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Applied Mathematics - Assignment III

A.1 Given: $z = f(x,y)$, $x = r\cosh\theta$ and $y = r\sinh\theta$

To Prove: $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

Proof: $\frac{\partial x}{\partial \theta} = r\sinh\theta \qquad \frac{\partial y}{\partial \theta} = r\cosh\theta \qquad \left[\begin{array}{l}\frac{d\sinh\theta}{d\theta} = \cosh\theta \\ \frac{d\cosh\theta}{d\theta} = \sinh\theta\end{array}\right]$

$\frac{\partial x}{\partial r} = \cosh\theta \qquad \frac{\partial y}{\partial r} = \sinh\theta$

By chain rule of partial derivates,

$$\frac{\partial z}{\partial r} = \left(\frac{\partial z}{\partial x}\right)\cdot\left(\frac{\partial x}{\partial r}\right) + \left(\frac{\partial z}{\partial y}\right)\cdot\left(\frac{\partial y}{\partial r}\right)$$

$$= \left(\frac{\partial z}{\partial x}\right)\cosh\theta + \left(\frac{\partial z}{\partial y}\right)\sinh\theta$$

and $$\frac{\partial z}{\partial \theta} = \left(\frac{\partial z}{\partial x}\right)\cdot\left(\frac{\partial x}{\partial \theta}\right) + \left(\frac{\partial z}{\partial y}\right)\cdot\left(\frac{\partial y}{\partial \theta}\right)$$

$$= r\sinh\theta\left(\frac{\partial z}{\partial x}\right) + r\cosh\theta\,\frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{1}{r}\frac{\partial z}{\partial \theta} = \left(\frac{\partial z}{\partial x}\right)\sinh\theta + \left(\frac{\partial z}{\partial y}\right)\cosh\theta$$

R.H.S: $\left(\frac{\partial z}{\partial r}\right)^2 - \left(\frac{1}{r}\frac{\partial z}{\partial \theta}\right)^2$

$$= \left(\frac{\partial z}{\partial x}\right)^2\cosh^2\theta + \left(\frac{\partial z}{\partial y}\right)^2\sinh^2\theta + \cancel{2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\cosh\theta\sinh\theta} - \left(\frac{\partial z}{\partial x}\right)^2\sinh^2\theta$$
$$- \left(\frac{\partial z}{\partial y}\right)^2\cosh^2\theta - \cancel{2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\cosh\theta\sinh\theta}$$

$$= \left[\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2\right]\left[\cosh^2\theta - \sinh^2\theta\right]$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \text{L.H.S} \qquad \left[\text{As } \cosh^2\theta - \sinh^2\theta = 1\right]$$

Hence Proved

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A.2 Euler's Theorem: If $u = y^m f\left(\frac{x}{y}\right)$ then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = mu$$

(i) $u = y^{n+\frac{1}{2}}\left(1+\left(\frac{x}{y}\right)^{1/2}\right)\left(1+\left(\frac{x}{y}\right)^{n}\right)$

Here $m = n+\frac{1}{2}$

To verify: $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(n+\frac{1}{2}\right)u$

$$u = x^{n+1/2} + x^{1/2}y^n + y^{1/2}x^n + y^{n+1/2}$$

$$\frac{\partial u}{\partial x} = \left(n+\frac{1}{2}\right)x^{n-1/2} + \frac{1}{2}x^{-1/2}y^n + nx^{n-1}y^{1/2}$$

$$x\frac{\partial u}{\partial x} = \left(n+\frac{1}{2}\right)x^{n+1/2} + \frac{1}{2}x^{1/2}y^n + nx^n y^{1/2}$$

Similarly $y\frac{\partial u}{\partial y} = \left(n+\frac{1}{2}\right)y^{n+1/2} + \frac{1}{2}y^{1/2}x^n + ny^n x^{1/2}$

L.H.S: $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(n+\frac{1}{2}\right)x^{n+1/2} + \left(\frac{1}{2}+n\right)x^{1/2}y^n + \left(n+\frac{1}{2}\right)y^{1/2}x^n + \left(n+\frac{1}{2}\right)y^{n+1/2}$

$$= \left(n+\frac{1}{2}\right)\left(x^{n+1/2} + x^{1/2}y^n + y^{1/2}x^n + y^{n+1/2}\right) = \left(n+\frac{1}{2}\right)u = R.H.S$$

Hence Verified

(ii) $u = y^0\left(\sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{x}{y}\right)\right)$ Here $m=0$

