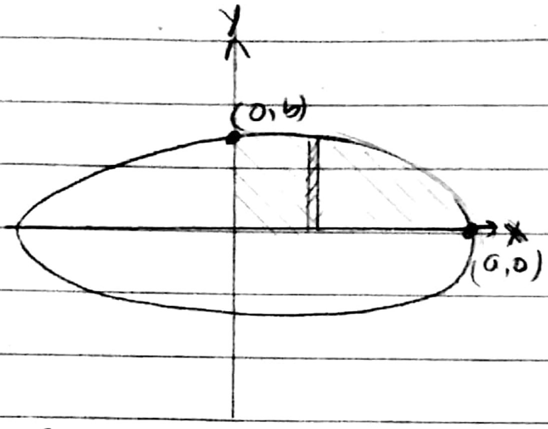
BATCH - A2
ROLL NO - DTU/2K17/A2/166

ASSIGNMENT

Q1 Evaluate the integral $\iint (x+y)\,dx\,dy$ over the region R in the positive quadrant bounded by ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Sol



$$I = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} (x+y)\,dx\,dy$$

$$I = \int_0^a \left[xy + \frac{y^2}{2}\right]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$I = \int_0^a \left\{(x)\left(\frac{b}{a}\right)\sqrt{a^2-x^2} + \frac{b^2}{2a^2}(a^2-x^2)\right\} dx$$

$$I = \frac{b}{a}\int_0^a x\sqrt{a^2-x^2}\,dx + \frac{b^2}{2a^2}\int_0^a (a^2-x^2)\,dx$$

Let $\int_0^a x\sqrt{a^2-x^2}\,dx = I_1$

and, $\int_0^a (a^2-x^2)\,dx = I_2$

So, $I = \left(\frac{b}{a}\right)(I_1) + \left(\frac{b^2}{2a^2}\right)(I_2)$ —— ①

$I_1 = \int_0^a x\sqrt{a^2-x^2}\,dx$

Let $a^2-x^2 = t$

$-2x\,dx = dt$

$I_1 = \int_{a^2}^0 \sqrt{t}\left(\frac{dt}{-2}\right)$

$I_1 = \left[\frac{2}{2}\left(\frac{t^{3/2}}{3}\right)\right]_0^{a^2}$

$I_1 = \frac{a^3}{3}$

$I_2 = \int_0^a (a^2-x^2)\,dx$

$= \left[a^2x - \frac{x^3}{3}\right]_0^a$

$I_2 = a^3 - \frac{a^3}{3}$

$I_2 = \frac{2a^3}{3}$

Substituting the value of I_1 and I_2 in ①

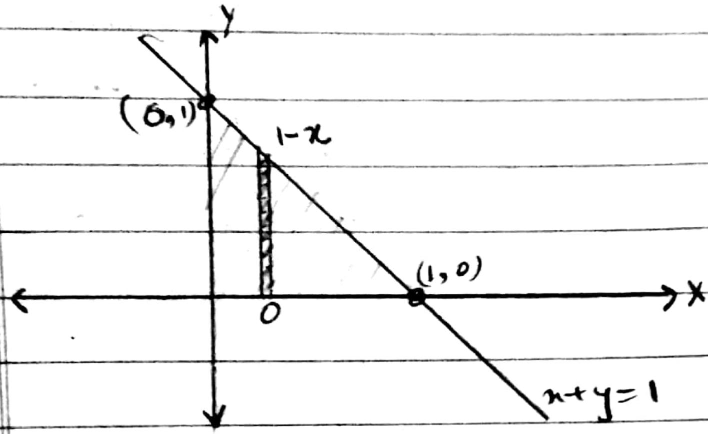
$I = \left(\frac{b}{a}\right)\left(\frac{a^3}{3}\right) + \left(\frac{b^2}{2a^2}\right)\left(\frac{2a^3}{3}\right)$

$= \frac{ba^2}{3} + \frac{b^2a}{3}$

$= \frac{ab(a+b)}{3}$

Q2 Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ in the positive quadrant for which $x+y \le 1$

Sol



$$I = \int_0^1 \int_0^{1-x} (x^2+y^2)\,dx\,dy$$

$$I = \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_0^{1-x} dx$$

$$I = \int_0^1 \left\{x^2(1-x) + \frac{(1-x)^3}{3}\right\} dx$$

$$I = \int_0^1 \left[\frac{3x^2-3x^3+1-x^3-3x+3x^2}{3}\right] dx$$

$$I = \frac{1}{3}\int_0^1 -4x^3+6x^2-3x+1\,dx$$

$$I = \frac{1}{3}\left[-x^4+2x^3-\frac{3x^2}{2}+x\right]_0^1$$

$$I = \frac{1}{3}\left[-1+2-\frac{3}{2}+1\right]$$

$$I = \frac{1}{6}$$

Q3 Evaluate $\int_0^1 \int_{\sqrt{y}}^{2-y} x^2 \, dx \, dy$.

