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Batch :- A11 (civil).

① soln $\vec{r} = t\hat{i} - t^2\hat{j} + (t-1)\hat{k}$

$\vec{s} = 2t^2\hat{i} + 6t\hat{k}$

$\int_0^2 \vec{r}\times\vec{s}\, dt = ?$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t & -t^2 & (t-1) \\ 2t^2 & 0 & 6t \end{vmatrix}$$

$\hat{i}(-6t^3) - \hat{j}(6t^2 - (t-1)2t^2) + \hat{k}(+2t^4)$

$= -6t^3\hat{i} - \hat{j}(6t^2 - 2t^3 + 2t^2) + 2t^4\hat{k}$

$\boxed{\vec{r}\times\vec{s} = -6t^3\hat{i} - \hat{j}(8t^2 - 2t^3) + 2t^4\hat{k}}$

so, $\int_0^2 -6t^3\hat{i} + (2t^3 - 8t^2)\hat{j} + 2t^4\hat{k}\, dt$

$\Rightarrow -\int_0^2 6t^3\hat{i}\, dt + \int_0^2 (2t^3 - 8t^2)\hat{j}\, dt + \int_0^2 2t^4\hat{k}\, dt$

$\Rightarrow \frac{-6}{4}[t^4]_0^2\hat{i} + \left[\frac{2t^4}{4} - \frac{8t^3}{3}\right]_0^2\hat{j} + \left[\frac{2t^5}{5}\right]_0^2\hat{k}$

$\Rightarrow \frac{-6}{4}\times 16\hat{i} + \left[\frac{2\times 16}{4} - \frac{8\times 8}{3}\right)\hat{j} + \left[\frac{2\times 32}{5}\right]\hat{k}$

$= -24\hat{i} + \left(\frac{-40}{3}\right)\hat{j} + \frac{64}{5}\hat{k}$

$= -24\hat{i} - \frac{40}{3}\hat{j} + \frac{64}{5}\hat{k}$ Ans.

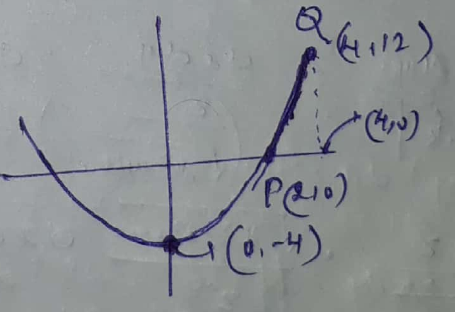
② soln $y = x^2 - 4$ from A(2,0) to B (4,12) in x-y plane

$\vec{F} = xy\hat{i} + (x^2+y^2)\hat{j}$

soln $x^2 = (y+4)$

$x^2 = y+4$

vertex (0,-4)



$y = x^2 - 4$

$dy = 2x\, dx$

$\int \vec{F}\cdot d\vec{r}$

$\int_{PQ} [xy\hat{i} + (x^2+y^2)\hat{j}]\cdot[dx\hat{i} + dy\hat{j}]$

$\Rightarrow \int_{PQ} xy\,dx + (x^2+y^2)\,dy$

$\Rightarrow \int_2^4 [x(x^2-4) + (x^2 + (x^2-4)^2)\cdot 2x]\,dx$

$= \int_2^4 [x^3 - 4x + 2x^3 + 2x\cdot(x^2-4)^2]\,dx$

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$$\int_2^4 [(3x^3-4x+2x(x^4+16-8x^2)]\,dx$$

$$\Rightarrow \int_2^4 [3x^3-4x+2x^5+32x-16x^3)\,dx \Rightarrow \int_2^4 [2x^5-13x^3+28x]\,dx$$

$$= \frac{2}{6}[x^6]_2^4 - \frac{13}{4}[x^4]_2^4 + \frac{28}{2}[x^2]_2^4$$

$$= \frac{1}{3}[4^6-2^6] - \frac{13}{4}[4^4-2^4] + 14[4^2-2^2]$$

$$= 732 \text{ Ans.}$$

(3)

Soln. $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k})\cdot ds$ s is surface of sphere in 1st octant, $x^2+y^2+z^2=a^2$ (sphere)

$\phi = x^2+y^2+z^2-a^2$

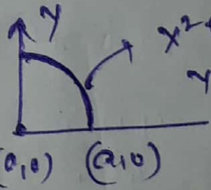
$\vec{n} = \nabla\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ (normal vector to surface)

unit vector $= \dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$ (normal to surface)

$$\iint_S (yz\hat{i}+zx\hat{j}+xy\hat{k})\cdot\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}\,ds$$

$$\Rightarrow \iint_S \frac{xyz+xyz+xyz}{\sqrt{x^2+y^2+z^2}}\,ds = \iint_S \frac{3xyz}{\sqrt{x^2+y^2+z^2}}\cdot ds$$

$$ds = \frac{dx\cdot dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)} = ds$$



