DELHI TECHNOLOGICAL UNIVERSITY
MA-102
ASSIGNMENT - 4
TOPIC: LAPLACE TRANSFORMATIONS
Name: RAJ KUMAR SINGH
Roll NO: DTU/2K17/A5/570
Batch: A5
Date: 12-04-2018
Submitted to: DR R. SRIVASTAVA

1. Find the laplace transform of:

(a) $\frac{\cos\sqrt{t}}{\sqrt{t}}$;

Let $f(t) = \sin\sqrt{t}$

$$f'(t) = \frac{1}{2\sqrt{t}} \cdot \cos\sqrt{t}$$

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$$

$$\frac{1}{2}\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = s \cdot \mathcal{L}\{\sin\sqrt{t}\} - 0$$

From expansion:

$$\sin\sqrt{t} = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \cdots$$

$$\therefore\ \mathcal{L}\{\sin\sqrt{t}\} = \mathcal{L}\left[t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \cdots\right]$$

from linearity of laplace transforms;

$$= \mathcal{L}[t^{1/2}] - \frac{1}{3!}\mathcal{L}[t^{3/2}] + \frac{1}{5!}\mathcal{L}[t^{5/2}] - \cdots$$

$$= \frac{\Gamma 3/2}{s^{3/2}} - \frac{\Gamma 5/2}{3!\, s^{5/2}} + \frac{\Gamma 7/2}{5!\, s^{7/2}} - \cdots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{2!(4s)^2} - \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} \cdot e^{-1/4s}$$

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = 2 \times s \times \mathcal{L}\{\sin\sqrt{t}\}$$

$$\boxed{\mathcal{L}\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}} \cdot e^{-1/4s}}$$ Ans

Scanned by CamScanner

(b) $\sinh^2 t$

from properties of hyperbolic functions,

$$\sinh^2 t = \frac{\cosh 2t - 1}{2}$$

$$\mathcal{L}\{\sinh^2 t\} = \frac{1}{2}\mathcal{L}\{\cosh 2t\} - \frac{1}{2}\mathcal{L}\{1\}$$

$$= \frac{1}{2}\left\{\frac{s}{s^2-4} - \frac{1}{s}\right\} \qquad \left\{\because \mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}\right\}$$

$$\boxed{\mathcal{L}\{\sinh^2 t\} = \frac{2}{s(s^2-4)}} \quad \text{Ans} \quad s > 2$$

(c) $f(t) = \begin{cases} 0 & 0 \le t < \pi \\ \sin t & t \geq \pi \end{cases}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\pi e^{-st} \cdot 0 \, dt + \int_\pi^\infty e^{-st} \cdot \sin t \cdot dt$$

$$= 0 + -\left[\frac{e^{-st} \cdot \cos t + s e^{-st} \sin t}{s^2+1}\right]_\pi^\infty$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{-e^{-s\pi}}{s^2+1}} \quad \text{Ans}$$

(d) $\sin^3 2t$

$\because$, $\sin 3t = 3\sin t - 4\sin^3 t$;

$$\sin^3 2t = \frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t \, ;$$

$$\mathcal{L}[\sin^3 2t] = \frac{3}{4}\mathcal{L}[\sin 2t] - \frac{1}{4}\mathcal{L}[\sin 6t] \qquad \{\text{Linearity}\}$$

$$= \frac{3}{4} \cdot \frac{2}{s^2+4} - \frac{1}{4} \cdot \frac{6}{s^2+36}$$

$$\boxed{\mathcal{L}[\sin^3 2t] = \frac{48}{(s^2+4) \cdot (s^2+36)}} \quad \text{Ans} \qquad \left\{\because \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}\right.$$

