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DTU/2K17/A9/1119

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ASSIGNMENT-'I'
MATRICES

Ans-1 a) $\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix}$ $R_2 \to -2R_1 + R_2$
$R_3 \to 2R_1 + R_3$

$\begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 14 & 12 \end{bmatrix}$ $R_3 \to R_3 + 2R_2$

$\begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore$ Rank of matrice $= 2$

b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - 8R_1$

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{bmatrix}$ $R_3 \to R_3 + R_2$
$R_4 \to R_4 - 5R_2$

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\therefore$ Rank of matrice $= 2$

c) $\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 4 \\ 4 & -1 & 7 \\ 2 & 1 & 5 \end{bmatrix}$ $R_2 \to 3R_2 - R_1$
$R_3 \to 3R_3 - 4R_1$
$R_4 \to 3R_4 - 2R_1$

$$\begin{bmatrix} 3 & 1 & 7 \\ 0 & 5 & 5 \\ 0 & -7 & -7 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 \to 5R_3 + 7R_2$

$R_4 \to R_4 - \frac{R_2}{5}$

$$\begin{bmatrix} 3 & 1 & 7 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of matrice is 2

d) $$\begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & 6 & 9 & 13 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 15 & 6 & 7 \end{bmatrix}$$

$R_3 \to 2R_3 - 15R_2$

$$\begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & -48 & -76 \end{bmatrix}$$

$\therefore$ Rank of matrice is 3

Ans-2 a) $$\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

Consider the augmented matrix,

$$[A|I] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 + 3R_1$

$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] \quad \begin{array}{l} R_3 \to -R_3/5 \\ R_2 \to R_2 - 7R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -13/5 & -2/5 & +7/5 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right] \quad R_1 \to R_1 - 2R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -3/5 & -2/5 & 2/5 \\ 0 & 2 & 0 & -13/5 & -2/5 & 7/5 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2/2 \\ R_1 \to R_1 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/10 & \frac{-2}{10} & -3/10 \\ 0 & 1 & 0 & \frac{-13}{10} & \frac{-2}{10} & \frac{7}{10} \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right] = [A \mid I]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 7/10 & -2/10 & -3/10 \\ -13/10 & -2/10 & 7/10 \\ 4/5 & \frac{1}{5} & -1/5 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 7 & -2 & -3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}$$

b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Consider the augmented matrix

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + 2R_3 \\ R_1 \to R_1 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & 1 & -3 & 2 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to -R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right] = [A \mid I]$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

c) $\begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{bmatrix}$

Consider the augment matrix

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 1 & 5 & -1 & 0 & 1 & 0 \\ 3 & 13 & -6 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 4 & 6 & -3 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right]$$

As $R_3 = 0$, $\therefore$ inverse doesn't exists, Also $\boxed{|A|=0}$

d) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 4 & 4 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Consider the augmented matrix.

$$[A/I] = \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 4 & 4 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 4R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & -4 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \quad R_1 \to R_1 + R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & -4 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \leftrightarrow R_4 \\ R_2 \to -R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -3 & -4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \end{array}\right] \quad R_4 \to R_4 - R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -3 & -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & -3 & 1 & -1 & 0 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 + R_4 \\ R_4 \to R_4/3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & +1 & 1/3 & -1/3 & 0 \end{array}\right] \quad R_1 \to R_1 - R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & -1 & -1/3 & 1/3 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1/3 & -1/3 & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1 & -1/3 & 1/3 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 1/3 & -1/3 & 0 \end{bmatrix}$$

