a^x$

$$f(x) = 1\,\cancel{f(0)} + x\ln(a) + \frac{x^2}{2!}(\ln a)^2 + \frac{x^3}{3!}(\ln a)^3 + \dots$$

(b) For $\log(1-x)$ $\quad f'(x) = \frac{-1}{(1-x)}$ $\quad f''(x) = \frac{-1}{(1-x)^2}$ $\quad f'''(x) = \frac{-2}{(1-x)^3}$

$$f(x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \dots$$

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5. $f\left(\frac{11}{10}\right) = f\left(1+\frac{1}{10}\right)$ $f(x) = x^3 + 3x^2 + 15x - 10$

$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \dots$

$f\left(1+\frac{1}{10}\right) = f(1) + \frac{1}{10} f'(1) + \frac{1}{100\times 2!} f''(1) + \dots$

$= 9 + \frac{1}{10}(24) + \frac{1}{200}(12) + \frac{1}{1000\times 6}(6)$

$= 11.461$

Q6. Curvature and Radius of Curvature

(i) $x^2 + y^2 = a^2$ $y = \sqrt{a^2 - x^2}$

Radius of curvature $= \frac{(1+y_1^2)^{3/2}}{y_2}$ $y_1 = \frac{1}{\cancel{2}\sqrt{a^2-x^2}}(-\cancel{2}x)$

$y_2 = \frac{-\sqrt{a^2-x^2} + x\cdot}{(a^2-x^2)}$

$$= \frac{\left[1 + \frac{x^2}{(a^2-x^2)}\right]^{3/2}}{\frac{-\sqrt{a^2-x^2} - \frac{x^2}{\sqrt{a^2-x^2}}}{(a^2-x^2)}}$$

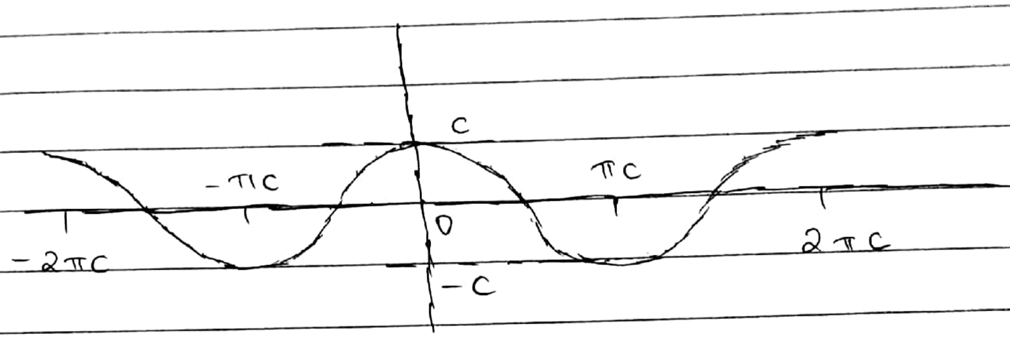
$$= \frac{\left[\frac{x^2 + a^2 - x^2}{a^2 - x^2}\right]^{3/2}}{\frac{(x^2 - a^2 - x^2)}{(a^2-x^2)^{3/2}}} = -a$$