$x^2+y^2=a^2$

$y=+\sqrt{a^2-x^2}$

$x^2+y^2=a^2$

$r=a$

$$\iint_R \frac{3xy\cancel{z}}{\cancel{\sqrt{x^2+y^2+z^2}}}\cdot\frac{\cancel{\sqrt{x^2+y^2+z^2}}}{\cancel{z}}\,dx\,dy$$

$\iint_R 3xy\,dx\,dy =$

changing to polar form

$$\int_0^{\pi/2}\int_0^a 3r^3\cos\theta\cdot\sin\theta\,dr\,d\theta$$

$$\Rightarrow \left(\frac{3}{2}\right)\int_0^{\pi/2}\int_0^a (r^3\sin 2\theta)\,dr\,d\theta = \left(\frac{3}{2}\right)\int_0^{\pi/2}\left[\left(\frac{r^4}{4}\right)\right]_0^a\cdot\sin 2\theta\,d\theta$$

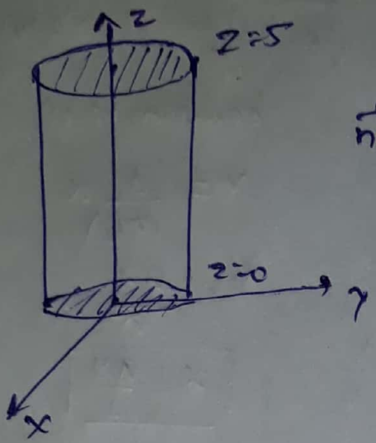
$$= \left(\frac{3}{8}\right)\int_0^{\pi/2} a^4\sin 2\theta\,d\theta = \frac{3a^4}{8}\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{3a^4}{8}\left(+\frac{1}{2}+\frac{1}{2}\right) = \frac{3a^4}{8}$$

(4) Soln $\iint_S \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = 2\hat{i} + x\hat{j} - 3y^2 z\hat{k}$

S is surface of cylinder $x^2+y^2=16$ included in 1st octant between $z=0$ and $z=5$

Soln



$\phi = x^2+y^2-16$

$\vec{n} = \nabla\phi = 2x\hat{i} + 2y\hat{j}$ (normal vector to surface)

$$\boxed{\hat{n} = \frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}}}$$

$$\iint_S (2x\hat{i} + x\hat{j} - 3y^2 z\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}}\right) ds \quad \Bigg| \quad ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$\Rightarrow \iint_{S'} \frac{2x+xy}{\sqrt{x^2+y^2}}\cdot dx\,dy$$

[wrong question → can't be solved with given data some error with vector function given].

Sol: after modification of ques

$A = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$ $\quad \vec{n} = 2x\hat{i} + 2y\hat{j}$

$\iint_S \vec{A}\cdot\hat{n}\, ds$ $\quad \hat{n} = \frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}}$

$$\iint_S (z\hat{i} + x\hat{j} - 3y^2 z\hat{k})\cdot\frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}}\cdot\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

$$\to \iint_S \left(\frac{xz+xy}{\cancel{\sqrt{x^2+y^2}}}\right)\cdot\frac{\cancel{\sqrt{x^2+y^2}}}{x}\, dy\,dz$$

$$\int_0^5\int_0^4 (z+y)\, dy\,dz$$

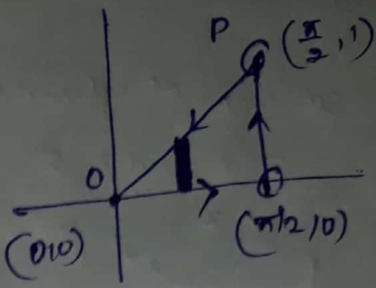
$$\to \int_0^5 \left[zy + \frac{y^2}{2}\right]_0^4 dz = \int_0^5 (4z+8)\, dz$$

$$= (2z^2+8z)_0^5$$

$$= (50+40) = 90 \text{ Ans.}$$

⑤ $\int_C (y-\sin x)\,dx + \cos x\,dy$

$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$



$P = (y - \sin x)$

$\frac{\partial P}{\partial y} = 1$

$Q = \cos x$

$\frac{\partial Q}{\partial x} = -\sin x$

putting values

$\iint (-\sin x - 1)\,dA.$

$\int_0^{\pi/2}\int_0^{2y/\pi} (-\sin x - 1)\,dx\,dy$

equation of OP

$y = \frac{1}{(\pi/2)}x$

$y = \frac{2x}{\pi}$

$\boxed{x = \frac{\pi y}{2}}$

$\Rightarrow \int_0^{\pi/2}\int_0^{2x/\pi} (\sin x - 1)\,dy\,dx$

$\Rightarrow \int_0^{\pi/2} \left[-\sin x \cdot y - y\right]_0^{2x/\pi} dx$

$\Rightarrow \int_0^{\pi/2} \left[-\sin x \cdot \left(\frac{2x}{\pi}\right) - \frac{2x}{\pi}\right] dx$

