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Assignment-5
Fourier Series and Fourier Transformation

Q1 Expand in Fourier Series $f(x) = x + x^2$, $-\pi < x < \pi$
Hence prove that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$

Ans: $f(x) = x + x^2$

$2l = 2\pi$

$\Rightarrow l = \pi$

According to Fourier Series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (x + x^2)\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x^2\,dx$$

$$a_0 = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cancel{f(x)}(x + x^2)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n}\Big|_0^{\pi} - 2\int_0^{\pi} x\frac{\sin nx\,dx}{n} \right]$$

$$= \frac{-4}{n\pi}\left[-x\frac{\cos nx}{n}\Big|_0^\pi + \int_0^\pi \frac{\cancel{\sin}\cos nx}{n}\,dx\right]$$

$$= -\frac{4}{n\pi}\left(\frac{-\pi(-1)^n}{n}\right)$$

$$a_n = \frac{4(-1)^n}{n\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[-x\frac{\cos nx}{n}\Big|_0^\pi + \int_0^\pi \frac{\cos nx}{n}\,dx\right]$$

$$b_n = \frac{2}{\pi}\left[\frac{-\pi(-1)^n}{n}\right] = \frac{2}{n}(-1)^{n+1}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$$

$$x+x^2 = \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \dots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \dots\right]$$

For $x = \cancel{0}\ \pi$

$$\pi + \pi^2 = \frac{\pi^2}{3} - 4\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right]$$

$$\frac{\pi}{4} + \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

Q2 Find the Fourier series for the even function

$$f(x) = \begin{cases} 1+\frac{2x}{\pi} & -\pi \le x \le 0 \\ 1-\frac{2x}{\pi} & 0 \le x < \pi \end{cases}$$

in $-\pi \le x \le \pi$. Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots = \frac{\pi^2}{8}$.

Ans2 $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ [Even Function]

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}\left(1+\frac{2x}{\pi}\right)dx + \int_{0}^{\pi}\left(1-\frac{2x}{\pi}\right)dx\right]$$

$$= \frac{1}{2\pi}\left[\pi + \frac{1}{\pi}(-\pi^2) + \pi - \frac{1}{\pi}(\pi)^2\right]$$

$$a_0 = 0$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{\sin nx}{n}\Big|_0^{\pi} - \frac{2}{\pi}\left[\frac{x\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,dx\right]\right]$$

$$= -\frac{4}{n^2\pi^2}\left[(-1)^n - 1\right] = \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)$$

$$a_n = \begin{cases} \dfrac{8}{n^2\pi^2} & n = \text{odd} \\ 0 & n = \text{even} \end{cases}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right]$$

For $x = 0$,

$$f(x) = 1$$

$$1 = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Q3 Obtain a Fourier series to represent the function $f(x) = |\sin x|$ for $-\pi < x < \pi$.

Ans 3

$$f(x) = \begin{cases} \sin x & 0 < x < \pi \\ -\sin x & -\pi < x < 0 \end{cases}$$

$f(x)$ is an even function.

$$\Rightarrow f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin x \, dx = -\frac{1}{\pi} \cos x \Big|_0^{\pi}$$

$$a_0 = -\frac{1}{\pi}\left[-1-1\right] = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^\pi \sin x \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^\pi \sin(n+1)x + \sin(1-n)x \, dx$$

$$= -\frac{1}{\pi}\left[\frac{\cos x(1+n)}{1+n} + \frac{\cos x(1-n)}{1-n}\right]_0^\pi \qquad (n \neq 1)$$

$$= -\frac{1}{\pi}\left[\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{1-n}}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right] \qquad (n \neq 1)$$

$$a_n = -\frac{1}{\pi}\left[\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{1-n}}{1-n} - \frac{2}{1-n^2}\right] \qquad (n \neq 1)$$

$$a_n = \begin{cases} \dfrac{-4}{\pi(n^2-1)} & n = \text{even} \\ 0 & n = \text{odd} \end{cases}$$

$$a_1 = \frac{2}{\pi}\int_0^\pi \sin x \cos x \, dx$$

$$a_1 = \frac{1}{\pi}\int_0^\pi \sin 2x \, dx = -\frac{1}{\pi} \cos 2x \Big|_0^\pi = 0$$

$$\therefore f(x) = a_0 + a_1 + \sum_{n=2}^{\infty} a_n \cos nx$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \cdots\right]$$

$$= \frac{2}{\pi}\left[1 - \frac{2\cos 2x}{3} - \frac{2\cos 4x}{15} - \frac{2\cos 6x}{35} + \cdots\right]$$