Curvature $= \frac{1}{\text{Radius of curvature}} = \frac{-1}{a}$

Q7. $y = c\cos(x/c)$

Symmetry $\rightarrow$ Symmetric about y-axis

Period $\rightarrow$ $2\pi c$

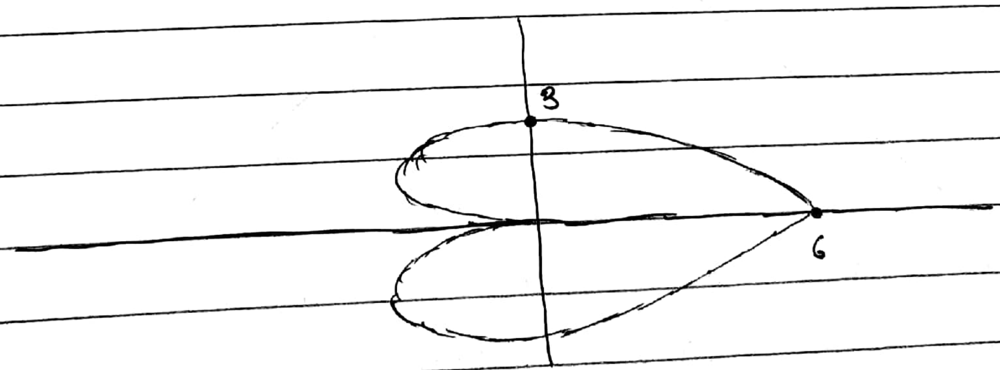


(ii) $x = 2(\theta + \sin\theta)$, $y = 2(1-\cos\theta)$

$\downarrow$ ODD $\qquad$ $\downarrow$ EVEN

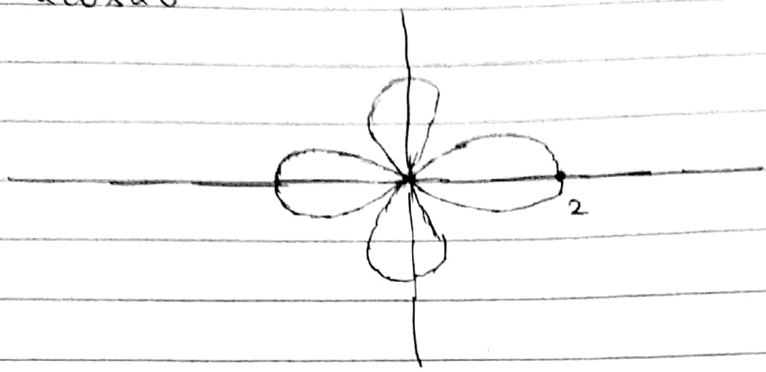
SYMMETRIC ABOUT Y-AXIS

(iii) $r = 3(1+\cos\theta)$



(iv)