Ans-3 a) $2x + y - 2z = 10$; $3x + 2y + 2z = 1$; $5x + 4y + 3z = 4$

Consider the augmented matrix

$$\tilde{A} = \begin{bmatrix} 2 & 1 & -2 & 10 \\ 3 & 2 & 2 & 1 \\ 5 & 4 & 3 & 4 \end{bmatrix} \quad \begin{array}{l} R_2 \to 2R_2 - 3R_1 \\ R_3 \to 2R_3 - 5R_1 \end{array}$$

$$= \begin{bmatrix} 2 & 1 & -2 & 10 \\ 0 & 1 & 10 & -28 \\ 0 & 3 & 16 & -42 \end{bmatrix} \quad R_3 \to R_3 - 3R_2$$

$$= \begin{bmatrix} 2 & 1 & -2 & 10 \\ 0 & 1 & 10 & -28 \\ 0 & 0 & -14 & 42 \end{bmatrix} \quad \therefore \text{Rank of matrix } \tilde{A} \text{ is } 3$$

As rank is equal to number of unknowns the equations are consistent.

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$$2x + y - 2z = \cancel{1}10$$
$$y + 10z = -28$$
$$-14z = 42$$

Using the backward substitution, we obtain

$$z = -3, \quad y = \cancel{}2, \quad x = 1$$

b) $2x + y - 3z = 1$; $5x + 2y - 6z = 5$; $3x - y - 4z = 7$

Consider the augmented matrix

$$\tilde{A} = \begin{bmatrix} 2 & 1 & -3 & 1 \\ 5 & 2 & -6 & 5 \\ 3 & -1 & -4 & 7 \end{bmatrix} \quad \begin{matrix} R_2 \to 2R_2 - 5R_1 \\ R_3 \to 2R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 2 & 1 & -3 & 1 \\ 0 & -1 & 3 & 5 \\ 0 & -5 & 1 & 11 \end{bmatrix} \quad R_3 \to R_3 - 5R_2$$

$$\begin{bmatrix} 2 & 1 & -3 & 1 \\ 0 & -1 & 3 & 5 \\ 0 & 0 & -14 & -14 \end{bmatrix} \quad \therefore \text{Rank of matrice is } 3$$

As rank is equal to number of ~~equation~~ variables eq^{n}s are consistent.

$$2x + y - 3z = 1$$
$$-y + 3z = 5$$
$$-14z = -14$$

Using the backward substitution, we obtain

$$z = 1, \quad y = -2, \quad x = 3$$

c) ~~x-z~~ $x - y + 5z - \omega = 0$; $x + y - 2z + 3\omega = 0$

$3x - y + 8z + \omega = 0$; $x + 3y - 9z + 7\omega = 0$

The coefficient matrix is,

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$$A = \begin{bmatrix} 1 & -1 & 5 & -1 \\ 1 & 1 & -2 & 3 \\ 3 & -1 & 8 & 1 \\ 1 & 3 & -9 & 7 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 5 & -1 \\ 0 & 2 & -7 & 4 \\ 0 & 2 & -7 & 4 \\ 0 & 4 & -14 & 8 \end{bmatrix} \quad \begin{array}{l} R_4 \to R_4 - 2R_2 \\ R_3 \to R_3 - R_2 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 5 & -1 \\ 0 & 2 & -7 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of matrix is 2

As rank of matrix is less than number of variables, thus there are infinite number of solutions

$$x - y + 5z - \omega = 0$$
$$2y - 7z + 4\omega = 0$$

$$x = \alpha, \; y = \frac{-1}{3}(7\alpha + 13\beta), \; z = \frac{-2}{3}(\alpha + \beta), \; \omega = \beta$$

where α & β are two parameters

a) $x + hy = 2$; $4x + 8y = k$

Coefficient matrix (A) $= \begin{bmatrix} 1 & h \\ 4 & 8 \end{bmatrix}$ & augmented matrix $(\tilde{A}) = \begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix}$

for unique solⁿ, rank of A should be 2, and for that $|A| \neq 0$

$\Rightarrow 8 - 4h \neq 0$, $\boxed{h \neq 2}$

If $h=2$, then system will have no solution for those values of k for which matrices A & $\tilde{A}$ have different ranks. for $k \neq 8$ rank of ~~matrix~~ matrix $\tilde{A}$ is 2