Sol.

$$I = \int_0^1 \int_{\sqrt{y}}^{2-y} x^2 \, dx \, dy$$

$$I = \int_0^1 \left[\frac{x^3}{3}\right]_{\sqrt{y}}^{2-y} dy$$

$$I = \int_0^1 \frac{(2-y)^3 - (\sqrt{y})^3}{3} \, dy$$

$$I = \left[2y - \frac{y^2}{2} - \frac{2y^{3/2}}{3}\right]_0^1$$

$$I = 2 - \frac{1}{2} - \frac{2}{3}$$

$$I =$$

$$3I = \int_0^1 8 - y^3 - 12y + 6y^2 - y^{3/2} \, dy$$

$$3I = \left[8y - \frac{y^4}{4} - \frac{12y^2}{2} + \frac{6y^3}{3} - \frac{2y^{5/2}}{5}\right]_0^1$$

$$3I = 8 - \frac{1}{4} - 6 + 2 - \frac{2}{5}$$

$$I = \frac{67}{60}$$

Q4 Evaluate $\displaystyle\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\,dr\,d\theta$

sol

$$I = \int_0^{\frac{\pi}{4}}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\,dr\,d\theta$$

Let $1+r^2 = t$

$r\,dr = \frac{dt}{2}$

$$I = \int_0^{\frac{\pi}{4}}\int_1^{1+\cos 2\theta} t^{-2}\left(\frac{dt}{2}\right) d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \left[\left(-\frac{1}{2}\right)\left(\frac{1}{t}\right)\right]_1^{1+\cos 2\theta} d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \left(-\frac{1}{2}\right)\left(\frac{1}{1+\cos 2\theta} - 1\right) d\theta$$

$$I = \int_{\pi/4}^{0} \frac{-\cos 2\theta}{2(1+\cos 2\theta)}\,d\theta$$

$$2I = \int_0^{\pi/4} \left(1 - \frac{1}{1+\cos 2\theta}\right) d\theta$$

Now, $\cos 2\theta = 2\cos^2\theta - 1$

$$2I = \int_0^{\pi/4} \left(1 - \frac{\sec^2\theta}{2}\right) d\theta$$

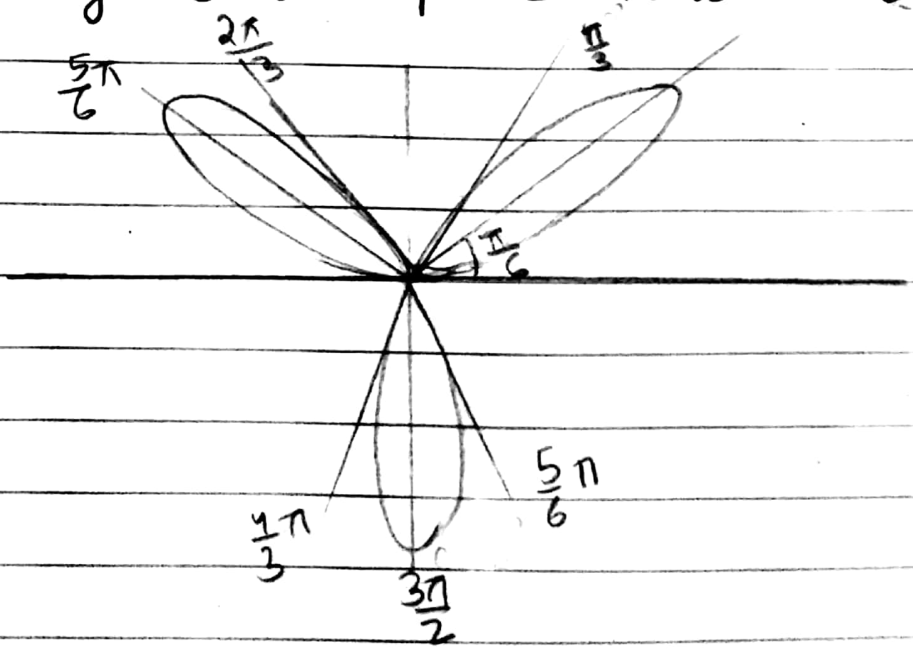
$$2I = \left[\theta - \frac{\tan\theta}{2}\right]_0^{\pi/4} \qquad \int \sec^2 x \, dx = \tan x + c$$

$$I = \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

$$I = \frac{\pi - 2}{8}$$

Q5 Use polar double integral to find the area enclosed by the three petaled rose $r = \sin 3\theta$

Sol



Sol. $r = \sin 3\theta$

Area of three petaled rose = 3 × Area of one petaled rose

Area of one petal is:

$$I = \int_0^{\frac{\pi}{3}} \int_0^{\sin 3\theta} r \, dr \, d\theta$$

$$I = \int_0^{\frac{\pi}{3}} \frac{\sin^2 3\theta}{2} \, d\theta$$

$$I = \frac{1}{2} \int_0^{\frac{\pi}{3}} \left(\frac{1}{2} - \frac{\cos 6\theta}{2} \right) d\theta$$

$$I = \frac{1}{2} \left[\frac{\theta}{2} - \frac{\sin 6\theta}{12} \right]_0^{\frac{\pi}{3}}$$

$$I = \left(\frac{1}{2}\right)\left(\frac{\pi}{3}\right)\left(\frac{1}{2}\right)$$

