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Roll NO : DTU/2K17/A5/570
ASSIGNMENT: MA-102 ASSIGNMENT-II
TOPIC : ORDINARY DIFFERENTIAL EQN
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DATE: 09-January'18

Question 1. Find the general solution of the following homogenous differential equations.

(a) $y'' + 1.75y' - 0.5y = 0$

Let $\frac{d}{dx} \equiv D$

then, homogenous equation can be represented as;

$\Rightarrow (D^2 + 1.75D - 0.5)y = 0$

The auxillary equation of the above DF Eqn is,

$m^2 + 1.75m - 0.5 = 0$

$\Rightarrow m = \frac{-1.75 \pm \sqrt{(1.75)^2 - 4 \times 1 \times (-0.5)}}{2}$

$\Rightarrow m = 0.25, -2$

The general solution of the above differential equation is:

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\boxed{y = c_1 e^{0.25x} + c_2 e^{-2x}}$

where, c_1 and c_2 are arbitrary constants.

(b) $y'' + 0.54y' - (0.0729 + \pi)y = 0$

The DF Eqn is,

$\Rightarrow (D^2 + 0.54D - (0.0729 + \pi))y = 0$

The auxillary eqn is,

$\Rightarrow m^2 + 0.54m - (0.0729 + \pi) = 0$

$m = \frac{-0.54 \pm \sqrt{(0.54)^2 - 4(0.0729 + \pi)}}{2 \times 1}$

$m = -0.27 \pm 1.77i$

The general solution is; (A & B are arbitrary constants)

$\boxed{y = (Ae^{i(1.77)x} + Be^{-i(1.77)x})e^{-0.27x}}$

or

$y = e^{-0.27x}[A(\cos 1.77x + i\sin 1.77x) + B(\cos 1.77x - i\sin 1.77x)]$

(c) $y^{(5)} - 3y^{(4)} + 3y''' - y'' = 0$

The DFEqn is,

$\Rightarrow (D^5 - 3D^4 + 3D^3 - D^2)y = 0$

The auxillary eqn is,

$\Rightarrow m^5 - 3m^4 + 3m^3 - m^2 = 0$

$\Rightarrow m^2(m-1)(m-1)(m-1) = 0$

$m = 0, 0, 1, 1, 1$

The general solution is,

$y = (c_1 + c_2 x)e^{0x} + (c_3 + c_4 x + c_5 x^2)e^{1x}$

$\boxed{y = c_1 + c_2 x + c_3 e^x + c_4 x e^x + c_5 x^2 e^x}$

where, c_1, c_2, c_3, c_4, c_5 are arbitrary constants

<u>Question 2.</u> Find the general solution of the following non-homogenous differential eqns with constant coefficients

(a) $(D^2 + a^2)y = \cot ax$

For the complementary function, $(D^2 + a^2)y = 0$

the auxillary eqn is,

$\lambda^2 + a^2 = 0$

$\lambda = \pm ia$

$C.F. = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

$C.F. = c_1 e^{iax} + c_2 e^{-iax}$

where, c_1 & c_2 are arbitrary constants.

For the particular Integral,

$$P.I. = \frac{1}{f(D)} \cot ax = \frac{1}{(D^2 + a^2)} \cot ax$$

Since, the solutions of the auxillary equations are linearly independent we apply the wronskian method for finding the particular integral.

$y_1 = \cos ax \quad ; \quad y_2 = \sin ax$

Let y_p represent the particular integral,

$$y_p = u \cos ax + v \sin ax$$

$$W = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a\cos^2 ax + a\sin^2 ax = a$$

$$W_1 = \begin{vmatrix} 0 & \sin ax \\ \cot ax & a\cos ax \end{vmatrix} = \cot ax \sin ax = \cos ax$$

$$W_2 = \begin{vmatrix} \cos ax & 0 \\ -a\sin ax & \cot ax \end{vmatrix} = \cos ax \cot ax = \frac{\cos^2 ax}{\sin ax}$$

$$u = \int \frac{W_1}{W}\,dx$$

$$u = \int \frac{\cos ax}{a}\,dx$$

$$u = \frac{1}{a}\cdot\frac{\sin ax}{a} = \frac{\sin ax}{a^2}$$

$$v = \int \frac{W_2}{W}\,dx$$

$$v = \int \frac{\cos^2 ax}{a\sin ax}\,dx$$

$$v = \frac{1}{a}\int \frac{1-\sin^2 ax}{\sin ax}\,dx$$

$$v = \frac{1}{a}\int \operatorname{cosec} ax\,dx - \frac{1}{a}\int \sin ax\,dx$$

$$v = \frac{1}{a^2}\{[\ln(\operatorname{cosec} ax - \cot ax) + \cos ax]\}$$

$$y_p = \frac{1}{a^2}\cdot \sin ax \cos ax + \frac{1}{a^2}\cdot\{\ln|\operatorname{cosec} ax - \cot ax| + \cos ax\}\sin ax$$