Therefore, for unique solⁿ $\Rightarrow$ $h \neq 2$, k can have any value

no solⁿ $\Rightarrow$ $h=2$, $k \neq 8$

infine no solⁿ $\Rightarrow$ $h=2$, $k=8$

b) $x+3y=2$; $3x+hy=k$

Coefficient matrix (A) $= \begin{bmatrix} 1 & 3 \\ 3 & h \end{bmatrix}$ & augmented matrix ($\tilde{A}$) $= \begin{bmatrix} 1 & 3 & 2 \\ 3 & h & k \end{bmatrix}$

for unique solⁿ, rank of A should be 2, and for that $|A| \neq 0$

$\Rightarrow h - 9 \neq 0$, $\boxed{h \neq 9}$

If $h=9$, then system will have no solⁿ for those values of k for which matrices A & $\tilde{A}$ have different ranks. for $k \neq 6$ rank of matrix $\tilde{A}$ is 2

Therefore, for unique solⁿ $\Rightarrow$ $h \neq 9$, k can have any value

no solⁿ $\Rightarrow$ $h=9$, $k \neq 6$

infinite no solⁿ $\Rightarrow$ $h=9$, $k=6$

Ans-5 a) $x+2y=0$ $3x+6y=0$ $2x+ky=0$

The coefficient matrix is

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \\ 2 & k \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & k-4 \end{bmatrix} R_3 \leftrightarrow R_2 \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & k-4 \\ 0 & 0 \end{bmatrix}$$

If $k=4$;
rank of matrix = 1 i.e less than no. of variables
therefore eqn has infinite solutions

$$x+2y=0$$
$$x=-2y$$
$$x=-2t,\ y=t$$

where t is a parameter

else
rank of matrix = 2 i.e equal to no. of variables
therefor eqn has trivial solution

$$x+2y=0$$
$$(k-4)y=0$$

as $k\neq 0 \Rightarrow y=0$

$$x=0,\ y=0$$

b)
$$x+y+z=1$$
$$x+2y+4z=k$$
$$x+4y+10z=k^2$$

Consider the augmented matrix

$$\tilde{A}=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 2 & 4 & k\\ 1 & 4 & 10 & k^2\end{bmatrix} \quad \begin{matrix}R_2\to R_2-R_1\\ R_3\to R_3-R_1\end{matrix}$$

$$\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 3 & k-1\\ 0 & 3 & 9 & k^2-1\end{bmatrix} \quad R_3\to R_3-3R_2$$

$$\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 3 & k-1\\ 0 & 0 & 0 & k^2-3k+2\end{bmatrix}$$

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the coefficient matrix,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \qquad |A| = 0$$

Therefore, for the system to have solⁿ it's possible only if there are infinite no. of solⁿs.

for infinite solutions, $R_3 = 0$ of $\tilde{A}$.

$\Rightarrow k^2 - 3k + 2 = 0$

$k = 1$ or $k = 2$

If $k = 1$;

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x + y + z &= 1 \\ y + 3z &= 0 \end{aligned}$$

$x = 1 + 2\beta$, $y = -3\beta$, $z = \beta$

where β is a parameter

else if $k = 2$;

$$\tilde{A} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x + y + z &= 1 \\ y + 3z &= 1 \end{aligned}$$

$x = 2\beta$, $y = 1 - 3\beta$, $z = \beta$

where β is a parameter

Ans-6 a) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 5 \end{bmatrix}$ characteristic eqⁿ $|A - \lambda I| = 0$ gives

$$\begin{vmatrix} 1-\lambda & 3 & 0 \\ 0 & 1-\lambda & 0 \\ 2 & 1 & 5-\lambda \end{vmatrix} = \lambda^3 - 7\lambda^2 + 11\lambda - 5I = 0$$

$$= (\lambda - 5)(\lambda - 1)^2 = 0$$

$\Rightarrow \lambda = 5, \lambda = 1$

eigenvalues are $(1, 1, 5)$

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for $\lambda=1$, if x is eigenvector we have $(A-I)x=0$ i.e

$$\begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x_2=0$ $\quad$ $2x_1+x_2+4x_3=0$

