Engineering Mathematics

Engineering Analysis

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Reference

1- Advance Engineering and Mathematic, Kreyszing
2- Advance Engineering and Mathematic, Alan Jeffery.
3- Advance Engineering and Mathematic, Wylie.
4- Numerical methods, Robert W. handbook.

Engineering Analysis

Solution of Ordinary Differential Equations

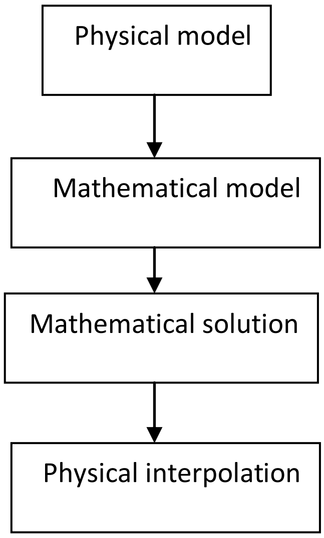
Definitions

Ordinary differential equations (ODE): has derivatives w.r.t. one independent variable.

Independent variable IV:- the variable that equation is Differential with respect to Such as x, t, r, θ, ...

Dependent variable DV:- it is the differentiated variable

Such as y, z, v, u, ∅, ...



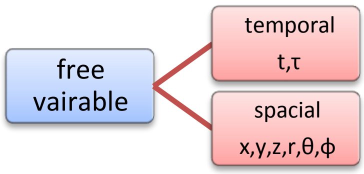
$\frac{\partial y}{\partial x} = 2y + \sin x$, Ex: y(x)

y :- dependent variable (DV); x :- independent variable (IV)

Note:-

$\frac{\partial y}{\partial x} = y'$; $\frac{d^2y}{\partial x^2} = y''$

$\frac{\partial y}{\partial t} = \dot{y}$; $\frac{d^2y}{\partial x^2} = \ddot{y}$



Dimensions: The dimensions of equation depend on spatial independent variable (x, y and z), but not temporal variable (t).

Order of differential equation n:- it's the highest derivative in the equation.

Ex:-

$y'' - 2y' + y = e^x$ 2nd order equation

$2\dot{y} = 3e^x$ 1st order equation

Degree of differential equation :- it's the exponent (degree) of the highest derivative in the equation.

Ex:-

$y'' - (2y')^2 - 5y = \sin x$ 1st degree

$\ddot{y} - 2\dot{y} = 3x^2$ 1st degree

$(y''')^2 + (y'')^3 + 2y' = y + x$ 2nd degree

Linear equation:- the equation in which all terms of the depended variable (derivative and variables) are zero or one degree.

$\ddot{y} + 2\dot{y} = t^2$ (linear)

$y'' - (2y')^2 - 5y = \sin x$ (non-linear)

$\ddot{y} + 3\dot{y}^2 + 3y = e^x$ (non-linear)

General form of Linear ODE

$$\frac{\partial^n y}{\partial x^n} + P(x)_0 \frac{\partial^{n-1} y}{\partial x^{n-1}} + \cdots + P(x)_{n-2} \frac{\partial y}{\partial x} + P(x)_{n-1} y = g(x)$$

Homogenous equation: - the equation in which all terms of the depended variable are of the same degree. g(x)=0 for linear equation

$\ddot{y} - 2\dot{y} = 0$ Homogenous

$(y''')^2 + (y'')^3 + 2y' - y$=0 non-homegenous

Particle solution: - a solution that does not contain arbitrary parameters.

General solution: - a solution that contain arbitrary parameters.

Ex:-

$\dot{y} = 2x$ $\frac{dy}{dx} = 2x$

y = x^2 (particular solution)

y = $x^2 + c$ (general solution)

Singular solution:- it's the particular solution that cannot be obtained from the general solution.

Completed solution:- it's the particular solution that contain all possible solutions including the Singular ones.