(e) $\mathcal{L}[t \sin at]$

$\Rightarrow$ using $\cdot$ $\mathcal{L}\{t \cdot e^{iat}\}$

$\because$, $\mathcal{L}\{t\} = \frac{1}{s^2}$

$\therefore$, $\mathcal{L}\{t \cdot e^{iat}\} = \frac{1}{(s-ia)^2}$

Img $\mathcal{L}\{t \cdot e^{iat}\}$ = Img $\frac{1}{(s-ia)^2}$

$= \frac{(s+ia)^2}{(s^2+a^2)^2}$

$$\boxed{\mathcal{L}\{t \cdot \sin at\} = \frac{2as}{(s^2+a^2)^2}}$$ Ans

(f) $\mathcal{L}[t^n \cdot e^{at}]$

$\mathcal{L}[t^n] = \frac{\Gamma n+1}{s^{n+1}} = \frac{n!}{s^{n+1}}$ $s > 0$

$\therefore$, $$\boxed{\mathcal{L}[t^n \cdot e^{at}] = \frac{n!}{(s-a)^{n+1}}}$$, $s > a$

Ans.

$\because$, $\mathcal{L}\{f(t)\} = \bar{f}(s)$ then,

$\mathcal{L}\{f(t) \cdot e^{at}\} = \bar{f}(s-a)$;

(g) $\mathcal{L}\left\{\frac{\sin at}{t}\right\}$

If $f(t)$ is a function whose,

$\mathcal{L}\{f(t)\} = \bar{f}(s)$;

then, $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\, ds \Rightarrow$ Provided the integral exist

$$\mathcal{L}\{\sin at\} = f(s) = \frac{a}{s^2+a^2}$$

$$\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{s^2+a^2}\, ds$$

$$= a \cdot \left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$

$$\boxed{\mathcal{L}\left\{\frac{\sin at}{t}\right\} = a\left[\pi/2 - \tan^{-1}\frac{s}{a}\right]}$$

for,
$$\mathcal{L}\left\{\frac{\cos at}{t}\right\} = \int_s^\infty \frac{s}{(s^2+a^2)}\, ds$$

$$= \frac{1}{2}\log(s^2+a^2)\Big|_s^\infty$$

$\Downarrow$

the integral does not exist;

therefore; $\mathcal{L}\left\{\frac{\cos at}{t}\right\} =$ D.N.E.

(2.) Find Laplace of $G(t)$

$$G(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0 & t < \frac{2\pi}{3} \end{cases}$$

$$\mathcal{L}[G(t)] = \int_0^\infty e^{-st}\cdot G(t)\, dt$$

$$= \int_0^{2\pi/3} e^{-st}\cdot 0\, dt + \int_{2\pi/3}^\infty e^{-st}\cos\left(t-\frac{2\pi}{3}\right) dt$$

let $t - \frac{2\pi}{3} = u$ then, $du = dt$

$$\mathcal{L}\{G(t)\} = \int_0^\infty e^{-s(u+\frac{2\pi}{3})} \cdot \cos u \, du$$

$$= e^{-s\cdot 2\pi/3} \, \mathcal{L}\{\cos u\}$$

$$\boxed{\mathcal{L}\{G(t)\} = e^{-s\cdot 2\pi/3} \cdot \frac{s}{s^2+1} \quad ; \quad s > 0}$$ Ans

③ $\mathcal{L}[f(t)] = \dfrac{(s^2-s+1)}{(2s+1)^2(s-2)}$

To prove,

$$\mathcal{L}[f(2t)] = \frac{(s^2 - 2s + 4)}{4(s+1)^2(s-2)}$$

solution;

Using the change of scale property;

If $\mathcal{L}\{f(t)\} = \bar{f}(s)$; then

$$\mathcal{L}\{f(at)\} = \frac{1}{a} \bar{f}\left(\frac{s}{a}\right)$$

$$\mathcal{L}\{F(at)\} = \int_0^\infty e^{-st} F(at) \, dt$$

$$at = u$$

$$dt = \frac{du}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-s/a \, u} F(u) \, du$$

$$\boxed{\mathcal{L}\{F(at)\} = \frac{1}{a} \bar{f}(s/a)}$$

As given in the problem;

