Name:-> Parth Johri

RollNo:-> 2K20/B17/33

MATHS ASSIGNMENT NO - 3

DELHI TECHNOLOGICAL UNIVERSITY

Parth johri
2K20/B17/33

Ques 7. Find a Series solution in powers of 'x' and 'x-1' of differential equation $y'' - xy = 0$

Clearly, $x=0$ is an ordinary point. Let $y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$

$\therefore \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0$

$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$

Recurrence relation $n(n-1)a_n - a_{n-3} = 0$

$a_n = \dfrac{a_{n-3}}{n(n-1)}$

$a_2 = 0,\ a_3 = \dfrac{a_0}{6},\ a_4 = \dfrac{a_1}{12},\ a_5 = 0,\ a_6 = \dfrac{a_0}{180},\ a_7 = \dfrac{a_1}{504}$

$y = a_0 + a_1 x + \dfrac{a_0 x^3}{6} + \cancel{a_3 x} \dfrac{a_1 x^4}{12} + \dfrac{a_0 x^6}{180} + \dfrac{a_1}{504} x^7 + \dots$

$y = a_0\left(1 + \dfrac{x^3}{6} + \dfrac{x^6}{180} + \dots\right) + a_1\left(x + \dfrac{x^4}{12} + \dfrac{x^7}{504} + \dots +\right)$ a_0 & a_1 are constant

Clearly, $x=1$ is an ordinary point

Let $y = \sum_{n=0}^{\infty} a_n (x-1)^n$

Now, $x-1 = t \Rightarrow x = t+1$

$dy/dx = dy/dt \cdot dt/dx = dy/dt$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}\left(dy/dt\right) = \dfrac{d}{dt}\left(\dfrac{dy}{dt}\right)\left(\dfrac{dt}{dx}\right) = \dfrac{d^2y}{dt^2}$

Substituting in differential equation, Parth Johri 2K20/B17/33

$$\frac{d^2y}{dt^2} - (t+1)y = 0$$

∴ Solution would be $y = \sum_{n=0}^{\infty} a_n t^n$

$$\frac{dy}{dt} = \sum_{n=0}^{\infty} n a_n t^{n-1} \quad \frac{d^2y}{dt^2} = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2}$$

$$\therefore \sum_{n=0}^{\infty} n a_n (n-1) t^{n-2} - \sum_{n=0}^{\infty} a_n t^{n+1} - \sum_{n=0}^{\infty} a_n t^n = 0$$

Recurrence relation

$$n(n-1)a_n - a_{n-3} - a_{n-2} = 0$$

$$a_n = \frac{a_{n-2} + a_{n-3}}{n(n-1)}$$

$$a_2 = \frac{a_0}{2}, \; a_3 = \frac{a_0 + a_1}{6}, \; a_4 = \frac{2a_1 + a_0}{24}$$

$$y = a_0 + a_1 + \frac{a_0}{2}t^2 + \left(\frac{a_0 + a_1}{6}\right)t^3 + \left[\frac{2a_1 + a_0}{24}\right]t^4 \ldots$$

$$y = a_0\left[1 + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \ldots\right] + a_1\left[t + \frac{t^3}{6} + \frac{t^4}{12} + \ldots\right)$$

$$y = a_0\left[1 + \frac{(x-1)^2}{2} + \frac{(x-1)^2}{6} + \frac{(x-1)^3}{24} + \ldots\right] + a_1\left[(x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \ldots\right]$$

where

a_0 and a_1 are constants

Ques 2) Classify the singular point of the following equations :→

Parthajoti
2K20/BO/133

a) $y'' + \frac{2x}{1-x^2} y' + \frac{n(n+1)}{1-x^2} y = 0$

$\to Q(x)$ Singular points at 1 & -1

At 1,

$\lim\limits_{x\to 1} (x-1)P(x) = -1$, $\lim\limits_{x \to 1} (x-1)^2 Q(x) = n(n+1)$

$\therefore$ n = 1 is a ~~regulator~~ regular singular point

At $x = -1$, $\lim\limits_{x \to -1} (x+1)P(x) = -1$, $\lim\limits_{x\to -1} (x+1)^2 Q(x) = 0$

$\therefore x = -1$ is a regular singular point

b) $y'' + \frac{1}{x-2} y' + \frac{6}{x^3(x-2)} y = 0$ Singular points at 0 and 2

$\frac{1}{x-2} \downarrow P(x)$, $\frac{6}{x^3(x-2)} \downarrow Q(x)$

At $x = 0$, $\lim\limits_{x\to 0} xP(x) = 0$ $\lim\limits_{x\to 0} x^2 Q(x) =$ does not exist

$\therefore x = 0$ is an irregular singular point.

