MAXWELL'S DISPLACEMENT CURRENT

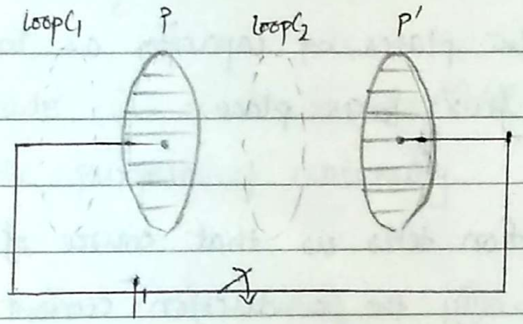
* Inconsistency of Ampere's Law:

Ampere's circuital law relating B & I is given by:

$$\oint_C B \cdot dl = \mu_0 I$$

Here, I is conduction current passing through a surface enclosed by loop C.

Maxwell noticed an inconsistency ie calculating mag. field outside capacitor at a point, in two differents ways ie. using two diff. loops, gives two different results



Clearly, I is the conduction current that flows through loop C_1 and no current flows through C_2 ∴ Applying ACL to two loops, we get,

$$\oint_{C_1} B \cdot dl = \mu_0 I$$

$$\oint_{C_2} B \cdot dl = 0$$

The above two equations will hold good until even if C_1 & C_2 are infinitesimally close to plate P.

But when C_1 & C_2 are too close to each other, then,

$$\oint_{C_1} B \cdot dl = \oint_{C_2} B \cdot dl.$$

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But, its not the case here. so, maxwell noticed the inconsistency.

To remove this contradiction, generalised ACL is :

$$\oint B\cdot dl = \mu_0\left(I + \varepsilon_0 \frac{d\phi_E}{dt}\right)$$ → Ampere-Maxwell law.

$$\oint B\cdot dl = \mu_0 (I + I_D)$$

where $I_D = \varepsilon_0 \frac{d\phi_E}{dt}$ = displacement current that exists b/w plates of capacitor as long as change in electric flux takes place.

* Modification tells us that source of Magnetic field is not only the conduction current but also rate of change of electric flux. In case of charging capacitor, both are present.

Ex 4.9>

$$E = \sigma/\varepsilon_0 = \frac{Q}{A\varepsilon_0}$$

$$\phi_E = E\cdot A = EA = \frac{Q}{\varepsilon_0}$$

$$I_D = \varepsilon_0 \frac{d\phi_E}{dt} = \varepsilon_0 \frac{d}{dt}\left(\frac{Q}{\varepsilon_0}\right) = \frac{dQ}{dt} = I.$$

* Continuity of current

It states that sum of conduction current & displacement current has the property of continuity.

Consider the previous fig,

for loop 1 : there is no. Elec. flux.

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$\therefore \phi_E = 0 \qquad \therefore I + I_D = I + 0 = I.$

For loop C_2 : there is no conduction current.

$\therefore I + I_D = 0 + \epsilon_0 \frac{d}{dt}\phi_E = \epsilon_0 \frac{d}{dt}(EA)$

$I + I_D = \epsilon_0 \frac{d}{dt}\left(\frac{Q}{A\epsilon_0} A\right) = I.$

So, property of continuity is verified.

* Characteristics of Displacement current.

i) It produces mag. field but is not related to the flow of charges.

ii) Its magnitude is equal to $\epsilon_0 \, d\phi_E/dt$.

iii) It serves the purpose of continuity.