The general solution = C.F. + P.I.

$$\boxed{y = A\cos ax + B\sin ax + \frac{1}{a^2}\cdot\sin ax\cos ax + \frac{1}{a^2}\{\ln|\operatorname{cosec} ax - \cot ax| + \cos ax\}\sin ax}$$

Ans

(6.) $(D^3 - D^2 - 6D)y = x^2+1$

For the complementary function,

the auxillary eqn;

$$m^3 - m^2 - 6m = 0$$

$$m^3 - 3m^2 + 2m^2 - 6m = 0$$

$$m^2(m-3) + 2m(m-3) = 0$$

$$(m^2+2m)(m-3) = 0$$

$$m(m+2)(m-3) = 0$$

$$m = 0, -2, 3$$

$$C.F. = c_1 e^{0x} + c_2 e^{-2x} + c_3 e^{3x}$$

where, c_1, c_2 and c_3 are complementary arbitrary constants.

For the particular integral;

$$PI = \frac{1}{f(D)}(x^2+1) = \frac{1}{D^3-6D-D^2}(x^2+1)$$

$$\Rightarrow \frac{1}{-6D\left[1 - \frac{(D^2-D)}{6}\right]}(x^2+1)$$

$$\Rightarrow -\frac{1}{6D}\left[1 + \left(\frac{D-D^2}{6}\right)\right]^{-1}(x^2+1)$$

$$\Rightarrow -\frac{1}{6}\int\left(1 - \left(\frac{D-D^2}{6}\right) + \left(\frac{D-D^2}{6}\right)^2\right](x^2+1)$$

$$\Rightarrow -\frac{1}{6}\int\left(x^2+1 - \frac{D}{6}(x^2+1) + \frac{D^2}{6}(x^2+1) + \frac{D^2}{36}(x^2+1)\right]$$

$$\Rightarrow -\frac{1}{6}\left[\frac{x^3}{3}+x - \frac{1}{6}(x^2+1) + \frac{D}{6}(x^2+1) + \frac{D}{36}(x^2+1)\right]$$

$$\Rightarrow -\frac{1}{6}\left[\frac{x^3}{3}+x - \frac{1}{6}(x^2+1) + \frac{(2x)}{6} + \frac{2x}{36}\right]$$

Ans ⇓

$$\Rightarrow P.I. = \boxed{-\frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25x}{10} - \frac{1}{6}\right]}$$

Ans $y = C.F. + P.I.$

(c.) $(D^4+D^2+1)y = e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$

For the complementary function,

the auxillary equation is,

$(m^4+m^2+1) = 0$

$$m^2 = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

$$m = \pm\sqrt{\frac{-1 \pm \sqrt{3}i}{2}} \Rightarrow m_1 = \sqrt{\frac{-1 \pm \sqrt{3}i}{2}}, \quad m_2 = -\sqrt{\frac{-1 \pm \sqrt{3}i}{2}}$$

$m_3 = -\,''\,-$ $\quad m_4 = -\,''\,-$

* $C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x}$

$$m_1 = \sqrt{\frac{-1+\sqrt{3}i}{2}}; \quad m_2 = \sqrt{\frac{-1-\sqrt{3}i}{2}}; \quad m_3 = -\sqrt{\frac{-1+\sqrt{3}i}{2}}, \quad m_4 = -\sqrt{\frac{-1-\sqrt{3}i}{2}}$$

For the particular integral;

$$P.I. = \frac{1}{f(D)} e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) = e^{-x/2}\frac{1}{f(D-\frac{1}{2})}\cos\frac{\sqrt{3}}{2}x$$

$$= e^{-x/2}\frac{1}{(D-\frac{1}{2})^4+(D-\frac{1}{2})^2+1}\cos\frac{\sqrt{3}}{2}x$$

$\left\{ \because \frac{1}{F(D)}\cos ax = \frac{1}{f(-a^2)}\cos ax \right.$

$$= e^{-x/2}\frac{1}{(D-\frac{1}{2})^2\{(D-\frac{1}{2})^2+1\}+1}\cos\frac{\sqrt{3}}{2}x$$

$$= e^{-x/2}\frac{1}{(D^2+\frac{1}{4}-D)\{(D^2+\frac{5}{4}-D)\}+1}\cos\frac{\sqrt{3}}{2}x$$

$\left\{ \frac{1}{f(D^2)} = \frac{1}{0} \because, \because, \frac{1}{f(D^2)}\cos ax = -\frac{x}{2a}\sin ax \right.$

* $$P.I. = e^{-x/2}\frac{-x}{2\left(\frac{\sqrt{3}}{2}\right)}\cdot\sin\frac{\sqrt{3}}{2}x$$

$\boxed{y = C.F. + P.I}$ Ans.