At $x = 2$, $\lim\limits_{x\to 2} (x-2)P(x) = 1$, $\lim\limits_{x\to 2} (x-2)^2 Q(x) = 0$

$\therefore x = 2$ is a regular singular point.

c) $y'' + \frac{\cos x}{\left(x - \frac{\pi}{2}\right)^2} y' + \frac{\sin x}{\left(x - \frac{\pi}{2}\right)^2} y = 0$

$\downarrow P(x)$, $\downarrow Q(x)$ Singular point at $x = \frac{\pi}{2}$

At $x = \pi/2$, $\lim\limits_{x \to \pi/2} (x - \pi/2) P(x) = 1$

$\lim\limits_{x\to \pi/2} (x - \pi/2)^2 Q(x) = 1$ $\therefore x = \frac{\pi}{2}$ is a regular singular point

(Ques 3.) Find power series solution of the following differential equation (d.e),

Parth Gulati
2K20/B17/33

a.) $y'' + (x-1)y' + y = 0$ about $x = 2$.

Clearly, $x=2$ is an ordinary point. Let solution by $y = \sum_{n=0}^{\infty} a_n (x-2)^n$

Now, $x-2 = t \Rightarrow t+2 = x$

$dy/dx = dy/dt \cdot dt/dx = dy/dt$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) + y = 0$$

$\therefore$ Its solution would be $y = \sum_{n=0}^{\infty} a_n t^n$

$$\therefore \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} + (t+1)\sum_{n=0}^{\infty} n a_n t^{n-1} + \sum_{n=0} a_n t^n = 0$$

Recurrence relation,

$$n(n-1)a_n + (n-1)a_{n-1} + a_{n-2}(n-1) = 0$$

$$a_n = -\left(\frac{a_{n-1} + a_{n-2}}{n}\right)$$

When, $n = 2, 3, 4$ ----- $a_2 = -\left(\frac{a_0 + a_1}{2}\right)t^2 + \left(\frac{a_0 - a_1}{6}\right)t^3 + \left[\frac{2a_1 + a_0}{12}\right]t^4 + \text{------}]$

$$y = a_0\left[1 - \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{12} + \text{---}\right] + a_1\left(t - \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{16} + \dots\right)$$

$$\therefore y = a_0\left[1 - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{6} + \frac{(x-2)^4}{12} + \text{---}\right] + a_1\left\{(x-2) - \frac{(x-2)^2}{2} - \frac{(x-2)^3}{6} + \dots\right\}$$

Parth Jolly
2K20/B17/33

b) $(1-x^2)y'' + 2xy' + y = 0$ about $x=0$

$dy/dx = \sum_{n=0}^{\infty} n a_n x^{n-1}$ $\quad \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$

Substituting this in the differential equation

$$(1-x^2)\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + 2x\sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Recurrence relation $n(n-1)a_n - (n-2)(n-3)a_{n-2}$
$+ 2(n-2)a_{n-2}$.

$$a_n = \frac{(a_{n-2})\cdot(n^2+7n+9)}{n(n-1)}$$

when, $n = 2, 3, 4 \ldots\ldots$

$a_2 = -\frac{a_0}{2}$, $a_3 = -\frac{a_1}{2}$, $a_4 = \frac{a_0}{8}$

$y = a_0 + a_1 x + a_2 \frac{x^2}{2} - \frac{a_1 x^3}{4} + \frac{a_4 x^4}{8} + \ldots$

$y = a_0\left[1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{16} + \ldots\right] + a_1\left(x - \frac{x^3}{2} + \ldots\right)$

c) $y'' + x^3 y = 0$ about $x=0$

Clearly, $x=0$ is an ordinary point. Let solution by $y = \sum_{n=0}^{\infty} a_n x^n$ differential equation,

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + x^3 \sum_{n=0}^{\infty} a_n x^n = 0$$

Recurrence relation $n(n-1)a_n + a_{n-5} = 0$

$$a_n = -\left(\frac{a_{n-5}}{n(n-1)}\right)$$

Now, putting $n = 2, 3, 4, \ldots$

Partha Jolai
2K20/B7/33

$a_2 = a_3 = a_4 = a_7 = a_8 = a_9 = 0$

$a_5 = \frac{-a_0}{20}, a_{10} = \frac{a_0}{180}, a_6 = \frac{-a_1}{30}, a_{11} = \frac{a_1}{3300}$

$y = a_0 + a_1 x + \frac{a_0 x^5}{20} - \frac{a_1 x^6}{30} + \frac{a_0}{180}(x^{10}) + \frac{a_1}{3300} x^{11} + \cdots$

$y = a_0 \left[1 - \frac{x^5}{20} + \frac{x^{10}}{180} + \cdots\right] + a_1 \left[x - \frac{x^6}{30} + \frac{x^{11}}{3300} + \cdots\right)$

Ques4) Find a power series solution of $(x^2-1)y'' + 3xy' + xy = 0$ subject to:->

a) $y(0) = 4, y'(0) = 6$

$x = 0$ is an ordinary point. Let solution be $y = \sum_{n=0}^{\infty} a_n x^n$

Substituting y'' and y' in equation, $(x^2-1)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$
$+ 3x \sum_{n=0}^{\infty} n a_0 x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n$.