$x_2=0$ $\quad$ $x_1+2x_3=0$

$\frac{x_1}{-2}=x_3$

eigenvector is $[-2,0,1]^T$

for $\lambda=5$, if x is eigenvector we have $(A-5I)x=0$ i.e

$$\begin{bmatrix} -4 & 3 & 0 \\ 0 & -4 & 0 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-4x_1+3x_2=0$

$-4x_2=0 \Rightarrow x_2=0$

$2x_1+x_2=0 \Rightarrow x_1=0$

eigenvector is $[0,0,1]^T$

b) $A=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad$ characteristic eqn given $|A-\lambda I|=0$

$\lambda^3-3\lambda^2+3\lambda-I$

$(\lambda-I)^3=0 \Rightarrow \lambda=1,1,1$

eigenvalues for A are $\lambda=1,1,1$

for $\lambda=1$, if x is eigenvector we have $(A-I)x=0$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2=0$, $x_3=0$

eigenvector is $[1,0,0]^T$

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c) $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ characteristic eqⁿ given $|A - \lambda I| = 0$

$$\lambda^3 - 3\lambda^2 + \lambda + 3I = 0$$

$$\lambda = -2$$

eigenvalues of A is -2

for $\lambda = -2$, if x is eigenvector we have $(A + 2I)x = 0$

$$\begin{bmatrix} 3 & 2 & 0 \\ -1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x_1 + 2x_2 = 0$
$-x_1 + 3x_2 + 2x_3 = 0$
$x_1 + 2x_2 + 3x_3 = 0$

On solving eqⁿs we get,

$$\frac{x_1}{2} = \frac{-x_2}{3} = \frac{x_3}{3}$$

Therefore the eigenvector is $[2, -3, 3]^T$

d) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ character eqⁿ given $|A - \lambda I| = 0$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

eigenvalues of A are $1, 2, 3$

for $\lambda = 1$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_2 + x_3 = 0$, $x_1 + x_3 = 0$

$x_1 = x_2 = -x_3$

Therefore the eigenvector is $[1, 1, -1]^T$

for $\lambda = 2$, $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = 0$, $x_3 = 0$, $2x_1 + x_3 = 0$

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Therefore the eigenvector is $[0,1,0]^T$

for $\lambda = 3$, $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = 0 \qquad -x_2 + x_3 = 0 \Rightarrow \boxed{x_2 = x_3}$

Therefore the eigenvector is $[0,1,1]^T$

e) $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ characteristic eqn of $|A - \lambda I| = 0$

$\lambda^2 - 3\lambda + (-10) = 0$

$\lambda = 5, \lambda = -2$

eigenvalues of A are $-2, 5$

for $\lambda = -2$, $\begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 + x_2 = 0$

$x_1 = -x_2$

Therefore the eigenvector is $[1,-1]^T$

for $\lambda = 5$, $\begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 - x_2 = 0$

$x_1 = x_2$

Therefore the eigenvector is $[1,1]^T$

Ans7 a) $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ The characteristic eqn of A is

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$(\lambda-1)(\lambda-2)(\lambda-3) = 0$

eigenvalues of A are $\lambda = 1, 2, 3$

for $\lambda=1$, $\begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$2x_1 + x_2 - x_3 = 0$$
$$x_1 = x_3$$
$$x_2 = -x_3$$
$$x_1 = -x_2 = x_3$$

therefore eigenvector is $[1, -1, 1]^T$

for $\lambda = 2$, $\begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$x_1 + x_2 - x_3 = 0$$
$$-2x_1 - x_2 + 2x_3 = 0$$
$$x_2 = 0$$

therefore eigenvector is $[1, 0, 1]^T$

for $\lambda = 3$, $\begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$x_2 = x_3$$
$$x_1 + x_2 - x_3 = 0 \Rightarrow x_1 = 0$$