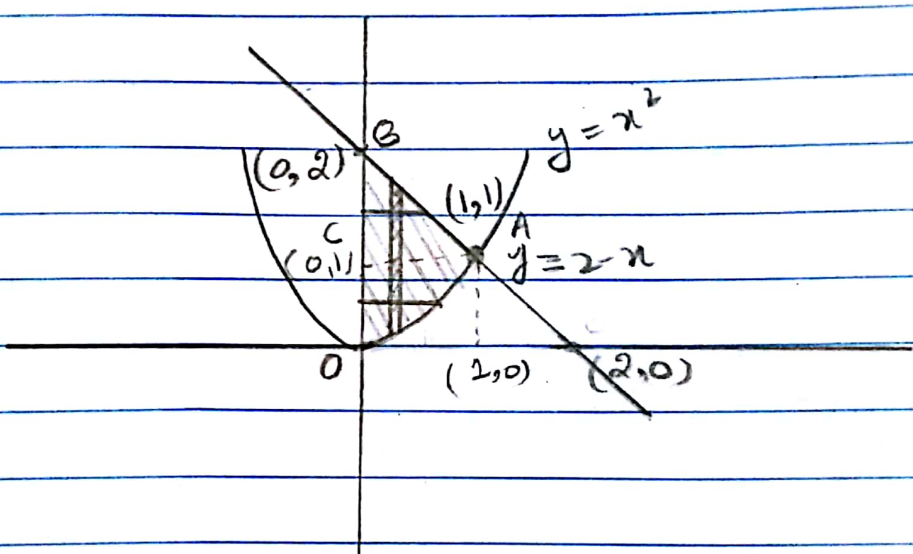
Area of three petaled rose $= 3(I)$

$$= \frac{\pi}{4}$$

(Q6) Change the order of integration:

$$I = \int_0^1 \int_{x^2}^{2-x} f(x,y)\,dy\,dx$$

Sol In the given integral, x varies from $x=0$ to $x=1$ and y varies from $y=x^2$ to $y=2-x$



The given integral represents area OABCO which can be split up into the sum of area OACO and area ABCA

$$ar(OABCO) = ar(OACO) + ar(ABCA)$$

Now, $ar(OABCO) = I$

$$ar(OACO) = \int_0^1 \int_0^{\sqrt{y}} f(x,y)\,dx\,dy \quad \ldots \text{(as } x = 0 \text{ to } x = \sqrt{y})$$

and $ar(ABCA) = \int_0^1 \int_0^{2-y} f(x,y)\, dy\, dx$ ... (as $x = 0$ to $2-2y$)

Hence,

$$I = \int_0^1 \int_0^{\sqrt{y}} f(x,y)\, dx\, dy + \int_0^1 \int_0^{2-y} f(x,y)\, dx\, dy$$

(Q) Change the order of integration of $\int_0^\infty \int_0^\infty e^{-xy} \sin(nx)\, dx\, dy$. Show that $\int_0^\infty \frac{\sin(nx)}{x}\, dx = \frac{\pi}{2}$

Sol $\int_0^\infty \int_0^\infty e^{-xy} \sin nx\, dx\, dy = \int_0^\infty dy \int_0^\infty e^{-xy} \sin nx\, dx$

$$= \int_0^\infty dy \left[\frac{e^{-xy}}{n^2+y^2} \{ -y \sin nx - n\cos nx \} \right]_0^\infty \ldots \left[\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C \right]$$

$$= \int_0^\infty dy \left[0 + \frac{n}{n^2+y^2} \right] = \int_0^\infty \frac{n}{n^2+y^2}\, dy = \left[\tan^{-1} \frac{y}{n} \right]_0^\infty = \frac{\pi}{2} \quad \text{--- (1)}$$

On changing the order of integration

$$\int_0^\infty \int_0^\infty e^{-xy} \sin nx\, dx\, dy = \int_0^\infty \sin nx\, dx \int_0^\infty e^{-xy}\, dy$$

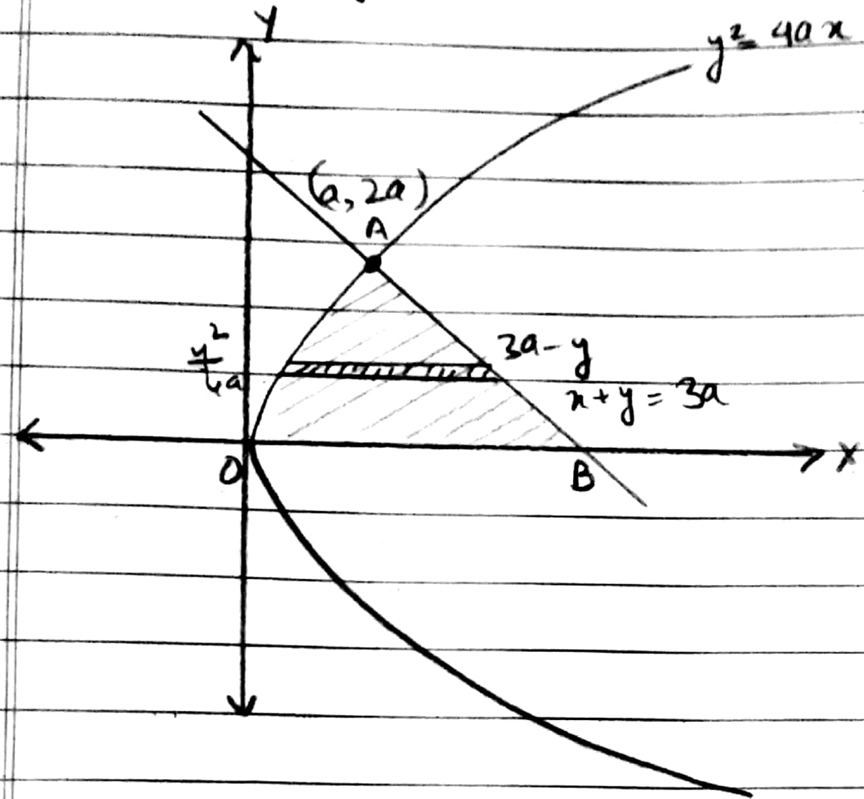
$$= \int_0^\infty \sin nx\, dx \left[\frac{e^{-xy}}{-x} \right]_0^\infty = \int_0^\infty \frac{\sin nx\, dx}{x} \left[-\frac{1}{e^{xy}} \right]_0^\infty$$

