1. POINT

FORMULAE – 1. POINT

1) If $A(x_1, y_1)$ and $B(x_2, y_2)$, then

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

2) The coordinates of circumcentre P of ΔABC can be obtained using the relation $PA^2 = PB^2 = PC^2$.

3) If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then

$$\text{area of } \Delta ABC = \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$

If the area of ΔABC is zero, then the points A, B and C are collinear.

Any point P is inside, on or outside ΔABC can be known from areas of ΔABC, ΔPAB, ΔPBC and ΔPCA.

4) If $A(x_1, y_1)$ and $B(x_2, y_2)$, then the coordinates of point P dividing $\overline{AB}$ in the ratio $\lambda : 1$ or $m : n$ from A (i.e., $AP : PB = \lambda : 1$ or $m : n$) are

$$\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1} \right) \text{ OR } \left(\frac{m x_2 + n x_1}{m + n}, \frac{m y_2 + n y_1}{m + n} \right)$$

λ or m/n are negative if P divides $\overline{AB}$ externally.

The midpoint of $\overline{AB}$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$.

5) The centroid G and incentre I of ΔABC are

$$G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right); \quad I\left(\frac{a x_1 + b x_2 + c x_3}{a + b + c}, \frac{a y_1 + b y_2 + c y_3}{a + b + c} \right)$$

where, $a = BC$, $b = CA$, $c = AB$.

6) If the origin is shifted to (h, k), then the new coordinates (x', y') of the point $P(x, y)$ are given by

$$x' = x - h \quad \text{and} \quad y' = y - k.$$

7) Distance function : $d : R^2 \times R^2 \to R^+ \cup \{0\}$, is defined as $d\,((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$.
d is called the distance function or metric function over $R^2 \times R^2$ and (R^2, d) is called the metric space.

Properties of Distance function :

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3) \in R^2$, then

(i) $d\,(A, B) \geq 0$,

(ii) $d\,(A, B) = 0 \Leftrightarrow A = B$,

(iii) $d\,(A, B) = d\,(B, A)$ and

(iv) $d\,(A, B) + d\,(B, C) \geq d\,(A, C)$.

<u>Proof</u> :

(ii) $d\,(A, B) = 0 \Leftrightarrow \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} = 0$

$$\Leftrightarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 = 0$$

$$\Leftrightarrow x_1 - x_2 = 0 \quad \text{and} \quad y_1 - y_2 = 0$$

$$\Leftrightarrow x_1 = x_2 \quad \text{and} \quad y_1 = y_2$$

$$\Leftrightarrow (x_1, y_1) = (x_2, y_2)$$

$$\Leftrightarrow A = B.$$

2. LINE

FORMULAE - 2. LINE

1) <u>Definition of slope :</u> "If the line is neither horizontal nor vertical and if it makes an angle θ with the positive direction of the X-axis, then slope of the line $= \tan\theta$, where $\theta \in (0, \pi) - \{\pi/2\}$. If the line is horizontal, its slope is defined to be zero. The slope of a vertical line is not defined.

2) The parametric equations of a line through (x_1, y_1) and (x_2, y_2) are
$x = tx_2 + (1-t)x_1, \; y = ty_2 + (1-t)y_1, \; t \in R.$

3) The parametric equations of a line through (x_1, y_1) and making an angle θ with the positive direction of X-axis are
$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r, \quad r \in R, \; \theta \in (0, \pi) - \{\pi/2\}$$
Here, θ is constant for a given line and r is the parameter. $|r|$ is the distance of $P(x, y)$ from (x_1, y_1). The horizontal and vertical lines cannot be defined by this equation.

4) The cartesian equation of a line through (x_1, y_1) and (x_2, y_2) is
$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

5) The equation of X-axis is $y = 0$.

6) The equation of Y-axis is $x = 0$.

7) The equation of a line parallel to Y-axis and making an intercept a on X-axis is $x = a$.

8) The equation of a line parallel to X-axis and

2. LINE

making an intercept b on Y-axis is $y = b$.

9) The equation of a line through (x_1, y_1) having slope m is $y - y_1 = m(x - x_1)$.

10) The equation of a line through (x_1, y_1) and (x_2, y_2) is
$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1)$$

11) The equation of a line making an intercept a on X-axis and b on Y-axis is $x/a + y/b = 1$ $(a, b \neq 0)$.

12) The general form of an equation of line in R^2 is the linear equation $ax + by + c = 0$ $(a^2 + b^2 \neq 0)$. The slope of this line, $m = -a/b$, the intercept on X-axis $= -c/a$ and the intercept on Y-axis $= -c/b$.

13) The general equation of any line parallel to the line $ax + by + c = 0$ is given by $ax + by + k = 0$, $k \in R - \{c\}$.

14) The general equation of any line perpendicular to the line $ax + by + c = 0$ is $bx - ay + k = 0$, $k \in R$.

15) The equation of a line with slope m and making an intercept c on Y-axis is $y = mx + c$.

16) $x\cos\alpha + y\sin\alpha = p$ is the equation of a line where p is the length of perpendicular from the origin on the line and α is the angle made by this perpendicular with the positive direction of X-axis.

17) The perpendicular distance from origin to the line $ax + by + c = 0$ is $p = \dfrac{|c|}{\sqrt{a^2 + b^2}}$ and the angle α which this perpendicular makes with the positive direction of X-axis is given by
$$\cos\alpha = \frac{-a}{\sqrt{a^2+b^2}} \quad \text{and} \quad \sin\alpha = \frac{-b}{\sqrt{a^2+b^2}} \quad \text{if } c > 0$$

and $\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$ and $\sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$ if $c<0$.

18) The slope m of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by $m = \dfrac{y_1-y_2}{x_1-x_2}$.

19) The lines with slopes m_1 and m_2 are parallel if $m_1 = m_2$.

20) The lines with slopes m_1 and m_2 are perpendicular if $m_1 m_2 = -1$.

21) The angle θ $(0 < \theta < \pi/2)$, between two lines having slopes m_1 and m_2 is given by $\theta = \tan^{-1}\left|\dfrac{m_1-m_2}{1+m_1m_2}\right|$

If one line is vertical and the other makes an angle θ_2 with the positive direction of X-axis, then $\theta = |\pi/2 - \theta_2|$.

If one line is horizontal and the other makes an angle θ_2 with the positive direction of X-axis, then $\theta = \theta_2$ or $\pi - \theta_2$ whichever is acute.

22) The necessary and sufficient condition for three distinct points of R^2 to be collinear is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

where (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the points considered.

