Final examination
Digital Signal Processing (S22)

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1 Task 1

1.1 Solution

We start by substituting numbers into the defined operations and we get:

$$(v|w) = (-1)^9 \cdot 8 \cdot v_1 w_1^* + 2000 \cdot v_2 w_2^*$$
$$(v|w) = -8 \cdot v_1 w_1^* + 2000 \cdot v_2 w_2^*$$

From Lecture notes, Topic 1, Slide 32, we know that: To check if any operation is dot-product we need to consider axioms of inner product. These axioms for all vectors $u, v, w \in \boldsymbol{C}$ any complex vector space and all scalars $a, b \in F$ any field of numbers are:

- Conjugate symmetry: $(u|v) = (v|u)^*$, where * is a conjugate operation on scalars.
- Linearity in the first argument: $(au + bv|w) = a(u|w) + b(v|w)$.
- Positive-definiteness: $(u|u) > 0$ assuming $u \neq 0$ and $(0|0) = 0$.

Now let us explain which dot-product axioms do hold and which do fail.:

1.1.1 Conjugate symmetry

Rewriting $(w|v)^*$ by definition of the given operation:

$$(w|v)^* = (-8 \cdot w_1 v_1^* + 2000 \cdot w_2 v_2^*)^*$$

Simplifying, using the properties of conjugates:

$$(w|v)^* = (-8 \cdot w_1 v_1^*)^* + (2000 \cdot w_2 v_2^*)^* =$$
$$= -8 \cdot w_1^* (v_1^*)^* + 2000 \cdot w_2^* (v_2^*)^* =$$
$$= -8 \cdot w_1^* v_1 + 2000 \cdot w_2^* v_2$$

Applying commutative law for multiplication:

$$(w|v)^* = -8 \cdot v_1 w_1^* + 2000 \cdot v_2 w_2^*$$

As a result,

$(v|w) = (w|v)^* \Rightarrow$ **Conjugate symmetry axiom holds.**

1.1.2 Linearity in the first argument

Taking scalars a and $b \in F$, vector $u \in \boldsymbol{C}$ and expression $(av + bw|u)$:

$$(av + bw|u) = -8(av_1 + bw_1)u_1^* + 2000(av_2 + bw_2)u_2^*$$

Applying distributive property of multiplication over addition:

$$(av + bw|u) = a(-8v_1u_1^* + 2000v_2u_2^*) + b(-8w_1u_1^* + 2000_2u_2^*)$$

As a result,

$$(av + bw|u) = a(v|u) + b(w|u) \Rightarrow \textbf{Linearity in the first argument axiom holds}$$

1.1.3 Positive-definiteness

Rewriting expression for $(v|v)$:

$$(v|v) = -8v_1v_1^* + 2000v_2v_2^*$$

Since v_1 and v_2 are complex numbers, they can be written as:

$$v_1 = x_1 + iy_1,\ v_2 = x_2 + iy_2$$

where $x_1, y_1, x_2, y_2 \in \mathbb{R}$ and i - iota: imaginary unit, which means

$$v_kv_k^* = (x_k + iy_k)(x_k - iy_k) = x_k^2 + y_k^2 = |v_k|^2$$

where $|v_k|$ argument of complex number v_k, $k \in \mathbb{N}$.

Using the above stated property:

$$(v|v) = -8|v_1|^2 + 2000|v_2|^2$$

The expression above is not always positive-definite because it contains negative number, that means it can be negative for some values of v_1 and v_2. For example, if $v_1 = 50$ and $v_2 = 1$ we get

$$(v|v) = -20000 + 2000 = -18000 < 0$$

As a result, **Positive-definiteness axiom fails**

1.2 Conclusion

The operations fails on one of the dot-product axioms, therefore the defined operation is **not a dot-product**.

2 Task 3

2.1 Notation

The day.month.year is given by 09.08.2000, therefore $a = \begin{pmatrix} \mathbf{0} & 9 & 0 & 8 \end{pmatrix}$ and $b = \begin{pmatrix} \mathbf{2} & 0 & 0 & 0 \end{pmatrix}$.