Question 3 solve the following initial value problem

(a) $(D^3 - 2D^2 - 9D + 18)y = e^{2x}$

The auxiliary eqn for the above DF eqn is;

$m^3 - 2m^2 - 9m + 18 = 0$

$m^2(m-2) - 9(m-2) = 0$

$(m^2 - 9)(m-2) = 0$

$m = 2, 3, -3;$

Therefore;
the $C.F. = c_1 e^{2x} + c_2 e^{3x} + c_3 e^{-3x}$ where c_1, c_2 & c_3 are arbitary constants.

for Particular Integral;

$P.I. = \frac{1}{f(D)} e^{2x} = \frac{1}{f(a)} e^{2x}$

$\Rightarrow \frac{1}{(D-3)(D+3)(D-2)} \cdot e^{2x} = \frac{1}{(-1)(5)(D-2)} e^{2x}$

$\Rightarrow -\frac{1}{5} \cdot \frac{1}{(D-2)} e^{2x} = -\frac{1}{5} \frac{x^1}{1!} e^{2x} = -\frac{1}{5} x e^{2x}$

Therefore;

$y = c_1 e^{2x} + c_2 e^{3x} + c_3 e^{-3x} - \frac{1}{5} x e^{2x}$

$y(0) = 4.5 \Rightarrow c_1 + c_2 + c_3 = 4.5$ —— ①

$y'(0) = 8.8 \Rightarrow 2c_2 + 3c_2 - 3c_3 = 9$ —— ②

$y''(0) = 17.2 \Rightarrow 4c_1 + 9c_2 + 9c_3 = 18$ —— ③

$c_1 = 4.5$; $c_2 = 0$; $c_3 = 0$

$\therefore$ $\boxed{y = 4.5e^{2x} - \frac{1}{5} x e^{2x}}$ Ans

(b) $(D^3+6D^2+11D+6)\,y = 2e^{-4x}$

for the complementary function,

the auxillary equation is,

$$m^3+6m^2+11m+6=0$$
$$(m+1)(m^2+5m+6)=0$$
$$(m+1)(m^2+3m+2m+6)=0$$
$$(m+1)(m+3)(m+2)=0$$
$$m=-1,-3,-2$$

$C.F. = c_1 e^{-x} + c_2 e^{-3x} + c_3 e^{-2x}$ where, c_1, c_2 and c_3 are complementary arbitrary constants;

for the Particular Integral.

$$P.I. = \frac{1}{f(D)}\,2e^{-4x} = \frac{1}{f(a)}\,2e^{-4x}$$

$$P.I. = 2\,\frac{1}{(D+1)(D+3)(D+2)}\,e^{-4x}$$

$$P.I. = \frac{2}{-3\cdot -1\cdot -2}\,e^{-4x} = -\frac{1}{3}e^{-4x}$$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-3x} + c_3 e^{-2x} - \frac{1}{3}e^{-4x}}$$

Given, $y(0) = -\frac{1}{3} \Rightarrow -\frac{1}{3} = c_1+c_2+c_3 - \frac{1}{3} \Rightarrow \boxed{c_1+c_2+c_3=0}$

$y'(0) = \frac{4}{3} \Rightarrow y'(0) = -c_1e^{-x} - 3c_2e^{-3x} - 2c_3e^{-2x} + \frac{4}{3}e^{-4x}$

$\frac{4}{3} = (-c_1-3c_2-2c_3) + \frac{4}{3} \Rightarrow \boxed{c_1+3c_2+2c_3=0}$

$y''(0) = \frac{16}{3} \Rightarrow y''(0) = c_1e^{-x} + 9c_2e^{-3x} + 4c_3e^{-2x} - \frac{16}{3}e^{-4x}$

$\frac{16}{3} = c_1+9c_2+4c_3 - \frac{16}{3} \Rightarrow \boxed{c_1+9c_2+4c_3=\frac{32}{3}}$

$c_1 = \frac{16}{3}$; $c_2 = \frac{16}{3}$; $c_3 = -\frac{32}{3}$;

$$\boxed{y = \frac{16}{3}e^{-x} + \frac{16}{3}e^{-3x} - \frac{32}{3}e^{-2x} - \frac{1}{3}e^{-4x}}$$

Ans

$\dots = (7D + 24)e^{t} - (9D+31)e^{2t}$

(c.) $(D^4 - 9D^2 - 400)y = 0$

The auxillary equation of the homogenous DF eqn is,

$$m^4 - 9m^2 - 400 = 0$$

$$m^4 + 16m^2 - 25m^2 - 400 = 0$$

$$m^2(m^2+16) - 25(m^2+16) = 0$$

$$(m^2-25)(m^2+16) = 0$$

$$m = 5, -5, +4i, -4i.$$

Solution of differential eqn is;

$$y = c_1 e^{5x} + c_2 e^{-5x} + c_3 e^{4ix} + c_4 e^{-4ix}$$

Given,

$y(0) = 0 \Rightarrow \boxed{c_1 + c_2 + c_3 + c_4 = 0} \longrightarrow c_1 = -c_3$

$y'(0) = 0 \Rightarrow 0 = 5c_1 e^{5x} - 5c_2 e^{-5x} + 4ic_3 e^{4ix} + 4ic_4 e^{-4ix}$

$$0 = 5c_1 - 5c_2 + 4ic_3 - 4ic_4$$

$$\Downarrow$$

equating Re and Img of LHS and RHS.