$$= \int_0^\infty \frac{\sin nx}{x}\, dx \quad \text{--- (2)}$$

From (1) and (2), we have $\int_0^\infty \frac{\sin(nx)}{x}\, dx = \frac{\pi}{2}$

(8) Find the area bounded by the parabola $y^2 = 4ax$ and the line $x + y = 2a$ in the positive quadrant

Sol



We have to find the area OABO.

Let us consider a strip parallel to x axis from curve $x = \frac{y^2}{4a}$ to $x = 3a - y$.

So the area is,

$$I = \int_0^{2a} \int_{y^2/4a}^{3a-y} dx\, dy$$

$$I = \int_0^{2a} \left(3a - y - \frac{y^2}{4a}\right) dy$$

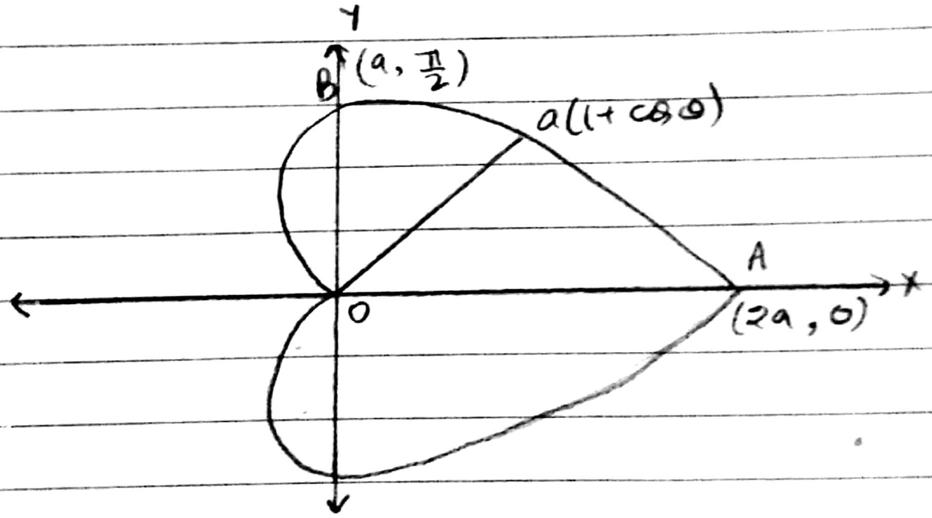
$$I = \left[3ay - \frac{y^2}{2} - \frac{y^3}{12a}\right]_0^{2a}$$

$$I = 6a^2 - 2a^2 - \frac{2a^2}{3}$$

$$I = \frac{10a^2}{3} \text{ sq units}$$

(Q) Find the area of the cardiod $r = a(1+\cos\theta)$ by double integration.

Sol



Area of cardiod = 2 × (area of OABO)

Now, area of OABO is $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$. Let this integral be I

$$I = \int_0^{\pi} \left[\frac{r^2}{2} \right]_0^{a(1+\cos\theta)} d\theta$$

$$I = \int_0^{\pi} \frac{(a(1+\cos\theta))^2}{2}\, d\theta$$

$$I = \frac{a^2}{2}\int_0^{\pi}(1+\cos^2\theta + 2\cos\theta)\, d\theta$$

$$I = \frac{a^2}{2}\int_0^{\pi}\left(\frac{3}{2} - \frac{\cos 2\theta}{2} + 2\cos\theta\right) d\theta$$

$$I = \left(\frac{a^2}{2}\right)\left[\frac{3}{2}\theta - \frac{\sin 2\theta}{4} + 2\sin\theta\right]_0^{\pi}$$

$$I = \left(\frac{a^2}{2}\right)\left[\frac{3}{2}\pi\right]$$

$$\cdot\, I = \frac{3a\pi^2}{4}$$

Hence the area OABO is $\frac{3a\pi^2}{4}$ sq units

Now, area of cardiod is $2\times\left(\frac{3a\pi^2}{4}\right) = \frac{3a\pi^2}{2}$ sq. units

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(Q)10 A circular hole of radius b is made centrally through a sphere of radius a. find the volume of the remaining sphere

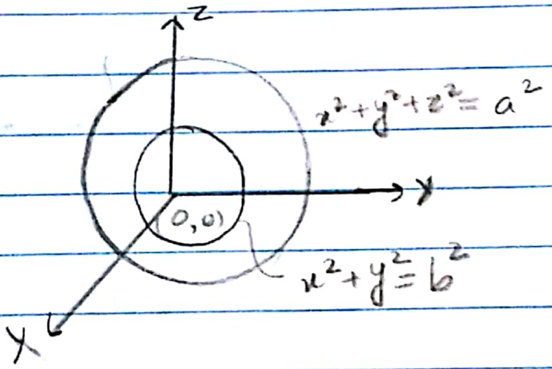
Sol Let the center of the sphere be taken as the origin and axis of the hole be taken as the z-axis.