therefore eigenvector is $[0, 1, 1]^T$

Since A has three linearly independent eigenvectors, thus it is diagnolizable

Modal matrix P of A is,

$$P = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad P^{-1} = \frac{1}{|P|} \operatorname{adj} P = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Special matrix D is $= P^{-1}AP = \begin{bmatrix} -1 & -1 & +1 \\ +2 & +1 & -1 \\ -1 & 0 & +1 \end{bmatrix}\begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$D = \begin{bmatrix} +1 & 0 & 0 \\ 0 & +2 & 0 \\ 0 & 0 & +3 \end{bmatrix}$$

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b) $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ characteristic eqn of A is

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$(\lambda-1)(\lambda-2)^2 = 0$

eigenvalues of A are $\lambda = 1, 2, 2$

for $\lambda = 1$, $\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_2 = -x_3$

$-x_1 + 2x_2 + x_3 = 0$

$x_1 = -x_3$

$x_1 = x_2 = -x_3$, thus eigenvector is $[1, 1, -1]^T$

for $\lambda = 2$, $\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_3 = 0$, $-x_1 + 2x_2 = 0$,

eigenvector is $[2, 1, 0]^T$

Modal matrix P of matrix A is, $P = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$|P| = 0$, thus matrix is not diagnozable

c) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ characteristic eqn of A is

$\lambda^3 - \lambda^2 + 9\lambda - 45 = 0$

$(\lambda-3)(\lambda^2 + 2\lambda + 15) = 0$

eigenvalues of A are $\lambda = 3$

for $\lambda = 3$, $\begin{bmatrix} -5 & 2 & -3 \\ 2 & -2 & -6 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-5x_1 + 2x_2 - 3x_3 = 0$; $2x_1 - 2x_2 - 6x_3$; $-x_1 - 2x_2 - 3x_3 =$

thus eigenvector is $[1, -2, -3]^T$

As there is only one eigenvector, ∴ matrix is not diagonazible

Ans8 Cayley-Hamilton theorem :- Every square matrix Satisfies its own characteristic equation.

Proof:- Let A be a square matrix of order $n \times n$, then its characteristic eqn is

$$|A - \lambda I| = 0 \quad \text{or,} \quad (-1)^n \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n = 0$$

Let P be the adjoint of the matrix $A - \lambda I$, then elements of P will be the polynomials of degree $(n-1)$ in λ. Thus matrix P is, $P = P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + \cdots P_{n-1}\lambda + P_n$

where P_i's are square matrices of order n, whose elements are functions of the elements of A.

As we know that, $[A - \lambda I] \, adj \cdot (A - \lambda I) = |A - \lambda I| \, I$,

therefore, $[A - \lambda I][P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + \cdots + P_n] = [(-1)^n \lambda^n + c_1 \lambda^{n-1} + \cdots c_n]$

Equating the coefficients of various powers,

$$-P_1 = (-1)^n I$$

$$AP_1 - P_2 = c_1 I$$

$$AP_2 - P_3 = c_2 I$$

$$\cdots$$

$$AP_{n-1} - P_n = c_{n-1} I$$

$$AP_n = c_n I$$

pre-multiplying these equations from top to bottom, resp. by $A^n, A^{n-1}, \ldots, A, I$ and adding the equations we get,

$$O = (-1)^n A^n + c_1 A^{n-1} + c_2 A^{n-2} + \cdots + c_{n-1} A + c_n$$

This eqn can be used to obtain A^n in terms of lower powers of n as,

$$A^n = (-1)^{n+1} [c_1 A^{n-1} + c_2 A^{n-2} + \cdots c_{n-1} A + c_n I]$$

a) $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$, the characteristic eqn $|A-\lambda I| = 0$ gives