Ex:-

$y'' = x$

$$\frac{\partial y}{\partial x} = \frac{1}{2}x^2 + c_1$$

$y = \frac{1}{6}x^3 + c_1x + c_2$ general solution and completed

$y = \frac{1}{6}x^3$ particular solution

$y = \frac{1}{6}x^3 + c_1x$ general solution but not completed

$y = \frac{1}{6}x^3 + c_2$ general solution but not completed

First Order Differential Equations

Separable Equations

$$\frac{dy}{dx} = F(x)G(y).$$

Separating of the variables gives

$$\frac{1}{G(y)}dy = F(x)dx, \qquad G(y) \neq 0$$

Ex: Find the general solution of ODE $\acute{u} = u \ \ sinx$ and specific solution at u(0)=1

$$\frac{du}{dx} = u\ sinx \quad \rightarrow \quad \frac{du}{u} = sinx\ dx$$

$$\ln(u) = -cosx + c$$

$$u(x) = c1\ e^{-cosx} \ \ general\ solution$$

At u(0)=1

$$u(x) = \ e^{1-cosx} \ \ specific\ solution$$

<u>Example</u>:-

Solve the initial value problem for the equation expressed in differential form

$$x^2y^2dx - (1 + x^2)dy = 0, \quad \text{given that } y(0) = 1.$$

Solution The equation is separable because it can be written

$$\frac{dy}{y^2} = \frac{x^2}{(1 + x^2)}dx.$$

Integration gives

$$\int \frac{dy}{y^2} = \int \frac{x^2}{(1 + x^2)}dx,$$

and after the integrations have been performed this becomes

$$-1/y = x - \text{Arctan}\, x + C,$$

where C is an arbitrary constant of integration. This general solution will satisfy the initial condition $y(0) = 1$ if $C = -1$, so the required solution is seen to be

$$y = 1/(\text{Arctan}\, x - x + 1).$$

Example: solve

$$r\frac{d^2T}{dr^2} + \frac{dT}{dr} = 0. \qquad \text{Setting } u = dT/dr$$

Equations Homogeneous

$f(tx, ty) = t^n f(x, y)$

The first order ODE in differential form

$$P(x, y)dx + Q(x, y)dy = 0$$

is called **homogeneous** if P and Q are homogeneous functions of the same degree or, equivalently, if when written in the form

$\dfrac{dy}{dx} = f(x, y),$ the function $f(x, y)$ can be written as $f(x, y) = g(y/x).$

Example:-

Solve

$$\frac{dy}{dx} = \frac{y^2}{xy - x^2}.$$

Solution The equation is homogeneous because it can be written

$$\frac{dy}{dx} = \frac{(y/x)^2}{(y/x) - 1}.$$

Making the substitution $y = ux$, and again using the result $dy/dx = u + xdu/dx$, reduces this to the separable equation

$$u + x\frac{du}{dx} = \frac{u^2}{u - 1}, \text{ or } \left(1 - \frac{1}{u}\right)du = \frac{dx}{x}.$$

Integration gives

$$u - \ln|u| = \ln|x| + \ln|C|,$$

where C is an arbitrary integration constant. Finally, substituting $u = y/x$ and simplifying the result we arrive at the following implicit solution for y:

$$y = Ce^{y/x}.$$

Example:-

Solve

$$(y^2 + 2xy)dx - x^2\,dy = 0.$$

Solution Both terms in the differential equation are homogeneous of degree 2, so the equation itself is homogeneous. Differentiating the substitution $y = ux$ gives

$$\frac{dy}{dx} = u + x\frac{du}{dx}, \quad \text{or} \quad dy = u dx + x du.$$

After substituting for y and dy in the differential equation and cancelling x^2, we obtain the variables separable equation

$$u(u+1)dx = x du, \quad \text{or} \quad \frac{du}{u(u+1)} = \frac{dx}{x}.$$

This has the general solution

$$u = \frac{Cx}{1 - Cx}, \quad \text{but} \quad y = ux \quad \text{and so } y = \frac{Cx^2}{1 - Cx},$$

where C is an arbitrary constant. In this case the general solution is simple and y is determined explicitly in terms of x. ■