2.2 Solution

2.2.1 Cross-Correlation

- Definition:
  From Lecture Notes, Topic 3, slide 10: The deterministic cross-correlation of two signals a and b is a sequence $c_{a,b} = \begin{pmatrix} ...c_{-3} & c_{-2} & c_{-1} & \mathbf{c_0} & c_1 & c_2 & c_3... \end{pmatrix}$, such that

$$c_n = \sum_{k=-\infty}^{k=\infty} a_kb_{k-n}^* = \langle a_k, b_{k-n} \rangle_k$$

For all $q < 0$, $q > 3$ $a_q = 0$, $b_q = 0$, where $q \in \boldsymbol{Z}$, hence $c_n = \langle a_q, b_{q-n} \rangle_q = 0$

In order to compute circular convolution we need to extend sequences to infinite sequences with *finite support*, therefore we add zeros in the beginning and the end of sequences a and b:

$$a = (...\ 0\ 0\ \mathbf{9}\ 0\ 8\ 0\ 0\ ...)$$

$$b = (...\ 0\ 0\ \mathbf{2}\ 0\ 0\ 0\ 0\ 0\ ...)$$

- Calculating c_n for $-3 \leq n \leq 3$, $n \in \boldsymbol{Z}$:
  $c_{-3} = a_0 b_3 + ... = 0 \cdot 0 + ... = 0$
  $c_{-2} = a_0 b_2 + a_1 b_3 + ... = 0 \cdot 0 + 9 \cdot 0 + ... = 0$
  $c_{-1} = a_0 b_1 + a_1 b_2 + a_2 b_3 + ... = 0 \cdot 0 + 9 \cdot 0 + 0 \cdot 0 + ... = 0$
  $c_0 = a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3 + ... = 0 \cdot 2 + 9 \cdot 0 + 0 \cdot 0 + 8 \cdot 0 + ... = 0$
  $c_1 = a_0 b_{-1} + a_1 b_0 + a_2 b_1 + a_3 b_2 + ... = 0 \cdot 0 + 9 \cdot 2 + 0 \cdot 0 + 8 \cdot 0 + ... = 18$
  $c_2 = a_2 b_0 + a_3 b_1 + ... = 0 \cdot 2 + 8 \cdot 0 + ... = 0$
  $c_3 = a_3 b_0 + ... = 8 \cdot 2 + ... = 16$

2.2.2 Convolution

- Definition:
  From Lecture Notes, Topic 3, Slide 31: The convolution $(a * b)$ between signals a and b is a sequence, such that:

$$(a * b)_n = z_n = \sum_{k \in \boldsymbol{Z}} a_{n-k} b_k = \sum_{k \in \boldsymbol{Z}} a_k b_{n-k}$$

  For all $q < 0$, $q > 3$ $a_q = 0$, $b_q = 0$, where $q \in \boldsymbol{Z}$, hence $z_q = 0$

  Just like last previous sub task, we will extend sequences a and b to infinite sequences with *finite support* and add zeros at the start and end.

  It is important to note that the product of elements a_i and b_i where $i < -3$ and $i > 3$ by any other element leads to 0, Therefore they can be excluded.

- Calculating z_n for non-negative n, $n \in \boldsymbol{Z}$:
  $z_0 = \sum_{k \in \boldsymbol{Z}} a_{0-k} b_k = ... + a_0 b_0 + a_{-1} b_1 + a_{-2} b_2 + a_{-3} b_3 + ... = a_0 b_0 = 0 \cdot 2 = 0$
  $z_1 = \sum_{k \in \boldsymbol{Z}} a_{1-k} b_k = ... + a_1 b_0 + a_0 b_1 + ... = 9 \cdot 2 + 0 \cdot 0 = 18$
  $z_2 = \sum_{k \in \boldsymbol{Z}} a_{2-k} b_k = ... + a_2 b_0 + a_1 b_1 + a_0 b_2 + ... = 0 \cdot 2 + 9 \cdot 0 + 0 \cdot 0 = 0$
  $z_3 = \sum_{k \in \boldsymbol{Z}} a_{3-k} b_k = ... + a_3 b_0 + a_2 b_1 + a_1 b_2 + a_0 b_3 + ... = 8 \cdot 2 + 0 \cdot 0 + 9 \cdot 0 + 0 \cdot 0 = 16$