3. LINES

FORMULAE - 3. LINES

1) For two lines

$$a_1 x + b_1 y + c_1 = 0, \quad a_1^2 + b_1^2 \neq 0 \quad \text{and}$$
$$a_2 x + b_2 y + c_2 = 0, \quad a_2^2 + b_2^2 \neq 0,$$

(i) if $a_1 b_2 - a_2 b_1 \neq 0$, then the two lines are distinct and not parallel and their point of intersection is

$$(x, y) = \left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1} \right).$$

(ii) if $a_1 b_2 - a_2 b_1 = 0$ but $b_1 c_2 - b_2 c_1 \neq 0$ or $c_1 a_2 - c_2 a_1 \neq 0$, then the lines are parallel and do not intersect.

(iii) if $a_1 b_2 - a_2 b_1 = b_1 c_2 - b_2 c_1 = c_1 a_2 - c_2 a_1 = 0$, then the lines are identical.

2) The necessary and sufficient conditions for the three distinct lines $a_i x + b_i y + c_i = 0$, $a_i^2 + b_i^2 \neq 0$, $i = 1, 2, 3$ to be concurrent are

(i) $a_1 b_2 - a_2 b_1 \neq 0$,
$a_2 b_3 - a_3 b_2 \neq 0$,
$a_3 b_1 - a_1 b_3 \neq 0$, and

(ii) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

3) For two lines

$$a_1 x + b_1 y + c_1 = 0, \quad a_1^2 + b_1^2 \neq 0 \quad \text{and}$$
$$a_2 x + b_2 y + c_2 = 0, \quad a_2^2 + b_2^2 \neq 0,$$

the equation $l(a_1 x + b_1 y + c_1) + m(a_2 x + b_2 y + c_2) = 0$, $l^2 + m^2 \neq 0$, $l, m \in R$ represents a family of lines

(i) passing through the point of intersection of the two given lines if they are not parallel, and

(ii) parallel to the two given lines if they are parallel.

If $l = 0$, then the above equation represents the line $a_2x + b_2y + c_2 = 0$. If $l \neq 0$, then the above equation can be written as

$$a_1x + b_1y + c_1 + \frac{m}{l}(a_2x + b_2y + c_2) = 0$$

or, $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0, \lambda \in R$

This equation includes all lines of the family except the line $a_2x + b_2y + c_2 = 0$.

4) To find the equation of a line passing through the point of intersection of the two given lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ if they are not parallel and also satisfying a given additional condition, first ascertain if any of the two given lines satisfies the given additional condition. If so, then the equation of that line is the required equation. If not, then find the equation of the required line by writing it in the form $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$ and determining the value of λ from the given additional condition.

5) The perpendicular distance, p, of the point (x_1, y_1) from the line $ax + by + c = 0$ is $p = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$.

6) The perpendicular distance, p, between two parallel lines $ax + by + c = 0$ and $ax + by + c' = 0$ $(c \neq c')$ is $p = \frac{|c - c'|}{\sqrt{a^2 + b^2}}$.

7) The homogeneous quadratic equation $ax^2 + 2hxy + by^2 = 0$, $a^2 + h^2 + b^2 \neq 0$, represents a pair of lines through the origin if $h^2 - ab > 0$.

The equations of the lines are $y = m_1 x$ and $y = m_2 x$, where m_1 and m_2 can be obtained by solving the equations $m_1 + m_2 = -2h/b$ and $m_1 m_2 = +a/b$. If these lines are perpendicular, $m_1 m_2 = a/b = -1$, i.e., $a + b = 0$. If the lines are not perpendicular, then the angle θ between them is given by

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{|a+b|}.$$

8) The general quadratic equation in R^2
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, $a^2 + b^2 + h^2 \neq 0$
represents a pair of lines, if and only if,
(i) $h^2 \geqslant ab$,
(ii) $g^2 \geqslant ac$, and (iv) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$
(iii) $f^2 \geqslant bc$,

The lines represented by the above equation are parallel to the lines given by $ax^2 + 2hxy + by^2 = 0$ and hence the angle θ between them is $\tan^{-1}\left(2\sqrt{h^2 - ab}/|a+b|\right)$.
The point of intersection (α, β) of the two lines represented by the above equation is

$$(\alpha, \beta) = \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$$

$$\text{i.e.,} = \left[\frac{\begin{vmatrix} h & g \\ b & f \end{vmatrix}}{\begin{vmatrix} a & h \\ h & b \end{vmatrix}}, \frac{\begin{vmatrix} g & a \\ f & h \end{vmatrix}}{\begin{vmatrix} a & h \\ h & b \end{vmatrix}}\right].$$

4. CIRCLE

FORMULAE - 4. CIRCLE

1) $(x-h)^2+(y-k)^2 = r^2$ is the cartesian equation of the circle with centre at (h, k) and radius, r.
2) $x^2+y^2=r^2$ is the standard equation of the circle with centre at the origin and radius, r.
3) $x^2+y^2=1$ is the equation of unit circle $(r=1)$.
4) $x^2+y^2+2gx+2fy+c=0$ is the general equation of the circle with centre $(-g,-f)$ and radius, $\sqrt{g^2+f^2-c}$.
5) $Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$ is the general quadratic equation in R^2 which represents a circle if $A=B\neq 0$ and $H=0$. For radius of this circle to be real, $G^2+F^2-AC>0$.
6) $x = h + r\cos\theta$, $y = k + r\sin\theta$, $\theta\in(-\pi,\pi]$ are parametric equations of the circle with centre at (h, k) and radius $= r$, θ being the parameter.
7) $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$ is the equation of the circle, the extremities of whose diameter are (x_1, y_1) and (x_2, y_2).
8) The position of point $P(x_1, y_1)$ relative to the circle $x^2+y^2+2gx+2fy+c=0$ is
outside the circle if $x_1^2+y_1^2+2gx_1+2fy_1+c>0$,
on the circle if $x_1^2+y_1^2+2gx_1+2fy_1+c=0$ and
inside the circle if $x_1^2+y_1^2+2gx_1+2fy_1+c<0$.
9) The line $ax+by+c=0$ and circle $x^2+y^2=r^2$
intersect in two points if $\frac{c^2}{a^2+b^2}<r^2$,
touch each other if $\frac{c^2}{a^2+b^2}=r^2$ and
do not intersect if $\frac{c^2}{a^2+b^2}>r^2$.

10) For two circles with centres C_1 and C_2 and radii r_1 & r_2,

(i) if $C_1C_2 = r_1 + r_2$, the circles touch externally,
(ii) if $C_1C_2 = |r_1 - r_2|$ " " " internally,
(iii) if $C_1C_2 < |r_1 - r_2|$ " " do not intersect and one circle is inside the other circle,
(iv) if $C_1C_2 > r_1 + r_2$ the circles do not intersect and the circular regions are disjoint sets,
(v) if $|r_1 - r_2| < C_1C_2 < r_1 + r_2$, the circles intersect in two points.