$\boxed{c_1 = c_2}$ $\quad c_3 - c_4 = 0 \Rightarrow \boxed{c_3 = c_4}$

$y''(0) = 41 \quad = \quad 41 = 25c_1 + 25c_2 - 16c_3 - 16c_4.$

$$41 = 25(c_1 + c_2) - 16(c_3 + c_4)$$

$$41 = 25(2c_1) - 16(2c_3)$$

$$41 = 50c_1 + 32c_1$$

$$41 = 82\,c_1$$

$$\frac{\cancel{41}}{\cancel{82}}\frac{1}{2} = c_1$$

Solution I,

$$\boxed{y = \frac{1}{2}(e^{5x} + e^{-5x}) - \frac{1}{2}(e^{4ix} + e^{-4ix})}$$

Question 4 Find the general solution of the following differential equations using the method of variation of parameters.

(a) $y'' + 16y = 32 \sec 2x$

$\Rightarrow (D^2+16)y = 32 \sec 2x$ —— ①

$m^2 + 16 = 0$

$m = \pm 4i$

$C.F. = [c_1 e^{4ix} + c_2 e^{-4ix}]$

$C.F. = (A \cos 4x + B \sin 4x)$

$y_1 = \cos 4x$ and $y_2 = \sin 4x$

Let;

$y_p = y_1 u + y_2 v$ —— ②

where $u = u(x)$ and $v = v(x)$;

$y_p = u \cos 4x + v \sin 4x$

Forming the wronskian;

$$W = \begin{vmatrix} \cos 4x & \sin 4x \\ -4\sin 4x & 4\cos 4x \end{vmatrix} = 4(\cos^2 4x + \sin^2 4x) = 4$$

$$W_1 = \begin{vmatrix} 0 & \sin 4x \\ 32\sec 2x & 4\cos 4x \end{vmatrix} = -\sin 4x \, 32 \sec 2x$$

$= (2\sin 2x \cos 2x) \cdot (32 \sec 2x)$

$= -64 \sin 2x$

$$W_2 = \begin{vmatrix} \cos 4x & 0 \\ -4\sin 4x & 32\sec 2x \end{vmatrix} = 32 \sec 2x \cos 4x$$

$$u = \int \frac{W_1}{W}\,dx = \int \frac{\overset{16}{\cancel{-64}} \sin 2x}{\cancel{4}}\,dx = +\frac{16}{2}\cos 2x = 8\cos 2x$$

$$V = \int \frac{W_2}{W}\,dx = \int 16 \cos 2x \,dx - \int 8 \sec 2x = 8 \sin 2x - (4 \ln|\sec 2x + \tan 2x|)$$

$y_p = u\cos 4x + v\sin 4x$

$y_p = 8\cos 2x \cos 4x + (8\sin 2x - 4\ln|\sec 2x + \tan 2x|)\sin 4x$)

Ans $\Rightarrow$ $y = C.F. + y_p$

$$\boxed{y = A\cos 4x + B\sin 4x + 8\cos 2x\cos 4x + \sin 4x\,(8\sin 2x - 4\ln|\sec 2x + \tan 2x|)}$$

(b) $y'' - 3y' + 2y = \dfrac{e^x}{(1+e^x)}$

$\Rightarrow (D^2 - 3D + 2)y = \dfrac{e^x}{1+e^x}$ —— (1)

$m^2 - 3m + 2 = 0$

$m^2 - 2m - m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$(m-1)(m-2) = 0$

$m = 1, 2$;

C.F. $= c_1 e^x + c_2 e^{2x}$

where c_1 & c_2 are arbitrary constants;

$y_1 = e^x$ $\quad y_2 = e^{2x}$

Let $y_p = u e^x + v e^{2x}$

where, $u = u(x)$ and $v = v(x)$

$$W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{2x}e^x - e^{2x}e^x = e^{2x}e^x$$

$$W_1 = \begin{vmatrix} 0 & e^{2x} \\ \frac{e^x}{1+e^x} & 2e^{2x} \end{vmatrix} = -e^{2x}\cdot\frac{e^x}{1+e^x}$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{1+e^x} \end{vmatrix} = e^x\cdot\frac{e^x}{1+e^x}$$