2.2.3 Circular Convolution

- Definition:
  From Lecture Notes, Topic 4, Slide 43: The circular convolution $(a^{(*)} b)$ between two finite (complex-valued) sequences a and b of some fixed length $m > 0$ is defined as the following sequence of the same length m:

$$(a^{(*)} b)_n = \sum_{k=0}^{k=m-1} a_k b_{(n-k) mod(m)} = \sum_{k=0}^{k=m-1} a_{(n-k) mod(m)} b_k$$

  The length of the signal is 4, i.e. $m = 4$, therefore formula of circular convolution for signals a and b is:

$$(a^{(*)} b)_n = \sum_{k=0}^{k=3} a_{(n-k)\text{mod}4} b_k = \sum_{k=0}^{k=3} b_{(n-k)\text{mod}4} a_k$$

- Calculating $(a^{(*)}b)_n$ for $0 \leq n \leq 3$:
  $(a^{(*)}b)_3 = a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0 = 0 \cdot 0 + 9 \cdot 0 + 0 \cdot 0 + 8 \cdot 2 = 16$
  $(a^{(*)}b)_2 = a_0b_2 + a_1b_1 + a_2b_0 + a_3b_3 = 0 \cdot 0 + 9 \cdot 0 + 0 \cdot 2 + 8 \cdot 0 = 0$
  $(a^{(*)}b)_1 = a_0b_1 + a_1b_0 + a_3b_2 + a_2b_3 = 0 \cdot 0 + 9 \cdot 2 + 8 \cdot 0 + 0 \cdot 0 = 18$
  $(a^{(*)}b)_0 = a_1b_3 + a_2b_2 + a_3b_1 + a_0b_0 = 9 \cdot 0 + 0 \cdot 0 + 8 \cdot 0 + 0 \cdot 2 = 0$

2.3 Conclusion

1. The cross-correlation of signals a and b is $c_{a,b} = \begin{pmatrix} ...0 & 0 & 0 & \mathbf{0} & 18 & 0 & 16... \end{pmatrix}$
2. The convolution of $(a * b)_0 = 0$, $(a * b)_1 = 18$, $(a * b)_2 = 0$, $(a * b)_3 = 16$
3. The circular convolution is as follows: $(a^{(*)}b)_0 = 0$, $(a^{(*)}b)_1 = 18$, $(a^{(*)}b)_2 = 0$, $(a^{(*)}b)_3 = 16$.

3 Task 4

3.1 Solution

Substituting numbers into formula for output signal, we get:

$$(y_n) = \left(\frac{8x_{n-1} + 9x_{n+1}}{2000}\right)$$

Exploring the properties of this system.

3.1.1 Linear

- Definition:
  From Lecture Notes, Topic 3, Slide 13: A linear system enjoys additive and scaling that together are known in engineering *as the superposition principle*: $T(ax + bu) = aT(x) + bT(u)$.

  Let us consider equation for $T(ax + bu)$:

$$T(ax + bu) = \frac{8(ax_{n-1} + bu_{n-1}) + 9(ax_{n+1} + bu_{n+1})}{2000} =$$

$$= \frac{a(8x_{n-1} + 9x_{n+1})}{2000} + \frac{b(8u_{n-1} + 9u_{n+1})}{2000} = aT(x) + bT(u)$$

  Hence we can conclude that this system is **linear**.

- Infinite matrix:

$$\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & \frac{9}{2000} & 0 & \cdots \\ \cdots & \frac{8}{2000} & 0 & \frac{9}{2000} & \cdots \\ \cdots & 0 & \frac{8}{2000} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

3.1.2 Memoryless

- Definition:
  From Lecture notes, Topic 3, Slide 15: A system T is *memoryless* if for every $k \in \mathbb{Z}$ and all input signals x and x' the following implication holds: $x_k = x'_k$ implies $(Tx)_k = (Tx')_k$.