11) If (x_1, y_1) is a point on the circle, then equation of the tangent at (x_1, y_1) to the circle
(i) $x^2 + y^2 = r^2$ is $x_1x + y_1y = r^2$, and
(ii) $x^2+y^2+2gx+2fy+c=0$ is
$$x_1x + y_1y + g(x+x_1) + f(y+y_1) + c = 0.$$
If (x_1, y_1) is a point outside the circle, then these equations represent chords of contact.

12) The equation of a normal to the circle $x^2+y^2+2gx+2fy+c=0$ passing through any point (x_1, y_1) outside, on or inside the circle other than its centre is the equation of the line passing through the centre $(-g, -f)$ of the circle and the given point (x_1, y_1).

13) The condition for the line $y = mx + c$ to be tangent to the circle $x^2+y^2 = r^2$ is $c^2 = r^2(1+m^2)$ and the coordinates of the point of contact are $\left(-\frac{r^2m}{c}, \frac{r^2}{c}\right)$. This condition can be used to find the equations of the two tangents to the circle $x^2+y^2=r^2$ (NOT THE GENERAL CIRCLE, $x^2+y^2+2gx+2fy+c=0$) from an external point (x_1, y_1).

14) To find the equations of the tangents to the circle $x^2+y^2+2gx+2fy+c=0$ from an external point (x_1, y_1), assume m to be the slope of the required

4. CIRCLE

tangent through (x_1, y_1) and equate the perpendicular distance from the centre $(-g, -f)$ of the circle to the tangent $y - y_1 = m(x - x_1)$ to the radius $\sqrt{g^2+f^2-c}$ of the circle to find two values m_1 and m_2 of the slope of the tangent. Using these values, find the equations of the tangents.

15) The length PT of a tangent from the point, $P(x_1, y_1)$ to the circle

(i) $x^2+y^2=r^2$ is $\sqrt{x_1^2+y_1^2-r^2}$, and

(ii) $x^2+y^2+2gx+2fy+c=0$ is $\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$.

16) If two circles, $S_1 : x^2+y^2+2g_1x+2f_1y+c_1=0$ and $S_2 : x^2+y^2+2g_2x+2f_2y+c_2=0$ intersect, then the equation $lS_1+mS_2=0$, $l+m \neq 0$ represents a circle passing through the points of intersection of S_1 and S_2.

17) If the circle $S: x^2+y^2+2gx+2fy+c=0$ and the line $L: lx+my+n=0$ intersect, then $S+\lambda L=0$, $\lambda \in R$ represent a circle passing through their points of intersection.

18) If two circles $S_1=0$ and $S_2=0$ intersect in two distinct points, then the equation of the line containing their common chord is $S_1-S_2=0$. If the two circles touch each other, then this equation represents the equation of their common tangent.

19) To find the equation of a circle passing through the points of intersection of the circles $S_1=0$ and $S_2=0$, or of the circle $S=0$ and the line $L=0$ and satisfying an additional condition, first check if any of the given circles satisfies the

given condition. If so, then the equation of that circle is the required equation. If not, then write the equation of the required circle as $S_1 + \lambda S_2 = 0$ or as $S + \lambda L = 0$ and find out the value of λ using the given additional condition substitution of which in the above equation gives the equation of the required circle.

20) If the angle between the tangents to the two circles at their points of intersection is a right angle, then the two circles are said to be orthogonal circles. The two circles, $S_1 : x^2+y^2+2g_1x+2f_1y+c_1=0$ and $S_2 : x^2+y^2+2g_2x+2f_2y+c_2=0$ intersect each other orthogonally if $2g_1g_2+2f_1f_2 = c_1+c_2$.

21) If a line through an external point P intersects the circle in A and B and $\overline{PT}$ is the tangent touching the circle at T, then $PA \cdot PB = PT^2$. This equation can be used to find the value of $PA \cdot PB$ using the formula for PT.

5. PARABOLA

FORMULAE - 5. PARABOLA

1) The standard equation of the parabola is $y^2 = 4ax$. The line through the focus and perpendicular to the directrix is called the axis of the parabola. The point of intersection of the parabola with its axis is called the vertex of the parabola. For the parabola $y^2 = 4ax$, the X-axis is the axis of the parabola, origin is the vertex, $S(a, 0)$ is the focus and $x = -a$ is the directrix. If the axis of the parabola is Y-axis and vertex the origin, then its equation is $x^2 = 4ay$ for which focus is $S(0, a)$ and the directrix is $y = -a$.
2) The parametric equations of a parabola $y^2 = 4ax$ are $x = at^2$, $y = 2at$, $t \in R$
3) The equation of tangent at the point (x_1, y_1) of the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$. The equation of tangent at the parametric point $P(at^2, 2at)$ is $y = \frac{1}{t}x + at$. Slope of the tangent is $\frac{1}{t}$.
4) The condition for the line $y = mx + c$ to be a tangent to the parabola $y^2 = 4ax$ is $c = a/m$. The coordinates of the point of contact are $\left(\frac{c}{m}, \frac{2a}{m}\right)$ or $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$. The equation of tangent of slope m to the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$.
5) If $y_1^2 - 4ax_1 > 0$, then two tangents can be drawn to the parabola $y^2 = 4ax$ from the point (x_1, y_1). If $y_1^2 - 4ax_1 = 0$, then the point (x_1, y_1) is on the parabola and only one tangent can be drawn. If $y_1^2 - 4ax_1 < 0$, then no tangent can be drawn to the parabola from (x_1, y_1).

6. ELLIPSE

FORMULAE - 6. ELLIPSE

1) The standard equation of the ellipse is
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ for which

(i) $C(0,0)$ is the centre of the ellipse.

(ii) X- and Y-axes are the axes of the ellipse. $A(a,0)$, $A'(-a,0)$, $B(0,b)$ and $B'(0,-b)$ are the vertices of the ellipse.

(iii) $\overline{AA'}$ is the chord of the ellipse passing through its centre having maximum length and is called the major axis of the ellipse. $AA' = 2a$. Similarly, $\overline{BB'}$ is the minor axis and $BB' = 2b$.

(iv) The foci of the ellipse are $S(ae, 0)$, $S'(-ae, 0)$.

(v) The directrix corresponding to S is $x = \frac{a}{e}$ and corresponding to S' is $x = -\frac{a}{e}$.

(vi) The eccentricity e is given by $b^2 = a^2(1-e^2)$.

2) In the above equation of the ellipse, if $b > a$, $\overline{BB'}$ will be major axis and $\overline{AA'}$ will be minor axis. The foci are $(0, \pm be)$ and the directrices are $y = \pm b/e$. The eccentricity will be given by $a^2 = b^2(1-e^2)$

3) The length of the latus rectum is
$\frac{2b^2}{a}$ $(a > b)$ and $\frac{2a^2}{b}$ $(b > a)$.