To Verify: $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0\times u = 0$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\frac{1}{y} + \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y} = \frac{1}{\sqrt{y^2-x^2}} + \frac{y}{y^2+x^2}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{y^2+x^2}$$

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$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot x \cdot \left(\frac{-1}{y^2}\right) + \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot x \cdot \left(\frac{-1}{y^2}\right) = \frac{-x}{y\sqrt{y^2-x^2}} - \frac{x}{y^2+x^2}$$

$$y\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} - \frac{xy}{y^2+x^2}$$

L.H.S : $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{y^2+x^2} - \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{y^2+x^2} = 0 = $ R.H.S

Hence Verified

A.3 $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right) \Rightarrow \tan u = x^2\left(\frac{1+\left(\frac{y}{x}\right)^3}{1-\left(\frac{y}{x}\right)}\right)$.

Let $z = \tan u = x^2 \frac{\left(1+\left(\frac{y}{x}\right)^3\right)}{\left(1-\left(\frac{y}{x}\right)\right)}$

then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \quad \text{[By Euler's theorem]}$$

$$\Rightarrow x\frac{\partial \tan u}{\partial x} + y\frac{\partial \tan u}{\partial y} = 2\tan u$$

$$x\left(\frac{\partial \tan u}{\partial u}\right)\frac{\partial u}{\partial x} + y\left(\frac{\partial \tan u}{\partial u}\right)\cdot\frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow \frac{1}{\cos^2 u}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\frac{\sin u}{\cos u}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u\cos u = \sin 2u$$

L.H.S = R.H.S Hence Proved

A.4 $w = \sin^{-1}u \Rightarrow u = \sin w$ Also, $u = \frac{x^2+y^2+z^2}{x+y+z}$

$$\frac{\partial u}{\partial x} = \frac{2x(x+y+z) - x^2-y^2-z^2}{(x+y+z)^2} = \frac{x^2+2xy+2xz-y^2-z^2}{(x+y+z)^2}$$

$$\Rightarrow \text{but } \frac{\partial u}{\partial x} = \frac{\partial \sin w}{\partial x} = \cos w\frac{\partial w}{\partial x}$$

$$\Rightarrow x\frac{\partial w}{\partial x} = \frac{1}{\cos w}\left(\frac{x^3+2x^2y+2x^2z-xy^2-xz^2}{(x+y+z)^2}\right)$$

Similarly $$y\frac{\partial w}{\partial y} = \frac{1}{\cos w}\left(\frac{y^3+2y^2x+2y^2z-yx^2-yz^2}{(x+y+z)^2}\right)$$

and $$z\frac{\partial w}{\partial z} = \frac{1}{\cos w}\left(\frac{z^3+2z^2x+2z^2y-zx^2-zy^2}{(x+y+z)^2}\right)$$

L.H.S: $$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = \frac{x^3+x^2y+x^2z+y^3+y^2x+y^2z+z^3+z^2x+z^2y}{\cos w\,(x+y+z)^2}$$

$$= \frac{x^2(x+y+z)+y^2(y+x+z)+z^2(z+x+y)}{\cos w\,(x+y+z)^2} = \frac{(x^2+y^2+z^2)}{\cos w\,(x+y+z)}$$

$$= \frac{u}{\cos w} = \frac{\sin w}{\cos w} = \tan w = \text{R.H.S}$$

Hence Proved

A.5 $u = \tan^{-1}\frac{y^2}{x} \Rightarrow \tan u = \frac{y^2}{x} = x\cdot\left(\frac{y}{x}\right)^2 = x^1 f\left(\frac{y}{x}\right)$ where $f\left(\frac{y}{x}\right) = \frac{y^2}{x^2}$

Let $z = \tan u = x^1 f\left(\frac{y}{x}\right)$ Clearly z is a homogeneous function of degree 1

$\therefore$ By Euler's theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1\times z$$

$$\Rightarrow x\frac{\partial \tan u}{\partial x} + y\frac{\partial \tan u}{\partial y} = \tan u$$

$$\Rightarrow \sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin u\cos u = \frac{\sin 2u}{2} \quad \text{—(i)}$$

Taking partial derivative of (i) w.r.t x, we get

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = \frac{2\cos 2u}{2}\frac{\partial u}{\partial x} = \cos 2u\frac{\partial u}{\partial x}$$

