Parth Tohri
2K20/B17/33

Maths Assignment No:- 1

(Q1.)

a.) $e^{-3t}(2\cos 5t - 3\sin 5t)$

We know

$L\{2\cos 5t - 3\sin 5t\} = 2L\{\cos 5t\} - 3L\{\sin 5t\}$

$\frac{2s}{s^2+5^2} - \frac{3.5}{s^2+5^2} = \frac{2s-15}{s^2+25}$

Using Shifting property

$L\{e^{-3t}(2\cos 5t - 3\sin 5t)\} = \frac{2(s+3)-15}{(s+3)^2+25}$

$= \frac{2s-9}{s^2+6s+34}$

b.) $\frac{\cos\sqrt{t}}{\sqrt{t}}$ → We know that

$f(t) = \sin\sqrt{t}$ $\quad L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}e^{-\frac{1}{4s}}}{2s^{3/2}}$

$f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$ $\quad f(0)=0$

$L(f'(t)\} = sL\{f(t)\} - f(0) = \frac{\sqrt{\pi}\lambda^{-\frac{1}{4s}}}{2s^{1/2}}$

$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}\,e^{-\frac{1}{4s}}}{\sqrt{s}}$

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c) $|t-1|+|t+1|;t\geq 0$

$$f(t)=\begin{cases}2 & 0\leq t<1\\ 2t & 1\leq t<\infty\end{cases}$$

$$L\{f(t)\}=\int_0^{\infty}e^{-st}f(t)\,dt$$

$$=\int_0^1 2e^{-st}\,dt+\int_1^{\infty}2t\,e^{-st}\,dt$$

$$=\frac{2}{s^2}\left[s+e^{-s}\right]$$

Q2.) a) $3u(t-2)$

$$\mathcal{L}^{-1}[3u(t-2)]=3L^{-1}[u(t-2)]$$

As $L^{-1}[u(t-a)]=\frac{e^{-as}}{s}$

$$3L^{-1}[u(t-2)]=\frac{3e^{-2s}}{s}$$

b.) $u(t-2)\sin(a(t-2))$

$$L^{-1}\{u(t-2)\sin(a(t-2))\}=e^{-2s}\left(\{\sin(a(t+2-2))\}\right)$$

As

$$\mathcal{L}^{-1}\{u(t-c)f(t)\}=e^{-cs}\mathcal{L}\{f(t+c)\}$$

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$$= \frac{ae^{-2s}}{s^2+a^2}$$

Q3.)
$$f(t) = \begin{cases} 0 & 0<t<1 \\ t^2-2t+2 & t \geq 1 \end{cases}$$

$$\mathcal{L}(f(t)) = \int_0^1 e^{-st}(0)\,dt + \int_1^{\infty} e^{-3t}(t^2-2t+2)\,dt$$

$$= \frac{(t^2-2t+2)e^{-3t}}{-s}\Bigg]_1^{\infty} + \frac{1}{s}\int_1^{\infty}(2t-2)e^{-st}\,dt$$

$$= \frac{e^{-s}}{s} + \frac{1}{s}\left\{\frac{(2t-2)e^{-st}}{s}\Bigg]_1^{\infty} - \frac{2e^{-st}}{s^2}\Bigg|_1^{\infty}\right\}$$

$$= \frac{e^{-s}(s^2+2)}{s^3}$$

Q4.) a.)
$$f(t) = \begin{cases} \sin\omega t & 0<t<\frac{\pi}{\omega} \\ 0 & \frac{\pi}{\omega}<t<\frac{2\pi}{\omega} \end{cases}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{sp}}\int_0^p e^{-st} f(t)\,dt \quad \text{where } p = \frac{2\pi}{\omega}$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st}\sin\omega t\,dt.$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left(\frac{\omega e^{-s\pi/\omega}+\omega}{s^2+\omega^2}\right)$$

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$$= \frac{\omega}{(s^2+\omega^2)\left(1-e^{-\frac{s\pi}{\omega}}\right)}$$

b.) $f(t) = \begin{cases} t & 0<t<\pi \\ \pi - t & \pi < t < 2\pi \end{cases}$ $\quad p = 2\pi$

$$\int_0^p e^{-st} f(t)\,dt = \int_0^{\pi} t e^{-st}\,dt + \int_{\pi}^{2\pi} \pi e^{-st}\,dt - \int_{\pi}^{2\pi} t e^{-st}\,dt$$

$$= \frac{\pi}{s}\left(e^{-2\pi s} - e^{-\pi s}\right) + \frac{1}{s^2}\left\{1 + e^{-2\pi s} - 2 s^{-\pi s}\right\}$$

$$\mathcal{L}(f(t)) = \frac{1}{1-e^{-sp}}\int_0^p e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{\pi}{s}\left(e^{-2\pi s} - e^{-\pi s}\right) + \frac{1}{s^2}\left(1 + e^{-2\pi s} - 2 s^{-\pi s}\right)\right]$$

Ans.