4) The parametric equations of the ellipse are
$x = a\cos\theta$, $y = b\sin\theta$ $\theta \in (-\pi, \pi]$

5) The equation of tangent at the point $P(x_1, y_1)$ of the ellipse is

$$y - y_1 = -\frac{b^2 x_1}{a^2 y_1}(x - x_1) \quad \text{or,} \quad \frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1.$$

6. ELLIPSE

6) The equation of tangent at the parametric point θ on the ellipse is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$.

7) The condition that the line $y = mx + c$ may be a tangent to the ellipse is $c^2 = a^2m^2 + b^2$ for the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the coordinates of the point of contact are $\left(-\frac{ma^2}{c}, \frac{b^2}{c}\right)$.

8) For any point $P(x_1, y_1)$ in the plane of the ellipse $x^2/a^2 + y^2/b^2 = 1$,

(i) if $x_1^2/a^2 + y_1^2/b^2 > 1$, P lies outside the ellipse and two tangents can be drawn to the ellipse from P.

(ii) if $x_1^2/a^2 + y_1^2/b^2 = 1$, P lies on the ellipse and there can be only one tangent through P.

(iii) if $x_1^2/a^2 + y_1^2/b^2 < 1$, no tangent can be drawn to the ellipse from P.

9) The circle described on major axis of the ellipse as diameter is called the auxiliary circle of the ellipse and its equation is $x^2 + y^2 = a^2$. Point $P(a\cos\theta, b\sin\theta)$ on the ellipse and $Q(a\cos\theta, a\sin\theta)$ on its auxiliary circle are called coherent points, where θ is the angle subtended by Q at the origin with the X-axis.

10) The set of all points from which the two tangents drawn to the ellipse are perpendicular to each other, is a circle concentric with the ellipse and is known as the director circle of the ellipse. For the ellipse $x^2/a^2 + y^2/b^2 = 1$, its equation is $x^2 + y^2 = a^2 + b^2$.

7. HYPERBOLA

FORMULAE - 7. HYPERBOLA

1) The standard equation of the hyperbola is
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$ for which

(i) $C(0,0)$ is the centre of the hyperbola.

(ii) $A(a,0)$ and $A'(-a,0)$ are the vertices of the hyperbola. $\overline{AA'}$ is called the transverse axis. $\overline{AA'} = 2a$.

(iii) $B(0,b)$ and $B'(0,-b)$ form the conjugate axis $\overline{BB'}$ which does not meet the curve. $BB' = 2b$.

(iv) The foci of hyperbola are $S(ae, 0)$, $S'(-ae, 0)$.

(v) The directrices corresponding to the foci S and S' are respectively the lines $x = a/e$ and $x = -\frac{a}{e}$.

(vi) The eccentricity, e, of the hyperbola is given by $b^2 = a^2(e^2-1)$.

(vii) For $a > b$ or $b > a$, the type of hyperbola is the same. But for hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, $\overline{BB'}$ will be the transverse axis and $\overline{AA'}$ will be the conjugate axis. For this type of hyperbola, foci, directrices, eccentricity and length of latus-rectum can be obtained by interchanging a and b.

(viii) The length of latus-rectum is $2b^2/a$.

2) The parametric equations of the hyperbola are
$$x = a\sec\theta, \quad y = b\tan\theta, \quad \theta \in (-\pi, \pi] - \{\pm \pi/2\}$$

3) The equation of tangent at the point $P(x_1, y_1)$ of the hyperbola is
$$y - y_1 = \frac{b^2 x_1}{a^2 y_1}(x - x_1) \quad \text{or,} \quad \frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1.$$

7. HYPERBOLA

4) The equation of tangent at the parametric point θ on the hyperbola is $\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} = 1$.

5) The condition that the line $y = mx + c$ may be a tangent to the hyperbola is $c^2 = a^2m^2 - b^2$ and the coordinates of the point of contact are $\left(-\frac{ma^2}{c}, -\frac{b^2}{c}\right)$.

6) The equations of asymptotes of the hyperbola are $y = \pm \frac{b}{a} x$, i.e., $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

7) The standard equation of the rectangular hyperbola is $x^2 - y^2 = a^2$ $(e = \sqrt{2})$.

8) The equation of auxiliary circle of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is $x^2 + y^2 = a^2$.

9) The set of all points from which the two tangents drawn to the hyperbola are perpendicular to each other, is a circle concentric with the hyperbola and is known as the director circle of the hyperbola. For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, its equation is $x^2 + y^2 = a^2 - b^2$, which does not exist for $a < b$ and is a point at the centre of the hyperbola for $a = b$.

FORMULAE - 8. GENERAL SECOND DEGREE CURVE

1) If the origin is shifted to (h, k) without changing the direction of axes and if (x, y) and (x', y') are the coordinates of a point with respect to old and new axes respectively, then $x = x' + h$ and $y = y' + k$.
If the axes are rotated about the origin through an angle θ, keeping the origin fixed, then

$$x = x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = x'\sin\theta + y'\cos\theta.$$

Also, $x' = x\cos\theta + y\sin\theta$ and $y' = -x\sin\theta + y\cos\theta$.

For $\theta = \pi/4$, $x = \dfrac{x'-y'}{\sqrt{2}}$ and $y = \dfrac{x'+y'}{\sqrt{2}}$.

Also, $x' = \dfrac{x+y}{\sqrt{2}}$ and $y' = \dfrac{y-x}{\sqrt{2}}$

2) To get rid of xy term from the equation,
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, rotate the axes by an angle θ given by $\tan 2\theta = \dfrac{2h}{a-b}$.
When $a = b$, $\theta = \pi/4$.

3) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, ... (I),

$a^2 + b^2 + h^2 \neq 0$ and $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, then

(i) if $D = 0$, $h^2 \geq ab$, $g^2 \geq ac$ and $f^2 \geq bc$, then I represents a pair of lines.

(ii) if $D \neq 0$, $h^2 = ab$, then I represents parabola.

(iii) " $h^2 < ab$ " " " ellipse.

(iv) " $h^2 > ab$ " " " hyperbola

(v) if $a = b \neq 0$, $h = 0$ and $g^2 + f^2 - ac > 0$, then I represents a circle.

9. VECTORS

FORMULAE - 9. VECTORS

1) For $\bar{x} = (x_1, x_2, x_3)$, $\bar{y} = (y_1, y_2, y_3)$ and
$\bar{z} = (z_1, z_2, z_3) \in R^3$ and $\alpha \in R$,

(i) $\bar{x} = \bar{y} \Leftrightarrow x_1 = y_1,\ x_2 = y_2$ and $x_3 = y_3$.

(ii) $\bar{x} + \bar{y} = (x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3)$.

(iii) $\alpha \bar{x} = (\alpha x_1, \alpha x_2, \alpha x_3)$.

(iv) Inner product of vectors $\bar{x}, \bar{y} \in R^3$ is written as $\bar{x} \cdot \bar{y}$ and it is defined by

$\bar{x} \cdot \bar{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$.