The volume of the upper half of the hole is $\iint_R z\,dx\,dy$, where

$z = \sqrt{a^2-x^2-y^2}$ and R is the orthogonal projection of the surface for the hollow portion $z=\sqrt{a^2-x^2-y^2}$ in the xy-plane, i.e., $R: x^2+y^2=b^2$.

Hence the volume V_1 of the circular hole is

$$V_1 = 2\iint_{x^2+y^2=b^2} \sqrt{a^2-x^2-y^2}\,dx\,dy$$



Using the polar coordinates, we obtain.

$$V_1 = 2\int_0^{2\pi}\int_0^b \sqrt{a^2-r^2}\,r\,dr\,d\theta = 4\pi\int_0^b \sqrt{a^2-r^2}\,r\,dr = 4\pi\left[-\frac{1}{3}(a^2-r^2)^{3/2}\right]_0^b$$

$$= \frac{4\pi}{3}\left[a^3-(a^2-b^2)^{3/2}\right]$$

Since the volume of the sphere is $V_2 = \frac{4\pi a^3}{3}$, hence the volume V of the remaining portion is

$$V = V_2 - V_1 = \frac{4}{3}\pi a^3 - \frac{4}{3}\pi\left[a^3-(a^2-b^2)^{3/2}\right] = \frac{4}{3}\pi(a^2-b^2)^{3/2}$$

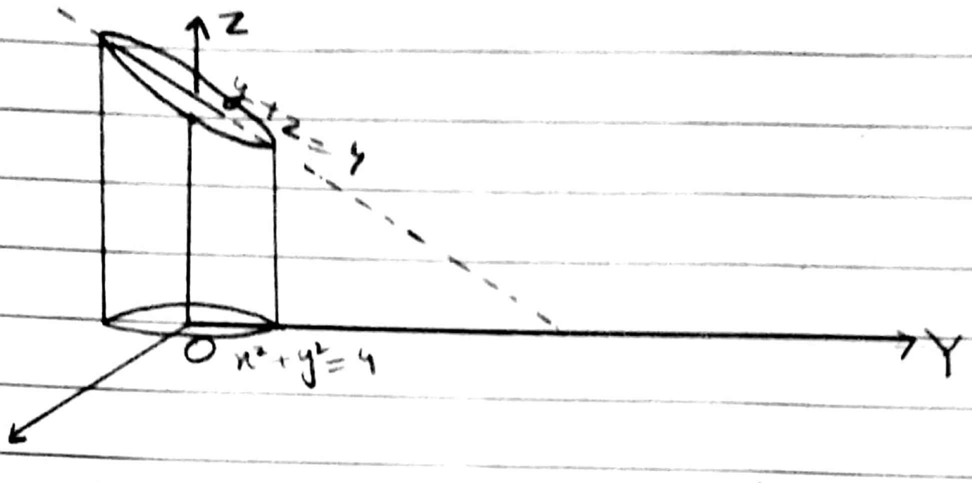
Q11 Find the volume bounded by cylinder $x^2+y^2=4$ and the plane $y+z=4$ and $z=0$

Sol

$$x^2+y^2=4 \quad \text{——(1)}$$

$$y+z=4 \quad \text{——(2)}$$

$$z=0 \quad \text{——(3)}$$



(i) z varies from 0 to $4-y$.

(ii) y varies from $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$

(iii) x varies from -2 to $+2$.

∴ Required volume $= \displaystyle\int_{-2}^{+2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-y} dx\,dy\,dz$

$$= \int_{-2}^{+2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\,[z]_0^{4-y}$$

$$= \int_{-2}^{+2} dx\left[4y-\frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} = \int_{-2}^{+2}\left[4\sqrt{4-x^2}-\frac{(4-x^2)}{2}+4\sqrt{4-x^2}+\frac{4-x^2}{2}\right]dx$$

$$= 8\int_{-2}^{+2}\sqrt{4-x^2}\,dx = 8\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{+2} = 8(2)\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = 16\pi$$

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Q12 Evaluate $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}}\, dx\, dy\, dz$

Sol As the limits of first integration is in terms of x and y, it represents limits of z, i.e, z varies from

$$z=0 \quad \text{to} \quad z=\sqrt{a^2-x^2-y^2}$$

Similarly for second integration, limits are of y so y varies from

$$y=0 \quad \text{to} \quad y=\sqrt{a^2-x^2}$$

and x varies from $x=0$ to $x=a$

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}}\, dz\, dy\, dx$$

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{a^2-x^2-y^2}}\right)\right]_0^{\sqrt{a^2-x^2-y^2}} dy\, dx$$

$$I = \int_0^a \left[\frac{\pi}{2} y\right]_0^{\sqrt{a^2-x^2}} dx \Rightarrow I = \int_0^a \frac{\pi}{2}\sqrt{a^2-x^2}\, dx$$

$$I = \left[\frac{\pi}{2}\left\{\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right\}\right]_0^a \Rightarrow \boxed{I = \frac{\pi^2 a^2}{8}}$$