Recurrence relation,

$n(n-1)a_n = (n-2)(n-3)a_{n-2} + 3(n-2)a_{n-2} + a_{n-3}$

$a_0 = \frac{a_{n-2}[n^2 - 5n + 3n] + a_{n-3}}{n(n-1)}$

where, $n = 2, 3, 4$, $a_2 = 0$, $a_3 = \frac{a_0 + 3a_1}{6}$, $a_4 = \frac{a_1}{12}$

$y = a_0 + a_1 x + \frac{a_0 + 3a_1}{6} x^3 + \frac{a_1}{12} x^4 + \cdots$ ——— (1)

$y = a_0 \left\{1 + \frac{x^3}{6} + \cdots\right\} + a_1 \left[x + \frac{x^3}{2} + \frac{x^4}{12} + \cdots\right]$ ——— (2)

Use $y(0) = 4$ in (2), $a_0 = 4$, Use $y'(0) = 6$ in (1)

$a_1 = 6$.

Parth Johri
2K20/B17/33

b.) $y(2)=4,\ y'(2)=6$

$x=2$ is an ordinary point. Let solution by $y=\sum_{n=0}^{\infty} a_n(x-2)^n$

Now, $x-2=t \Rightarrow x=t+2$

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right)=\frac{d^2y}{dt^2}$$

Substituting in differential equation (d.e.),

$$(t+1)(t+3)\frac{d^2y}{dt^2}+3(t+2)\frac{dy}{dt}+(t+2)y=0$$

∴ It's solution would be $y=\sum_{n=0}^{\infty} a_n t^n$

∴ The differential Equation would be :-> $(t^2+4t+3)\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2}+$

$$(3t+6)\sum_{n=0}^{\infty} n a_n t^{n-1}+(t+2)\sum_{n=0}^{\infty} a_n t^n=0$$

Recurrence relation:-> $(n-2)(n-3)a_{n-2}+4(n-1)(n-2)a_{n-1}+3(n-2)a_{n-2}$

$\Rightarrow 3n(n-1)a_n+(n-2)(n-3)a_{n-2}+4(n-1)(n-2)a_{n-1}+3(n-2)a_{n-2}$

$+6(n-1)a_{n-1}+a_{n-3}+2a_{n-2}=0$

$$a_n=\frac{\left[a_{n-1}(-4n^2+6n-2)+a_{n-2}(-n^2+2n-2)-a_{n-3}\right]}{3n(n-1)}$$

when, $n=2,3;$

$a_2=-\frac{(2a_0+6a_1)}{6}t^2,\quad a_3=\frac{17a_0+45a_1}{54}$

$\therefore y=a_0+a_1t+-\frac{(2a_0+6a_1)}{6}t^2+\left(\frac{17a_0+45a_1}{54}\right)t^3+\ldots\ldots \text{ ——(1)}$

$y=a_0\left(1-\frac{(x-2)^2}{3}+\frac{17}{54}(x-2)^3+\ldots\right)+a_1\left((x-2)-(x-2)^2+\frac{45}{54}(x-2)^3+\ldots\right)$

Put $x=2 \Rightarrow 4=a_0$, use $y'(2)=6$ in (1), $6=a_1$

$y=4\left(1-(x-2)^2/3+17/54(x-2)^3+\ldots\right)+6\left((x-2)-(x-2)^2+\frac{45}{54}(x-2)^3+\ldots\right)$

Ques 5.) Show that differential equation $(1-x^2)y''(x) - xy'(x) + a^2 y(x) = 0$ with $a \in (0,\infty)$ has the following linearly independent power series solutions.