Ex 4.10> $E = Q/A\epsilon_0 \Rightarrow \phi_E = E \times \frac{A}{3} = \frac{Q}{A\epsilon_0} \times \frac{A}{3} = \frac{Q}{3\epsilon_0}$

$I_D = \epsilon_0 \frac{d}{dt}\left(\frac{Q}{3\epsilon_0}\right) = \frac{1}{3} I$

Ex 4.11> Time constant of R-C circuit is :

$\tau = RC = 0.5 \times 10^6 \times 2 \times 10^{-9} = 10^{-3}\,s.$

Charge on capacitor at any instant,

$q(t) = CV\left[1 - e^{-t/\tau}\right]$

$= 2 \times 10^{-9} \times 3 \left[1 - e^{-t/10^{-3}}\right]$

$= 6 \times 10^{-9} \left[1 - e^{-t/10^{-3}}\right].$

Electric field in between plates $\Rightarrow E = \frac{q(t)}{A\epsilon_0}$

$E = \frac{q(t)}{\pi (1)^2 \epsilon_0} = \frac{q(t)}{\pi \epsilon_0}.$

* Maxwell's Equation.

The entire theory of EM waves is contained in equations called Maxwell's Equation.

i) $$\oint E \cdot ds = \frac{Q}{\epsilon_0}.$$

This is Gauss's law in electrostatics which states that surface net electric flux through a closed surface S is equal to total charge Q contained in the surface S divided by ϵ_0.

ii) $$\oint B \cdot ds = 0.$$

This is Gauss's law in Magnetostatics.

⇒ monopoles do not exist.

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iii) $$\oint E \cdot dl = -\frac{d\phi_B}{dt} = -\frac{d}{dt}\oint B \cdot ds.$$

This is faraday's laws of electromagnetic induction.

iv) $$\oint B \cdot dl = \mu_0 (I) + \mu_0 \epsilon_0 \frac{d}{dt} \oint E \cdot ds$$

$$\oint B\, dl = \mu_0 (I + I_D)$$

where $I_D = \epsilon_0 \frac{d}{dt}\phi_E = \epsilon_0 \frac{d}{dt} \oint E \cdot ds.$

It is Ampere - Maxwell's law.

* Differential form of Maxwell's Equation.

i) $\text{div}. E = \nabla \cdot E = \rho/\epsilon_0$

where ρ = charge density ie. charge per unit volume

or $\text{div}. D = \nabla \cdot D = \rho.$

where $D = (\epsilon E)$ = electric displacement vector $\left(= \frac{Q}{4\pi r^2}\right)$

ii) $\text{div}. B = \nabla \cdot B = 0.$

iii) $\text{curl } E = \nabla \times E = -\frac{\partial B}{\partial t}.$

iv) $\text{curl } B = \nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial}{\partial t} E.$

$\because B = \mu_0 H.$

$\text{curl } H = \nabla \times H = J + \epsilon_0 \frac{\partial}{\partial t} D.$

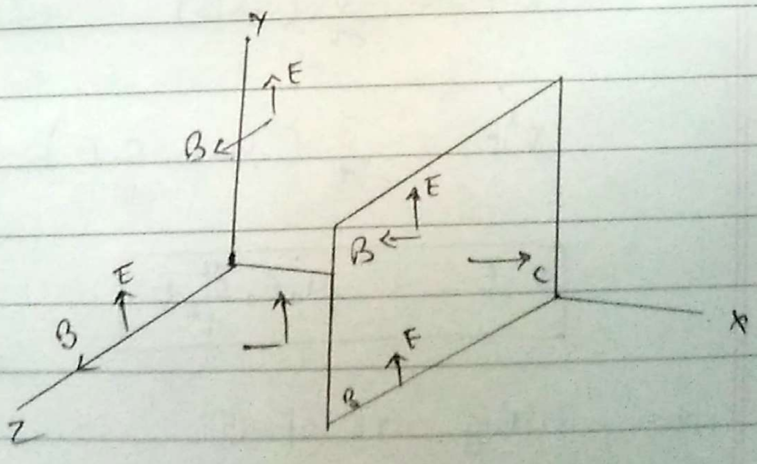
* WAVE EQUATION FOR PLANE - ELECTROMAGNETIC WAVE IN VACCUM.

We have seen in derivation for mechanical wave that a fn $y(x,t)$ representing displacement at any point in mechanical wave propagating in +x-direction must satisfy:

$$\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{dx^2} \quad \text{—— (1)}$$

equation (1) is called wave equation.
v is speed of propagation of wave.