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$$\Rightarrow x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = -(1-\cos 2u)\frac{\partial u}{\partial x} = -2\sin^2 u\frac{\partial u}{\partial x}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = -2\sin^2 u\, x\frac{\partial u}{\partial x} \quad \text{--- (ii)}$$

Similarly, taking partial derivative of (i) w.r.t y, we get

$$y\frac{\partial^2 u}{\partial y^2} + x\frac{\partial^2 u}{\partial x \partial y} = -2\sin^2 u\frac{\partial u}{\partial y}$$

$$y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial^2 u}{\partial x \partial y} = -2\sin^2 u\, y\frac{\partial u}{\partial y}$$

L.H.S: $\left(x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y}\right) + \left(xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right)$

$= -2\sin^2 u\, x\frac{\partial u}{\partial x} + (-2\sin^2 u)\, y\frac{\partial u}{\partial y}$ [From (ii) & (iii)]

$= -2\sin^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$

$= -2\sin^2 u \cdot \frac{\sin 2u}{2} = -\sin^2 u \sin 2u =$ R.H.S

Hence Proved

A.6 $u = \sin^0 \frac{y}{x} + x\sin^{-1} \frac{y}{x}$

Let $z_1 = x^0 \sin\left(\frac{y}{x}\right)$ — (i) and $z_2 = x^1 \sin^{-1}\left(\frac{y}{x}\right)$ — (ii) $\therefore u = z_1 + z_2$

$\Rightarrow z_2 = u - z_1$ — (iii)

Clearly, z_1 & z_2 are homogeneous functions of degree 0 and 1 respectively.

So, by Euler's theorem

$$x\frac{\partial z_1}{\partial x} + y\frac{\partial z_1}{\partial y} = 0 \times z_1 = 0 \quad \text{— (iv)}$$

and $$x\frac{\partial z_2}{\partial x} + y\frac{\partial z_2}{\partial y} = 1 \times z_2 = z_2$$

$$\Rightarrow x\frac{\partial}{\partial x}(u - z_1) + y\frac{\partial}{\partial y}(u - z_1) = u - z_1 \quad \text{[From (iii)]}$$

$$\Rightarrow x\left[\frac{\partial u}{\partial x} - \frac{\partial z_1}{\partial x}\right] + y\left[\frac{\partial u}{\partial y} - \frac{\partial z_1}{\partial y}\right] = u - z_1$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} - \left(x\frac{\partial z_1}{\partial x} + y\frac{\partial z_1}{\partial y}\right) = u - z_1$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u - z_1 \quad \text{— (v)} \quad \text{[From — (iv)]}$$

Taking partial derivative of (v) w.r.t x, we get

$$\cancel{\frac{\partial u}{\partial x}} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = \cancel{\frac{\partial u}{\partial x}} - \frac{\partial z_1}{\partial x}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = -x\frac{\partial z_1}{\partial x} \quad \text{— (vi)}$$

Similarly, taking partial derivative of (v), w.r.t y, we get

$$y\frac{\partial^2 u}{\partial y^2} + x\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial z_1}{\partial y}$$

$$\Rightarrow y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial^2 u}{\partial x \partial y} = -y\frac{\partial z_1}{\partial y} \quad \text{— (vii)}$$

L.H.S: $\left(x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y}\right) + \left(xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right)$

$= -x\frac{\partial z_1}{\partial x} - y\frac{\partial z_1}{\partial y}$ [From (vi) & (vii)]

$= -\left(x\frac{\partial z_1}{\partial x} + y\frac{\partial z_1}{\partial y}\right) = 0 = R.H.S$ [From (iv)]

Hence, Proved

A.7 $z = f(x,y)$, $x = e^{2u} + e^{-2v}$, $y = e^{-2u} + e^{2v}$

$$\frac{\partial x}{\partial u} = 2e^{2u}, \quad \frac{\partial x}{\partial v} = -2e^{-2v}, \quad \frac{\partial y}{\partial u} = -2e^{-2u}, \quad \frac{\partial y}{\partial v} = 2e^{2v}$$

By chain rule of partial derivatives,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= 2e^{2u}\left(\frac{\partial z}{\partial x}\right) - 2e^{-2u}\left(\frac{\partial z}{\partial y}\right)$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= -2e^{-2v}\left(\frac{\partial z}{\partial x}\right) + 2e^{2v}\left(\frac{\partial z}{\partial y}\right)$$