$u = \int \frac{W_1}{W}\,dx$ $\qquad$ $u = \int \frac{W_2}{W}\,dx$

$$u = \int -\frac{e^{2x}e^{x}}{1+e^{x}} \cdot \frac{1}{e^{x}e^{2x}}\,dx = -\int \frac{1}{1+e^{x}}\,dx = \ln\left|\frac{1+e^{x}}{e^{x}}\right|$$

$$v = \int e^{x} \cdot \frac{e^{x}}{1+e^{x}} \cdot \frac{1}{e^{x}e^{2x}}\,dx = \int \frac{1}{e^{x}(1+e^{x})}\,dx = \log\left(\frac{e^{x}}{1+e^{x}}\right) + \frac{1}{1+e^{x}}$$

$$y_P = \log\left(\frac{1+e^{x}}{e^{x}}\right)\cdot e^{x} + \left(\log\left(\frac{e^{x}}{1+e^{x}}\right) + \frac{1}{1+e^{x}}\right)e^{2x}$$

$$\boxed{\text{Ans} \Rightarrow \quad c_1 e^{x} + c_2 e^{2x} + \log\left(\frac{1+e^{x}}{e^{x}}\right)\cdot e^{x} + \log\left(\left(\frac{e^{x}}{1+e^{x}}\right) + \frac{1}{1+e^{x}}\right)e^{2x}}$$

Question 5. Solve the following Differential Equations.

a. $(x^2D^2 + xD + 1)\,y = \ln x \sin(\ln x)$

$\Rightarrow$ substitute $x = e^{z}$

$\Downarrow$

$$\frac{d}{dx} = \frac{1}{x}\frac{d}{dz}$$

$$z = \ln x$$

$$\boxed{x\frac{d}{dx} = \frac{d}{dz}} \Longrightarrow x\frac{dy}{dx} = \frac{dy}{dz}$$

Let, $\theta = \dfrac{d}{dz}$ Then, $\boxed{xDy = \theta y}$

and

$$\boxed{x^2D^2y = \theta^2 y - \theta y}$$

and

$$\boxed{x^3D^3y = \theta(\theta-1)(\theta-2)\,y}$$

Substituting these terms in the original DF eqn we have,

$$(\theta(\theta-1) + \theta + 1)\,y = z\sin z$$

$$(\theta^2 - \theta + \theta + 1)\,y = z\sin z$$

$$(\theta^2+1)\,y = z\sin z$$

The auxiliary equation is,

$$m^2+1=0$$
$$m=\pm i$$

C.F. $= c_1 e^{ix} + c_2 e^{-ix}$ or $A\cos x + B\sin x$

where coefficients c_1, c_2 or A, B are arbitrary constants;

Particular Integral (P.I.) $= \frac{1}{f(D)} z\sin z$

$$\Rightarrow \frac{1}{(D^2+1)} z\sin z$$

$$\Rightarrow z\cdot\frac{1}{(D^2+1)}\sin z - \frac{2D}{(D^2+1)^2}\sin z$$

$$\Rightarrow z\cdot\left(-\frac{z}{2}\cos z\right) - \frac{2D}{D^4+1+2D^2}\sin z$$

$$\Rightarrow \frac{1}{f(D)} z\sin z \Rightarrow \mathrm{Img}\left(\frac{1}{f(D)} z\sin z\right) \quad e^{iz}$$

$$\Rightarrow \mathrm{Img}\left(\frac{1}{(D^2+1)}\cdot z e^{iz}\right) \Rightarrow \mathrm{Img}\left(e^{iz}\frac{1}{((D+i)^2+1)}\cdot z\right)$$

$$\Rightarrow \mathrm{Img}\left(e^{iz}\frac{1}{D^2+i^2+2iD+1} z\right) \Rightarrow \mathrm{Img}\left(\frac{e^{iz}}{2iD}\left[1+\frac{D}{2i}\right]^{-1} z\right]$$

$$\Rightarrow \mathrm{Img}\ \frac{e^{iz}}{2i}\int\left(1+\frac{D}{2i}\right) z\, dz$$

$$\Rightarrow \mathrm{Img}\ -2i e^{iz}\left[\frac{z^2}{2}-\frac{z}{2i}\right] \Rightarrow \boxed{+z^2\sin z + \cos z}$$

$$\Downarrow$$

$$z = \ln x$$

solution;

$$\boxed{y = A\cos x + B\sin x + (\ln x)^2 \sin\ln x + \cos\ln x}$$

(b) $(x^3 D^3 + 3x^2 D^2 - 2xD + 2)y = 0$ ... $t - (9n+3)e^{2t}$

$\Rightarrow$ Let $x = e^z$; $z = \ln x$

$$\frac{dz}{dx} = \frac{1}{x}$$

Let $\frac{d}{dz} \equiv \theta$; $\theta^n = \frac{d^n}{dz^n}$

$\therefore$, $xDy = \theta y$

$x^2 D^2 y = \theta(\theta - 1)y$

$(\theta(\theta-1)(\theta-2) + 3\theta(\theta-1) - 2\theta + 2)y = 0$

$(\theta^3 - 3\theta + 2)y = 0$ — ②

Considering the auxillary equation;

$m^3 - 3m + 2 = 0$;

$(m-1)^2(m+2) = 0$

$m = 1, 1, -2$

Therefore; solution to DFeqn ② is;

$y = c_1 e^z + c_2 x e^z + c_3 e^{-2z}$

Thus,

the solution for the original differential equation is;

$$\boxed{y = c_1 x + c_2 x^2 + c_3 e^{-2\ln x}}$$

Question 6 show that the solutions e^x, e^{-x}, e^{2x} of the ..... the given equation.