Parth John
2K20/B0/33

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{(2n)!}\left[\prod_{k=0}^{n-1}(4k^2-a^2)\right]x^{2n} \text{ and }.$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{1}{(2n+1)!}\left[\prod_{k=0}^{n-1}\left[4k^2+4k+1-a^2\right]\right]x^{2n+1}$$

The point $x=0$ is an ordinary point so we look for a solution of the form $y(x)=\sum_{n=0}^{\infty} a_n x^n$. Plug the series into the equation to get

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} \alpha^2 a_n x^n = 0$$

Upon Rearrangement we get,

$$2a_2 + \alpha^2 a_0 + (6a_3 - a_1 + \alpha^2 a_0)x + \sum_{n=2}^{\infty}\{(n+2)(n+1)a_{n+2} - [n(n+1)+n-\alpha^2]a_n\}x^n = 0$$

Equating the powers of 'x' yields,

$$2a_2 + \alpha^2 a_0 = 0$$

$6a_3 - a_1 + \alpha^2 a_0 = 0 \quad (n+2)(n+1)a_{n+2} - (n^2-\alpha^2)a_n = 0,$

$n = 2, 3, \ldots$

Solving for the a_n's we get,

Parth Tobri
2K20/B12/33

$a_2 = \frac{-\alpha^2 a_0}{2}$

$a_3 = \frac{1-\alpha^2}{3!} a_1$ Where $n = 2, 3, \ldots$

$a_{n+2} = \frac{n^2 - \alpha^2}{(n+2)(n+1)} a_n$

Writing out the first few explicitly yields,

$a_4 = \frac{(2^2-\alpha^2)(-\alpha^2) a_0}{4!}$, $a_5 = \frac{(3^2-\alpha^2)(1-\alpha^2) a_1}{5!}$, $a_6 =$

$a_6 = \frac{(4^2-\alpha^2)(2^2-\alpha^2)(-\alpha^2) a_0}{6!}$

$a_{2n} = \frac{[(2n-2)^2-\alpha^2][(2n-4)^2-\alpha^2] \cdots (2^2-\alpha^2)(-\alpha^2) a_0}{2n!}$

where
$n = 1, 2, \ldots$

$a_{2n+1} = \frac{[(2n+1)^2-\alpha^2][(2n-3)^2-\alpha^2] \cdots [3^2-\alpha^2](1-\alpha^2) a_1}{(2n+1)!}$

where
$n = 1, 2, \ldots$

Hence, two nontrivial solutions are given by

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{[(2n-2)^2-\alpha^2][(2n-4)^2-\alpha^2] \cdots (2^2-\alpha^2)(-\alpha^2) x^{2n}}{(2n)!}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{[(2n-1)^2-\alpha^2][(2n-3)^2-\alpha^2] \, [3^2-\alpha^2](1-\alpha^2) x^{2n+1}}{(2n+1)!}$$

Ques 6) Use the method of frobenius to find solutions of the following differential equation in some interval $0<x<R$.

Parth Johri
2K20/B1/33

a) $2x^2y'' - xy' + (x-5)y = 0$

$y'' - \frac{1}{2x}y' + \frac{x-5}{2x^2}y = 0$ so $x=0$ is a regular singular point since, $\lim_{x\to 0} xP(x) = \frac{-1}{2}$ and $\lim_{x\to 0} x^2Q(x) = \frac{-5}{2}$.

Let the solution of differential equation be

$y = (x-0)^r \sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} C_n x^{n+r}$.

$y' = \sum_{n=0}^{\infty} (n+r) C_n x^{n+r-1}$, $y'' = (n+r)(n+r-1) C_n x^{n+r-2}$

$+(x-5)\sum_{n=0}^{\infty} C_n x^{n+r} = 0$

$\sum_{n=0}^{\infty} [2(n+r)(n+r-1) - (n+r) - 5] C_n x^{n+r} + \sum_{n=0}^{\infty} C_n x^{n+r+1} = 0$

Equating to zero the coefficient of lowest power, $n=0$.

$2r(r-1) - r - 5 = 0 \Rightarrow 2r^2 - 3r - 5 = 0 \Rightarrow r = 5/2$ and -1

Indicial Equation.

Recurrence relation $[2(n+r)(n+r-1) - (n+r) - 5] C_n + C_{n-1}$

To obtain $y_1(x)$, put $r = 5/2$. Then (1) reduces to

$C_n = \frac{-C_{n-1}}{2n^2+7n}$ For $n = 1, 2, 3 \ldots$

$C_1 = \frac{-C_0}{9}$ $C_2 = \frac{C_0}{198}$ $C_3 = \frac{-C_0}{7722}$

Parth Johri 2K20/BT/33

$\therefore y_1(x) = C_0 x^{5/2}\left(1 - \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \dots\right]$

To obtain $y_2(x)$, put $r = -1$ in (1)