L.H.S: $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = 2e^{2u}\left(\frac{\partial z}{\partial x}\right) - 2e^{-2u}\left(\frac{\partial z}{\partial y}\right) + 2e^{-2v}\left(\frac{\partial z}{\partial x}\right) - 2e^{2v}\left(\frac{\partial z}{\partial y}\right)$

$$= 2\left[\left(e^{2u}+e^{-2v}\right)\frac{\partial z}{\partial x} - \left(e^{-2u}+e^{2v}\right)\frac{\partial z}{\partial y}\right]$$

$$= 2\left(x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right) = R.H.S$$

Hence Proved

A.8 $u = f(x^2+2yz, y^2+2zx)$

let $t_1 = x^2+2yz$ and $t_2 = y^2+2zx$

then $u = f(t_1, t_2)$

By chain rule of partial derivatives,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x\frac{\partial u}{\partial t_1} + 2z\frac{\partial u}{\partial t_2}$$

$$(y^2-zx)\frac{\partial u}{\partial x} = (2y^2x-2x^2z)\frac{\partial u}{\partial t_1} + (2y^2z-2z^2x)\frac{\partial u}{\partial t_2} \quad -(i)$$

Also, $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial y} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial y}$

$= 2z\frac{\partial u}{\partial t_1} + 2y\frac{\partial u}{\partial t_2}$

$(x^2-yz)\frac{\partial u}{\partial y} = (2x^2z-2z^2y)\frac{\partial u}{\partial t_1} + (2x^2y-2y^2z)\frac{\partial u}{\partial t_2}$ — (ii)

and $\frac{\partial u}{\partial z} = 2y\frac{\partial u}{\partial t_1} + 2x\frac{\partial u}{\partial t_2}$

$(z^2-xy)\frac{\partial u}{\partial z} = (2z^2y-2y^2x)\frac{\partial u}{\partial t_1} + (2z^2x-2x^2y)\frac{\partial u}{\partial t_2}$ — (iii)

L.H.S: $(y^2-zx)\frac{\partial u}{\partial x} + (x^2-yz)\frac{\partial u}{\partial y} + (z^2-xy)\frac{\partial u}{\partial z}$

$= (2y^2x-2x^2z)\frac{\partial u}{\partial t_1} + (2y^2z-2z^2x)\frac{\partial u}{\partial t_2} + (2x^2z-2z^2y)\frac{\partial u}{\partial t_1} + (2x^2y-2y^2z)\frac{\partial u}{\partial t_2}$

$+ (2z^2y-2y^2x)\frac{\partial u}{\partial t_1} + (2z^2x-2x^2y)\frac{\partial u}{\partial t_2}$ [From (i), (ii) and (iii)]

$= \cancel{2y^2x-2x^2z+2y^2z-2z^2x+2x^2z-2z^2y}\ (2y^2x-2x^2z+2x^2z-2z^2y+2z^2y-2y^2x)\frac{\partial u}{\partial t_1}$

$+ (2y^2z-2z^2x+2x^2y-2y^2z+2z^2x-2x^2y)\frac{\partial u}{\partial t_2}$

$= 0 = \text{R.H.S}$

Hence Proved

A.9 $f(x,y) = e^x\cos y \qquad f(1,\pi/4) = \frac{e}{\sqrt2}$

$\left(\frac{\partial f}{\partial x}\right) = e^x\cos y,\ \left(\frac{\partial f}{\partial x}\right)_{1,\pi/4} = \frac{e}{\sqrt2},\ \left(\frac{\partial f}{\partial y}\right) = -e^x\sin y,\ \left(\frac{\partial f}{\partial y}\right)_{1,\pi/4} = \frac{-e}{\sqrt2}$

$\left(\frac{\partial^2 f}{\partial x^2}\right) = e^x\cos y,\ \left(\frac{\partial^2 f}{\partial x^2}\right)_{1,\pi/4} = \frac{e}{\sqrt2}\quad \left(\frac{\partial^2 f}{\partial x\partial y}\right) = -e^x\sin y \quad \left(\frac{\partial^2 f}{\partial x\partial y}\right)_{1,\pi/4} = \frac{-e}{\sqrt2}$

$\left(\frac{\partial^2 f}{\partial y^2}\right) = -e^x\cos y \quad \left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{-e}{\sqrt2}$

Quadratic Taylor Series Expansion can be given as:

$$f(x_0+h, y_0+k) = f(x_0,y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x_0,y_0) + \frac{1}{2!}\left(h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x\partial y} + k^2\frac{\partial^2}{\partial y^2}\right)f(x_0,y_0)$$