Solution :

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$$

$(D^3 - 2D^2 - D + 2)y = 0.$

The auxiliary equation is,

$$m^3 - 2m^2 - m + 2 = 0$$

$$m^2(m-2) - 1(m-2) = 0$$

$$(m^2-1)(m-2) = 0$$

$$(m-1)(m+1)(m-2) = 0$$

$$m = 1, -1, 2 ;$$

Therefore, the solution of the differential eqn is;

$$\boxed{y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x}}$$

where, c_1, c_2 and c_3 are arbitrary constants.

To prove the linear independence of the solution, we frame the wronskian determinant;

$$\begin{vmatrix} e^{x} & e^{-x} & e^{2x} \\ e^{x} & -e^{-x} & 2e^{2x} \\ e^{x} & e^{-x} & 4e^{2x} \end{vmatrix}$$

$$\Rightarrow e^{x}(-4e^{x} - 2e^{x}) - e^{-x}(4e^{3x} - 2e^{3x})$$
$$+ e^{2x}(1+1)$$

$$\Rightarrow -6e^{2x} - 2e^{2x} + 2e^{2x}$$

$$\Rightarrow -6e^{2x} \neq 0 \quad \text{for all values of } x$$

Therefore, the the solution of the DE eqn is linearly independent

Question 7. solve the following systems of differential eqns.

(a) $\frac{dx}{dt} = ax + y$ —— ①, $\frac{dy}{dt} = bx + ay$ —— ②

From eqn ①, the value of y is;

$y = \frac{dx}{dt} - ax$ —— substitute in eqn ②, We have;

$$y = \left(\frac{d}{dt} - a\right)x$$

$$\Rightarrow \frac{d}{dt}\left(\frac{d}{dt} - a\right)x = bx + a\left(\frac{d}{dt} - a\right)x$$

$$\Rightarrow \left(\frac{d^2}{dt^2} - 2a\frac{d}{dt} + (a^2-b)\right)x = 0$$

Let $\frac{d}{dt} \equiv D$

Then;

$$(D^2 - 2aD + (a^2-b))x = 0$$

The auxillary eqn. of the above DEqn is;

$$m^2 - 2am + (a^2-b) = 0$$

$$m_1 = \frac{2a + \sqrt{4a^2 - 4(a^2-b)}}{2} \qquad m_2 = \frac{2a - \sqrt{4a^2-4(a^2-b)}}{2}$$

$$m_1 = \frac{2a + 2\sqrt{b}}{2} = a + \sqrt{b} \qquad m_2 = a - \sqrt{b}$$

$$\boxed{x = c_1 e^{(a+\sqrt{b})t} + c_2 e^{(a-\sqrt{b})t}}$$ Ans

where c_1 and c_2 are arbitrary const

substituting the value of x in eqn y;

$$\frac{dx}{dt} - ax = y$$

$$\boxed{y = c_1(a+\sqrt{b})e^{(a+\sqrt{b})t} + c_2(a-\sqrt{b})e^{(a-\sqrt{b})t} - a\left(c_1 e^{(a+\sqrt{b})t} + c_2 e^{(a-\sqrt{b})t}\right)}$$

Ans.

(b.) $$4\frac{dx}{dt} + 9\frac{dy}{dt} + 11x + 31y = e^t$$

$$3\frac{dx}{dt} + 7\frac{dy}{dt} + 8x + 24y = e^{2t}$$

P.T.O

$$(4D+11)x + (9D+31)y = e^t \quad — ①$$
$$(3D+8)x + (7D+24)y = e^{2t} \quad — ②$$

multiplying ① by $(3D+8)$ and ② by $(4D+11)$ we have after subtracting the two equations,

$$(3D+8)(9D+31)y - (4D+11)(7D+24)y = (3D+8)e^t - (4D+11)e^{2t}$$

$$\Rightarrow (D^2+8D+16)y = 19e^{2t} - 11e^t$$

The auxillary eqn for above DFeqn;