Q4 a) Q. Determine the half-range sine series for the function defined by

$$f(x) = \begin{cases} x & 0 < x < \pi/2 \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

Ans 4 a) Half range Fourier Sine Series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left[-x\frac{\cos nx}{n}\Big|_0^{\pi/2} + \int_0^{\pi/2}\frac{\cos nx \, dx}{n}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{\sin nx}{n^2}\Big|_0^{\pi/2}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$b_n = -\frac{1}{n}\cos\frac{n\pi}{2} + \frac{2}{\pi n^2}\sin\frac{n\pi}{2}$$

$$b_n = \begin{cases} \frac{2}{n^2\pi}\sin\frac{n\pi}{2} & \text{when } n = \text{odd} \\ -\frac{1}{n}\cos\frac{n\pi}{2} & \text{when } n = \text{even} \end{cases}$$

$$f(x) = \frac{2}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \cdots\right]$$

$$+ \left[\frac{\sin 2x}{2} - \frac{\sin 4x}{4} + \frac{\sin 6x}{6} - \cdots\right]$$

(b) Obtain the half range cosine and sine series for the function

$$f(x) = \begin{cases} 0 & 0<x<\pi/2 \\ 1 & \frac{\pi}{2}<x<\pi \end{cases}$$

Ans b) Half range cosine series is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_{\pi/2}^{\pi} 1\, dx$$

$$a_0 = \frac{1}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_{\pi/2}^{\pi} \cos nx\, dx$$

$$a_n = \frac{2}{\pi}\left.\frac{\sin nx}{n}\right|_{\pi/2}^{\pi} = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$a_n = \begin{cases} -\frac{2}{n\pi}\sin\frac{n\pi}{2} & n = \text{odd} \\ 0 & n = \text{even} \end{cases}$$

Half range Fourier Cosine Series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{1}{2} \mp \frac{2}{\pi}\left[\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \cdots\right]$$

Half Range Fourier Sine Series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx$$

$$= \frac{2}{\pi}\int_{\pi/2}^{\pi} \sin nx \, dx$$

$$= -\frac{2}{\pi} \left.\frac{\cos nx}{n}\right|_{\pi/2}^{\pi}$$

$$b_n = -\frac{2}{n\pi}\left[(-1)^n - \cos\frac{n\pi}{2}\right]$$

$$f(x) = -\frac{2}{\pi}\left[-\sin x - \frac{\sin 3x}{3} - \frac{\sin 5x}{5} - \dots\right] - \frac{4}{\pi}\left[\frac{\sin 2x}{2} + \frac{\sin 6x}{6} + \dots\right]$$

$$f(x) = \frac{2}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right] - \frac{2}{\pi}\left[\frac{\sin 2x}{1} + \frac{\sin 6x}{3} + \dots\right]$$

Q5 Using complex form find the Fourier series of the function:

a) $f(x) = x^2$, defined on the interval $[-1, 1]$

Ans $2l = 2 \Rightarrow l = 1$

$$f(x) = x^2 = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x}$$

$$C_n = \frac{1}{2}\int_{-1}^{1} x^2 e^{-in\pi x} \, dx$$

$$= \frac{1}{2}\left[\frac{x^2 e^{-in\pi x}}{-in\pi}\Big|_{-1}^{1} + \int_{-1}^{1} \frac{2x\, e^{-in\pi x}}{in\pi} dx \right]$$

$$= \frac{1}{2}\left[-\frac{e^{-in\pi}}{in\pi} + \frac{e^{in\pi^{-1}}}{in\pi} + \frac{2}{in\pi}\left[\frac{x e^{-in\pi x}}{-in\pi}\Big|_{-1}^{1} - \int_{-1}^{1} \frac{e^{-in\pi x}}{-in\pi} dx \right]\right]$$

$$= \frac{\sin n\pi}{n\pi} + \frac{1}{in\pi}\left[-\frac{e^{-in\pi}}{in\pi} - \frac{e^{in\pi}}{in\pi} + \frac{e^{-in\pi x}}{n^2\pi^2}\Big|_{-1}^{1} \right]$$

$$= \frac{\sin n\pi}{n\pi} + \frac{2}{n^2\pi^2}\cos n\pi - \frac{2}{n^3\pi^3}\sin n\pi$$

$$= \frac{2}{n^2\pi^2}\cos n\pi$$

$$C_n = \frac{2(-1)^n}{n^2\pi^2} \qquad n \neq 0$$

$$C_0 = \frac{1}{2}\int_{-1}^{1} x^2\, dx = \int_0^1 x^2 dx = \frac{x^3}{3} = \frac{1}{3}$$

$$f(x) = x^2 = \frac{1}{3} + \sum_{n=-\infty}^{\infty} \frac{2}{n^2\pi^2}(-1)^n e^{in\pi x}$$