Then RR reduces to,

$$C_n = \frac{-C_{n-1}}{2n^2 - 7n}$$

For $n = 1, 2, 3 \dots$

$C_1 = \frac{C_0}{5}$, $C_1 = \frac{C_0}{30}$, $C_3 = \frac{C_0}{90}$

$\therefore y_2(x) = C_0 x^{-1}\left(1 + \frac{x}{5} + \frac{x^2}{30} + \frac{x^3}{90} + \dots\right\}$

Hence, complete solution will be $y = C_1 y_1 + C_2 y_2$

$y = A x^{5/2}\left(1 - \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \dots\right] + B x^{-1}\left[1 + \frac{x}{5} + \frac{x^2}{30} + \frac{x^3}{90}\right)$

b.) $2x^2y'' + xy' + (x^2 - 3)y = 0$

$y'' + \frac{1}{2x}y' + \left(\frac{x^2 - 3}{2x^2}\right)y = 0$, So, $x = 0$ is a regular singular point since, $\lim_{x \to 0} xP(x) = \frac{1}{2}$ and $\lim_{x \to 0} x^2 Q(x) = -\frac{3}{2}$

Let the solution of differential equation be

$y = \cancel{\sum_{x \to 0}^{\infty} x^2 Q(x) = -\frac{3}{2}}$

$y = \sum_{n=0}^{\infty} C_n x^{n+r}$. $y' = \sum_{n=0}^{\infty} (n+r)\, C_n x^{n+r-1}$

$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, C_n x^{n+r-2}$

Substitute y, y', y'' in the differential equation,

Parth John
2K20/18/7135

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) C_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r) C_n x^{n+r-1}$$

$$+(x^2-3) \sum_{n=0}^{\infty} C_n x^{n+r} = 0.$$

$$\sum_{n=0}^{\infty} [2(n+r)(n+r-1) + n+r-3] C_n x^{n+r} + \sum_{n=0}^{\infty} C_n x^{n+r+2} = 0$$

Equating to zero the coefficient of lowest power, $n=0$.

Indicial equation $2r^2 - r - 3 = 0 \Rightarrow r = \frac{3}{2}$ and -1.

Recurrence relation $[2(n+r)(n+r-1) + (n+r) - 3] C_n + C_{n-2} = 0$

To obtain $y_1(x)$ put $r = \frac{3}{2}$. Then RR reduces to,

$$C_n = -\frac{C_{n-2}}{2n^2+5n}$$

For $n = 1, 2, 3 \ldots$ $C_1 = 0$, $C_2 = \frac{-C_0}{18}$, $C_3 = \frac{C_0}{936}$ $C_3 = 0$

$$\therefore y_1(x) = C_0 x^{\frac{3}{2}} \left[1 - \frac{x^2}{18} + \frac{x^4}{936} + \ldots\right]$$

To obtain $y_2(x)$, put $r = -1$ in ①. Then RR reduces to,

$$C_n = \frac{-C_{n-2}}{2n^2-5n}.$$

For $n=1,2,3 \ldots\ldots$ $C_1=0, C_2=\frac{C_0}{2}, C_3=0, C_4=\frac{-C_0}{24}$

Parth Johri
2K20/B17/33

$\therefore y_2(x)= C_0 x^{-1}\left(1+\frac{x^2}{2}-\frac{x^4}{24}+2 - - -\right]$

Hence, complete solution will be $y=C_1y_1+C_2y_2$

$$y= A x^{3/2}\left(1-\frac{x^2}{18}+\frac{x^4}{936}+ - - -\right]+Bx^{-1}\left(1+\frac{x^2}{2}-\frac{x^4}{24}+ - -\right).$$

c) $x^2y'' - xy' - (x^2+\frac{5}{4})y=0$

$y''-y'/x - \frac{(x^2+5/4)}{x^2}y=0$, So, $x=0$ is a point

Since, $\lim_{x\to 0} xP(x)=-1$ and $\lim_{x\to 0} x^2Q(x)=-\frac{5}{4}$.

Let the solution of Differential Equation be

$$y=\sum_{n=0}^{\infty} C_n x^{n+r}$$

$y'=\sum_{n=0}^{\infty}(n+r)C_n x^{n+r-1}$ $y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n x^{n+r-2}$

Substitute y, y', y'' in the differential Equation

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n x^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)C_n x^{n+r-1}$$

$$-(x^2+5/4)\sum_{n=0}^{\infty}C_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)-(n+r)-\frac{5}{4}\right]C_n x^{n+r} - \sum_{n=0}^{\infty}C_n x^{n+r+2}=0$$

Equating to zero the coefficient of lowest power of $n=0$.