* What is meant by a plane wave?
Consider a wavefront located in a plane ⊥ x-axis, let E has only y-component & B has only z component. both fields move together in +x-direction. Field values are zero at all points in a wavefront on right side of it and field values are uniform at all points on left side of it. Such a wave, at any instant, in which, fields are uniform over any plane ⊥ to dirn of propagation, is called a plane wave.



NOTE : There is definite ratio between magnitudes of E & B:

$E = cB$.

\+ Wave Equation for E & B.

In vaccum, there is no charge or current.

$\therefore$ $J = 0$ and $\rho = 0$.

maxwell's eqn becomes :

$$\nabla \cdot E = 0 \quad \text{--- (2)}$$

$$\nabla \cdot B = 0 \quad \text{--- (3)}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \text{--- (4)}$$

$$\nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \quad \text{--- (5)}$$

Taking curl of (4),

$$\nabla \times (\nabla \times E) = \nabla \times \left(-\frac{\partial B}{\partial t}\right).$$

$\because$ $\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E$

and using (2) $\hookrightarrow = -\nabla^2 E$.

$$\therefore \quad -\nabla^2 E = \nabla \times \left(-\frac{\partial B}{\partial t}\right)$$

$$-\nabla^2 E = -\frac{\partial}{\partial t}(\nabla \times B)$$

$$\nabla^2 E = \frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial}{\partial t} E\right)$$

$$\boxed{\nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} E} \quad \text{--- (6)}.$$

Now, taking curl of (5),

$$\nabla \times (\nabla \times B) = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\nabla(\nabla \cdot B) - \nabla^2 B = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times E)$$

$$-\nabla^2 B = \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial}{\partial t} B\right)$$

$$-\nabla^2 B = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} B$$

$$\boxed{\nabla^2 B = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}} \quad \text{--- (7)}.$$

Equation (6) and (7) are second order differential eqns. satisfying general wave eqn. (1) $\therefore$ E and B behaves as waves and maxwell's eqn are supporting the propagation of em waves through vaccum with constant speed.

$$1/v^2 = \mu_0 \epsilon_0 \qquad \text{(on comparing)}$$

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Putting values : $v = 3 \times 10^8$ m/s.

(= speed of light in vaccum).

* SINUSODIAL ELECTROMAGNETIC WAVES

In sinusodial em waves, field vectors E and B at any point in space are sinusodial function of time and their variations are sinusodial too. Some sinusodial em waves are plane waves.

See fig 4.22, consider a linearly polarized em waves travelling in +X-direction. Note that E & B are in same phase.

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Sinusodial waves can be represented by wave functions, just like waves on a string. A transverse wave travelling in +x -direction along a stretched string is given by:

$$y = A\cos(kx-\omega t)$$

Let E_{max} and B_{max} be amplitudes of two fields & E_y and $B_z(x,t)$ be instantaneous values of y component of E and z component of B resp.

$$\therefore \left.\begin{aligned} E_y &= E_{max}\cos(kx-\omega t) \\ B_z &= B_{max}\cos(kx-\omega t)\end{aligned}\right\} - ①$$

or

$$\left.\begin{aligned} E &= \hat{j}\, E_{max}\cos(kx-\omega t) \\ B &= \hat{k}\, B_{max}\cos(kx-\omega t)\end{aligned}\right\} - ②$$

if wave is travelling in $-x$ direction, then,

$$\left.\begin{aligned} E(x,t) &= E_{max}\cos(kx+\omega t) \\ B(x,t) &= -B_{max}\cos(kx+\omega t)\end{aligned}\right\} - ③$$

In terms of exponential, eq ② becomes,

$$\left.\begin{aligned} E(x,t) &= \hat{j}\, E_{max}\, e^{i(kx-\omega t)} \\ B(x,t) &= \hat{k}\, B_{max}\, e^{i(kx-\omega t)}\end{aligned}\right\} - ④$$