(b) $f(x) = \dfrac{a \sin x}{1 - 2a\cos x + a^2}, \quad |a| < 1$

$$= \frac{a\frac{(e^{ix} - e^{-ix})}{2i}}{1 - 2a\frac{(e^{ix} + e^{-ix})}{2} + a^2} = \frac{1}{2i}\frac{a(e^{ix} - e^{-ix})}{1 - ae^{ix} - ae^{-ix} + a^2}$$

$$= \frac{1}{2i}\frac{a(e^{ix} - e^{-ix})}{(1 - ae^{ix} - ae^{-ix} + a^2 e^{ix} e^{-ix})}$$

$$= \frac{1}{2i}\frac{a(e^{ix} - e^{-ix})}{(1 - ae^{ix})(1 - ae^{-ix})}$$

$$f(x) = \frac{1}{2i}\left[\frac{+1}{1-ae^{ix}} - \frac{1}{1-ae^{-ix}}\right]$$

$$(1-ae^{ix})^{-1} = \sum_{n=0}^{\infty} a^n e^{inx}$$

and

$$(1-ae^{-ix})^{-1} = \sum_{n=0}^{\infty} a^n e^{-inx}$$

$$f(x) = \frac{1}{2i}\left[\sum_{n=0}^{\infty} a^n e^{inx} - \sum_{n=0}^{\infty} a^n e^{-inx}\right]$$

$$= \frac{1}{2i}\left[\sum_{n=0}^{\infty} a^n (e^{inx} - e^{-inx})\right]$$

$$f(x) = \frac{\cancel{1}}{\cancel{2i}} \sum_{n=0}^{\infty} a^n \sin nx$$

This is the fourier series.

Q6 Find the fourier integral representation of the function

$$f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

Ans6 $$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Using fourier integral,

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(u)\cos w(u-x)\,du\,dw$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_{-1}^{1} \cos w(u-x)\,du\,dw$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin\omega(u-x)}{\omega}\Big|_{-1}^{1} d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \frac{1}{\omega}\left(\sin\omega(1-x) - \sin\omega(1+x)\right) d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \frac{2}{\omega}\left(\sin\omega\cos\omega x\right) d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega} d\omega$$

Q7 Find the Fourier transform of

a) $$f(x) = \begin{cases} x & |x| \le a \\ 0 & |x| > a \end{cases}$$

Ans
$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{i\omega u} du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} u e^{i\omega u} du$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{u e^{i\omega u}}{i\omega}\Big|_{-a}^{a} - \int_{-a}^{a}\frac{e^{i\omega u}}{i\omega} du\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a e^{i\omega a}}{i\omega} + \frac{a e^{-i\omega a}}{i\omega} + \frac{e^{i\omega u}}{\omega^2}\Big|_{-a}^{a}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2a}{i\omega}\cos\omega a + \frac{2i}{\omega^2}\sin\omega a\right]$$

$$F(\omega) = \frac{i}{\omega^2}\sqrt{\frac{2}{\pi}}\left[\sin\omega a - \omega\cos\omega a\right]$$

(b) $f(x) = e^{-x^2}$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^{i\omega x}\, dx \quad \text{——} \ (1)$$

$$f'(x) = -2xe^{-x^2} = -2x f(x)$$

Taking Fourier Transform on both sides,

$$i\omega F(\omega) = -2F[x f(x)]$$

$$i\omega F(\omega) = -2i F'(\omega)$$

$$\frac{F'(\omega)}{F(\omega)} = -\frac{\omega}{2}$$

Integrating Both sides

$$\ln|F(\omega)| = -\frac{\omega^2}{4} + \ln C$$

$$F(\omega) = Ce^{-\omega^2/4}$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^{i\omega x}\, dx = Ce^{-\omega^2/4}$$

Let $\omega = 0$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2}\, dx = C$$

$$x^2 = t$$

$$2x\, dx = dt$$

$$C = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{t^{-1/2}}{2} e^{-t}\, dt = \frac{\Gamma\frac{1}{2}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2}}$$

$$\therefore\ F(\omega) = \frac{1}{\sqrt{2}} e^{-\omega^2/4}$$

c) $f(x) = \frac{1}{1+x^2}$

Ans c) $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1+x^2} e^{i\omega x} dx$