$r = 2\cos 2\theta$

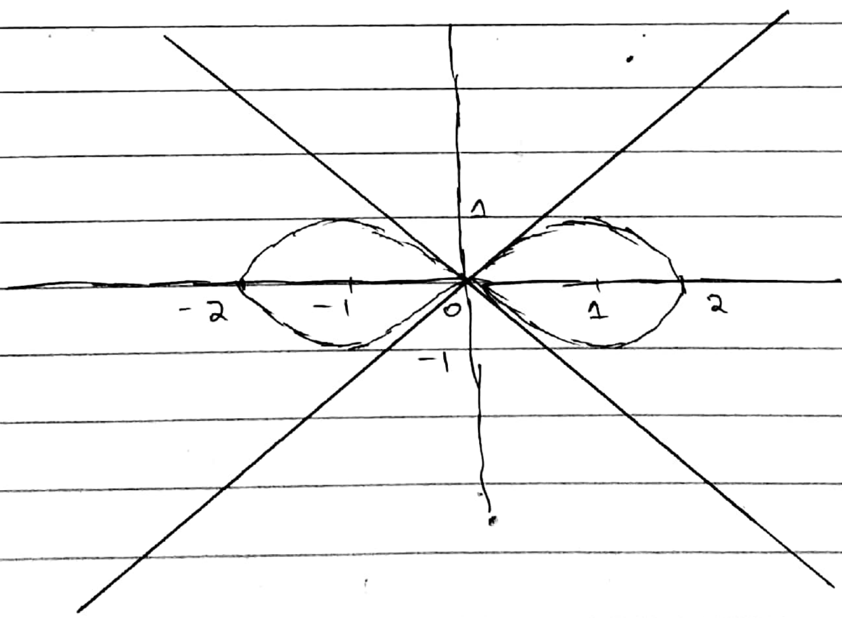


(v) $(x^2+y^2)^2 = 4(x^2-y^2)$

Symmetric about x-axis and Y-axis both

when $y^2 > x^2$ → Graph doesnot exist

when $y=0$ $x = \pm 2, 0$ and when $x=0, y=0$



Differentiating the above function

$2(x^2+y^2)(2x + 2y\frac{dy}{dx}) = 4(2x - 2y\frac{dy}{dx})$

For max/min values put $\frac{dy}{dx} = 0$ ⇒

$x=-1$ $y=1$

$x=1$ $y=-1$

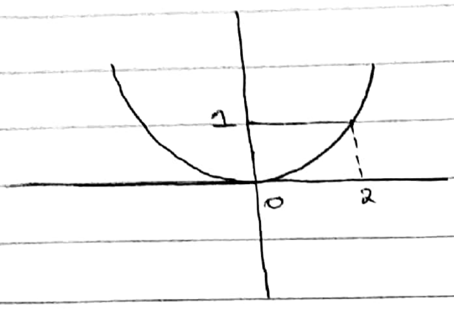
$x=1$ $y=1$

$x=-1$ $y=-1$

Q8. Length of the curve $y = \left(\frac{x}{2}\right)^2$ from $x = 0$ to $x = 2$

$x^2 = 4y$

$\frac{dy}{dx} = \frac{x}{2}$



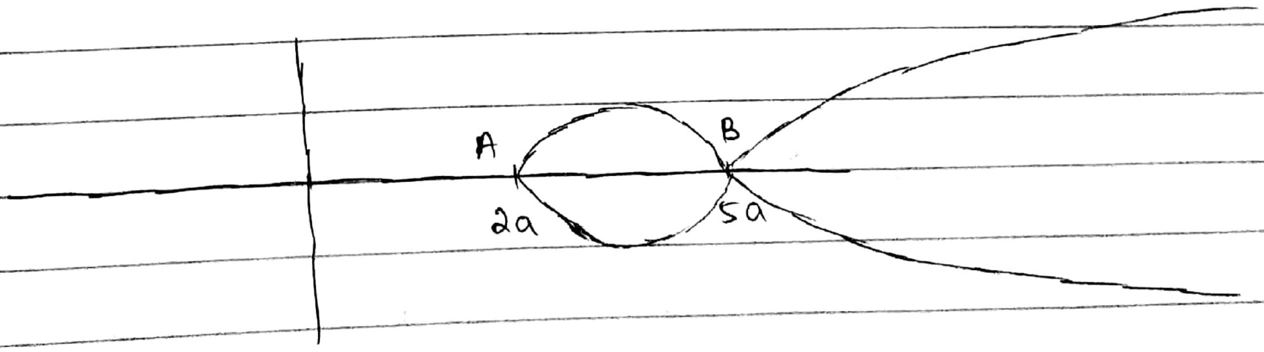
$$\int_0^2 \sqrt{1 + (dy/dx)^2} = \int_0^2 \sqrt{1 + x^2/4}$$

$$= \frac{1}{2}\int_0^2 \sqrt{4+x^2} = \frac{1}{2}\left[\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|\right.$$

$$= \frac{1}{2}\left[\sqrt{8} + 2\log(2+\sqrt{8}) - 2\log(2)\right]$$

$$= \frac{1}{2}\left(\sqrt{8} + 2\log(1+\sqrt{2})\right)$$

Q9 $9ay^2 = (x-2a)(x-5a)^2.$



Length of loop $= 2 \times AB$

$$y = \sqrt{\frac{(x-2a)(x-5a)^2}{9a}}$$

$$3\sqrt{a}\, y = (x-5a)\sqrt{(x-2a)}$$

$$3\sqrt{a}\, \frac{dy}{dx} = \sqrt{(x-2a)} + \frac{(x-5a)}{2\sqrt{(x-2a)}}$$

$$\frac{dy}{dx} = \frac{\sqrt{(x-2a)} + \frac{(x-5a)}{2\sqrt{(x-2a)}}}{3\sqrt{a}}$$

$$= \frac{3x - 9a}{6\sqrt{a}\sqrt{(x-2a)}} = \frac{x-3a}{2\sqrt{a}\sqrt{(x-2a)}}$$

$$\sqrt{1 + \frac{(x-3a)^2}{4a(x-2a)}}$$

$$\sqrt{\frac{4ax - 8a^2 + x^2 + 9a^2 - 6ax}{4a(x-2a)}} = \sqrt{\frac{x^2 + a^2 \overset{-2ax}{+\cancel{4ax} - \cancel{6a}}}{4a(x-2a)}}$$

$$= \int_{2a}^{5a} \sqrt{\frac{(x-a)^2}{4a(x-2a)}}$$

put $x - 2a = t$

$dx = dt$

$$= \int_{0}^{3a} \sqrt{\frac{(t+a)^2}{4a \times t}}$$

$$= \frac{1}{2\sqrt{a}} \int_{0}^{3a} \left(\sqrt{t} + \frac{a}{\sqrt{t}} \right) dt$$

$$= \frac{1}{2\sqrt{a}} \left(\frac{2}{3} t^{3/2} + 2\sqrt{t}\, a \right)_0^{3a} = \frac{1}{2\sqrt{a}} \left(\frac{2}{3} \times 3^{3/2} a^{3/2} + 2\sqrt{} \right.$$

$\Rightarrow$ LOOP AB $= 4\sqrt{3}a$ //   ARC AB $= 2\sqrt{3}\, a$

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Q10.