$$\mathcal{L}[f(t)] = \frac{(s^2 - s + 1)}{(2s+1)^2 (s-2)}$$

$\therefore$,

$$\mathcal{L}[f(2t)] = \frac{1}{2} \cdot \bar{f}(s/2)$$

$$\Rightarrow \frac{1}{2} \cdot \frac{\left(\frac{s}{2}\right)^2 - \frac{s}{2} + 1}{\left(\frac{2s}{2} + 1\right)^2 \left(\frac{s}{2} - 2\right)}$$

$$\Rightarrow \frac{1}{2} \cdot \frac{1}{4} \frac{(s^2 - 2s + 4)}{(s+1)^2 \frac{(s-4)}{2}}$$

$$\Rightarrow \frac{1}{4} \cdot \frac{(s^2 - 2s + 4)}{(s+1)^2 (s-4)}$$

$\therefore$,

Hence, Proved.

$$\mathcal{L}[f(2t)] = \frac{s^2 - 2s + 4}{4(s+1)^2 (s-2)}$$

P.T.O

Que 4 $\mathcal{L}[\sin\sqrt{t}] = \frac{\sqrt{\pi}}{2s^{3/2}} \cdot e^{-1/4s}$

then show that $\mathcal{L}\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$

Let $f(t) = \sin\sqrt{t}$

$$f'(t) = \frac{1}{2\sqrt{t}} \cdot \cos\sqrt{t}$$

$$\mathcal{L}\{f'(t)\} = s \cdot \mathcal{L}\{f(t)\} - f(0)$$

$$\frac{1}{2}\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = s \cdot \mathcal{L}\{\sin\sqrt{t}\} - 0 \quad \text{— (A)}$$

$$\mathcal{L}\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}} \cdot e^{-1/4s}$$ Given;

Substituting in eqn (A);

$$\boxed{\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\, e^{-1/4s}.}$$

Hence Proved!

(5) Find:

a) $\mathcal{L}^{-1}\left[\frac{3s-2}{s^{5/2}} - \frac{7}{2s+2}\right]$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{3s-2}{s^{5/2}}\right] - \mathcal{L}^{-1}\left[\frac{7}{2s+2}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{3}{s^{3/2}}\right] - \mathcal{L}^{-1}\left[\frac{2}{s^{5/2}}\right] - \frac{7}{2} \cdot e^{-t}$$

$$\Rightarrow L^{-1}\left[\frac{\Gamma 3/2}{s^{3/2}} \cdot \frac{3}{\Gamma 3/2}\right] - L^{-1}\left[\frac{\Gamma 5/2}{s^{5/2}} \cdot \frac{2}{\Gamma 5/2}\right] - \frac{7}{2}e^{-t}$$

$$\because \mathcal{L}^{-1}\left[\frac{\Gamma n+1}{s^{n+1}}\right] = t^n$$

Scanned by CamScanner

$$\frac{3}{\Gamma 3/2} \cdot t^{1/2} - \frac{2}{\Gamma 5/2} \cdot t^{3/2} - \frac{7}{2} e^{-t} = f(t)$$

$$\Rightarrow \boxed{6 \cdot \left(\frac{t}{\pi}\right)^{1/2} - \frac{8t}{3}\left(\frac{t}{\pi}\right)^{1/2} - \frac{7}{3} e^{-2t/3}}$$ Ans

(b) $$\mathcal{L}^{-1}\left[\frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}\right]$$

$$\Rightarrow s^3 - 6s^2 + 11s - 6 = (s-1)(s-2)(s-3)$$

$$\mathcal{L}^{-1}\left[\frac{2s^2 - 6s + 5}{(s-1)(s-2)(s-3)}\right]$$

$$\frac{2s^2 - 6s + 5}{(s-1)(s-2)(s-3)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

$$= \frac{1}{2} \cdot \frac{1}{s-1} + \frac{1}{(s-2)} + \frac{5}{3(s-3)}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{2} \cdot \frac{1}{(s-1)} + \frac{1}{(s-2)} + \frac{5}{3(s-3)}\right\}$$

$$\left\{\because \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}\right\}$$