Parthi Johni
22K20/2/33

Indicial Equation $4x^2-8x-5=0 \Rightarrow r=\frac{1}{2} \,\&\, \frac{5}{2}$

Recurrence relation $[(n+r)(n+r-1)-(n+r)-\frac{5}{4}]$
$C_n - C_{n-2} = 0 \longrightarrow ①$

To obtain $y_1(x)$, put $r=\frac{-1}{2}$. Then RR reduces to
$$C_n = \frac{(C_{n-2})}{(n^2-3n)}$$

For $n=1,2,3,$ ---- $C_1=0$, $C_2=-\frac{C_0}{2}$, $C_3=0$, $C_4=\frac{-C_0}{28}$

$\therefore y_2(x) = C_0 x^{5/2}\left(1+x^2/10+x^4/280+ - - \right)$

Hence, complete solution will be $y=C_1y_1+C_2y_2$

$$y = A x^{-1/2}\left[1-\frac{x^2}{2}-\frac{x^4}{2\cdot 4}-\sum_{n=3}^{\infty}\frac{x^{2n}}{(2\cdot 4\cdot 6\ldots 2n)(3\cdot 5\cdot 7\ldots 2n)}\right]$$
$$+Bx^{5/2}\times\left[1+\sum_{n=1}^{\infty}\frac{x^{2n}}{(2\cdot 4\cdot 6\ldots 2n)(5\cdot 7\cdot 9\ldots 2n+3)}\right]$$

d.) $x^2y''+(x^2-3x)y'+y=0$

$y''+\left(\frac{x^2-3x}{x^2}\right)y'+y/x^2=0$, so, $x=0$ is a regular singular point since, $\lim_{x\to 0} xP(x)=-3$ and $\lim_{x\to 0} x^2Q(x)=1$

Let the solution of differential equation be

Parth Tolia
2K20/BT/33;

$$y = \sum_{n=0}^{\infty} C_n x^{n+r}$$

Substitute y, y', y'' in the differential equation

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) C_n x^{n+r-2} + (x^2 - 3x) \sum_{n=0}^{\infty} (n+r) C_n x^{n+r-1}$$

$$+ \sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) - 3(n+r)+3] C_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) C_n x^{n+r+1} = 0$$

Equating to zero the coefficient of lowest power, $n=0$.

Indicial Equation $[(n+r)(n+r-1) - 3(n+r)+3] C_n + (n+r-1) C_{n-1} = 0$

To obtain $y_1(x)$, put $r = 2+\sqrt{3}$. Then RR reduces to

$$C_n = \frac{-C_{n-1} \times (n+1+\sqrt{3})}{n(n+2\sqrt{3})}$$

For $n = 1, 2, \ldots$ $C_1 = \frac{-C_0(2+\sqrt{3})}{(1+2\sqrt{3})}$ $C_2 = \frac{C_0(9+5\sqrt{3})}{4(7+3\sqrt{3})}$

$$\therefore y_1(x) = C_0 x^{2+\sqrt{3}} \left(1 - \frac{x(2+\sqrt{3})}{(1+2\sqrt{3})} + \frac{x^2(9+5\sqrt{3})}{4(7+3\sqrt{3})} + \cdots\right)$$

To obtain $y_2(x)$, put $r = 2-\sqrt{3}$, RR reduces to

Parth Johri
2K20/B17/33

$$C_n = \frac{-C_{n-1}(n+1-\sqrt{3})}{n(n-2\sqrt{3})}$$

For $n = 1, 2, 3 \ldots\ldots$ $C_1 = \frac{-C_0(2-\sqrt{3})}{1-2\sqrt{3}}$, $C_2 = \frac{C_0(9-5\sqrt{3})}{4(7-3\sqrt{3})}$

$$\therefore y_2(x) = C_0 x^{2-\sqrt{3}}\left[1 - \frac{x(2-\sqrt{3})}{1-2\sqrt{3}} + \frac{x^2(9-5\sqrt{3})}{4(7-3\sqrt{3})} \ldots\right]$$

Hence, complete solution will be

$$y = C_1 y_1 + C_2 y_2$$

$$y = Ax^{2+\sqrt{3}}\left(1 - \frac{x(2+\sqrt{3})}{1+2\sqrt{3}} + \frac{x^2(9+5\sqrt{3})}{4(7+3\sqrt{3})}\right) + Bx^{2-\sqrt{3}}$$

$$\left[1 - \frac{x(2-\sqrt{3})}{1-2\sqrt{3}} + \frac{x^2(9-5\sqrt{3})}{4(7-3\sqrt{3})} + \ldots\right]$$

Ques 7) Define Legendre Polynomial $P_n(x)$.
If m and n are non negative integers then show that,