$$m^2+8m+16=0$$
$$(m+4)^2=0$$
$$m=-4,-4$$

$$C.F. = (c_1+c_2t)e^{-4t}$$

$$P.I. = \frac{1}{(D^2+8D+16)}19e^{2t} - \frac{1}{(D^2+8D+16)}11e^t$$

$$P.I. = \frac{19}{4+16+16}e^{2t} - \frac{11}{25}e^t$$

$$= \frac{19}{36}e^{2t} - \frac{11}{25}e^t$$

$$\boxed{y = (c_1+c_2t)e^{-4t} + \frac{19}{36}e^{2t} - \frac{11}{25}e^t}$$

substituting in DFeqn ① we have;
$(4D+11)x + 9c_1(-4)e^{-4t}$ will be tough procedure
therefore we try to eliminate y simultaneously from the two equations

multiplying ① by $(7D+24)$ and ② by $(9D+31)$ we have after subtracting the two equations;

$$(4D+11)(7D+24)x + (9D+31)(3D+8)x = (7D+24)e^t - (9D+31)e^{2t}$$

$$\Rightarrow \{(4D+11)(7D+24) - (9D+31)(3D+8)\}x = (7D+24)e^t - (9D+31)e^{2t}$$

$$\Rightarrow \quad (D+4)^2 x = 31e^t - 49e^{2t}$$

The auxillary equation;

$$(m+4)^2 = 0$$

$$\boxed{m = -4, -4;}$$

$$x = (c_1 + c_2 t)e^{-4t} = C.F.$$

for particular integral;

$$P.I. = 31 \cdot \frac{1}{f(D)} e^t - 49 \cdot \frac{1}{f(D)} e^{2t}$$

$$P.I. = 31 \cdot \frac{1}{f(1)} e^t - 49 \cdot \frac{1}{f(2)} e^{2t}$$

$$P.I = 31 \cdot \frac{1}{25} e^t - 49 \cdot \frac{1}{36} e^{2t}$$

$$P.I. = \frac{31}{25} e^t - \frac{49}{36} e^{2t}$$

$\therefore$,

$$\boxed{x = (c_1 + c_2 t)e^{-4t} + \frac{31}{25}e^t - \frac{49}{36}e^{2t}}$$ Ans

<u>Question 8.</u> solve the system of differential equations;

$$\frac{d^2x}{dt^2} - 2\frac{dy}{dt} - x = e^t \cos t \quad \text{—— (1)}$$

$$\frac{d^2y}{dt^2} + 2\frac{dx}{dt} - y = e^t \sin t \quad \text{—— (2)}$$

<u>solution</u>

Let $\frac{d}{dt} \equiv D$; then

$$(D^2-1)x - 2Dy = e^t \cos t \quad \text{—— (1)}$$

$$(D^2-1)y + 2Dx = e^t \sin t \quad \text{—— (2)}$$

Multiplying first eqn by 2D and second eqn by (D^2-1) and subtracting ② from ① in order to eliminate x we have;

$$(D^2+1)^2 y = e^t \sin t$$

The auxillary equation is;

$$(m^2+1)^2 = 0$$

$$m^2 = -1, -1$$

$$m = \pm i, \pm i;$$

$$C.F. = (c_1 + c_2 t)\cos t + (c_3 + c_4 t)\sin t$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

for particular integral;

$$P.I. = \frac{1}{F(D)} e^t \sin t = e^t \frac{1}{F(D+1)} \sin t$$

$$P.I. = e^t \cdot \frac{1}{((D+1)^2+1)^2} \sin t$$

$$\Rightarrow e^t \frac{1}{(D^2+2D+2)^2} \sin t = e^t \cdot \frac{1}{(2D+1)^2} \sin t$$

$$\Rightarrow e^t \frac{4D+3}{16D^2-9} \sin t = -\frac{e^t}{25}(4\cos t + 3\sin t)$$

$$\therefore \boxed{y = (c_1 + c_2 t)\cos t + (c_3 + c_4)\sin t - \frac{e^t}{25}(4\cos t + 3\sin t)}$$

Substituting the value of y in ② eqn and evaluating for x we have;

$$(D^2-1)y + 2Dx = e^t \sin t$$

$$\Rightarrow \boxed{(D^2+1)^2 x = 2e^t \cos t}$$

solving for x in the same fashion as for y we have,

$$(D^2+1)^2 x = 2e^t \cos t$$

$$P.I. = \frac{1}{f(D)} 2e^t \cos t = 2e^t \frac{1}{f(D+1)} \cos t$$

$$= 2e^t \frac{4D+3}{-25} \cos t$$

$$= -\frac{2}{25} e^t (-4 \sin t + 3\cos t)$$

we have;

$$\boxed{x = (c_1 + c_2 t)\cos t + (c_3 + c_4 t)\sin t + \frac{8}{25} e^t \sin t - \frac{6}{25} e^t \cos t.}$$

where, c_1, c_2, c_3, c_4 are arbitrary constants.