Ques 5)

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a.) $t^3 e^{-3t}$ $\mathcal{L}\{e^{-3t}\} = \frac{1}{s+3}$

Using multiplication property

$$\mathcal{L}\{t^3 e^{-3t}\} = (-1)^3 \frac{\partial^3}{\partial s^3}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{6}{(s+3)^4}$$

b.) $te^{-t}\sin 3t$ $\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$

Using multiplication property

$$\mathcal{L}\{t\sin 3t\} = (-1)^1 \frac{\partial}{\partial s}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$$

Using Shifting property

$$\mathcal{L}\{te^{-t}\sin 3t\} = \frac{6(s+1)}{(s^2+2s+10)^2}$$

Q6) a.) $\frac{1-e^t}{t}$

$$\mathcal{L}\{1-e^t\} = \frac{1}{s} - \frac{1}{s-a} = f(s)$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds \Rightarrow \mathcal{L}\left\{\frac{1-e^t}{t}\right\} = \int_s^\infty \mathcal{L}(1-e^t)\,ds$$

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$$= \log\left(\frac{s}{s-a}\right)\Big]_s^{\infty} = \log\left[\frac{s-1}{s}\right]$$

b) $\dfrac{\cos at - \cos bt}{t}$

$$L(\cos at - \cos bt) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^{\infty}\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^{\infty} = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Q7)

a.) $$\int_0^{\infty} \frac{\sin at}{t}\, dt = \int_0^{\infty} e^{-0t}\, \frac{\sin at}{t}\, dt$$

$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^{\infty} \frac{a\, ds}{s^2+a^2} = \tan^{-1}\left[\frac{s}{a}\right]_s^{\infty}$$

$$= \cot^{-1}\left(\frac{s}{a}\right)$$; Here, $s = 0$

$$\int_0^{\infty} \frac{\sin at}{t} = \frac{\pi}{2}.$$

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b.) $\int_0^\infty e^{-t}\left(\frac{\cos at - \cos bt}{t}\right)dt$

$\mathcal{L}\left[\frac{\cos at - \cos bt}{t}\right]$ at $s = 1$

$\mathcal{L}\left(\frac{\cos at - \cos bt}{t}\right) = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds =$

$\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

Answer $= \frac{1}{2}\log\left(\frac{1+b^2}{1+a^2}\right)$

(Ques 8.)

a.) $\frac{s^2-3s+4}{s^3} = \frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}$

$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$

$\mathcal{L}^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\} = 1 - 3t + 2t^2$

b.) $\frac{s+2}{s^2-4s+13} = \frac{s+2}{(s-2)^2+9} = \frac{(s-2)+4}{(s-2)^2+3^2}$

$\mathcal{L}^{-1}\left\{\frac{(s-2)+4}{(s-2)^2+3^2}\right\} = \mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2+3^2}\right\} + \frac{4}{3}\left\{\frac{1\times 3}{(s-2)^2+3^2}\right\}$

Using Shifting property

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We get

Answer $= e^{2t}\left\{\cos(3t)+\frac{4}{3}\sin(3t)\right\}$

c) $\frac{4s+5}{(s-1)^2(s+2)}=\frac{A}{(s-1)}+\frac{B}{(s-1)^2}+\frac{C}{s+2}$

$4s+5=A(s^2+s-2)+B(s+2)+C(s^2-2s+1)$

$A+C=0$
$A+B-2C=4 \longrightarrow A=\frac{1}{3}, B=3, C=\frac{-1}{3}$
$-2A+2B+C=5$

$\frac{4s+5}{(s-1)^2(s-2)}=\frac{1}{3}\left[\frac{1}{s-1}\right]+\frac{3}{(s-1)^2}-\frac{1}{3}\left[\frac{1}{s+2}\right]$

Taking ILT

Answer $=\frac{e^t}{3}+3te^t-\frac{1}{3}e^{-2t}$

d) $\frac{(s+2)^2}{(s^2+4s+8)^2}$

We know $\frac{1}{(s^2+4s+8)^2}=\frac{-1}{2(s+2)}\left\{\frac{d}{ds}(s^2+4s+8)^{-1}\right\}$

$\mathcal{L}^{-1}\left\{\frac{(s+2)^2}{(s^2+4s+8)^2}\right\}=\mathcal{L}^{-1}\left\{(s+2)^2\times\frac{-1}{2(s+2)}\frac{d}{ds}(s^2+4s+8)^{-1}\right\}$