$\bar{x} \cdot \bar{x} = |\bar{x}|^2 \geq 0$, $|\bar{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

$\bar{x} \cdot (\bar{y} + \bar{z}) = \bar{x} \cdot \bar{y} + \bar{x} \cdot \bar{z}$.

$\bar{x} \cdot (\alpha \bar{y}) = \alpha (\bar{x} \cdot \bar{y}) = (\alpha \bar{x}) \cdot \bar{y}$

If $|\bar{x}| = 1$, $\bar{x}$ is called a unit vector.

$\bar{i} = (1, 0, 0)$, $\bar{j} = (0, 1, 0)$ and $\bar{k} = (0, 0, 1)$ are standard unit vectors.

(v) The outer product of vectors $\bar{x}, \bar{y}$ of R^3 is denoted by $\bar{x} \times \bar{y}$ and it is defined by

$$\bar{x} \times \bar{y} = (x_2 y_3 - x_3 y_2)\,\bar{i} - (x_1 y_3 - x_3 y_1)\,\bar{j} + (x_1 y_2 - x_2 y_1)\,\bar{k}.$$

The above result is obtained by expanding the determinant

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$ by first row.

9. VECTORS

(vi) The box product $\bar{x}\cdot(\bar{y}\times\bar{z})$ of vectors $\bar{x}, \bar{y}, \bar{z}$ of R^3 is denoted as $[\bar{x}\ \bar{y}\ \bar{z}]$ and it is defined by

$$[\bar{x}\ \bar{y}\ \bar{z}] = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}.$$

Also, $[\bar{x}\ \bar{y}\ \bar{z}] = [\bar{y}\ \bar{z}\ \bar{x}] = [\bar{z}\ \bar{x}\ \bar{y}]$.

(vii) $\bar{x}\times(\bar{y}\times\bar{z}) = (\bar{x}\cdot\bar{z})\,\bar{y} - (\bar{x}\cdot\bar{y})\,\bar{z}$.

(viii) $(\bar{x}\times\bar{y})\times\bar{z} = (\bar{x}\cdot\bar{z})\,\bar{y} - (\bar{y}\cdot\bar{z})\,\bar{x}$.

2) A unit vector in the direction of $\bar{x}$ is $\hat{x} = \dfrac{\bar{x}}{|\bar{x}|}$.

3) Vector form of Lagrange's identity is

$$|\bar{x}|^2|\bar{y}|^2 - (\bar{x}\cdot\bar{y})^2 = |\bar{x}\times\bar{y}|^2.$$

4) The result $|\bar{x}\cdot\bar{y}| \le |\bar{x}||\bar{y}|$ is known as Schwartz's inequality.

5) For $\bar{x}, \bar{y} \in R^3$, $|\bar{x}+\bar{y}| \le |\bar{x}| + |\bar{y}|$ is known as triangular inequality.

6) The angle between two non-null vectors $\bar{x}$ and $\bar{y}$ is given by

$$\alpha = \cos^{-1}\frac{\bar{x}\cdot\bar{y}}{|\bar{x}||\bar{y}|}. \qquad 0 \le \alpha \le \pi.$$

(i) If $\bar{x}\cdot\bar{y} = 0$, then $\alpha = \pi/2$ and the two vectors are said to be mutually orthogonal or perpendicular.

9. VECTORS

(ii) If $\bar{x} = k\bar{y}$, then $\cos\alpha = \dfrac{(k\bar{y})\cdot\bar{y}}{|k\bar{y}||\bar{y}|} = \dfrac{k}{|k|}$.

$\therefore$ $\cos\alpha = \pm 1$ or $\alpha = 0$ or π and the two vectors are parallel.

7) $\bar{i} = (1,0,0)$, $\bar{j} = (0,1,0)$ and $\bar{k} = (0,0,1)$ are unit vectors in the directions of X-, Y-, Z-axis respectively.

8) If $\bar{x} = (a, b)$, then $\bar{x}$ makes with the direction of positive X-axis angle $\alpha = \cos^{-1}\dfrac{a}{\sqrt{a^2+b^2}}$.

$\bar{x}$ is in upper open half plane of X-axis if $b>0$ and in lower open half plane of X-axis if $b<0$. If $b=0$, then $\bar{x}$ will be in direction of positive X-axis if $a>0$ or along negative X-axis if $a<0$.

9) If $\bar{r} = (a, b, c)$ and $\bar{r}$ makes angles α, β, γ with X-, Y- and Z-axis respectively, then

$$\cos\alpha = \frac{a}{|\bar{r}|},\ \cos\beta = \frac{b}{|\bar{r}|} \text{ and } \cos\gamma = \frac{c}{|\bar{r}|}.$$

The angles α, β, γ made with positive direction of the axes are called 'direction angles of the vector' and $\cos\alpha, \cos\beta, \cos\gamma$ are called the 'direction cosines of $\bar{r}$. They are usually denoted by letters l, m, n respectively.
$l^2+m^2+n^2 = 1$.

Unit vector, $\hat{r} = \left(\dfrac{a}{|\bar{r}|}, \dfrac{b}{|\bar{r}|}, \dfrac{c}{|\bar{r}|}\right) = (l, m, n)$.

Thus, components of $\hat{r}$ are direction cosines of $\bar{r}$.
$a:b:c$ or $l:m:n$ are direction ratios of $\bar{r}$.

10) i) If for vectors $\bar{x}$, $\bar{y}$ and scalars $\alpha, \beta \in R$, $\alpha\bar{x} + \beta\bar{y} = \theta \Rightarrow \alpha = 0$ and $\beta = 0$, then $\bar{x}$ and $\bar{y}$ are said to be non-collinear vectors.

If for vectors $\bar{x}$, $\bar{y}$ and $\bar{z}$ and scalars $\alpha, \beta, \gamma \in R$,

$\alpha\bar{x} + \beta\bar{y} + \gamma\bar{z} = \theta \Rightarrow \alpha = 0, \beta = 0$ and $\gamma = 0$, then $\bar{x}$, $\bar{y}$ and $\bar{z}$ are said to be non-coplanar vectors. Such vectors are also said to be linearly independent.

ii) The necessary and sufficient condition that non-null vectors $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$ of R^2 be collinear is that $x_1 y_2 - x_2 y_1 = 0$.

iii) The necessary and sufficient condition that two non-null vectors $\bar{x}$, $\bar{y}$ of R^3 be collinear is that $\bar{x} \times \bar{y} = \theta$.

iv) The necessary and sufficient condition for three non-null vectors $\bar{x}, \bar{y}, \bar{z}$ to be coplanar is that $\bar{x} \cdot \bar{y} \times \bar{z} = 0$.