$$\lambda^3 - 9\lambda^2 + (-9\lambda) + 81 = 0$$

To verify cayley-hamilton theorem we are to verify

$$A^3 - 9A^2 - 9A + 81 = 0 \quad -①$$

$$A^2 = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 17 & -16 & -16 \\ -16 & 41 & 32 \\ -16 & 32 & 41 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 17 & -16 & -16 \\ -16 & 41 & 32 \\ -16 & 32 & 41 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 81 & -144 & -180 \\ -144 & 333 & 324 \\ -180 & 324 & 315 \end{bmatrix}$$

putting in eqn ①,

$$\Rightarrow \begin{bmatrix} 81 & -144 & 80 \\ -144 & 333 & 324 \\ -180 & 324 & 315 \end{bmatrix} - \begin{bmatrix} 17\times9 & -16\times9 & -16\times9 \\ -16\times9 & 41\times9 & 32\times9 \\ -16\times9 & 32\times9 & 41\times9 \end{bmatrix} - \begin{bmatrix} 9 & 0 & -36 \\ 0 & 45 & 36 \\ -36 & 36 & 27 \end{bmatrix} + 81I$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$, Hence verified

b) $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$, the characteristic eqn of A is,

$$\lambda^3 - 8\lambda^2 + 20\lambda - 16 = 0$$

To verify the theorem,

$$A^3 - 8A^2 + 20A - 16 = 0 \quad -①$$ must be true

$$A^2 = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 12 & 6 \\ 0 & 4 & 0 \\ 6 & 12 & 10 \end{bmatrix}$$

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$$A^3 = \begin{bmatrix} 10 & 12 & 6 \\ 0 & 4 & 0 \\ 6 & 12 & 10 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 36 & 56 & 28 \\ 0 & 8 & 0 \\ 28 & 56 & 36 \end{bmatrix}$$

putting in eqⁿ ① we have,

$$\Rightarrow \begin{bmatrix} 36 & 56 & 28 \\ 0 & 8 & 0 \\ 28 & 56 & 36 \end{bmatrix} - \begin{bmatrix} 80 & 96 & 78 \\ 0 & 32 & 0 \\ 48 & 96 & 80 \end{bmatrix} + \begin{bmatrix} 60 & 40 & 20 \\ 0 & 40 & 0 \\ 20 & 40 & 60 \end{bmatrix} - \begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$, Hence verified

Ans 9 a) $3x+y+2z=0;\ x-2y+3z=0;\ x+5y-4z=0$

Coefficient matrix $= A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -2 & 3 \\ 1 & 5 & -4 \end{bmatrix}$

$|A| = 3(-7) - (-7) + 2(7) = 0$

Thus system has non-trivial solution as $|A| = 0$

on solving the eqⁿs,

$-x = y = z$

$x = -\alpha,\ y = \alpha,\ z = \alpha$ where α is a parameter

b) $a+2b+3c+4d=0;\ a+b+c+d=0;\ a+2b+6c+12d=0$

Coefficient matrix $= A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 6 & 12 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

that gives, $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 3 & 8 \end{bmatrix}$ $|A| =$ i.e not defined

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~~x + 2y + 3z +~~ $a + 2b + 3c + 4d = 0$

$-b - 2c - 3d = 0$

$3c + 8d = 0$

Solving the eqn's we get,

$a = \frac{-2d}{3}, \quad b = \frac{7d}{3}, \quad c = \frac{-8d}{3}$

$a = \frac{-2\alpha}{3}, \quad b = \frac{7\alpha}{3}, \quad c = \frac{-8\alpha}{3}, \quad d = \alpha$ where α is a parameter

Ans-10 a) $3x + 3y + 7z = 4; \quad 3x + 26y + 2z = 9; \quad 7x + 2y + 10z = 5$

Consider the augmented matrix, $\tilde{A} = \begin{bmatrix} 3 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to \frac{3R_3 - 7R_1}{3}$

$\tilde{A} = \begin{bmatrix} 3 & 3 & 7 & 4 \\ 0 & 23 & -5 & 5 \\ 0 & -15 & -19 & -13 \end{bmatrix}$ $R_3 \to 5R_3 - 19R_2$