Writing equations for $(Tx)_k$ and $(Tx')_k$:

$$(Tx)_k = \Big(\frac{8x_{k-1} + 9x_{k+1}}{2000}\Big),\ (Tx')_k = \Big(\frac{8x'_{k-1} + 9x'_{k+1}}{2000}\Big)$$

The formulas for $(Tx)_k$ and $(Tx')_k$ contain elements x_{k-1}, x'_{k-1}, x_{k+1} and x'_{k+1} . We do not have information if they are equal to each other, therefore we cannot surely say that this system is memory-less. Hence, the above mentioned system is **not memory-less**.

3.1.3 Causal

- Definition:
  From Lecture notes, Topic 3, Slide 15: A system T is called *causal* if for every $k \in \mathbb{Z}$ and all input signals x and x' , the following implication holds: $x_{(-\infty,k]} = x'_{(-\infty,k]}$ implies $(Tx)x_{(-\infty,k]} = (Tx')x_{(-\infty,k]}$.

Similar to the previous sub task(memory-less), equations for $(Tx)_k$ and $(Tx')_k$ contain elements x_{k+1} and x'_{k+1} that do not belong to $x_{(-\infty,k]}$ and $x'_{(-\infty,k]}$ respectively, therefore we cannot surely say that these elements are equal. Hence the above mentioned system is **not causal**

3.1.4 Shift-invariant

- Definition:
  From Lecture notes, Topic 3, Slide 16: A linear system T is *shift-invariant* (LSI) if for every input signal T (shifted x) is shifted Tx, i.t. if $\big((Tx)_{n-k}\big) = T(x_{n-k})$ for every input signal x and $k \in \mathbb{Z}$.

Let us consider equations for $\big((Tx)_{n-k}\big)$ and $T(x_{n-k})$:

$$\big((Tx)_{n-k}\big) = (y)_{n-k} = \frac{8x_{n-k-1} + 9x_{n-k+1}}{2000}$$

$$T(x_{n-k}) = \frac{8x_{n-k-1} + 9x_{n-k+1}}{2000}$$

As a result, it is clear that these equations are equal to each other, therefore $\big((Tx)_{n-k}\big) = T(x_{n-k})$. Hence the above mentioned system is **shift-invariant**

3.1.5 BIBO-stable

- Definition:
  From Lecture notes, Topic 3, Slide 18: A system T is called *bounded-input, bounded-output stable* (BIBO-stable) if a bounded input always produces bounded output: $Tx \in l^\infty$ fir all $x \in l^\infty$.

From the same lecture notes, according to the theorem, the LSI system is BIBO-stable iff its impulse response is absolutely summable. The impulse response of this system is:

$$(..., 0, 0, \frac{9}{2000}, 0, \frac{8}{2000}, 0, 0, ...)$$

Therefore the sum of the response can be computed:

$$\frac{9}{2000} + \frac{8}{2000} = \frac{17}{2000}$$

As a result, it is clear that the impulse response is summable. Hence the above mentioned system is **BIBO-stable**.

3.2 Conclusion

The given system is linear, shift-invariant and BIBO-stable, however, it is not memory-less and causal.

4 Task 5

4.1 Notation

For the calculations below, the signal $a = \begin{pmatrix} 0 & 9 & 0 & 8 \end{pmatrix}$

4.2 Solution

4.2.1 DTFT

- Definition:
  From Lecture notes, Topic 4, Slide 9, The Discrete-Time Fourier Transform maps each filter a to the frequency response spectrum function

$$A(e^{j\omega}) = \sum_{k \in \boldsymbol{Z}} e^{-j\omega k} a_k$$

  of real argument ω

- Computing DTFT $A(e^{j\omega})$:

$$A(e^{j\omega}) = \sum_{k \in \boldsymbol{Z}} e^{-j\omega k} a_k = ... + a_0 e^{-0j\omega} + a_1 e^{-j\omega} + a_2 e^{-2j\omega} + a_3 e^{-3j\omega} + ...$$

$$A(e^{j\omega}) = ... + 0 \cdot 1 + 9e^{-j\omega} + 0e^{-2j\omega} + 8e^{-3j\omega} + ...$$

  For all $k < 0$ or $k > 3$ $a_k = 0$, therefore their sum is zero. Hence:

$$A(e^{j\omega}) = 9e^{-j\omega} + 8e^{-3j\omega}$$

4.2.2 IDTFT

- Definition:
  From Lecture Notes, Topic 4, Slide 10: The Inverse-DTFT of a 2π-periodic function $A(e^{j\omega})$ of the real argument is the following (two-side infinite) sequence $a = \begin{pmatrix} ...a_{-3} & a_{-2} & a_{-1} & \boldsymbol{a_0} & a_1 & a_2 & a_3 \end{pmatrix}$, where