Putting $x_0=1$ $y_0=\pi/4$ $h=x-1$ & $k=y-\pi/4$, we get

$$f\left(x-1+1, y-\tfrac{\pi}{4}+\tfrac{\pi}{4}\right) = f(x,y)$$

$$= f\left(1,\frac{\pi}{4}\right) + \left[(x-1)\left(\frac{\partial f}{\partial x}\right)_{1,\pi/4} + \left(y-\frac{\pi}{4}\right)\left(\frac{\partial f}{\partial x}\right)_{1,\pi/4}\right] + \frac{1}{2!}\left[(x-1)^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{1,\pi/4}\right.$$

$$\left. + 2(x-1)\left(y-\frac{\pi}{4}\right)\left(\frac{\partial^2 f}{\partial x\partial y}\right)_{1,\pi/4} + \left(y-\frac{\pi}{4}\right)^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{1,\pi/4}\right]$$

$$= \frac{e}{\sqrt{2}} + \left[(x-1)\frac{e}{\sqrt{2}} - \left(y-\frac{\pi}{4}\right)\frac{e}{\sqrt{2}}\right] + \frac{1}{2!}\left[(x-1)^2\frac{e}{\sqrt{2}} - 2(x-1)\left(y-\frac{\pi}{4}\right)\frac{e}{\sqrt{2}} - \left(y-\frac{\pi}{4}\right)^2\frac{e}{\sqrt{2}}\right]$$

A.10 $f(x,y) = \cos x\cos y$ $\quad f(0,0) = 1$

$\frac{\partial f}{\partial x} = -\sin x\cos y \quad \left(\frac{\partial f}{\partial x}\right)_{0,0} = 0 \quad \frac{\partial f}{\partial y} = -\sin y\cos x \quad \left(\frac{\partial f}{\partial y}\right)_{0,0} = 0$

$\frac{\partial^2 f}{\partial x^2} = -\cos x\cos y \quad \left(\frac{\partial^2 f}{\partial x^2}\right)_{0,0} = -1 \quad \frac{\partial^2 f}{\partial x\partial y} = \sin x\sin y \quad \left(\frac{\partial^2 f}{\partial x\partial y}\right)_{0,0} = 0$

$\frac{\partial^2 f}{\partial y^2} = -\cos x\cos y \quad \left(\frac{\partial^2 f}{\partial y^2}\right)_{0,0} = -1$

Quadratic Approximation of $f(x,y)$ using Taylor's Series:

$$f(0+x, 0+y) = f(x,y) = f(0,0) + \left[x\left(\frac{\partial f}{\partial x}\right)_{0,0} + y\left(\frac{\partial f}{\partial x}\right)_{0,0}\right] + \frac{1}{2!}\left[x^2\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial^2}{\partial x\partial y} + y^2\frac{\partial^2}{\partial y^2}\right]f(0,0)$$

$$\Rightarrow \cos x \cos y = 1 + [x\times 0 + y\times 0] + \frac{1}{2!}\left[x^2(-1) + 2xy\times 0 + y^2(-1)\right]$$

$$= 1 - \frac{x^2}{2} - \frac{y^2}{2}$$

A.11 $f(x,y) = \tan^{-1} xy \qquad f(1,1) = \tan^{-1}(1) = 0.7854$

$$\frac{\partial f}{\partial x} = \frac{y}{1+x^2y^2} \qquad \frac{\partial f}{\partial y} = \frac{x}{1+x^2y^2} \qquad \left(\frac{\partial f}{\partial x}\right)_{1,1} = \frac{1}{2} \qquad \left(\frac{\partial f}{\partial y}\right)_{1,1} = \frac{1}{2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-y\cdot 2xy^2}{(1+x^2y^2)^2} = \frac{-2xy^3}{(1+x^2y^2)^2} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{-2x^3y}{(1+x^2y^2)^2} \qquad \left(\frac{\partial^2 f}{\partial x^2}\right)_{1,1} = -\frac{1}{2} \qquad \left(\frac{\partial^2 f}{\partial y^2}\right)_{1,1} = -\frac{1}{2}$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{1+x^2y^2 - y\cdot 2yx^2}{(1+x^2y^2)^2} = \frac{1-x^2y^2}{(1+x^2y^2)^2} \qquad \left(\frac{\partial^2 f}{\partial x\partial y}\right)_{1,1} = 0$$