$$\Rightarrow \boxed{\frac{e^t}{2} - e^{2t} + \frac{5e^{3t}}{3}}$$ Ans

(c) $$\mathcal{L}^{-1}\left[\frac{s^2}{s^4 + 4a^2}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s^2}{s^4 + 4a^4}\right]$$

$$\Rightarrow \frac{s^2}{s^4 + 4a^4} = s \cdot \frac{s}{(s^2 + 2a^2)^2 - 4a^2 s^2}$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s^2}{(s^2+2a^2)^2-4a^2s^2}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s^2}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s}{4a}\cdot\left\{\frac{(s+a)^2-(s-a)^2}{((s-a)^2+a^2)((s+a)^2+a^2)}\right\}\right]$$

$$\Rightarrow \frac{1}{4a}\mathcal{L}^{-1}\left[\frac{s}{(s-a)^2+a^2}-\frac{s}{(s+a)^2+a^2}\right]$$

$$\Rightarrow \frac{1}{4a}\mathcal{L}^{-1}\left[\frac{(s-a)}{(s-a)^2+a^2}+\frac{a}{(s-a)^2+a^2}+\frac{a}{(s+a)+a^2}-\frac{s+a}{(s+a)^2+a^2}\right]$$

$$\Rightarrow \frac{1}{4a}\left[e^{at}(\sin at+\cos at)+e^{-at}(\sin at-\cos at)\right]$$

$$\Rightarrow \frac{1}{2a}\left[\left(\frac{e^{at}+e^{-at}}{2}\right)\cdot\sin at+\left(\frac{e^{at}-e^{-at}}{2}\right)\cos at\right]$$

$$\Rightarrow \frac{1}{2a}\cdot\left[\cosh at\cdot\sin at+\sinh at\cos at\right]$$

$$\therefore \boxed{\mathcal{L}^{-1}\left[\frac{s^2}{s^4+4a^4}\right]=\frac{1}{2a}\left[\cosh at\cdot\sin at+\sinh at\cos at\right]}$$

Ans

(d.) $\mathcal{L}^{-1}\left[\frac{e^{-4s}}{(s-3)^4}\right]$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-3)^4}\right] = \frac{1}{3!}\mathcal{L}^{-1}\left[\frac{3!}{(s-3)^4}\right]$$

$$= \frac{1}{3!} e^{3t} \cdot t^3$$

$\because$, $\mathcal{L}^{-1}[e^{-as}\bar{f}(s)] = u(t-a)\, f(t-a)$

$\therefore$, $$\boxed{\mathcal{L}^{-1}\left[\frac{e^{-4s}}{(s-3)^4}\right] = \frac{1}{6}\cdot (t-4)^3 e^{3(t-4)} H(t-4)}$$ Ans

(e) $\mathcal{L}^{-1}\left[\log \frac{1+s}{s}\right]$

$\because$, $$t\, f(t) = \mathcal{L}^{-1}\left[-\frac{d}{ds}\bar{f}(s)\right]$$

$$= \mathcal{L}^{-1}\left[-\frac{d}{ds}\{\log(1+s) - \log(s)\}\right]$$

$$= \mathcal{L}^{-1}\left[-\frac{1}{(1+s)} + \frac{1}{s}\right]$$

$$\boxed{\mathcal{L}^{-1}\left[\log\frac{(1+s)}{s}\right] = f(t) = \frac{1-e^{-t}}{t}}$$ Ans

Que 6. State and prove convolution theorem and hence solve the following.

(a) $\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

P.T.O

CONVOLUTION THEOREM

If, $\mathcal{L}\{f_1(t)\} = \bar{f_1}(s)$ and $\mathcal{L}\{f_2(t)\} = \bar{f_2}(s)$

then,

$$\bar{f_1}(s)\cdot\bar{f_2}(s) = \mathcal{L}\{\underbrace{\int_0^t f_1(u)\cdot f_2(t-u)\,du}_{F(t)}\}$$