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$$= \frac{-1}{2}\mathcal{L}^{-1}\left\{(s+2)\frac{d}{ds}(s^2+4s+8)^{-1}\right\} - ①$$

As $v\,du = d(uv) - v\,du$ Eq ① becomes.

$$\Rightarrow \frac{-1}{2}\mathcal{L}^{-1}\left[\frac{d}{ds}\left(\frac{s+2}{s^2+4s+8}\right) - \frac{1}{s^2+4s+8}\,d/ds\,(s+2)\right]$$

$$\Rightarrow -\frac{1}{2}\left\{\mathcal{L}^{-1}\left[\frac{d}{ds}\left(\frac{s+2}{s^2+4s+8}\right)\right]\right\} - \mathcal{L}^{-1}\left(\frac{1}{(s+2)^2+2^2}\right)$$

As $\mathcal{L}^{-1}\left[\frac{d^n}{ds^n}f(s)\right] = (-1)^n t^n f(s)$

$$\Rightarrow \frac{-1}{2}\left[(-1)(t)\,e^{-2t}\cos 2t - e^{-2t}\frac{\sin 2t}{2}\right]$$

$$\Rightarrow e^{-2t}\left[\frac{\sin 2t}{4} - \frac{t\cos 2t}{4}\right]$$

Ques 9.) a.)

$$\mathcal{L}^{-1}\left\{\left(\frac{s}{s^2+a^2}\right)^2\right\} \qquad \frac{s}{(s^2+a^2)^2} = \underbrace{\frac{1}{(s^2+a^2)}}_{f(s)} \cdot \underbrace{\frac{s}{(s^2+a^2)}}_{g(s)}$$

$$\mathcal{L}^{-1}(f(s)) = \frac{1}{a}(\sin(at))$$

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$\mathcal{L}^{-1}\{g(s)\} = \cos at$

Now, $f(u) = \frac{1}{a}\sin(au)$

$g(t-u) = \cos(a(t-u))$

By convultion theorem

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \frac{1}{a}\sin au \cdot \cos a(t-u)\,du$$

$$= \frac{1}{2a} t \sin at$$

b.) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\}$ $f(s) = \frac{1}{s^2+1}$, $g(s) = \frac{1}{s^2+9}$

$f(t) = \mathcal{L}^{-1}\{F(s)\} = \sin(t)$

$g(t) = \mathcal{L}^{-1}\{g(s)\} = \frac{1}{3}\sin(3t)$

Using convultion Theorem

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\} = \int_0^t \left(\frac{1}{3}\sin 3u\right)\sin(t-u)\,du$$

$$= \frac{1}{24}(3\sin t - \sin 3t)$$

(Q10)

a) $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t \quad x(0)=2,\ x'(0)=-1$

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Taking $\mathcal{L}T$ on both sides

$$[s^2\bar{x} - s\,x(0) - x'(0)] - 2[s\bar{x} - x(0)] + \bar{x} = \frac{1}{s-1}$$

$$\bar{x} = \frac{2s^2 - 7s + 6}{(s-1)^3}$$

Taking $I\mathcal{L}T$ both sides

$$\frac{2s^2 - 7s + 6}{(s-1)^3} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3}$$

$A = 2,\ B = -3,\ C = 1$

$$x = 2\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - 3\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s-1)^3}\right)$$

$$x = e^t\left(2 - 3t + \frac{t^2}{2}\right)$$

b) $t\frac{d^2y}{dt} + 2\frac{dy}{dt} + ty = \cos t \quad y(0)=1$

Taking $\mathcal{L}T$ on both sides

$$-\frac{d}{ds}\mathcal{L}(y'') + 2\mathcal{L}\{y'\} - \frac{d}{ds}\mathcal{L}\{y\} = \frac{s}{s^2+1}$$

$$-\frac{d}{ds}\left[s^2\mathcal{L}\{y\}-sy(0)-y'(0)\right]+2s\mathcal{L}\{y\}-2-$$

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$$\frac{d}{ds}\mathcal{L}\{y\}=\frac{s}{s^2+1}$$

Putting values and sharing

$$(-s^2-1)\frac{d}{ds}\mathcal{L}\{y\}-1=\frac{s}{s^2+1}$$

$$-\frac{d}{ds}\mathcal{L}\{y\}=\frac{1}{s^2+1}+\frac{s}{(s^2+1)^2}$$

Taking ILT both sides

$$\mathcal{L}^{-1}\left\{-\frac{d}{ds}\mathcal{L}\{y\}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}+\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

$$ty=\sin t+\frac{t\sin t}{2}$$

As $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=\frac{t\sin at}{2a}$

$$y=\frac{1}{2}\left(1+\frac{2}{t}\right)\sin t \text{ Ans}$$