$$r^2 = 2a^2 \cos(2\theta) \qquad 0 \le \theta \le \pi/4$$

$$S = \int 2\pi (r \sin\theta) \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

$$S = \int 2\pi (2a^2 \cos 2\theta \sin\theta) \sqrt{\cancel{4a^4 \cos^2(2\theta)} + \cancel{16a^4 \sin^2(2\theta)}}$$

$$\cancel{S = 8\pi a^4 \int_0^{\pi/4} \cos 2\theta \sin\theta \sqrt{1 + 3\sin^2(2\theta)}}$$

$$S = \int_0^{\pi/4} \left[2\pi (\sqrt{2}a \sqrt{\cos(2\theta)} \sin\theta\right] \sqrt{2a^2 \cos 2\theta + \left(\frac{dr}{d\theta}\right)^2}$$

$$S = \int_0^{\pi/4} (2\pi \times \cancel{\sqrt{2}a}) (\sqrt{\cancel{\cos(2\theta)}} \sin\theta \sqrt{(2a^2\cos 2\theta)^2 + \frac{4a^2 \sin^2 2\theta}{\cancel{r^2}}}$$

$$S = \int_0^{\pi/4} 2\pi \sin\theta \sqrt{4} = \int_0^{\pi/4} 2 \times 2\pi \sin\theta$$

$$= -4\pi \left[\cos\theta\right]_0^{\pi/4} \qquad \frac{dr}{d\theta} = \frac{-2a^2 \sin 2\theta}{r}$$

$$= -4\pi \left(\frac{\cancel{\pi}}{2} \frac{1}{\sqrt{2}} - 1\right)$$

$$= 4\pi \left(1 - \frac{1}{\sqrt{2}}\right)$$

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Q11

$$r = a(1+\cos\theta)$$

$$V = \int_0^{\pi/2} \frac{2\pi}{3} r^3 \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{2\pi}{3} a^3 (1+\cos\theta)^3 \cos\theta \, d\theta$$

$$= \frac{2\pi a^3}{3} \int_0^{\pi/2} (1 + \cos^3\theta + 3\cos^2\theta + 3\cos\theta)\cos\theta \, d\theta$$

$$= \frac{2\pi a^3}{3} \int_0^{\pi/2} (\cos\theta + 3\cos^3\theta + 3\cos^2\theta + \cos^4\theta) \, d\theta$$

$$= \frac{2\pi a^3}{3} \boxed{1 + 3(3)(1) + \frac{3(3)\pi}{2} + \frac{(4)(2)\pi}{2}}$$

$$= \frac{2\pi a^3}{3} \left[1 + \frac{\cancel{3}\times 2}{\cancel{3}} + \frac{\cancel{3}}{\cancel{2}} \times \frac{\pi}{2} + \frac{3\times 1}{4\times 2}\frac{\pi}{2} \right]$$

$$= \frac{\cancel{2}\pi a^3}{\cancel{3}} \left[2\cancel{3} + \frac{\cancel{3}}{8\cancel{4}}\left(\frac{5\pi}{4}\right) \right]$$

Ans = 2x

Q12.

$r = 1 + \cos(\theta)$

$$S = \left| \int_{\pi/2}^{\pi} 2\pi (r\sin\theta) \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \right| + \int_{0}^{\pi/2}$$

$=$

$$\int_{\pi/2}^{\pi} 2\pi (1+\cos\theta)\sin\theta \sqrt{1+\cos^2\theta + 2\cos\theta + \sin^2\theta} + \int_{0}^{\pi/2}$$

$$2\sqrt{2}\pi \int_{\pi/2}^{\pi} (1+\cos\theta)^{3/2} \sin\theta$$

$1+\cos\theta = t$

$-\sin\theta \, d\theta = dt$

$$\left| -2\sqrt{2}\pi \int_{1}^{0} t^{3/2} \, dt \right| + \left| -\int_{2}^{1} 2\sqrt{2}\pi \, t^{3/2} dt \right|$$

$$2\sqrt{2}\pi \left[\frac{2}{5} \right] + 2\sqrt{2}\pi \left[\frac{2}{5} (2^{3/2} - 1) \right]$$