RHS

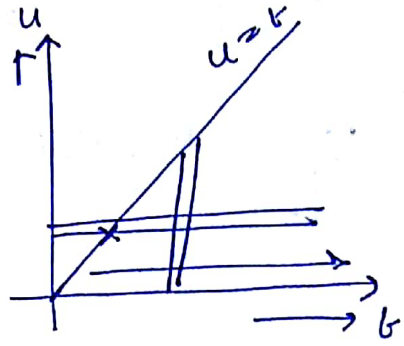
$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\cdot\int_0^t f_1(u)\cdot f_2(t-u)\,du\cdot dt$$

$$= \int_0^\infty\int_0^t e^{-st} f_1(u)\cdot f_2(t-u)\,du\,dt$$

Changing the order of integration;

$u=0$ to $u=t$;

$t=0$ to $t=\infty$;



$$= \int_0^\infty\int_u^\infty e^{-st}\cdot f_1(u)\, f_2(t-u)\,dt\,du$$

Let $t-u=v \Rightarrow dt = dv$

$$e^{-s(u+v)}\cdot f_1(u)\cdot f_2(v)\,dv\,du$$

$$\Rightarrow \int_0^\infty\int_0^\infty e^{-su}\cdot f_1(u)\cdot e^{-sv} f_2(v)\,dv\,du$$

$$\Rightarrow \int_0^\infty e^{-su} f_1(u)\,du\cdot\int_0^\infty e^{-sv} f_2(v)\,dv$$

LHS $\Rightarrow$ $\bar{f_2}(s)\,\bar{f_1}(s)$

Hence Proved

(a) $\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)} \cdot \frac{1}{a} \cdot \frac{a}{(s^2+a^2)}\right]$$

$\Rightarrow$ Let,

$$\bar{f}_1(s) = \frac{s}{(s^2+a^2)} \qquad \bar{f}_2(s) = \frac{1}{a} \cdot \frac{a}{(s^2+a^2)}$$

$$\mathcal{L}^{-1}\{\bar{f}_1(s)\} = f_1(t) \qquad \mathcal{L}^{-1}\{\bar{f}_2(s)\} = f_2(t)$$

$$f_1(t) = \cos at \qquad f_2(t) = \frac{1}{a}\sin at$$

Using convolution theorem;

$$\Rightarrow \frac{1}{a}\int_0^t \cos au \cdot \sin a(t-u)\,du$$

$$\Rightarrow \frac{1}{a}\int_0^t \cos au \cdot (\sin at \cdot \cos au - \cos at \sin au)\,du$$

$$\Rightarrow \frac{1}{a}\sin at \int_0^t (\cos au)^2\,du - \frac{1}{a}\frac{\cos at}{2}\int_0^t \sin 2au\,du$$

$$\Rightarrow \frac{t\sin at}{2a} + \frac{\sin at \sin 2at}{4a^2} - \frac{\cos at \cos 2at}{4a^2} + \frac{\cos at}{4a^2}$$

$$\Rightarrow \boxed{\frac{t\sin at}{2a}} \quad \text{Ans}$$

(b) Find $f(t)$ as solution of integral solution

$$f(t) = t + e^{-2t} + \int_0^t f(\tau) \cdot e^{2(t-\tau)}\,d\tau$$

Taking Laplace Transform of the above eqn;

$$\bar{f}(s) = \frac{1}{s} + \frac{1}{(s+2)} + \mathcal{L}\left[\int_0^t f(\tau) \cdot e^{2(t-\tau)}\,d\tau\right]$$

Let, $\mathcal{L}\{f(t)\} = \bar{f}(s)$

Using convolution theorem;

$$\mathcal{L}\left[\int_0^t f(\tau)\, e^{2(t-u)}\, du\right] = \bar{f}(s) \cdot \frac{1}{(s-2)}$$

$$\Rightarrow \bar{f}(s) = \frac{1}{s} + \frac{1}{s+2} + \bar{f}(s) \cdot \frac{1}{(s-2)}$$

$$\bar{f}(s)\left[1 - \frac{1}{(s-2)}\right] = \frac{2(s+1)}{s(s+2)}$$

$$\bar{f}(s)\, \frac{(s-3)}{(s-2)} = \frac{2(s+1)}{s(s+2)}$$

$$\bar{f}(s) = \frac{2(s+1)(s-2)}{s(s+2)(s-3)}$$

$$f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\left\{\frac{2(s+1)(s-2)}{s(s+2)(s-3)}\right\}$$