Let $\hat{k}$ be propagation (wave) vector pointing in dir^n of propagation of wave & $\hat{n}$ be polarization vector (dir^n of E) $\therefore$

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$$\left.\begin{aligned} E(x,t) &= \hat{n}\, E_{max}\, e^{i(\hat{k}\cdot r-\omega t)} \\ B(x,t) &= (\hat{k}\times\hat{n})\, B_{max}\, e^{i(k\cdot r-\omega t)}\end{aligned}\right\} - ⑤.$$

or

$$\left.\begin{aligned} E(x,t) &= \hat{n}\, E_{max}\cos(k\cdot\hat{r}-\omega t) \\ B(x,t) &= (\hat{k}\times\hat{n})\, B_{max}\cos(k\cdot r-\omega t)\end{aligned}\right\} - ⑥.$$

* Energy and momentum in em waves.

Electromagnetic waves carry energy.

The energy density associated with E is:

$$u_E = \frac{1}{2}\varepsilon_0 E^2$$

where ε_0 = permitivity of free space

The energy density associated with B is

$$u_B = \frac{1}{2\mu_0}B^2$$

where μ_0 = permeability of free space

In a region of free space, where both E & B exists,

$$u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2.$$

for electromagnetic waves in free space, $B = \frac{E}{c} = \sqrt{\mu_0\varepsilon_0}\, E$

$$\therefore u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}\mu_0\varepsilon_0 E^2 = \underbrace{\frac{1}{2}\varepsilon_0 E^2} + \underbrace{\frac{1}{2}\varepsilon_0 E^2}$$

$\therefore$ energy density associated with E field of an em wave = energy density associated with B field of same em wave in free space.

* u depends on position & time.

* energy flow and Poynting vector :

Energy is transfered as a wave moves from one region to another.

let us consider a stationery plane perpendicular to x-axis, that coincides with a wave front at some instant. after a time dt, wavefront moves distance $dx = cdt$ to right of stationery plane. The energy to right of plane must have passed through the plane. The volume of this region is $A \times cdt$ and energy du is given by:

$$dU = u \cdot dV$$
$$= 2\times\frac{1}{2}\epsilon_0 E^2 Acdt.$$

$$S = \frac{dU}{Adt} = 2\times\frac{1}{2}\epsilon_0 E^2 c = \epsilon_0 E^2 c$$

$\because c = 1/\sqrt{\mu_0\epsilon_0}$ $\therefore$ $S = \sqrt{\frac{\epsilon_0}{\mu_0}} E^2 = \frac{EB}{\mu_0}$ $(E = cB)$.

So, here we define a vector S called pointing vector which gives magnitude & direction of energy flow.

$$S = \frac{1}{\mu_0} E \times B.$$ (fig 4.23 in Book)

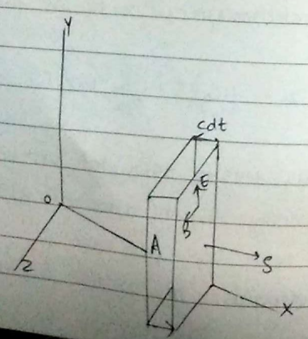

Fig 4.23.

S is always in dirn. of propagation of wave.

The total energy flow per unit time or Power is:

$$P = \oint S \cdot dA.$$

Due to high frequency of em waves, time variation of ponting vector is so rapid that sometimes, we deal with its average value.

The magnetide of average value of S at a point is called intensity of radiation at that point.