10. APPLICATION OF VECTORS

FORMULAE - 10. APPLICATION OF VECTORS

1) If $A(\bar{a}) = (x_1, y_1, z_1)$ and $B(\bar{b}) = (x_2, y_2, z_2)$,
then $\overrightarrow{AB} = \bar{b} - \bar{a} = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$
$\therefore AB = |\overrightarrow{AB}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$.

2) $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ is the triangle law of vectors.

3) For parallelogram OACB, $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$ is known as the parallelogram law for addition of vectors.

4) If $A(\bar{a}) = (x_1, y_1, z_1)$ and $B(\bar{b}) = (x_2, y_2, z_2)$ and $P(\bar{r})$ divides $\overline{AB}$ in the ratio $\lambda : 1$ from A $(\lambda \neq -1)$, then

$$\bar{r} = \frac{\bar{a} + \lambda \bar{b}}{\lambda + 1} = \left(\frac{x_1 + \lambda x_2}{1 + \lambda},\ \frac{y_1 + \lambda y_2}{1 + \lambda},\ \frac{z_1 + \lambda z_2}{1 + \lambda} \right).$$

If P is the mid-point of $\overline{AB}$, then

$$\bar{r} = \frac{\bar{a} + \bar{b}}{2} = \left(\frac{x_1 + x_2}{2},\ \frac{y_1 + y_2}{2},\ \frac{z_1 + z_2}{2} \right).$$

5) If $A(\bar{a}) = (x_1, y_1, z_1)$, $B(\bar{b}) = (x_2, y_2, z_2)$ and $C(\bar{c}) = (x_3, y_3, z_3)$ are the vertices of ΔABC, then centroid of the triangle is

$$G(\bar{g}) = \frac{\bar{a} + \bar{b} + \bar{c}}{3} = \left(\frac{x_1 + x_2 + x_3}{3},\ \frac{y_1 + y_2 + y_3}{3},\ \frac{z_1 + z_2 + z_3}{3} \right).$$

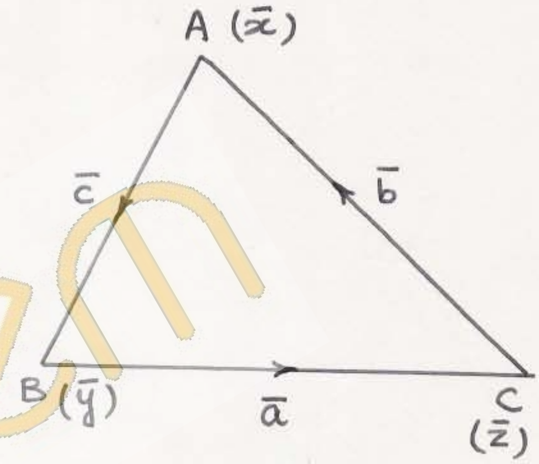
6) If $A(\bar{x}) = (x_1, y_1, z_1)$, $B(\bar{y}) = (x_2, y_2, z_2)$ and $C(\bar{z}) = (x_3, y_3, z_3)$ are vertices of ΔABC and $AB = c$, $BC = a$ and $CA = b$, then in-centre of

the triangle ABC is

$$I = \frac{a\bar{x} + b\bar{y} + c\bar{z}}{a+b+c}$$

$$= \left[\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}, \frac{az_1 + bz_2 + cz_3}{a+b+c} \right]$$

7) If $A(\bar{x})$, $B(\bar{y})$ and $C(\bar{z})$ are vertices of ΔABC, then taking $\overrightarrow{BC} = \bar{a}$, $\overrightarrow{CA} = \bar{b}$ and $\overrightarrow{AB} = \bar{c}$,



$BC = a = |\bar{a}| = |\bar{z} - \bar{y}|$,

$CA = b = |\bar{b}| = |\bar{x} - \bar{z}|$,

$AB = c = |\bar{c}| = |\bar{y} - \bar{x}|$.

$$\therefore A = \cos^{-1}\left[\frac{-\bar{b}\cdot\bar{c}}{|\bar{b}||\bar{c}|}\right] = \pi - \cos^{-1}\left[\frac{\bar{b}\cdot\bar{c}}{|\bar{b}||\bar{c}|}\right];$$

$$B = \pi - \cos^{-1}\left[\frac{\bar{c}\cdot\bar{a}}{|\bar{c}||\bar{a}|}\right]; \quad C = \pi - \cos^{-1}\left[\frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}\right].$$

8) Area of triangle ABC is

$$\Delta = \frac{1}{2}\left[|\bar{c}|^2|\bar{a}|^2 - (\bar{c}\cdot\bar{a})^2\right]^{1/2}$$

$$= \frac{1}{2}\left[|\bar{a}|^2|\bar{b}|^2 - (\bar{a}\cdot\bar{b})^2\right]^{1/2}$$

$$= \frac{1}{2}\left[|\bar{b}|^2|\bar{c}|^2 - (\bar{b}\cdot\bar{c})^2\right]^{1/2}, \text{ for } R^2 \,\&\, R^3.$$

$$= \frac{1}{2}|\bar{a}\times\bar{b}| = \frac{1}{2}|\bar{b}\times\bar{c}| = \frac{1}{2}|\bar{c}\times\bar{a}| \text{ for } R^3.$$

The last formula can also be used for R^2 by taking z-coordinate of the vertices as zero.

9) Area of parallelogram OACB $= |\overrightarrow{OA} \times \overrightarrow{OB}|$

$= |\bar{a} \times \bar{b}|$.

where $A(\bar{a})$, $B(\bar{b})$ w.r.t. $O(\theta)$.

10. APPLICATION OF VECTORS

10) If α is the angle between non-null vectors $\bar{a}$ and $\bar{b}$, then magnitude of
the projection of vector $\bar{a}$ on $\bar{b} = |\bar{a}|\cos\alpha = \bar{a}\cdot\hat{b}$,
the projection of vector $\bar{b}$ on $\bar{a} = |\bar{b}|\cos\alpha = \bar{b}\cdot\hat{a}$.

11) If $A(\bar{a})$, $B(\bar{b})$, $C(\bar{c})$ are vertices of prism $OABC-O'A'B'C'$ with respect to $O(\theta)$, where $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ are coterminus edges of the prism, then its volume, $V = |(\bar{a}\times\bar{b})\cdot\bar{c}|$.

12) If $A(\bar{a})$, $B(\bar{b})$ and $C(\bar{c})$ are vertices of a tetrahedron $O-ABC$ with respect to $O(\theta)$, then its volume, $V = \frac{1}{6}|(\bar{a}\times\bar{b})\cdot\bar{c}| = \frac{1}{6}|[\bar{a}\ \bar{b}\ \bar{c}]|$.