Let $g(x) = e^{-|x|}$

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{i\omega x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{x} e^{i\omega x} dx + \int_{0}^{\infty} e^{-x} e^{i\omega x} dx \right] \cancel{dx}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(1+i\omega)x}}{1+i\omega} \Big|_{-\infty}^{0} + \frac{e^{(-1+i\omega)x}}{-1+i\omega} \Big|_{0}^{\infty} \right].$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1+i\omega} + \frac{1}{1-i\omega} \right]$$

$$G(\omega) = \sqrt{\frac{2}{\pi}} \left(\frac{1}{1+\omega^2} \right)$$

Taking inverse Fourier Transform,

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2} e^{-i\omega x} d\omega$$

Put $x = -t$

$$g(-t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+\omega^2} e^{i\omega t} d\omega$$

$$e^{-|-t|} = \cancel{\frac{1}{\pi} \sqrt{2\pi} F(\omega)} \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2} dx$$

$$\pi e^{-|\omega|} = F(\omega)$$

Q8 Find the Fourier Sine and Cosine Transform of $f(x)$ if,

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 < x < 2 \\ 0 & x > 2 \end{cases}$$

Ans 8

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \sin \omega x \, dx + \int_1^2 (2-x) \sin \omega x \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{x \cos \omega x}{\omega} + \frac{\sin \omega x}{\omega^2} \Big|_0^1 - \frac{2 \cos \omega x}{\omega} \Big|_1^2 + \frac{x \cos \omega x}{\omega} - \frac{\sin \omega x}{\omega^2} \Big|_1^2 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos \omega}{\omega} + \frac{\sin \omega}{\omega^2} - \frac{2 \cos 2\omega}{\omega} + \frac{2 \cos \omega}{\omega} + \frac{2 \cos 2\omega}{\omega} - \frac{\sin 2\omega}{\omega^2} - \frac{\cos \omega}{\omega} + \frac{\sin \omega}{\omega^2} \right]$$

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \left(\frac{2 \sin \omega - \sin 2\omega}{\omega^2} \right)$$

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \cos \omega x \, dx + \int_1^2 (2-x) \cos \omega x \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{x \sin \omega x}{\omega} + \frac{\cos \omega x}{\omega^2} \Big|_0^1 + \frac{2 \sin \omega x}{\omega} \Big|_1^2 - \frac{x \sin \omega x}{\omega} - \frac{\cos \omega x}{\omega^2} \Big|_1^2 \right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin\omega}{\omega} + \frac{\cos\omega}{\omega^2} - \frac{1}{\omega^2} + \frac{2\sin 2\omega}{\omega} - \frac{2\sin\omega}{\omega} - \frac{2\sin 2\omega}{\omega} - \frac{\cos 2\omega}{\omega^2} + \frac{\sin\omega}{\omega} + \frac{\cos\omega}{\omega^2}\right]$$

$$F_c(\omega) = \sqrt{\frac{2}{\pi}}\left[\frac{2\cos\omega - \cos 2\omega - 1}{\omega^2}\right]$$

Q9 Find $F(x)$ if $\bar{F}_s(s) = s^n e^{-as}$ and $\bar{F}_c(s) = s^n e^{-as}$.

Ans9 $F_s(s) = s^n e^{-as}$ $\qquad F_c(s) = s^n e^{-as}$

Taking Fourier Sine inverse,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \omega^n e^{-a\omega} \sin\omega x \, d\omega \quad \text{—— (1)}$$

Taking Fourier Cosine inverse,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \omega^n e^{-a\omega} \cos\omega x \, d\omega \quad \text{—— (2)}$$

(2) − i(1)

$$(1-i) f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \omega^n e^{-a\omega}(\cos\omega x - i\sin\omega x) \, d\omega$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{1-i}\int_{-\infty}^{\infty} \omega^n e^{-a\omega} e^{-i\omega x} \, dx$$

Comparing with $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} \, d\omega$.

$$F(\omega) = \frac{\omega^n e^{-a\omega}}{1-i}$$

$$F(\omega) = \frac{\omega^n e^{-a\omega}(1+i)}{2}$$

$$F(x) = \frac{1}{2} x^n e^{-ax}(1+i)$$

Q10 Find the solution of the differential equation

$$y' - 2y = H(t)\,e^{-2t} \qquad -\infty < t < \infty$$

using Fourier transforms, where $H(t) = U_0(t)$ is the unit step function.