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$$

Case I: When $m \neq n$
We know that $P_m(x)$ and $P_n(x)$ are the solution to the equations

$$(1-x^2)u'' - 2xu' + m(m+1)u = 0 \rightarrow (1)$$

$$(1-x^2)v'' - 2xv' + n(n+1)v = 0 \longrightarrow \textcircled{2}$$

Multiplying ① by 'v' and ② by 'u' and substarting we get

Partha [Ohm] 2K20/B17/35

$$(1-x^2)(u''v - v''u) - 2x(u'v - v'u) + [m(m+1) - n(n+1)]uv = 0$$

$$(n-m)(n+m+1)uv = \frac{d}{dx}\left[(1-x^2)(u'v - v'u)\right]_{-1}^{1} = 0$$

Hence, $\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$, since $m \neq n$.

Case II: When $m = n$.

We know that $(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$

Squaring both sides, we get

$$(1-2xh+h^2)^{-1} = \sum_{n=0}^{\infty}\left[h^n P_n(x)\right]^2 = \sum_{n=0}^{\infty} h^{2n}\left[P_n(x)\right]^2 + 2\sum_{\substack{m=0 \\ m\neq n}}^{\infty}\sum_{n=0}^{\infty} h^{m+n} P_m(x) P_n(x)$$

Integrating w.r.t. x between the limits -1 to 1 we have

$$\sum_{n=0}^{\infty}\int_{-1}^{1} h^{2n}(P_n(x))^2\,dx + 2\sum_{\substack{m=0 \\ m\neq n}}^{\infty}\sum_{n=0}^{\infty}\int_{-1}^{1} h^{m+n} P_m(x) P_n(x)\,dx$$

$$= \int_{-1}^{1} \frac{dx}{1-2xh+h^2}$$

$$\sum_{n=0}^{\infty}\int_{-1}^{1} h^{2n}[P_n(x)]^2\,dx = \int_{-1}^{1}\frac{dx}{1-2xh+h^2}$$

Since other integrals on the L.H.S vanish by Case I as $m \neq n$.

$$= \frac{-1}{2h}\left[\log(1-2xh+h^2)\right]_{-1}^{1} = \frac{-1}{2h}\left[\log(1-h)^2 - \log(1+h)^2\right]$$

$$= \frac{1}{h}\left[\log(1+h) - \log(1-h)\right] = \frac{1}{h}\left[\left[h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + \cdots\right] + \left(h + \frac{h^2}{2} + \frac{h^3}{3} + \cdots\right)\right]$$

Paath Johri
2K2018/17/33

$$= \frac{2}{h}\left(h + \frac{h^3}{3} + \frac{h^5}{5} + \dots\right)$$

$$\sum_{n=0}^{\infty} h^{2n} \int_{-1}^{1} [P_n(x)]^2 dx = 2\left(1 + \frac{h^2}{3} + \frac{h^4}{5} + \dots + \frac{h^{2n}}{2n+1} + \dots\right)$$

Equating the coefficients of h^{2n} on two sides, we get

$$\int_{-1}^{1} [P_n(x)]^2 dx = \frac{2}{2n+1}$$

Ques 8) Show that for $n = 0, 1, 2, 3$ the corresponding legendre polynomial is given by

$$P_n(x) = \frac{1}{2^n n!}\left\{\frac{d^n}{dx^n}(x^2-1)^n\right\}$$

Let $v = (x^2-1)^n$, then $v_1 = \frac{dv}{dx} = n(x^2-1)^{n-1} \cdot 2x$

Multiply both sides by (x^2-1)

$(x^2-1)v_1 = 2nx(x^2-1)^n = 2nxv \Rightarrow (1-x^2)v_1 + 2nxv = 0$

Differentiating $(n+1)$ times by Leibnitz's theorem, we have

$$\left[(1-x^2)v_{n+2} + (n+1)(-2x)v_{n+1} + \frac{(n+1)n}{2!}(-2)v_n\right] + 2n\left[xv_{n+1} + (n+1)v_n\right] = 0$$

$$(1-x^2)v_{n+2} - 2xv_{n+1} + n(n+1)v_n = (1-x^2)\frac{d^2v_n}{dx^2} - 2x\frac{dv_n}{dx} + n(n+1)v_n = 0$$

Parthjou 2K20/B17/135

which is a legendre's equation and V_n is ~~distributed~~ its solution but the solutions of the legendre's equation are $P_n(x)$ and $Q_n(x)$.