Near homogenous

An equation of the form

$$\boxed{\frac{dy}{dx} = \frac{ax + by + c}{px + qy + r}}$$

is called **near-homogeneous**, because it can be transformed into a homogeneous equation by means of a variable change that shifts the origin to the point of intersection of the two lines

$$ax + by + c = 0 \quad \text{and} \quad px + qy + r = 0.$$

Exaple:

Solve the initial value problem

$$\frac{dy}{dx} = \frac{y+1}{x+2y} \quad \text{with } y(2) = 0.$$

Solution The equation is near-homogeneous and the lines $y + 1 = 0$ and $x + 2y = 0$ intersect at the point $x = 2$ and $y = -1$, so we make the variable change $x = X + 2$ and $y = Y - 1$, as a result of which the equation becomes the homogeneous equation

$$\frac{dY}{dX} = \frac{Y}{X + 2Y}.$$

Solving this as in Example 5.9 by setting $Y = uX$ leads to the equation

$$-\left(\frac{1 + 2u}{2u^2}\right)du = \frac{dX}{X},$$

with the solution

$$1/u = 2\ln|CuX|,$$

where C is an arbitrary integration constant. If we set $u = Y/X$, this becomes

$$X = 2Y\ln|CY|,$$

where C is an arbitrary constant. Returning to the original variables by substituting $X = x - 2$, $Y = y + 1$, we arrive at the required general solution

$$x = 2 + 2(y + 1)\ln|C(y + 1)|.$$

Although this is an implicit solution for y, if we regard y as the independent variable and x as the dependent variable, solution curves (integral curves) are easily graphed. Substituting the initial condition $y = 0$ when $x = 2$ in the general solution shows that $C = 1$, so the solution of the initial value problem is

$$x = 2 + 2(y + 1)\ln|y + 1|.$$

Exact Equations

The first order ODE

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be **exact** if a function $F(x, y)$ exists such that the total differential

$$d[F(x, y)] = M(x, y)dx + N(x, y)dy.$$

is exact if and only if $\partial M/\partial y = \partial N/\partial x$.

$F = \int M dx + f(y)$

or $F = \int N dy + g(x)$

Example:-

Show the following equation is exact and find its general solution:

$$\{3x^2 + 2y + 2\cosh(2x + 3y)\}dx + \{2x + 2y + 3\cosh(2x + 3y)\}dy = 0.$$

Solution In this equation $M(x, y) = 3x^2 + 2y + 2\cosh(2x + 3y)$, and $N(x, y) = 2x + 2y + 3\cosh(2x + 3y)$, so as $M_y = N_x = 2 + 6\sinh(2x + 3y)$ the equation is exact:

$$F(x, y) = \int M(x, y)dx = \int \{3x^2 + 2y + 2\cosh(2x + 3y)\}dx$$
$$= x^3 + 2xy + \sinh(2x + 3y) + f(y) + C,$$

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and

$$F(x, y) = \int N(x, y)dy = \int \{2x + 2y + 3\cosh(2x + 3y)\}dy$$
$$= 2xy + y^2 + \sinh(2x + 3y) + g(x) + D.$$

For these two expressions to be identical, we must set $f(y) \equiv y^2$, $g(x) \equiv x^3$, and $D = C$, so $F(x, y)$ is seen to be

$$F(x, y) = x^3 + 2xy + y^2 + \sinh(2x + 3y) + C,$$

and so the general solution is

$$x^3 + 2xy + y^2 + \sinh(2x + 3y) = C,$$

where as C is an arbitrary constant we have chosen to write C rather than $-C$ on the right of the solution. ■

$$check \;\; dF(x,y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

Linear First Order Equations

-population

-physics

-biology

The **standard form** of the **linear first order differential equation** is

standard form of linear first order equation

$$\frac{dy}{dx} + P(x)y = Q(x),$$

Rule for solving linear first order equations

STEP 1 If the equation is not in standard form and is written

$$a(x)\frac{dy}{dx} + b(x)y = c(x),$$

divide by $a(x)$ to bring it to the standard form

$$\frac{dy}{dx} + P(x)y = Q(x),$$

with $P(x) = b(x)/a(x)$ and $Q(x) = c(x)/a(x)$

STEP 2 Find the integrating factor

$$\mu(x) = \exp\left\{\int P(x)dx\right\}.$$

STEP 3 Rewrite the original differential equation in the form

$$\frac{d(\mu y)}{dx} = \mu Q(x).$$

STEP 4 Integrate the equation in Step 3 to obtain

$$\mu(x)y(x) = \int \mu(x)Q(x)dx + C.$$

STEP 5 Divide the result of Step 4 by $\mu(x)$ to obtain the required general solution of the linear first order differential equation in Step 1.