Ans10

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

$$y' - 2y = H(t)\,e^{-2t}$$

$y = f(t)$

Taking Fourier Transform on both sides

$$F\{y'\} - 2F\{y\} = F\{H(t)e^{-2t}\}$$

$$i\omega F(f(t)) - 2F(f(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(t)e^{-2t} e^{-i\omega t}\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t(2+i\omega)}\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-t(2+i\omega)}}{-(2+i\omega)} \right|_0^\infty$$

$$F(f(t)) = \frac{1}{\sqrt{2\pi}} \frac{1}{(2+i\omega)(-2+i\omega)}$$

$$F(f(t)) = \frac{-1}{\sqrt{2\pi}(\omega^2+4)}$$

Taking inverse fourier transform on both sides

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{-1}{\sqrt{2\pi}(\omega^2+4)} e^{-i\omega t} d\omega$$

$$= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2+4} d\omega$$

$$= -\frac{1}{2\pi}\left(\frac{1}{2}\sqrt{\frac{\pi}{2}} e^{-2|t|}\right) \quad \cancel{\left[\because \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2+a^2} d\omega = \frac{1}{a}\sqrt{\frac{\pi}{2}} e^{-|a|}\right]}$$

$$= -\frac{1}{4\sqrt{2\pi}} e^{-2|t|} \quad \left[\because \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{a^2+\omega^2} d\omega = \frac{1}{a}\sqrt{\frac{\pi}{2}} e^{-a|t|}\right]$$

$$\therefore \quad y = f(t) = -\frac{1}{4\sqrt{2\pi}} e^{-2|t|}$$

Q12 If $F(\bar{s})$ is the Fourier transform of $f(x)$, then prove that $F[f(x)\, e^{-iax}] = F(\overline{s-a})$.

Ans12

$$F(\bar{s}) = F(f(x)) = F(\omega)$$

$$F(f(x)\, e^{-iax}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iax} e^{i\omega x} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x)\, e^{i(\omega-a)x} dx$$

$$F(f(x)\, e^{-iax}) = F(s-a)$$

Q13 Find the Fourier Sine transform of e^{-x} $(x>0)$ and show that $\int_0^\infty \frac{x \sin mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m}$, $m>0$

Ans13 $f(x) = e^{-x}$

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{e^{-x}}{1+\omega^2} \left[\sin \omega x + \omega \cos \omega x \right] \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[+\frac{\omega}{1+\omega^2} \right]$$

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \frac{\omega}{1+\omega^2}$$

Taking Fourier Sine inverse

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{\omega}{1+\omega^2} \sin \omega x \, d\omega$$

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{\omega}{1+\omega^2} \sin \omega x \, d\omega$$

Put $x = m$ and $\omega = x$

$$e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{x}{1+x^2} \sin mx \, dx$$

$$\int_0^\infty \frac{x}{1+x^2} \sin mx \, dx = \frac{\pi}{2} e^{-m}$$

Hence proved.

Q14 Use Fourier Integral to prove that

$$\int_0^\infty \frac{\sin \pi\lambda \sin \lambda x}{1-\lambda^2}\, d\lambda = \begin{cases} \frac{\pi}{2}\sin x & 0<x<\pi \\ 0 & x>\pi \end{cases}$$

Ans14
$$f(x) = \begin{cases} \frac{\pi}{2}\sin x & 0<x<\pi \\ 0 & x>\pi \end{cases}$$

Take Fourier sine integral

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin \lambda x \int_0^\infty \sin \lambda t\, f(t)\, dt\, d\lambda$$

$$\text{Let } I = \int_0^\infty f(t) \sin \lambda t\, dt$$

$$= \frac{\pi}{2}\int_0^\pi \sin t \sin \lambda t\, dt$$

$$= \frac{\pi}{4}\left[\int_0^\pi \cos(\lambda-1)t - \cos(\lambda+1)t\, dt\right]$$

$$= \frac{\pi}{4}\left[\frac{\sin(\lambda-1)\pi}{\lambda-1} - \frac{\sin(\lambda+1)\pi}{\lambda+1}\right].$$

$$= -\frac{\pi}{4(1-\lambda^2)}\left[-\lambda\left[\sin(1-\lambda)\pi + \sin(1+\lambda)\pi\right] + 1\left[\sin(\lambda-1)\pi + \sin(\lambda+1)\pi\right]\right]$$

$$= \frac{-\pi}{4(1-\lambda^2)}\left[\lambda(2\sin\pi\cos\lambda\pi) + 2\sin\lambda\pi\cos\pi\right]$$

$$I = \frac{\pi \sin\lambda\pi}{2(1-\lambda^2)}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x\, \frac{\pi}{2}\, \frac{\sin\lambda\pi}{(1-\lambda^2)}\, d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin \pi\lambda \sin \lambda x}{(1-\lambda^2)}\, d\lambda$$

Hence proved.