Since, $V_n = \frac{d^n}{dx^n}(x^2-1)^n$ contains only positive powers of x, it must be a constant multiple of $P_n(x)$

i.e., $V_n = CP_n(x) \Rightarrow CP_n(x) = \frac{d^n}{dx^n}(x^2-1)^n \longrightarrow (1)$

$$= P_n(x) = \frac{d^n}{dx^n}[(x-1)^n(x+1)^n] = (x-1)^n \frac{d^n}{dx^n}(x+1)^n + {}^nC_1\, n(x-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}(x+1)^n +$$

$$+ \; - - - + (x+1)^n \frac{d^n}{dx^n}(x-1)^n$$

$$= (x-1)^n n! + {}^nC_1 (x-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}(x+1)^n + - - + (x+1)^n n!$$

$= n!(x+1)^n +$ terms containing powers of $(x+1)$

Putting $x=1$ on both sides,

$$C P_n(1) = n!\,2^n \text{ or } C = 2^n n!, \text{ Since } P_n(1) = 1$$

Substituting in (i) $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$.

Putting $n = 0, 1, 2, 3$ we get legendre's polynomials.

Thus,

$$P_0(x) = 1, \; P_1(x) = \frac{1}{2}\frac{d(x^2-1)}{dx} = x$$

$$P_2(x) = \frac{1}{2^2\, 2!}\frac{d^2(x^2-1)^2}{dx^2} = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{48}\frac{d^3}{dx^3}(x^6 - 3x^4 + 3x^2 - 1) = \frac{1}{2}(5x^3 - 3x)$$

Ques 9.) Express $x^4+2x^3+2x^2-x-3$ in terms of Legendre's polynomials.

Parth Johri
2K20/B17/33

$P_0(x)=1, P_1(x)=x \longrightarrow ①$ $P_3(x)=\frac{5x^3-3x}{2} \rightarrow ③$

$P_2(x)=\frac{3x^2-1}{2} \longrightarrow ②$ $P_4(x)=\frac{35x^4+30x^2+3}{8} ④$

~~$P_3(x)$~~

from ④, $x^4=\frac{8}{35}P_4(x)+\frac{6x^2}{7}-\frac{3}{35}$ from ②, $x^2=\frac{2P_2(x)}{3}+\frac{1}{3}$

from ③, $x^3=\frac{2}{5}P_3(x)-\frac{3}{5}x$, from ①, $x=P_1(x)$, $1=P_0$

$f(x)=\frac{8}{35}P_4(x)+\frac{6}{7}x^2-\frac{3}{35}+2x^3+2x^2-x-3$

$=\frac{8}{35}P_4(x)+2x^3+\frac{20}{7}x^2-x-\frac{108}{35}$

$=\frac{8}{35}P_4(x)+2\left[\frac{2}{5}P_3(x)+\frac{3}{5}x\right]+\frac{20}{7}x^2-x-\frac{108}{35}$

$=\frac{8}{35}P_4(x)+\frac{4}{5}P_3(x)+\frac{20}{7}\left[\frac{2}{3}P_2(x)+\frac{1}{3}\right]+\frac{x}{5}-\frac{108}{35}$

$f(x)=\frac{8}{35}P_4(x)+4/5\,P_3(x)+\frac{40}{41}P_2(x)+\frac{x}{5}-\frac{224}{105}$

$f(x)=\frac{8}{35}P_4(x)+\frac{4}{5}P_3(x)+\frac{40}{41}P_2(x)+\frac{P_1(x)}{5}-\frac{224}{105}P_0(x)$

Ques 10) Prove that $\int_{-1}^{1}x^2P_{n-1}(x)P_{n+1}(x)\,dx=\frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$

Recurrence relation (i) is

$(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1} \Rightarrow (2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$ (1)

Partha John
2k2018(7/33)

Replacing 'n' by 'n+1' and 'n-1' respectively in (1), we get

$$(2n+3)xP_{n+1} = (n+2)P_{n+2} + (n+1)P_n \longrightarrow (2)$$

$$(2n-1)xP_{n-1} = nP_n + (n-1)P_{n-2}$$

Multiplying (2) and (3) and integrating within limits -1 and 1,

$$(2n+3)(2n-1)\int_{-1}^{1} x^2 P_{n+1} P_{n-1}\,dx = n(n+1)\int_{-1}^{1} P_n^2\,dx + n(n+2)\int_{-1}^{1} P_n P_{n+2}\,dx + (n^2-1)\int_{-1}^{1} P_{n-2}P_n\,dx + (n+2)(n-1)\int_{-1}^{1} P_{n-1}P_{n+2}\,dx$$

$$= n(n+1)\frac{2}{2n+1}$$ [Using Orthogonal Properties

$$\int_{-1}^{1} x^2 P_{n+1} P_{n-1}\,dx = \frac{2n(n+1)}{(2n+1)(2n-1)(2n+3)}$$

X——X——X——X

Finish

Pjohn