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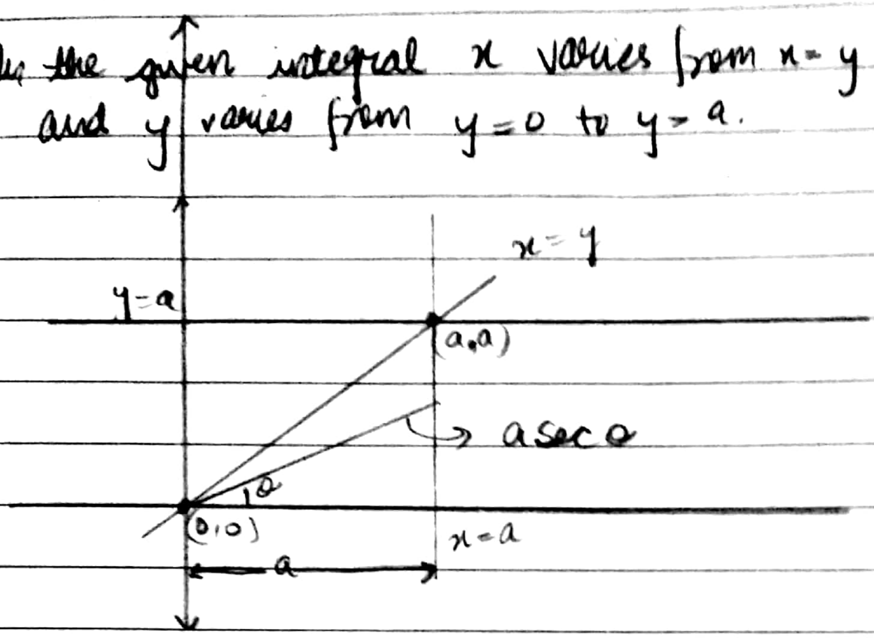
Q13 By changing into polar coordinates, evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$.

Sol In the given integral x varies from $x=y$ to $x=a$ and y varies from $y=0$ to $y=a$.



Converting the given function in polar coordinates.
Substituting $x = r\cos\theta$ and $y = r\sin\theta$

$$I = \int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$$

$$I = \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} \frac{r\cos\theta}{(r\cos\theta)^2+(r\sin\theta)^2}\, r\, dr\, d\theta \quad \ldots (dx\,dy = r\,dr\,d\theta)$$

$$I = \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} \cos\theta\, dr\, d\theta$$

$$I = \int_0^{\pi/4} \Big[r \Big]_0^{a\sec\theta} \cos\theta \, d\theta$$

$$I = \int_0^{\pi/4} a\sec\theta\cos\theta \, d\theta$$

$$I = \frac{a\pi}{4}$$

Q14 Evaluate the following by changing into polar coordinates

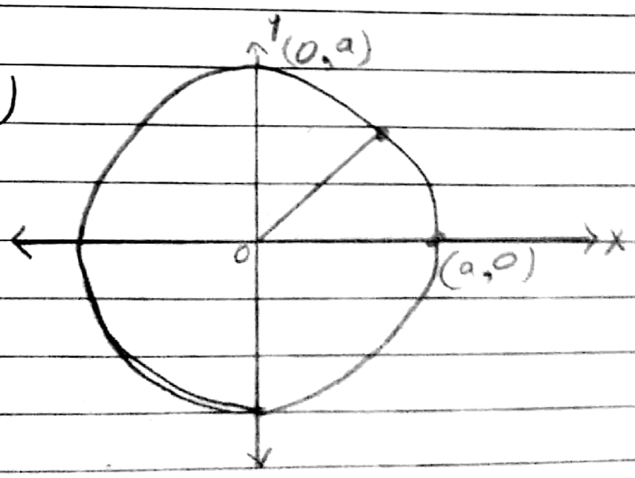
$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\, dx\, dy$$

Sol In the given integral x varies from $x=0$ to $x=\sqrt{a^2-y^2}$ and y varies from $y=0$ to $y=a$.

$x=0$ to $x=\sqrt{a^2-y^2}$ $(x>0)$

$\Rightarrow x^2 = a^2-y^2$

$\Rightarrow x^2+y^2 = a^2$



Converting in polar coordinates
Substituting $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

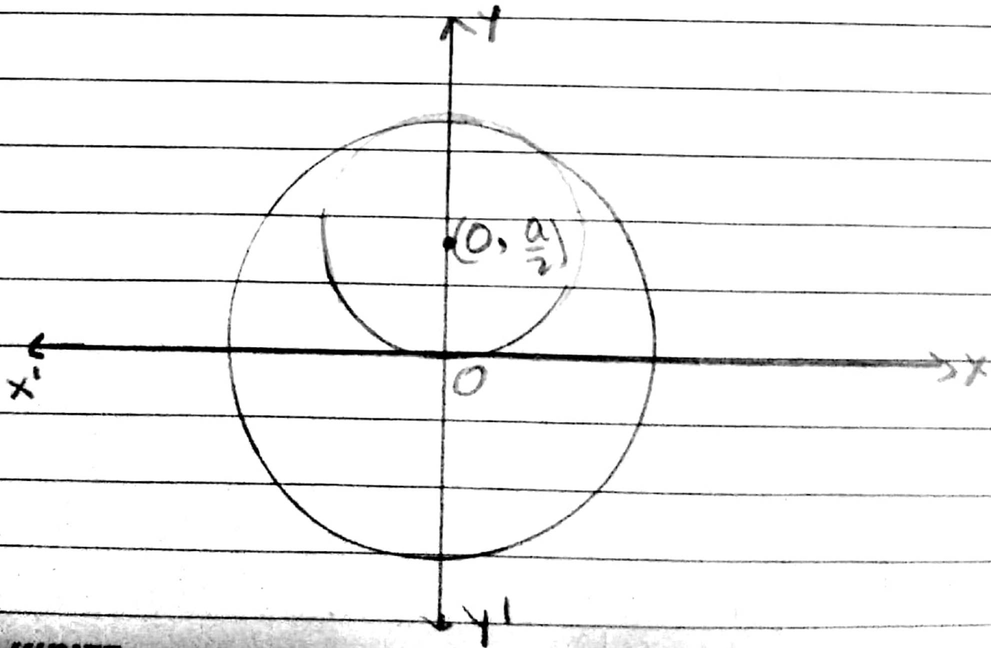
$$I = \int_0^{\pi/2}\int_0^{a} r^2\, r\, dr\, d\theta$$