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(e^{j\omega}) e^{j\omega n} d\omega$$

  for all $n \in \boldsymbol{Z}$ Find a_n for $-3 \leq n \leq 3$:

  Let us start by calculating a_{-3}:

$$a_{-3} = \frac{1}{2\pi} \int_{-\pi}^{\pi} (9e^{-j\omega} + 8e^{-3j\omega}) e^{-3j\omega} d\omega =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (9e^{-4j\omega} + 8e^{-6j\omega}) d\omega = \frac{9}{8\pi}(e^{-4j\pi} - e^{4j\pi}) + \frac{2}{3\pi}(e^{-6j\pi} - e^{6j\pi}) =$$

$$= \frac{9}{8\pi}(cos(-4\pi) + jsin(-4\pi)) - \frac{9}{8\pi} \cdot 1 + \frac{2}{3\pi}(cos(-6\pi) + jsin(-6\pi)) - \frac{2}{3\pi} \cdot 1 =$$

$$a_{-3} = \frac{9}{8\pi} - \frac{9}{8\pi} + \frac{2}{3\pi} - \frac{2}{3\pi} = 0$$

Similarly we can calculate a_{-2}, a_{-1}, and a_0 as follows:

$$a_{-2} = -\frac{3}{2\pi} + \frac{3}{2\pi} - \frac{4}{5\pi} + \frac{4}{5\pi} = 0$$

$$a_{-1} = \frac{9}{4\pi} - \frac{9}{4\pi} + \frac{1}{\pi} - \frac{1}{\pi} = 0$$

$$a_0 = -\frac{1}{2\pi} + \frac{1}{2\pi} - \frac{4}{3\pi} + \frac{4}{3\pi} = 0$$

Now we can calculate a_1 as follows:

$$a_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(9e^{-j\omega} + 8e^{-3j\omega})e^{j\omega}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}(9 + 8e^{-2j\omega})d\omega =$$

$$= \frac{9}{2\pi}(\pi - (-\pi)) + \frac{2}{\pi}(e^{-2j\pi} - e^{2j\pi}) =$$

$$= 9 + \frac{2}{\pi}(cos(-2\pi) + jsin(-2\pi)) - \frac{2}{\pi}\cdot 1 =$$

$$a_1 = 9 + \frac{2}{\pi} - \frac{2}{\pi} = 9$$

Similarly calculating a_2 as follows:

$$a_2 = -\frac{9}{2\pi} + \frac{9}{2\pi} - \frac{4}{\pi} + \frac{4}{\pi} = 0$$

Finally calculating a_3 as follows:

$$a_3 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(9e^{-j\omega} + 8e^{-3j\omega})e^{3j\omega}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}(9e^{2j\omega} + 8)d\omega =$$

$$= \frac{9}{4\pi}(e^{2j\pi} - e^{-2j\pi}) + \frac{4}{\pi}(\pi + \pi) =$$

$$= \frac{9}{4\pi}(cos(2\pi) + jsin(2\pi)) - \frac{9}{4\pi}(cos(-2\pi) + jsin(-2\pi)) + 8 =$$

$$a_3 = \frac{9}{4\pi} - \frac{9}{4\pi} + 8 = 8$$

As a result, $a = \begin{pmatrix}...a_{-3} & a_{-2} & a_{-1} & \boldsymbol{a_0} & a_1 & a_2 & a_3...\end{pmatrix} = \begin{pmatrix}...0 & 0 & 0 & \mathbf{0} & 9 & 0 & 8...\end{pmatrix}$, that is exactly the same as the given signal $a = \begin{pmatrix}\mathbf{0} & 9 & 0 & 8\end{pmatrix}$

4.3 Conclusion

After computing DTFT and IDTFT we can validate that in this case IDTFT is the inverse for DTFT indeed. Therefore, it is clear that IDTFT is indeed the inverse of $A(e^{j\omega})$.