$\because$ $E(x,t) = \hat{j} E_{max}\cos(kx - \omega t)$
$B(x,t) = \hat{k} B_{max}\cos(kx - \omega t)$

$\therefore$ $S = \frac{1}{\mu_0} E(x,t) \times B(x,t)$

$$= \frac{1}{\mu_0}\left[\hat{j} E_{max}\cos(kx-\omega t)\right] \times \left[\hat{k} B_{max}\cos(kx-\omega t)\right]$$

It points in $+x$-dirn $\therefore$

$$S_x(x,t) = \frac{1}{\mu_0} E_{max}B_{max}\cos^2(kx-\omega t)$$

$$= \frac{E_{max}B_{max}}{2\mu_0}\left[1 + \cos 2(kx-\omega t)\right]$$

$\because$ avg. value of $[\cancel{1+}\cos 2(kx-\omega t)] = 0$ $\therefore$

avg. value of poynting vector is $S_{avg} = \hat{i} S'_{avg}$ where S'_{avg} is its magnitude,

$$I = S'_{avg} = \frac{E_{max}B_{max}}{2\mu_0} = \frac{E_{max}^2}{2\mu_0 c} = \frac{1}{2}\epsilon_0 c E_{max}^2$$

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* momentum flow and radiation pressure.

$\because$ Em wave carries energy, it also carries momentum, with corresponding momentum density,

$$\frac{dp}{dV} = \frac{EB}{\mu_0 c^2} = \frac{S}{c^2}$$

$$\frac{dP}{Adtc} = \frac{EB}{\mu_0 c^2} = \frac{S}{c^2}$$

$$\frac{dP}{Adt} = \frac{EB}{\mu_0 c} = \frac{S}{c}$$

momentum transfered per unit area per unit time.

When a wave is completely absorbed by a surface momentum is also transferred to the surface. The rate at which momentum is transferred is equal to force on the surface.

The avg. value of force per unit area due to wave or radiation pressure P_{rad} is avg. value of dP/dt divided by absorbing area A.

$$P_{rad} = \frac{S_{avg}}{c} = \frac{I}{c}.$$

If complete reflection occurs,

$$P_{rad} = \frac{2S_{avg}}{c} = \frac{2I}{c}.$$

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* POYNTING'S THEOREM & ENERGY OF ELECTROMAGNETIC WAVES

Let us consider an electromagnetic wave travelling in +X-direction and E and B are confined to y & z axis respectively. The direction of propagation is also given by : (E×B).

Taking divergence of poynting vector in free space.

$$\nabla\cdot(E\times B) = B\cdot(\nabla\times E) - E\cdot(\nabla\times B)$$

$$= -B\frac{\partial B}{\partial t} - \mu_0\epsilon_0 E\frac{\partial E}{\partial t} \quad (\text{Maxwell eq}^n\text{s}).$$

$$\because \quad B\frac{\partial B}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(B^2) \text{ and } E\frac{\partial E}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2)$$

$$\therefore \quad \nabla\cdot(E\times B) = -\frac{1}{2}\frac{\partial}{\partial t}(B^2) - \frac{1}{2}\mu_0\epsilon_0\frac{\partial}{\partial t}(E^2)$$

$$= -\frac{\partial}{\partial t}\left[\frac{1}{2}\mu_0\epsilon_0 E^2 + \frac{1}{2}B^2\right]$$

$$\therefore \quad \frac{1}{\mu_0}\nabla\cdot(E\times B) = \frac{-\partial}{\partial t}\left[\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right]$$

Consider surface S bounding volume V, integrating above eq^n over volume V,

$$\frac{1}{\mu_0}\int_V \nabla\cdot(E\times B) = -\frac{\partial}{\partial t}\left[\int_V\left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right)dV\right]$$

Applying divergence theorem on LHS,

$$\frac{1}{\mu_0}\int_S (E\times B)\cdot da = -\frac{\partial}{\partial t}\left[\int_V\left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right)dV\right] \text{—(A)}$$

The RHS term is called poynting vector and it represents energy transfered per unit time per unit Area

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The pointing vector S is : $S = \frac{1}{\mu_0} E \times B$
magnitude is given : $\frac{EB}{\mu_0}$

The instantaneous energy associated with E,

$$u_E = \frac{1}{2}\epsilon_0 E^2$$

and that associated with B,

$$u_B = \frac{1}{2\mu_0} B^2$$

The sum of two terms on RHS ie. $\left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2\right)$ gives energy associated with em wave.

$$E = cB \quad \text{and} \quad E = \frac{1}{2}\epsilon_0 E^2 = \frac{1}{2}\epsilon_0 c^2 B^2$$

$$= \frac{1}{2}\epsilon_0 \frac{1}{\epsilon_0 \mu_0} B^2 = \frac{1}{2\mu_0} B^2$$

∴ instantaneous energy density associated with E = that with B.