$$I = \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^a d\theta$$

$$I = \left[\frac{a^4\theta}{4}\right]_0^{\pi/2}$$

$$I = \frac{a^4\pi}{8}$$

(Q)15 Find the volume of the portion of the sphere $x^2+y^2=a^2$ lying inside the cylinder $x^2+y^2=ay$.



Sol Cylindrical co-ordinates

$x = r\cos\theta$; $y = r\sin\theta$

$x^2+y^2+z^2 = a^2$

$\Rightarrow r^2+z^2 = a^2$

$x^2+y^2 = ay$

$\Rightarrow r^2 = ar\sin\theta$

$\Rightarrow r = a\sin\theta$

$$\text{volume} = \iiint dx\,dy\,dz$$

$$= \iiint (r\,d\theta\,dr)\,dz$$

$$= 2\int_0^{\frac{\pi}{2}} d\theta \int_0^{a\sin\theta} r\,dr \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} dz$$

$$= 2(2)\int_0^{\frac{\pi}{2}} d\theta \int_0^{a\sin\theta} r\,dr\,\sqrt{a^2-r^2}$$

$$= 2(2)\int_0^{\frac{\pi}{2}} d\theta \left[\left(-\frac{1}{2}\right)\frac{2}{3}(a^2-r^2)^{3/2}\right]_0^{a\sin\theta}$$

$$= 4\left(-\frac{1}{3}\right)\int_0^{\frac{\pi}{2}} d\theta\left[\left(a^2 - a^2 \cancel{\sin^2\theta} \sin^2\theta\right)^{3/2} - a^3\right]$$

$$= \frac{4}{3}\int_0^{\frac{\pi}{2}} (a^3 - a^3\cos^3\theta)\, d\theta$$

$$= \frac{4a^3}{3}\int_0^{\pi/2} (1-\cos^3\theta)\, d\theta$$

$$= \frac{4a^3}{3}\left[\int_0^{\frac{\pi}{2}} d\theta - \int_0^{\frac{\pi}{2}} \cos^3\theta\, d\theta\right]$$

Let $\int_0^{\frac{\pi}{2}} \cos^3\theta\, d\theta$ be I_1

$$I_1 = \int_0^{\frac{\pi}{2}} \cos^3\theta\, d\theta$$

$$I_1 = \int_0^{\frac{\pi}{2}} (1-\sin^2\theta)\cos\theta\, d\theta$$

Let $\sin\theta = t$

$\cos\theta\, d\theta = dt$

$$I_1 = \int_0^{1} (1-t^2)\, dt$$

$$I_1 = \left[t - \frac{t^3}{3}\right]_0^1 \Rightarrow I_1 = \frac{2}{3}$$

$$I = \left(\frac{4a^3}{3}\right)\left[\left(\frac{\pi}{2}\right) - \left(\frac{2}{3}\right)\right]$$

$$I = \frac{4a^3}{3}\left(\frac{3\pi}{6} - \frac{4}{6}\right)$$

$$I = \frac{2a^3(3\pi-4)}{9}$$

Q16 Use triple integration in cylindrical coordinates to find the volume of the solid R that is bounded above by the hemisphere $z = \sqrt{25 - x^2 - y^2}$, below by the xy-plane, and laterally by the cylinder $x^2+y^2=9$.

Q17 By using gamma function evaluate the integral

$$\int_0^1 x^5\left[\log\left(\frac{1}{x}\right)\right]^3 dx$$

Sol Put $\log_e \frac{1}{x} = t$ or $x = e^{-t}$

$\therefore \quad dx = -e^{-t}dt$

$$\int_0^1 x^5\left[\log_e\left(\frac{1}{x}\right)\right]^3 dx = \int_\infty^0 (e^{-t})^5 [t]^3(-e^{-t}\,dt)$$

$$= \int_0^\infty e^{-6t}\, t^3\, dt$$

Put $6t = u$ or $t = \frac{u}{6}$

$\Rightarrow dt = \frac{du}{6}$

$$\Rightarrow \int_0^\infty e^{-u}\left(\frac{u}{6}\right)^3 \frac{du}{6} = \frac{1}{6^4}\int_0^\infty e^{-u}\, u^3\, du$$

Now, as $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$

$$\Gamma 4 = \int_0^\infty e^{-u}\, u^{4-1}\, du$$

$$\Rightarrow \frac{1}{6^4}\int_0^\infty e^{-u}\, u^{4-1}\, du = \frac{1}{6^4}\Gamma 4 = \frac{3!}{6^4}$$ (as $\Gamma n+1 = n!$)