<u>Question 9.</u> $y'' + 4y' + 3y = x \sin 2x$

Solution:

The DF eqn can be represented as;

$$(D^2 + 4D + 3) y = x \sin 2x$$

The auxillary equation can be represented as;

$$m^2 + 4m + 3 = 0$$

$$(m+1)(m+3) = 0$$

$$m = -1, -3$$

$C.F. = c_1 e^{-x} + c_2 e^{-3x}$ where,

c_1 and c_2 are arbitrary constants.

for the particular integral we have;

$$P.I. = \frac{1}{F(D)} x \sin 2x$$

$$= x \frac{1}{F(D)} \sin 2x - \frac{F'(D)}{(F(D))^2} \sin 2x$$

$$\Rightarrow x \cdot \frac{1}{4D-1} \sin 2x - \frac{2D+4}{(4D-1)^2} \sin 2x$$

$$\Rightarrow \quad x \cdot \frac{4D+1}{(4D-1)(4D+1)} \sin 2x \quad + \quad \frac{2D+4}{(8D+63)} \sin 2x$$

$$\Rightarrow \quad x \cdot \frac{4D+1}{16D^2-1} \sin 2x \quad + \quad \frac{(2D+4)(8D-63)}{64D^2-(63)^2} \sin 2x$$

$$\Rightarrow -\frac{x}{65}(8\cos 2x + \sin 2x) \quad + \quad \frac{(188 \cos 2x + 252 \sin 2x)}{4225}$$

Therefore;

$$\boxed{y = c_1 e^{-x} + c_2^{-2x} - \frac{x}{65}(8\cos 2x + \sin 2x) + \frac{(188\cos 2x + 252 \sin 2x)}{4225}}$$

Ans

<u>Question 10</u> solve $(D^2+1)^2 y = 24x\cos x$ given that,

$y = Dy = D^2y = 0$, $D^3 y = 12$, when $x = 0$.

$\Rightarrow$ The auxillary equation is;

$$(m^2+1)^2 = 0$$

$$m = i, i, -i, -i.$$

$$C.F. \Rightarrow e^{ix}(c_1 + c_2 x) + e^{-ix}(c_3 + c_4 x)$$

$\Rightarrow (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$ where

c_1, c_2, c_3 and c_4 are arbitrary constants.

for particular integral we have;

$$P.I. = \frac{1}{F(D)} 24x \cos x$$

$$= 24 \frac{1}{(D^2+1)^2} x \cos x$$

$$\Rightarrow 24 \, Re\left(e^{ix} \cdot \frac{1}{((D+i)^2+1)^2} x\right)$$

$$\Rightarrow 24 \, Re\left(e^{ix} \cdot \frac{1}{(D^2+2iD)^2} x\right)$$

$$\Rightarrow 24 \, Re[2x e^{ix}] \Rightarrow \boxed{48x \cos x} = P.I.$$

Therefore, the general solution of the differential eqn;

$$y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x + 48x\cos x$$

$$y = (48 + c_2)x\cos x + (c_3 + c_4 x)\sin x + c_1\cos x$$

Given;

$y(0) = 0 \Rightarrow \boxed{c_1 = 0}$

$Dy = 0 \Rightarrow (48 + c_2)\cos x - (48 + c_2)x\sin x + c_4\sin x + (c_3 + c_4 x)\cos x$

$$c_2 + c_3 + 48 = 0$$

$$\boxed{c_2 + c_3 = -48}$$

$D^2 y = 0 \Rightarrow 2c_4 = 0$

$$\boxed{c_4 = 0}$$

$D^3 y = 12 \Rightarrow 12 = -96 - 2c_2 - 48 - c_2 - c_3$

$$\boxed{c_3 = +6}$$

$$y = -54x\cos x + 6\sin x + 48x\cos x$$

$$\boxed{y = 6(\sin x - x\cos x)} \quad \text{Ans}$$

<u>Question 11</u> Find the time required . . . . from the origin.

<u>Solution</u>:

The DF Eqn for SHM,

$$(D^2 + \omega^2)x = 0$$

$\omega^2 = \frac{k}{m}$; Aux eqn; $\lambda^2 + \omega^2 = 0$

$$\lambda = \pm i\omega$$

$$x = A\cos\omega t + B\sin\omega t$$

$$x = A\cos(\omega t + \phi)$$

Given: Amplitude = 20 cm

Time Period = $\frac{2\pi}{\omega}$ = 4s $\Rightarrow$ $\boxed{\omega = \pi/2}$

$x = A\cos(\omega t + \phi)$

$\Rightarrow$ $15 = 20\cos(\omega t_1 + \phi)$ —— ①

$5 = 20\cos(\omega t_2 + \phi)$ —— ②

$t_1 = \frac{2}{\pi}\{\cos^{-1}(3/4) - \phi\}$

$t_2 = \frac{2}{\pi}\{\cos^{-1}(1/4) - \phi\}$

$t_2 - t_1 = \frac{2}{\pi}\{\cos^{-1}(1/4) - \cos^{-1}(3/4)\}$ s

$(t_2 - t_1) = \frac{2}{\pi}\{0.595\} = \underline{0.38 \text{ s.}}$ Ans.