Eq (A) is called poynting theorem which tells that energy flow per unit time (= power) out of any closed surface is integral of S over the surface.

$$P = \oint S \cdot da.$$

NOTE: rate of flow of energy per unit area
= energy per unit time per unit area $= S = EB/\mu_0$

* $P = IA$
* Force $= P_{rad} \times A$

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* ELECTROMAGNETIC WAVES IN DIELECTRIC MEDIUM.

Consider an em wave propagating in an isotropic, homogeneous, non-conducting or dielectric medium having permittivity ϵ and permeability μ.
For such a medium, $J = \sigma E = 0$ and $\rho = 0$

∴ Maxwell equations :-

$$\nabla \cdot E = 0 \qquad \text{— (i)}$$

$$\nabla \cdot B = 0 \qquad \text{— (ii)}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad \text{— (iii)}$$

$$\nabla \times B = \mu\epsilon \frac{\partial E}{\partial t} \qquad \text{— (iv)}$$

(I)

Differentiating eq (I) (iv),

$$\frac{\partial}{\partial t}(\nabla \times B) = \frac{\partial}{\partial t}\left(\mu\epsilon \frac{\partial E}{\partial t}\right)$$

$$\nabla \times \frac{\partial B}{\partial t} = \mu\epsilon \frac{\partial^2 E}{\partial t^2}$$

From (iii), $$-\nabla \times (\nabla \times E) = \mu\epsilon \frac{\partial^2 E}{\partial t^2}$$

$$-\left[\nabla(\nabla \cdot E) - \nabla^2 E\right] = \mu\epsilon \frac{\partial^2 E}{\partial t^2}$$

$$+\left[\nabla^2 E\right] = \mu\epsilon \frac{\partial^2 E}{\partial t^2} \qquad \text{— (A)}$$

Similarly, $$\nabla^2 B = \mu\epsilon \frac{\partial^2 B}{\partial t^2} \qquad \text{— (B)}$$

Comparing with standard wave equation, speed of em wave in dielectric is:

$$v = \frac{1}{\sqrt{\mu \epsilon}}$$

Let μ_r be relative permeability of dielectric and ϵ_r be relative permittivity of dielectric or dielectric constant such that $\mu = \mu_r \mu_0$

$$\epsilon = \epsilon_r \epsilon_0$$

$$\therefore \quad v = \frac{1}{\sqrt{\mu_r \mu_0 \epsilon_r \epsilon_0}} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \frac{1}{\sqrt{\mu_r \epsilon_r}}$$

$$v = \frac{c}{\sqrt{\mu_r \epsilon_r}}$$

Mostly, μ_r is nearly equal to one

$$\therefore \quad v = \frac{c}{\sqrt{\epsilon_r}}$$

$\because$ $\epsilon_r > 1$ (mostly) $\therefore$ speed of em wave in dielectric is less than that of same em wave in vaccum.

The ratio of c to v is called refractive index of material.

$$\frac{c}{v} = n = \sqrt{\mu_r \epsilon_r} \simeq \sqrt{\epsilon_r}$$

ϵ_r is dependent on frequency of field.