(Q)13 Given that $\int_0^\infty \frac{x^{n-1}}{1+x} = \frac{\pi}{\sin(n\pi)}$. Show that $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin(n\pi)}$

Sol As we know that:

$$\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\frac{\Gamma m\,\Gamma n}{\Gamma m+n} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Putting $m+n=1$ or $m=1-n$

$$\frac{\Gamma 1-m\,\Gamma n}{\Gamma 1} = \int_0^\infty \frac{x^{n-1}}{(1+x)^1}\,dx$$

$$\Gamma 1-n\,\Gamma n = \int_0^\infty \frac{x^{n-1}}{1+x}\,dx \qquad \left[\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{\pi}{\sin n\pi}$$

Q14 Show that

(i) β function is a symmetric function i.e. $\beta(m,n) = \beta(n,m)$

Sol

$$\beta(n,m) = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx \qquad \left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)^{n-1}\left[\{1-(1-x)\}^{m-1}\right]dx$$

$$= \int_0^1 (1-x)^{n-1} x^{m-1} \, dx$$

$$= \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$$

$$= \beta(m,n)$$

(ii) Calculate $\Gamma\frac{7}{2}$.

sol As $\Gamma n+1 = n\Gamma n$ ... $(n>0)$

$$\Gamma\frac{7}{2} = \frac{5}{2}\Gamma\frac{5}{2}$$

$$= \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\Gamma\frac{3}{2} = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2} = \frac{15}{8}\sqrt{\pi} \quad \ldots \left(\Gamma\frac{1}{2} = \sqrt{\pi}\right)$$

(Q) 20 Prove that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

sol Consider $I = \int_0^\infty e^{-x^2} dx$

Let $x^2 = t$

$\Rightarrow 2x\,dx = dt$

$$I = \frac{1}{2}\int_0^\infty e^{-t} t^{-1/2} \, dt$$

$$I = \frac{\Gamma 1/2}{2} \Rightarrow I = \frac{\sqrt{\pi}}{2}$$

Q21 Show that $\int_0^{\pi/2} \sqrt{\tan x} = \frac{\pi}{\sqrt{2}}$

Sol As,

$$\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x \, dx = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}} \quad \text{——} \quad (1)$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\, dx = \int_0^{\pi/2} \frac{\sin^{1/2} x}{\cos^{1/2} x}\, dx$$

$$= \int_0^{\pi/2} \sin^{1/2} x \cos^{-1/2} x \, dx$$

On applying formula (1), we have

$$= \frac{\Gamma\frac{\frac{1}{2}+1}{2}\,\Gamma\frac{-\frac{1}{2}+1}{2}}{2\,\Gamma\frac{\frac{1}{2}-\frac{1}{2}+2}{2}}$$

$$= \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}}{2\,\Gamma 1}$$

$$= \frac{1}{2}\,\Gamma\frac{1}{4}\,\Gamma\frac{3}{4} \quad \text{——} \quad (2)$$

(2)

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Now as $\Gamma(x)\,\Gamma 1-x = \dfrac{\pi}{\sin(\pi x)}$

So from the above identity, for $x = \frac{1}{4}$, we get

$$\Gamma\left(\tfrac{1}{4}\right)\Gamma\tfrac{3}{4} = \frac{\pi}{\sin\left(\frac{\pi}{4}\right)} = \sqrt{2}\,\pi$$

$$\Gamma\tfrac{1}{4} = \frac{\pi\sqrt{2}}{\Gamma 3/4}$$

Substituting the value of $\Gamma\frac{1}{4}$ in (2)

$$\int_0^{\pi/2} \sqrt{\tan x}\,dx = \left(\tfrac{1}{2}\right)(\pi\sqrt{2})$$

$$= \frac{\pi}{\sqrt{2}}$$

(Q) Prove that $\dfrac{B(m+1,n)}{B(m,n)} = \dfrac{m}{m+n}$ where $m>0, n>0$

Sol As, relation between Beta and Gamma function is:

$$B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n} \quad \text{—— (1)}$$

$$\Rightarrow B(m+1,n) = \frac{\Gamma m+1\,\Gamma n}{\Gamma m+n+1} \quad \text{—— (2)}$$

Dividing (2) by (1), we get

$$\frac{B(m+1,n)}{B(m,n)} = \frac{\left(\dfrac{\Gamma m+1\,\Gamma n}{\Gamma m+n+1}\right)}{\left(\dfrac{\Gamma m\,\Gamma n}{\Gamma m+n}\right)}$$

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Now, as $\Gamma{n+1} = n\Gamma{n}$ .... $(n>0)$

$$\frac{\beta(m+1, n)}{\beta(m, n)} = \frac{\left(\dfrac{(m\Gamma{m})(\Gamma{n})}{(m+n)\Gamma{m+n}}\right)}{\dfrac{\Gamma{m}\,\Gamma{n}}{\Gamma{m+n}}}$$

$\left(\begin{array}{l}\Gamma{m+1} = m\Gamma{m} \\ \Gamma{m+n+1} = (m+n)\Gamma{m+n}\end{array}\right)$

$$\boxed{\frac{\beta(m+1, n)}{\beta(m, n)} = \frac{m}{m+n}}$$