Taking Partial fractions;

$$f(t) = \mathcal{L}^{-1}\left[\frac{2}{3}\frac{1}{s} + \frac{4}{5}\cdot\frac{1}{(s+2)} + \frac{8}{15}\cdot\frac{1}{(s-3)}\right]$$

$$f(t) = \frac{1}{45}\left[30t + 36e^{-2t} + 24e^{3t}\right] + k$$

$f(0) = 1$ from original eqn;

$$f(0) \Rightarrow 1 = \frac{1}{45}[36 + 24] + k$$

$$1 = \frac{4}{3} + k \Rightarrow k = \frac{-1}{3}$$

$$\boxed{f(t) = \frac{1}{45}\left[30t + 36e^{-2t} + 24e^{3t} - 15\right]}$$ Ans

Que 8. $(D+1)^2 y = t$, if $y(0) = -3$, $y'(1) = -1$

Let,

$y(0) = y_0$, $y'(0) = y_1$

considering Laplace transform of DF eqn;

If $\mathcal{L}\{y\} = \bar{y}$

and

$\mathcal{L}\{f(t)\} = \bar{f}(s)$

Then,

$$(s+1)^2 \bar{y} = \frac{1}{s^2} + (s+2) y_0 + y_1$$

$$\bar{y} = \frac{1}{s^2(s+1)^2} + \frac{(s+2)(-3)}{(s+1)^2} + \frac{y_1}{(s+1)^2}$$

$$\mathcal{L}^{-1}\{\bar{y}\} = y = \mathcal{L}^{-1}\left\{ \frac{1}{s^2(s+1)^2} + \frac{-3(s+2)}{(s+1)^2} + \frac{y_1}{(s+1)^2} \right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} ds$$

$$= (t-2) + (t+2)e^{-t}$$

$$\mathcal{L}^{-1}\left\{\frac{-3(s+2)}{(s+1)^2}\right\} = -3 \cdot \left\{ \frac{(s+1)}{(s+1)^2} + \frac{1}{(s+1)^2} \right\}$$

$$= -3 \cdot \{e^{-t} + t \cdot e^{-t}\}$$

$$\mathcal{L}^{-1}\left\{\frac{y_1}{(s+1)^2}\right\} = y_1 \cdot t \cdot e^{-t}$$

$$\mathcal{L}^{-1}\{\bar{y}\} = y = (t-2) + (t+2)e^{-t} - 3\cdot\{e^{-t} + t\cdot e^{-t}\} + y_1 t e^{-t}$$

$\mathcal{L}^{-1}\{\bar{y}\} = y \Rightarrow y(1) = -1$ given

substituting ; $\boxed{y_1 = 3}$

$$\boxed{y = (t-2) + (t-1)e^{-t}}$$ Ans

Scanned by CamScanner

9. $(tD^2 + (1-2t)D - 2)y = 0$, if $y(0) = 1$, $y'(0) = 2$

Taking Laplace Transform of the DF eqn;

$$\mathcal{L}\{t \cdot y''\} + \mathcal{L}\{y'\} - 2\mathcal{L}\{ty'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

From property, $\mathcal{L}\{t \cdot f(t)\} = -\frac{d}{ds}\bar{f}(s)$

If, $\mathcal{L}\{f(t)\} = \bar{f}(s)$

and $y(0) = y_0$ and $y'(0) = y_1$

Then,

$$-\frac{d}{ds}\mathcal{L}\{y''\} + \mathcal{L}\{y'\} - 2\cdot\left(-\frac{d}{ds}\mathcal{L}\{y'\}\right) - 2\mathcal{L}\{y\} = 0$$