Q15 Find the inverse Fourier Transform of $F(\omega) = \dfrac{1}{(4+\omega^2)(9+\omega^2)}$

Ans15 Let $g(x) = F^{-1}\left\{\dfrac{1}{9+\omega^2}\right\}$ and $h(x) = F^{-1}\left\{\dfrac{1}{4+\omega^2}\right\}$

$$g(x) = \frac{1}{6}\sqrt{2\pi}\, e^{-3|x|} \qquad h(x) = \frac{1}{4}\sqrt{2\pi}\, e^{-2|x|}$$

$$f(x) = F^{-1}\{F(\omega)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{(4+\omega^2)(9+\omega^2)} e^{i\omega x}\, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} (g * h)(x)$$

$$= \frac{\sqrt{2\pi}}{24} e^{-2|x|} * e^{-3|x|}$$

$$= \frac{\sqrt{2\pi}}{24}\int_{-\infty}^{\infty} e^{-2|x-t|} e^{-3|t|}\, dt$$

For $x > 0$

$$\int_{-\infty}^{\infty} e^{-2|x-t|} e^{-3|t|}\, dt = \int_{-\infty}^{0} e^{-2|x-t|} e^{-3|t|} + \int_0^x e^{-2|x-t|} e^{-3|t|}\, dt$$

$$+ \int_x^\infty e^{-2|x-t|} e^{-3|t|}\, dt$$

$$= \int_{-\infty}^{0} e^{-2(x-t)} e^{3t}\, dt + \int_0^x e^{-2(x-t)} e^{-3t}\, dt + \int_x^\infty e^{-2(t-x)} e^{-3t}\, dt$$

$$= \frac{6}{5} e^{-2x} - \frac{4}{5} e^{-3x}$$

For $x<0$,

$$\int_{-\infty}^{\infty} e^{-2|x-t|} e^{-3|t|} dt = \frac{6}{5} e^{2x} - \frac{4}{5} e^{3x}$$

For $x=0$, $\int_{-\infty}^{\infty} e^{-2|x-t|} e^{-3|t|} dt = \frac{2}{5}$.

$$\therefore f(x) = \begin{cases} \sqrt{2\pi}\left(\frac{1}{20} e^{2x} - \frac{1}{30} e^{3x}\right) & x<0 \\ \frac{\sqrt{2\pi}}{60} & x=0 \\ \sqrt{2\pi}\left(\frac{1}{20} e^{-2x} - \frac{1}{30} e^{-3x}\right) & x>0. \end{cases}$$

Q11 By Applying an integral transform, solve the boundary problem $f''(x) - f(x) = 3e^{-2x}$ $(0<x<\infty)$, $f(0) = x_0$, $f(\infty)$ is bounded}.

Ans 11 $f''(x) - f(x) = 3e^{-2x}$

Taking Fourier Sine Transform on both sides

$$F_s\{f''(x)\} - F_s\{f(x)\} = 3 F_s\{e^{-2x}\}$$

$$-\omega^2 F_s(\omega) + \omega\sqrt{\frac{2}{\pi}} f(0) - F_s(\omega) = 3\sqrt{\frac{2}{\pi}} \frac{\omega}{\omega^2+4}$$

$$F_s(\omega)\left[-\omega^2 - 1\right] = \sqrt{\frac{2}{\pi}}\,\omega\left[\frac{3}{\omega^2+4} - x_0\right]$$

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \frac{\omega}{\omega^2+1}\left(x_0 - \frac{3}{\omega^2+4}\right)$$

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \left[(x_0 - 1) \frac{\omega}{\omega^2+1} + \frac{\omega}{\omega^2+4} \right]$$

Taking inverse fourier sine transform on both sides,

$$F_s^{-1}\{F_s(\omega)\} = \sqrt{\frac{2}{\pi}}\, F_s^{-1}\left\{ (x_0-1)\frac{\omega}{\omega^2+1} + \frac{\omega}{\omega^2+4} \right\}$$

$$f(x) = (x_0 - 1) e^{-x} + e^{-2x}$$