STEP 6 If an initial condition $y(x_0) = y_0$ is given, the required solution of the i.v.p. is obtained by choosing the arbitrary constant C in the general solution found in Step 5 so that $y = y_0$ when $x = x_0$.

Example:-

Solve the initial value problem

$$\cos x \frac{dy}{dx} + y = \sin x, \text{ subject to the initial condition } y(0) = 2.$$

Solution We follow the steps in the above rule.

STEP 1 When written in standard form the equation becomes

$$\frac{dy}{dx} + \frac{1}{\cos x} y = \tan x,$$

so $P(x) = 1/\cos x$ and $Q(x) = \tan x$.

STEP 2 The integrating factor

$$\mu(x) = \exp\left\{\int \frac{dx}{\cos x}\right\} = \exp\{\ln|\sec x + \tan x|\}$$

$$= \sec x + \tan x = \frac{1 + \sin x}{\cos x}.$$

STEP 3 The original differential equation can now be written

$$\frac{d}{dx}\left[\left(\frac{1+\sin x}{\cos x}\right) y(x)\right] = \left(\frac{1+\sin x}{\cos x}\right) \tan x.$$

STEP 4 Integrating the result of Step 3 gives

$$\left(\frac{1+\sin x}{\cos x}\right) y(x) = \int \left(\frac{1+\sin x}{\cos x}\right) \tan x dx + C$$

$$\left(\frac{1+\sin x}{\cos x}\right)y(x) = \int \left(\frac{1+\sin x}{\cos x}\right)\tan x dx + C$$

$$= \int \sec x \tan x dx + \int \tan^2 x dx + C$$

$$= \sec x + \tan x - x + C = \frac{1+\sin x}{\cos x} - x + C.$$

STEP 5 Dividing the result of Step 4 by the integrating factor $\mu(x) = (1+\sin x)/\cos x$ shows that the required general solution is

$$y(x) = \frac{C\cos x}{1+\sin x} + 1 - \frac{x\cos x}{1+\sin x},$$

for x such that $1+\sin x \neq 0$.

The *complementary function* is seen to be

$$y_c(x) = \frac{C\cos x}{1+\sin x},$$

and the *particular integral* is

$$y_p(x) = 1 - \frac{x\cos x}{1+\sin x}.$$

STEP 6 The initial condition requires that $y = 2$ when $x = 0$, and the general solution is seen to satisfy this condition if $C = 1$, so the solution of the i.v.p. is

$$y(x) = 1 + \frac{(1-x)\cos x}{1+\sin x}.$$

The Bernoulli Equation

The **Bernoulli equation** is a nonlinear first order differential equation with standard form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \quad (n \neq 1).$$

The substitution

$$u = y^{1-n}$$

The reduces of the above equation to the linear first order O.D.E.

$$\frac{1}{(1-n)}\frac{du}{dx} + P(x)u = Q(x),$$

Example:

Find the general solution of

$$\frac{dy}{dx} - 2y = xy^{1/2}.$$

Let $u = y^{1/2}$

$$\frac{du}{dx} = \frac{1}{2}y^{-1/2}\frac{dy}{dx} = \frac{1}{2u}\frac{dy}{dx}, \quad \text{so} \quad \frac{dy}{dx} = 2u\frac{du}{dx}.$$

$$\frac{du}{dx} - u = \frac{1}{2}x.$$

$$u(x) = Ce^x - (1/2)(1 + x),$$

so as $u = y^{1/2}$, the required general solution of the Bernoulli equation is

$$y(x) = [Ce^x - (1/2)(1 + x)]^2.$$

Where

i.v.p. initial value problem