* ELECTROMAGNETIC WAVES IN A CONDUCTING MEDIUM

Let us consider propagation of electromagnetic wave in a homogeneous conducting medium of permittivity ϵ, permeability μ and free charge density ρ_f.

$$\therefore \quad J_f = \sigma E$$

Using Maxwell eq^n,

$$\nabla \cdot E = \rho_f / \epsilon \quad \text{— (i)}$$

$$\nabla \cdot B = 0 \quad \text{— (ii)}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \text{— (iii)} \qquad (1)$$

$$\nabla \times B = \mu \sigma E + \mu \epsilon \frac{\partial E}{\partial t} \quad \text{— (iv)}$$

using continuity eq^n for free charge

$$\nabla \cdot J_f = -\frac{\partial \rho_f}{\partial t}$$

$$\sigma (\nabla \cdot E) = -\frac{\partial \rho_f}{\partial t}$$

$$\therefore \quad -\frac{\partial \rho_f}{\partial t} = \frac{\sigma \rho_f}{\epsilon}$$

Integrating, we get,

$$\rho_f(t) = \rho_f(0)\, e^{-\left(\frac{\sigma}{\epsilon}\right)t}$$

ie. any intial free charge dissipates in characteristic time $\tau = \epsilon / \sigma$.

in conductors, any free charge quickly dissappears and interior of conductor may be regarded as free of free charges. ie. there is no sustainable free charge in conductor.

$\therefore$ Maxwell eq^n becomes:

$\nabla \cdot E = 0$ —(i)

$\nabla \cdot B = 0$ —(ii)

$\nabla \times E = -\frac{\partial B}{\partial t}$ —(iii) ②

$\nabla \times B = \mu \sigma E + \mu \epsilon \frac{\partial E}{\partial t}$ —(iv)

Taking curl of 2 (iii),

$\nabla \times (\nabla \times E) = \nabla \times \left(-\frac{\partial B}{\partial t}\right)$

$-\nabla^2 E + \nabla \cdot (\nabla \cdot E) = -\frac{\partial}{\partial t}(\nabla \times B)$

$-\nabla^2 E = -\frac{\partial}{\partial t}\left(\mu \sigma E + \mu \epsilon \frac{\partial E}{\partial t}\right)$ using (2(iv)).

$\nabla^2 E = \mu \sigma \frac{\partial E}{\partial t} + \mu \epsilon \frac{\partial^2 E}{\partial t^2}$ —③.

Taking curl of 2(iv),

$\nabla \times (\nabla \times B) = \nabla \times \left(\mu \sigma E + \mu \epsilon \frac{\partial E}{\partial t}\right)$

$\nabla \cdot (\nabla \cdot B) - \nabla^2 B = \mu \sigma (\nabla \times E) + \mu \epsilon \frac{\partial}{\partial t}(\nabla \times E)$

$-\nabla^2 B = \mu \sigma \left(-\frac{\partial B}{\partial t}\right) + \mu \epsilon \frac{\partial}{\partial t}\left(-\frac{\partial B}{\partial t}\right)$

$\nabla^2 B = \mu \sigma \frac{\partial B}{\partial t} + \mu \epsilon \frac{\partial^2 B}{\partial t^2}$ —④.

Eq ③ & ④ represents wave eqn for E & B vectors, resp. Solution of these eqn, yields.

$E(x,t) = E_{max}\, e^{i(kx - \omega t)}$ —⑤

$B(x,t) = B_{max}\, e^{i(kx - \omega t)}$ —⑥.

Put ⑤ in ③,

$\nabla^2 E + \left(-\mu \sigma \frac{\partial E}{\partial t}\right) - \mu \epsilon \frac{\partial^2 E}{\partial t^2} = 0$

$(ik)^2 E - \mu \sigma (-i\omega) E - \mu \epsilon (-i\omega)(-i\omega) E = 0$

$-k^2 E + \mu \sigma i \omega E + \mu \epsilon \omega^2 E = 0.$

$(-k^2 + \mu \sigma i \omega + \mu \epsilon \omega^2) E = 0.$

as $E \neq 0$ $\therefore$ $-k^2 + \mu \sigma i \omega + \mu \epsilon \omega^2 = 0$

or $k^2 = \mu \sigma i \omega + \mu \epsilon \omega^2.$ —⑦

Let $k = \alpha + i\beta$

$k^2 = \alpha^2 - \beta^2 + 2\alpha\beta i$ --⑧

Comparing ⑦ & ⑧,

$\alpha^2 - \beta^2 = \mu \epsilon \omega^2$ —⑨

$2\alpha\beta = \mu \sigma \omega.$ —⑩

Solving ⑨ and ⑩,

$$\alpha = \sqrt{\epsilon \mu}\, \omega \left\{ \frac{\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon}\right)^2} - 1}{2} \right\}^{1/2} \quad —⑪$$

$$\beta = \sqrt{\epsilon \mu}\, \omega \left\{ \frac{\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon}\right)^2} + 1}{2} \right\}^{2} \quad —⑫.$$