$\tilde{A} = \begin{bmatrix} 3 & 3 & 7 & 4 \\ 0 & 23 & -5 & 5 \\ 0 & -512 & 0 & -160 \end{bmatrix}$ Therefore rank of $\tilde{A} = 3$ hence eqn's are consistent & have unique soln

~~$3x + 3y + 7z = 4$~~

~~$23y - 5z = 5$~~

~~$-512y = -160$~~

~~on backward substitution we get,~~

~~$y = \frac{...}{1632}$, $z =$~~

$3x + 3y + 7z = 4$

$23y - 5z = 5$

$-512y = -160$

on backward substitution we get,

$x = 0, \quad y = \frac{5}{16}, \quad z = \frac{7}{16}$

Alwayz ahead

b) $-x+2y+2z=2;\ 3x-y+z=6;\ -x+3y+4z=4$

Consider the augmented matrix,

$$\tilde{A} = \begin{bmatrix} -1 & 2 & 2 & 2 \\ 3 & -1 & 1 & 6 \\ -1 & 3 & 4 & 4 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\tilde{A} = \begin{bmatrix} -1 & 2 & 2 & 2 \\ 0 & 5 & 7 & 12 \\ 0 & 1 & 2 & 2 \end{bmatrix} \quad R_3 \to 5R_3 - R_2$$

$$\cancel{\tilde{A} = \begin{bmatrix} -1 & 2 & 2 & 2 \\ 0 & 5 & 7 & 12 \\ 1 & -1 & 0 & 0 \end{bmatrix}} \qquad \tilde{A} = \begin{bmatrix} -1 & 2 & 2 & 2 \\ 0 & 5 & 7 & 12 \\ 0 & 0 & 3 & -2 \end{bmatrix}$$

$$-x+2y+2z=2$$
$$5y+7z=12$$
$$3z=-2$$

on backward substitution we get,

$$x = \frac{10}{3},\ y = \frac{10}{3},\ z = \frac{-2}{3}$$

Ans-11 If A & B are invertible matrices of same order, then $(AB)^{-1} = B^{-1}A^{-1}$

Proof- $(AB)(AB)^{-1} = I$

Pre multiplying by A^{-1}

$\Rightarrow A^{-1}(AB)(AB)^{-1} = A^{-1}I$

$\Rightarrow I\,B(AB)^{-1} = A^{-1}$

Pre multiplying by B^{-1}

$\Rightarrow B^{-1}B(AB)^{-1} = B^{-1}A^{-1}$

$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$

Hence proved.

Ans12 If λ is an eigenvalue of matrix A, then eigenvalue of A^{-1} is λ^{-1}.

Proof:- Let x be the eigenvector of A corresponding to the eigenvalue λ, then $Ax = \lambda x$,

$$(A-\lambda)x = 0$$

$$\Rightarrow Ax = \lambda x$$

Pre multiplying by A^{-1},

$$\Rightarrow A^{-1}Ax = A^{-1}\lambda x$$

$$\Rightarrow Ix = \lambda(A^{-1}x)$$

$$\Rightarrow \frac{x}{\lambda} = A^{-1}x$$

Thus $\frac{1}{\lambda} = \lambda^{-1}$ is an eigenvalue of A^{-1}

An13 The matrices A and A^T has the same eigenvalues.

λ is an eigenvalue of A if

$$|A-\lambda I| = 0$$

Since $|A| = |A^T|$

$$|A-\lambda I| = |(A-\lambda I)^T| = 0$$

$$|A-\lambda I| = |A^T - \lambda I|$$

So, λ is an eigenvalue of A^T.

Ans-14 The modal and spectral matrices of A are, respectively

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \quad \& \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|}\,\mathrm{adj}(P) = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

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$$A = PDP^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A = \frac{1}{2}\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A = \frac{1}{2}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$