13) If forces $\vec{F_1}, \vec{F_2}, \ldots \vec{F_n}$ act on a particle, then the resultant force, $\vec{F} = \vec{F_1} + \vec{F_2} + \cdots + \vec{F_n}$.
$|\vec{F}|$ is the magnitude of the resultant force.
If $\hat{F} = (a, b)$, then its direction is $\alpha = \cos^{-1}a$ with East towards North for $b \geq 0$ and with East towards South for $b < 0$. If $\hat{F} = (a, b, c)$, then $\vec{F}$ makes angles $\alpha = \cos^{-1}a$, $\beta = \cos^{-1}b$ and $\gamma = \cos^{-1}c$ with positive directions of X, Y, Z axes, respectively.

14) If a particle gets displaced from A to B under the effect of force $\vec{F}$, then work done by $\vec{F}$ is given by $W = \vec{F}\cdot\vec{AB} = |\vec{F}|\,AB\cos\theta$, where θ is the angle between $\vec{F}$ and $\vec{AB}$.

15) If the velocities of particles A and B are $\bar{u}$ and $\bar{v}$ respectively, then the velocity of B relative to A is $\bar{v} - \bar{u}$ and the velocity of A relative to B is $\bar{u} - \bar{v}$.

11. LINE IN SPACE

FORMULAE - 11. LINE IN SPACE

1) Equation of a line through $A(\bar{a})$ having direction $\bar{l}$.

   a) Vector equation : $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$.

   b) Cartesian equation : Taking $\bar{r} = (x, y, z)$, $\bar{a} = (x_1, y_1, z_1)$ and $\bar{l} = (a, b, c)$, we get $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c}$ where $a : b : c$ are direction ratios of $\bar{l}$. If the direction cosines of $\bar{l}$ are (l, m, n), then

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}, \quad l^2+m^2+n^2 = 1.$$

2) Equation of a line through $A(\bar{a})$ and $B(\bar{b})$.

   a) Vector equation : $\bar{r} = \bar{a} + k(\bar{b}-\bar{a})$, $k \in R$.

   b) Cartesian equation : $\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1}$

3) Condition for $A(\bar{a})$, $B(\bar{b})$, $C(\bar{c})$ to be collinear points in space.

   a) Necessary and sufficient condition is

$$\bar{c} - \bar{a} = k(\bar{b} - \bar{a}) \text{ for some } k \in R.$$

   b) The condition $[\bar{a}\ \bar{b}\ \bar{c}] = 0$ is necessary, but not sufficient. This is a necessary and sufficient condition for three <u>vectors</u>, $\bar{a}$, $\bar{b}$ and $\bar{c}$ to be <u>coplanar</u>.

   c) $l\bar{a} + m\bar{b} + n\bar{c} = \theta$ and $l + m + n = 0$ for some non-zero real numbers l, m, n.

4) Angle between two distinct lines,

   $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and $\bar{r} = \bar{b} + k\bar{m}$, $k \in R$.

   The acute angle α between the lines is given by

$$\cos\alpha = \frac{|\bar{l}\cdot\bar{m}|}{|\bar{l}|\,|\bar{m}|}, \quad 0 < \alpha < \frac{\pi}{2}$$

   The formula is valid for $\alpha = 0$ and $\pi/2$ also.

11. LINE IN SPACE

If $\bar{l} \cdot \bar{m} = 0$, the lines are perpendicular.
If $\bar{l} = k\bar{m}$ or $\bar{l} \times \bar{m} = \theta$, the lines are parallel or same.

5) A necessary condition that two non-parallel lines $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and $\bar{r} = \bar{b} + k\bar{m}$, $k \in R$ intersect is that $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$.
Taking $\bar{a} = (x_1, y_1, z_1)$, $\bar{b} = (x_2, y_2, z_2)$, $\bar{l} = (l_1, m_1, n_1)$ and $\bar{m} = (l_2, m_2, n_2)$, the cartesian form of the above condition is

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

6) The above two lines are coplanar if they are either parallel or intersecting for which $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$. If this condition is not satisfied, then the lines are non-coplanar or skew.

7) The perpendicular distance of the point $P(\bar{p})$ from the line $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ is $|(\bar{p} - \bar{a}) \times \hat{l}|$.
Taking $\bar{p} = (x_1, y_1, z_1)$, $\bar{a} = (a, b, c)$, $\hat{l} = (l, m, n)$, perpendicular distance = magnitude of the vector

$$\begin{vmatrix} i & j & k \\ x_1 - a & y_1 - b & z_1 - c \\ l & m & n \end{vmatrix}$$

8) The distance between parallel lines $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and $\bar{r} = \bar{b} + k\bar{l}$, $k \in R$ is $|(\bar{b} - \bar{a}) \times \hat{l}|$.

9) The shortest distance between two skew lines, $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and $\bar{r} = \bar{b} + k\bar{m}$, $k \in R$, given as MN is

$$MN = |(\bar{b} - \bar{a}) \cdot \bar{u}| \quad \text{where } \bar{u} = \frac{\bar{l} \times \bar{m}}{|\bar{l} \times \bar{m}|}.$$

12. PLANE

FORMULAE - 12. PLANE

1) Equation of a plane through non-collinear points $A(\bar{a})$, $B(\bar{b})$, $C(\bar{c})$.

a) Vector equation : $\bar{r} = \bar{a} + m(\bar{b}-\bar{a}) + n(\bar{c}-\bar{a})$, $m, n \in R$.

b) Parametric vector equation :

$$\bar{r} = l\bar{a} + m\bar{b} + n\bar{c}, \quad l+m+n = 1, \quad l, m, n \in R.$$

c) Cartesian equation : Taking $\bar{r} = (x, y, z)$, $\bar{a} = (x_1, y_1, z_1)$, $\bar{b} = (x_2, y_2, z_2)$ and $\bar{c} = (x_3, y_3, z_3)$, the vector equation $(\bar{r}-\bar{a}).[(\bar{b}-\bar{a}) \times (\bar{c}-\bar{a})] = 0$

$$\Rightarrow \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0.$$

d) Parametric cartesian equations :

$$\left.\begin{aligned} x &= lx_1 + mx_2 + nx_3 \\ y &= ly_1 + my_2 + ny_3 \\ z &= lz_1 + mz_2 + nz_3 \end{aligned}\right\} \quad l+m+n = 1, \quad l, m, n \in R.$$

2) Equation of the plane through $A(\bar{a})$ having normal $(\bar{n})$.

a) Vector equation : $\bar{r}.\bar{n} = d$ where $d = \bar{a}.\bar{n}$.

b) Cartesian equation : Taking $\bar{r} = (x, y, z)$, $\bar{a} = (x_1, y_1, z_1)$, and $\bar{n} = (l, m, n)$, $\bar{r}.\bar{n} = \bar{a}.\bar{n}$

$\Rightarrow lx + my + nz = p$, where $p = lx_1 + my_1 + nz_1$.

c) General equation : $ax+by+cz+d = 0$ or, $ax+by+cz+1 = 0$.