Taylor's series expansion upto second degree term:

$$f(1+x-1, 1+y-1) = f(x,y)$$

$$= f(1,1) + \left[(x-1)\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y}\right] f(1,1) + \frac{1}{2!}\left[(x-1)^2\frac{\partial^2}{\partial x^2} + 2(x-1)(y-1)\frac{\partial^2}{\partial x\partial y} + (y-1)^2\frac{\partial^2}{\partial y^2}\right] f(1,1)$$

$$= 0.7854 + \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{2}\left[-\frac{1}{2}(x-1)^2 - \frac{1}{2}(y-1)^2\right]$$

$$= 0.7854 + \frac{1}{2}(x-1) + \frac{1}{2}(y-1) - \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2$$

$$f(1.1, 0.8) = 0.7854 + \frac{1}{2}(1.1-1+0.8-1) - \frac{1}{4}\left[(1.1-1)^2 + (0.8-1)^2\right]$$

$$= 0.7854 - \frac{0.1}{2} - \frac{1}{4}[0.01 + 0.04]$$

$$= 0.7854 - 0.05 - 0.0125 = 0.7229$$

A.12 Consider point (x_1, y_1) on given line and (x_2, y_2) on given ellipse. Let d be the distance between them.

$y_1 = 10 - 2x_1$ and $y_2 = \frac{3}{2}\sqrt{4 - x_2^2}$

$z = d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

For d to be minimum, $d^2 (= z)$ shall also be min^m

$$z = (x_2 - x_1)^2 + \left(\frac{3}{2}\sqrt{4-x_2^2} + 2x_1 - 10\right)^2$$

$$\frac{\partial z}{\partial x_1} = -2(x_2 - x_1) + 2\left(\frac{3}{2}\sqrt{4-x_2^2} + 2x_1 - 10\right)2$$

For stationary point, $\frac{\partial z}{\partial x_1} = 0$

$$-(x_2 - x_1) + 2\left(\frac{3}{2}\sqrt{4-x_2^2} + 2x_1 - 10\right) = 0 \quad \text{- (i)}$$

$$\frac{\partial z}{\partial x_2} = 2(x_2 - x_1) + 2\left(\frac{3}{2}\sqrt{4-x_2^2} + 2x_1 - 10\right)\frac{3}{2}\cdot\frac{1}{\cancel{2}}\cdot\frac{-\cancel{2}x_2}{\sqrt{4-x_2^2}}$$

$$= 2(x_2 - x_1) - 2\left(\frac{3}{2}\sqrt{4-x_2^2} + 2x_1 - 10\right)\cdot\frac{3x_2}{2\sqrt{4-x_2^2}}$$

$$\frac{\partial z}{\partial x_2} = 0$$

$$\Rightarrow (x_2 - x_1) - \frac{3x_2}{2\sqrt{4-x_2^2}}\left(\frac{3}{2}\sqrt{4-x_2^2} + 2x_1 - 10\right) = 0 \quad \text{- (ii)}$$

$$\left(2 - \frac{3x_2}{2\sqrt{4-x_2^2}}\right)\left(\frac{3}{2}\sqrt{4-x_2^2} + 2x_1 - 10\right) = 0 \quad \text{[Adding (i) \& (ii)]}$$

$$\Rightarrow 2 - \frac{3x_2}{2\sqrt{4-x_2^2}} = 0$$

$$\Rightarrow 3x_2 = 4\sqrt{4-x_2^2} \quad (\Rightarrow x_2 > 0)$$

$$\Rightarrow 9x_2^2 = 64 - 16x_2^2 \quad \text{[S.B.S]}$$

$$\Rightarrow 25x_2^2 = 64 \Rightarrow x_2 = 8/5 \quad (\text{As } x_2 > 0)$$

$$x_1 - \frac{8}{5} + 3\sqrt{4 - \frac{64}{25}} + 4x_1 - 20 = 0 \quad [\text{from (i)}]$$

$$5x_1 - \frac{8}{5} + 3\,\frac{\sqrt{100-64}}{5} - 20 = 0$$

$$5x_1 = 20 - \frac{18+8}{5} = 20 - 2 = 18$$

$$\Rightarrow x_1 = \frac{18}{5}$$

$$d^2 = \left(\frac{18}{5} - \frac{8}{5}\right)^2 + \left(\frac{3}{2} \times \frac{8}{5} + 2 \times \frac{18}{5} - 10\right)^2$$