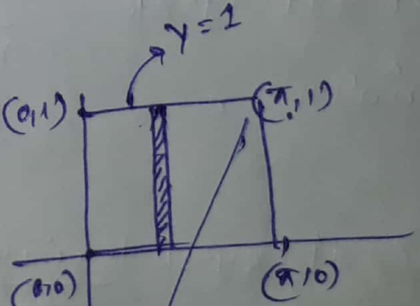
$\Rightarrow \left(\frac{-2}{\pi}\right)\int_0^{\pi/2} x \cdot \sin x - \frac{2}{\pi}\int_0^{\pi/2} x\,dx$

$\Rightarrow \left(\frac{-2}{\pi}\right)\left[(-x\cos x)_0^{\pi/2} + \int_0^{\pi/2} \cos x\,dx\right] - \frac{2}{\pi}\cdot\left[\frac{x^2}{2}\right]_0^{\pi/2}$

$\Rightarrow \left(\frac{-2}{\pi}\right)\left[0 + 1\right] - \frac{1}{\pi} \times \left(\frac{\pi^2}{4}\right)$

$\Rightarrow \left(\frac{-2}{\pi}\right) - \frac{\pi}{4}$ Ans

⑥ $\int (x^2 - \cosh y)\,dx + (y + \sin x)\,dy$



Soln $\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$P = x^2 - \cosh y$

$\frac{\partial P}{\partial x} = 2x$

$Q = (y + \sin x)$

$\frac{\partial Q}{\partial x} = \cos x$

$\Rightarrow \int_0^{\pi}\int_0^{1} (\cos x - 2x)\,dx\,dy$

$\Rightarrow \int_0^{\pi} \left[\cos x \cdot y - 2xy\right]_0^1 dx$

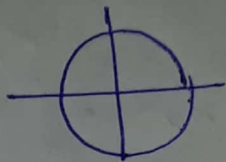
$\Rightarrow \int_0^{\pi} \left[\cos x - 2x\right] dx = \left[\sin x - x^2\right]_0^{\pi}$

$\left[0 - \pi^2\right]$

$$= (12x^2 - \cancel{x^2} + \cancel{x^2}) = (12x^2)$$

$$\boxed{\operatorname{div}\vec{F} = 12x^2}$$

$$\int_{-a}^{a}\int_{-a}^{a}\int_{0}^{b} (12x^2)\, dx\, dy\, dz$$



(11) $\vec{F} = 4x^3\hat{i} - x^2y\hat{j} + x^2z\hat{k}$

$$\operatorname{div}F = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) = (12x^2 - \cancel{x^2} + \cancel{x^2})$$

$$\operatorname{div}\vec{f} = \underline{12x^2}$$

$x^2 + y^2 = a^2 \rightarrow$ (Cylinder)

$$\iiint_V (12x^2)\, dV$$

$$\Rightarrow \int_{-a}^{a}\int_{+\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}}\int_{0}^{b} 12x^2\, dx\, dy\, dz \Rightarrow \int_{-a}^{a}\int_{+\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}}\int_{0}^{b} 12x^2\, dz\, dy\, dx$$

$$= \int_{-a}^{a}\int_{\pm\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} 12x^2 b\, dy\, dx$$

$$= 2\int_{-a}^{a}\left[12x^2 b\right]\left[\sqrt{a^2-x^2}\right] dx$$

$$= 4\int_{0}^{a} (12b)\left(x^2\cdot\sqrt{a^2-x^2}\right) dx$$

Put $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$ | when $x=0$, $\theta = 0$; $x = a$, $\theta = \pi/2$

$$4\int_{0}^{\pi/2} (12b)\cdot a^2\sin^2\theta \cdot a\cos\theta\, a\cos\theta\, d\theta$$

$$48a^4b\int_{0}^{\pi/2} \sin^2\theta\cdot\cos^2\theta\, d\theta =$$

↳ solving it we get $\left(\frac{\pi}{16}\right)$

$$(48a^4b)\times\frac{\pi}{16} = 3\pi a^4 b$$

(10) $\iint_S \vec{r} \cdot ds = \iiint_V \operatorname{div} r \, dV$ divergence theorem

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ let (say)

$\operatorname{div} \vec{r} = (1+1+1) = 3$ $\quad \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$

$= \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 3$

now $\iiint_V 3 \, dV$

Volume of sphere $= \frac{4}{3}\pi r^3$

$x^2 + y^2 + z^2 = (3)^2$

radius $= 3$ unit

$= 3 \iiint_V dV$

volume $= \frac{4}{\cancel{3}} \times \pi \times 3 \times 3 \times \cancel{3}$

$= 36\pi$ (unit)^3

$= (3 \times 36\pi) = 108\pi$ Ans.