For plane passing through origin : $ax+by+cz = 0$.

Here, (a, b, c) is normal to the plane.

3) $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ is the equation of the plane making intercepts a, b, c on X-, Y-, Z- axes.

4) $x\cos\alpha + y\cos\beta + z\cos\gamma = p$ is the equation of a plane where, p = perpendicular distance from origin to the plane and $\cos\alpha, \cos\beta, \cos\gamma$ are the direction

cosines of the perpendicular from the origin.

5) Plane through two parallel lines $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and $\bar{r} = \bar{b} + k\bar{l}$, $k \in R$. $\bar{a}$ is not in $\bar{r} = \bar{b} + k\bar{l}$.

a) Vector equation : $(\bar{r} - \bar{a}) \cdot \bar{n} = 0$

where $\bar{n} = (\bar{b} - \bar{a}) \times \bar{l}$.

b) Cartesian equation : Taking $\bar{r} = (x, y, z)$, $\bar{a} = (x_1, y_1, z_1)$, $\bar{b} = (x_2, y_2, z_2)$ and $\bar{l} = (l, m, n)$, the cartesian equation is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l & m & n \end{vmatrix} = 0$$

6) Plane through two intersecting lines $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and $\bar{r} = \bar{b} + k\bar{m}$, $k \in R$.

a) Vector equation : $(\bar{r} - \bar{a}) \cdot \bar{n} = 0$ where $\bar{n} = \bar{l} \times \bar{m}$

b) Cartesian equation : Taking $\bar{r} = (x, y, z)$, $\bar{l} = (l_1, m_1, n_1)$, $\bar{m} = (l_2, m_2, n_2)$, the cartesian equation is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

7) The acute angle α between the planes $(\bar{r} - \bar{a}) \cdot \bar{n}_1 = 0$ and $(\bar{r} - \bar{b}) \cdot \bar{n}_2 = 0$ is the angle between their normals given by

$$\cos\alpha = \frac{|\bar{n}_1 \cdot \bar{n}_2|}{|\bar{n}_1|\,|\bar{n}_2|}, \quad 0 < \alpha < \frac{\pi}{2}$$

a) If $\bar{n}_1 = k\bar{n}_2$ or $\bar{n}_1 \times \bar{n}_2 = \theta$, then the planes are parallel or same.

b) If $\bar{n}_1 \cdot \bar{n}_2 = 0$, then the planes are perpendicular to each other.

8) Two planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and

$a_2 x + b_2 y + c_2 z + d_2 = 0$

(a) are perpendicular if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$,

12. PLANE

(b) are parallel (or same) if $(a_1, b_1, c_1) = k(a_2, b_2, c_2)$ or if $a_1b_2 - a_2b_1 = b_1c_2 - b_2c_1 = c_1a_2 - c_2a_1 = 0$.

(c) In all other cases, the acute angle α between the planes is given by

$$\cos\alpha = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} \qquad 0 < \alpha < \frac{\pi}{2}$$

9) Condition for four points $A(\bar{a})$, $B(\bar{b})$, $C(\bar{c})$ and $D(\bar{d})$, no three of which are collinear, to be coplanar is $(\bar{d} - \bar{a}) \cdot [(\bar{b} - \bar{a}) \times (\bar{c} - \bar{a})] = 0$. Taking $\bar{a} = (x_1, y_1, z_1)$, $\bar{b} = (x_2, y_2, z_2)$, $\bar{c} = (x_3, y_3, z_3)$ and $\bar{d} = (x_4, y_4, z_4)$, the condition is

$$\begin{vmatrix} x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

10) The perpendicular distance from $A(\bar{a})$ to the plane $\bar{r} \cdot \bar{n} = d$ is $|d - \bar{a} \cdot \bar{n}| / |\bar{n}|$ and the foot of perpendicular $M(\bar{m})$ is given by $\bar{m} = \bar{a} + k_1\bar{n}$, $k_1 = (d - \bar{a} \cdot \bar{n})/|\bar{n}|^2$. The perpendicular distance from $\bar{a} = (\alpha, \beta, \gamma)$ to the plane $ax + by + cz + d = 0$ is $|a\alpha + b\beta + c\gamma + d| / \sqrt{a^2+b^2+c^2}$. The perpendicular distance between parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $|d_1 - d_2| / \sqrt{a^2+b^2+c^2}$.

11) The angle α between the line $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and the plane $\bar{r} \cdot \bar{n} = d$ is $\sin^{-1}(|\bar{l} \cdot \bar{n}| / |\bar{l}||\bar{n}|)$, $0 \le \alpha \le \pi/2$. If $\bar{l} \times \bar{n} = \theta$, the line is perpendicular to the plane. If $\bar{l} \cdot \bar{n} = 0$ and $\bar{a} \cdot \bar{n} \ne d$, the line is parallel to the plane. If $\bar{l} \cdot \bar{n} = 0$ and $\bar{a} \cdot \bar{n} = d$, line is on the plane.

12) If $A(\bar{a})$ is a point common to two non-parallel planes $\bar{r} \cdot \bar{n}_1 = d_1$ and $\bar{r} \cdot \bar{n}_2 = d_2$, then the equation of common line of intersection of the planes is $\bar{r} = \bar{a} + k\bar{n}$, $k \in R$, where $\bar{n} = \bar{n}_1 \times \bar{n}_2$.

13. SPHERE

FORMULAE - 13. SPHERE

1) A sphere is a set of all points of R^3 which are at a constant distance, called radius of the sphere, from a given fixed point called the centre of the sphere.

2) The vector equation of a sphere with centre $\bar{c}$ and radius of length a is $|\bar{r}|^2 - 2\bar{r}\cdot\bar{c} + |\bar{c}|^2 - a^2 = 0$.

3) Taking $\bar{r} = (x, y, z)$ and $\bar{c} = (\alpha, \beta, \gamma)$, we get cartesian equation of the sphere as

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = a^2.$$

For centre at origin, the equation is $x^2+y^2+z^2=a^2$.

4) The vector equation of a sphere, the extremities of whose diameter is $A(\bar{a})$ and $B(\bar{b})$ is given by

$$(\bar{r}-\bar{a}).(\bar{r}-\bar{b}) = 0$$

5) Taking $\bar{r} = (x, y, z)$, $\bar{a} = (x_1, y_1, z_1)$ and $\bar{b} = (x_2, y_2, z_2)$, the cartesian form of the above equation is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0$$

6) $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is a general form of equation of any sphere with centre $(-u, -v, -w)$ and radius of length $\sqrt{u^2+v^2+w^2-d}$.

7) Most of the remaining formulae of sphere are similar to those of circles and the problems of sphere can be solved in the same way as the problems of the circle.

8) For locus problems, assume (x, y, z) to be the coordinates of the moving point and using the conditions of the problem, establish the relationship between x, y, z which becomes the equation of the required locus.