$$= 4 + \left(\frac{45}{5} - 10\right)^2 = 4 + 1 = 5$$

$$\Rightarrow d = \sqrt{5}$$

A.13 The extremum of function $f(x,y) = d^2 = (x-0)^2 + (y-0)^2 = x^2 + y^2$ have to be found given the condition:

$$x^2 + xy + y^2 - 16 = 0$$

$$F(x,y) = x^2 + y^2 + \lambda(x^2 + xy + y^2 - 16)$$

Using Langrange's Method,

$$\frac{\partial F}{\partial x} = 2x + 2\lambda x + \lambda y = 0 \Rightarrow \lambda = \frac{-2x}{2x+y} \quad \text{— (i)}$$

and $$\frac{\partial F}{\partial y} = 2y + 2\lambda y + \lambda x = 0 \Rightarrow \lambda = \frac{-2y}{2y+x} \quad \text{— (ii)}$$

$$\Rightarrow \frac{\cancel{-}2x}{2x+y} = \frac{\cancel{-}2y}{2y+x} \quad [\text{From (i) \& (ii)}]$$

$$\Rightarrow 2xy + x^2 = 2xy + y^2$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow x = y \ \& \ x = -y$$

Case (i) $x = y$

$$x^2 + x(x) + (x)^2 - 16 = 0 \Rightarrow 3x^2 = 16 \Rightarrow x^2 = \frac{16}{3} \Rightarrow x = \frac{4}{\sqrt{3}}, \frac{-4}{\sqrt{3}}$$

Points $\left(\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right)$ & $\left(-\frac{4}{\sqrt{3}}, -\frac{4}{\sqrt{3}}\right)$

$d_1^2 = x^2+y^2 = 2x^2 = \frac{32}{3}$

Case (iii) $x=-y$

$x^2 + x(-x) + (-x)^2 - 16 = 0 \Rightarrow x^2 = 16 \Rightarrow x = 4, -4$

Points: $(4,-4)$ and $(-4,4)$

$d_2^2 = x^2 + (-x)^2 = 2x^2 = 32$

Clearly $d_2^2 > d_1^2$

Hence, the farthest points are $(4,-4)$ and $(-4,4)$ with $d = 4\sqrt{2}$ $(d^2 = 32)$ and closest points are $\left(\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right)$ and $\left(-\frac{4}{\sqrt{3}}, -\frac{4}{\sqrt{3}}\right)$ with $d = 4\sqrt{\frac{2}{3}}$ $\left(d^2 = \frac{32}{3}\right)$

A.14 Let P be (x_1, y_1, z_1) and Q be (x_2, y_2, z_2).

Extremum of function $f = d^2 = (x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2$ have to be found given the conditions:

$x_1 + y_1 + z_1 - 2a = 0$

and $x_2^2 + y_2^2 + z_2^2 - a^2 = 0$

$F = (x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2 + \lambda_1(x_1+y_1-2a) + \lambda_2(x_2^2+y_2^2+z_2^2-a^2)$

Using Lagrange's method,

$\frac{\partial F}{\partial x_1} = 2(x_1-x_2) + \lambda_1 = 0 \Rightarrow \lambda_1 = -2(x_1-x_2)$ — (i)

$\frac{\partial F}{\partial y_1} = 2(y_1-y_2) + \lambda_1 = 0 \Rightarrow \lambda_1 = -2(y_1-y_2)$ — (ii)

$\frac{\partial F}{\partial z_1} = 2(z_1-z_2) + \lambda_1 = 0 \Rightarrow \lambda_1 = -2(z_1-z_2)$ — (iii)

$\frac{\partial F}{\partial x_2} = -2(x_1-x_2) + 2\lambda_2 x_2 = 0 \Rightarrow \lambda_2 = \frac{2(x_1-x_2)}{x_2}$ — (iv)

$\frac{\partial F}{\partial y_2} = -2(y_1-y_2) + 2\lambda_2 y_2 = 0 \Rightarrow \lambda_2 = \frac{2(y_1-y_2)}{y_2}$ — (v)

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$$\frac{\partial F}{\partial z_2} = -2(z_1 - z_2) + 2\lambda_2 z_2 = 0 \Rightarrow \lambda_2 = \frac{2(z_1 - z_2)}{z_2} \quad \text{— (vi)}$$

$x_1 - x_2 = y_1 - y_2 = z_1 - z_2$ [From (i), (ii) & (iii)]