Now $\because$ $k = \alpha + i\beta$

put it in ⑤ and ⑥,

$E(x,t) = E_{max}\, e^{-\beta x}\, e^{i(\alpha x - \omega t)}$

$B(x,t) = B_{max}\, e^{-\beta x}\, e^{i(\alpha x - \omega t)}$

$\vec{E}$ and $\vec{B}$ amplitudes are attenuated due to term $e^{-\beta x}$. Quantity β measures attenuation or absorption constant.

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$\because \quad k = \frac{\omega}{v} \rightarrow$ wave velocity.

$v = \frac{\omega}{k} = \frac{\omega}{\alpha}$ { for conductor, or conducting medium, $k=\alpha$ }.

$$v = \frac{1}{\sqrt{\mu\epsilon}} \left\{ \frac{\sqrt{1+(\sigma/\omega\epsilon)^2} - 1}{2} \right\}^{-1/2}$$

Two cases for propagation of wave.

i) poor conductor :

$$\frac{\sigma}{\omega\epsilon} << 1$$

ii) good conductor :

$$\frac{\sigma}{\omega\epsilon} >> 1$$

i) Poor conductor : using $\sigma/\omega\epsilon << 1$ in (11)

$$\alpha = \sqrt{\mu\epsilon}\,\omega \left\{ \frac{1 + \frac{\sigma^2}{2\omega^2\epsilon^2} - 1}{2} \right\}^{1/2}$$

$$\alpha = \sqrt{\mu\epsilon}\,\omega \left\{ \frac{\sigma}{2\omega\epsilon} \right\}$$

$$\alpha = \sqrt{\frac{\mu}{\epsilon}}\,\frac{\sigma}{2}$$

Now, put same $\sigma/\omega\epsilon << 1$ in (12) or put α in (12).

or $\quad \beta = \sqrt{\mu\epsilon}\,\omega \left\{ \frac{1 + \frac{\sigma^2}{2\omega^2\epsilon^2} + 1}{2} \right\}^{1/2}$

$$\beta = \sqrt{\mu\epsilon}\,\omega \left\{ 1 + \frac{\sigma^2}{4\omega^2\epsilon^2} \right\}^{1/2}$$

$$\beta = \sqrt{\mu\epsilon}\,\omega \left\{ 1 + \frac{\sigma^2}{8\omega^2\epsilon^2} \right\}.$$

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ii) Good conductor : Put $\frac{\sigma}{\omega\epsilon} >> 1$ in (11),

$$\alpha = \sqrt{\mu\epsilon}\,\omega \left\{ \frac{\sigma^{1/2}}{\sqrt{2}\,\omega^{1/2}\epsilon^{1/2}} \right\} = \sqrt{\mu\omega\sigma/2}$$

$$\beta = \sqrt{\mu\epsilon}\,\omega \left\{ \frac{\sigma^{1/2}}{\sqrt{2}\,\omega^{1/2}\epsilon^{1/2}} \right\} = \sqrt{\mu\omega\sigma/2}.$$

• Skin depth or Penetration depth.

The distance (in direction of propagation) where the amplitude of field vector reduces to $1/e$ is called 'skin depth'.

i.e. $\quad e^{-\beta x} = e^{-1}$

$$-\beta x = -1$$

$$\boxed{x = 1/\beta}$$

However, for good conductor,

$$x = \sqrt{\frac{2}{\mu\omega\sigma}}$$