$$\cancel{2\sqrt{2}}\pi \times \frac{2}{5} \times \cancel{2^{3/2}}\ 4$$

$\frac{16\pi}{5}$

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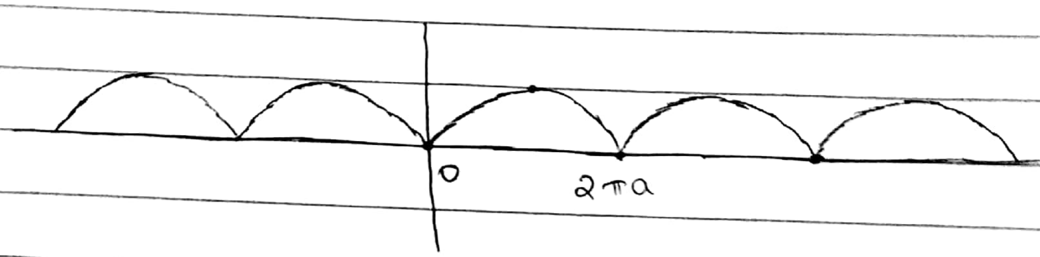
Q 13. $x = a(t - \sin t)$ $y = a(1 - \cos t)$ → Symmetrical about Y-AXIS.

Length of curve from $t=0$ to $t = 2\pi$



$$\int_0^{2\pi} a\sqrt{(1-\cos t)^2 + \sin^2 t}\; dt$$

$$a\int_0^{2\pi} \sqrt{2 - 2\cos t}\; dt$$

$$\sqrt{2}a \int_0^{2\pi} \sqrt{2}\sin\left(\frac{t}{2}\right) dt$$

$$-2a\left[2\cos(t/2)\right]_0^{2\pi}$$

$$-2a\left[2(-1-1)\right] = \underline{\underline{8a}}$$

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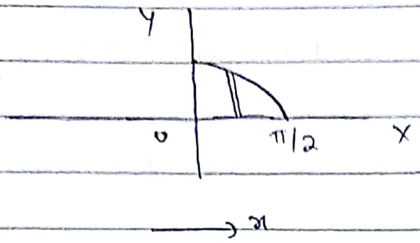
Q14.



~~Ote~~ volume of an element dx at ~~an~~ a distance x from the origin is

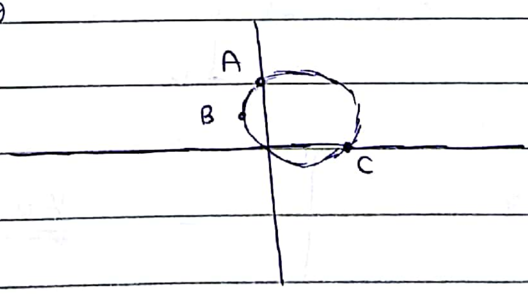
$$= \frac{\sqrt{3}}{4} (\cos^2 x)\, dx$$

$$\text{Net volume} = \int_0^{\pi/2} \frac{\sqrt{3}}{4} (\cos^2 x)\, dx$$

$$= \frac{\sqrt{3}}{4} \left[\frac{\sin 2x}{4} + \frac{x}{2} \right]_0^{\pi/2}$$

$$= \frac{\sqrt{3}\pi}{16}$$

Q15. $r = \cos\theta + \sin\theta$



$$\text{Length} = 4 \times AB + AC$$

$$= 4 \int_{\pi/2}^{3\pi/4} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta + \int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

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$$4\int_{\pi/2}^{3\pi/4} \sqrt{\cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta + \cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta}$$

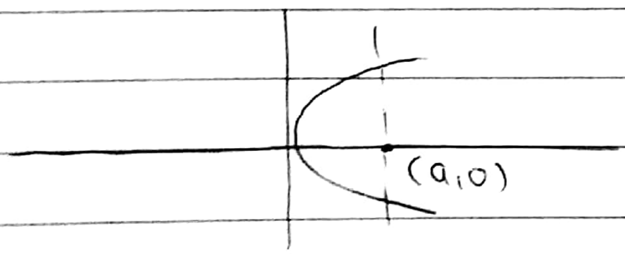
$$+ \int_{0}^{\pi/2} \sqrt{\cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta + \cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta}$$

$$= 4\int_{\pi/2}^{3\pi/4} \sqrt{2}\, d\theta + \int_{0}^{\pi/2} \sqrt{2}\, d\theta$$

$$= 4\sqrt{2} \times \frac{\pi}{4} + \sqrt{2} \times \frac{\pi}{2}$$

$$= \frac{3\pi}{\sqrt{2}}$$

16. $y^2 = 4ax$



$$V = \int_{0}^{a} \pi y^2\, dx$$

$$= \int_{0}^{a} \pi (4ax)\, dx$$

$$= \cancel{2a\pi}\left(\cancel{a^2 - 0}\right) = \cancel{2a^3\pi} \quad 2a\pi(a^2 - 0)$$