$\frac{x_1 - x_2}{x_2} = \frac{y_1 - y_2}{y_2} = \frac{z_1 - z_2}{z_2}$ [from (iv), (v) & (vi)]

$\Rightarrow x_2 = y_2 = z_2$

$\Rightarrow x_1 = y_1 = z_1$

$x_1 + x_1 + x_1 - 2a = 0$

$\Rightarrow x_1 = \frac{2a}{3} \quad \therefore y_1 = \frac{2a}{3} \; \& \; z_1 = \frac{2a}{3}$

$x_2^2 + y_2^2 + z_2^2 = a^2$

$\Rightarrow 3x_2^2 = a^2 \Rightarrow x_2 = \frac{a}{\sqrt{3}}, -\frac{a}{\sqrt{3}}$

Points: $P\left(\frac{2a}{3}, \frac{2a}{3}, \frac{2a}{3}\right)$ & $Q\left(\pm\frac{a}{\sqrt{3}}, \pm\frac{a}{\sqrt{3}}, \pm\frac{a}{\sqrt{3}}\right)$

For $Q\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$

$$d_1^2 = 3\left(\frac{2}{3} - \frac{1}{\sqrt{3}}\right)^2 a^2$$

$$= 3\left(\frac{2 - \sqrt{3}}{3}\right)^2 a^2$$

$$= \frac{3}{9}(4 + 3 - 4\sqrt{3})a^2$$

$$\Rightarrow d_1 = \frac{a}{\sqrt{3}}\sqrt{7 - 4\sqrt{3}}$$

For $Q\left(-\frac{a}{\sqrt{3}}, -\frac{a}{\sqrt{3}}, -\frac{a}{\sqrt{3}}\right)$

$$d_2^2 = 3\left(\frac{2}{3} + \frac{1}{\sqrt{3}}\right)^2 a^2 \Rightarrow \frac{3}{9}(4 + 3 + 4\sqrt{3})a^2$$

$$d_2 = \frac{a}{\sqrt{3}}\sqrt{7+4\sqrt{3}}$$

So, closest points $P\left(\frac{2}{3}a, \frac{2}{3}a, \frac{2}{3}a\right)$ & $Q\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$

with $d_1 = \frac{a}{\sqrt{3}}\sqrt{7-4\sqrt{3}}$

and furthest points ~~&~~ $P\left(\frac{2}{3}a, \frac{2}{3}a, \frac{2}{3}a\right)$ & $Q\left(\frac{-a}{\sqrt{3}}, \frac{-a}{\sqrt{3}}, \frac{-a}{\sqrt{3}}\right)$

with $d_2 = \frac{a}{\sqrt{3}}\sqrt{7+4\sqrt{3}}$.

A.15 $f(x,y) = y^2 + x^2y + x^4$

$$\frac{\partial f}{\partial x} = 2xy + 4x^3 = 0 \Rightarrow 2x(2x^2+y) = 0 \quad -(i) \Rightarrow x=0 \text{ or } x^2 = -\frac{y}{2}$$

$$\frac{\partial f}{\partial y} = 2y + x^2 = 0 \quad -(ii)$$

~~Clearly (0,0) satisfies of both equation (i) & (ii)~~

~~$\therefore$ It is a stationary point~~

~~$r = \frac{\partial^2 f}{\partial x^2} = 2y + 12x^2 \; r_{0,0} = 0$~~

Putting $x=0$ in (ii), we get

$$2y + 0 = 0$$
$$\Rightarrow y = 0$$

Putting $x^2 = -\frac{y}{2}$ in (ii), we get

$$2y - \frac{y}{2} = 0 \Rightarrow \frac{3y}{2} = 0 \Rightarrow y = 0 \;\therefore x = 0$$

i.e. $(0,0)$ is the only stationary point of $f(x,y)$

$f(x,y)$ can be treated as a quadratic ~~equation~~ function of y with $a = 1 > 0$ for a ~~given~~ particular x $(x=k)$

~~g(x,y)~~ $f(k,y) = y^2 + k^2y + k^4$

$D = (k^2)^2 - 4(1)(k^4)$

$= -3k^4 \leq 0$

~~The vertex is~~ The vertex can be given as:

$V\left(-\frac{k^2}{2}, -\frac{3}{4}k^4\right)$

Clearly $F(k,y) \geq 0$ with minimum vale of it occuring at $k=0$

i.e. $x=0$ and $y=0$ is minima of $f(x,y)$