If $\mathcal{L}\{y\} = \bar{y}$

and using Diffrn Laplace Transform,

$$-\frac{d}{ds}(s^2\bar{y} - sy_0 - y_1) + (s\bar{y} - y_0) + 2\frac{d}{ds}(s\bar{y} - y_0) - 2\bar{y} = 0$$

$$\Rightarrow -\frac{d}{ds}(s^2\bar{y} - s - 2) + (s\bar{y} - 1) + 2\frac{d}{ds}(s\bar{y} - 1) - 2\bar{y} = 0$$

$$\Rightarrow -s^2\frac{d}{ds}\bar{y} + 2s\frac{d}{ds}\bar{y} + s\bar{y} - 2 = 0$$

$$\Rightarrow (2s - s^2)\frac{d}{ds}\bar{y} + s\bar{y} = 2$$

$$\frac{d}{ds}\bar{y} + \frac{s}{(2s - s^2)}\bar{y} = \frac{2}{2s - s^2} \quad \text{—— (A)}$$

(A) is a linear differential eqn with Integrating factor

Integrating factor $= e^{\int \frac{1}{2-x} dx}$ $\left\{ \begin{array}{l} s \to x \\ ds \to dx \end{array} \right\}$

$\Rightarrow (x-2)^{-1}$

Multiplying and integrating throughout

$$\frac{1}{(x-2)} \cdot y = \int \frac{1}{(x-2)} \cdot \frac{2}{x(2-x)} dx$$

$$\frac{1 \cdot 2}{(x-2)\, x\, (2-x)} = -\left[\frac{A}{(x-2)} + \frac{Ax+B}{(x-2)^2} + \frac{C}{x} \right]$$

Taking $x = 1, -1, 4$

$$B + C = 2$$
$$12A + 3B + C = 2$$
$$12A + 2B + 2C = 1$$

we get,

$A = -1/4$, $B = 3/2$; $C = 1/2$

$x \to s$

$$\frac{1}{(s-2)} \cdot y = -\int \left[\frac{-1}{4(s-2)} + \frac{-\frac{1}{4}s + \frac{3}{2}}{(s-2)^2} + \frac{1}{2s} \right] ds$$

$\Rightarrow$ simplifying ;

$$(2s - s^2)\, \bar{y}'(s) = s\bar{y}$$

$$\bar{y} = \frac{c}{s-2}$$

$$L^{-1}\{\bar{y}\} \Rightarrow y = e^{2t} \cdot c$$

$y(0) = 1$; $c = 1$

$\boxed{y(t) = e^{2t}}$ Ans

Que. 10. Solve $\frac{dx}{dt} = 2x - 3y$, $\frac{dy}{dt} = y - 2x$;

if $x(0) = 8$ and $y(0) = 3$.

The two equations can be written as;

$$(D-2)x + 3y = 0$$
$$2x + (D-1)y = 0$$

If $L\{x\} = \bar{x}$
$L\{y\} = \bar{y}$

$$(s-2)\bar{x} + 3\bar{y} = 0 + 8 + 0 \quad \text{——} ①$$
$$2\bar{x} + (s-1)\bar{y} = 0 + 0 + 3 \quad \text{——} ②$$

Multiplying first eqn by 2 and second eqn by $(s-2)$;

$$\Rightarrow \quad 2(s-2)\bar{x} + 6\bar{y} = 16$$
$$- \; 2(s-2)\bar{x} + (s-1)(s-2)\bar{y} = 3(s-2)$$

$$[6 - (s-1)(s-2)]\bar{y} = 16 - 3s + 6$$

$$\bar{y} = \frac{3s-22}{s^2-3s-4}$$

$$\bar{y} = \frac{3s-22}{(s+1)(s-4)}$$

$$L^{-1}\{\bar{y}\} = \boxed{y = 5e^{-t} - 2e^{4t}.}$$ Ans

for $\bar{x}$ eliminate $\bar{y}$ from the above eqns to get

$$[(s-2)(s-1) - 6]\bar{x} = 8(s-1) - 9$$

$$\bar{x} = \frac{8(s-1)-9}{[(s-2)(s-1)-6]}$$

Taking Inverse Laplace transform;

$$\boxed{x = 3e^{4t} + 5e^{-t}}$$ Ans